CHAPTER

03

Area of Bounded Regions

Learning Part

Session 1

- Sketching of Some Common Curves
- Some More Curves which Occur Frequently in Mathematics in Standard Forms
- Asymptotes
- Areas of Curves Given by Cartesian Equations

Session 2

- Area Bounded by Two or More Curves

Practice Part

- JEE Type Examples
- Chapter Exercises

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Session 1

Sketching of Some Common Curves, Some More Curves which Occur Frequently in Mathematics in Standard Forms, Asymptotes, Areas of Curves Given by Cartesian Equations

Sketching of Some Common Curves

For finding the area of a given region, we require the knowledge of some standard curves.

(i) Straight Line

Every first degree equation in x, y represents a straight line. So, the general equation of a line is $ax + by + c = 0$. To draw a straight line find the points, where it meets with the coordinate axes by putting $y = 0$ and $x = 0$ respectively in its equation.

By joining these two points we get the sketch of the line. Sometimes the equation of a line is given in the form $y = mx$. This equation represents a line passing through the origin and inclined at an angle $\tan^{-1} m$ with the positive direction of X-axis. The equation of the form $x = a$ and $y = b$ represents straight lines parallel to Y-axis and X-axis, respectively.

Region Represented by a Linear Inequality

To find the region represented by linear inequality $ax + by \le c$ and $ax + by \ge c$, we proceed as follows

(i) Convert the inequality into equality to obtain a linear equation in x, y.

(ii) Draw the straight line represented by it.

(iii) The straight line obtained in (ii) divides the XY-plane in two parts.

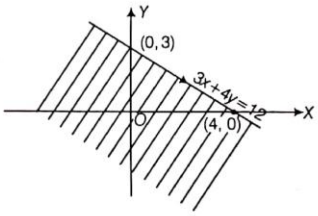
To determine the region represented by the inequality choose some convenient points; e.g. origin or some points on the coordinate axes.

If the coordinates of a point satisfy the inequality, then region containing the points is the required region, otherwise the region not containing the point is required region.

| Example 1 Mark the region represented by $3x + 4y \le 12$.

Sol. Converting the inequality into equation, we get $3x + 4y = 12$.

This line meets the coordinate axes at (4, 0) and (0, 3), respectively. Join these points to obtain straight line represented by $3x + 4y = 12$.



This straight line divides the plane in two parts. One part contains the origin and the other does not contains the origin. Clearly, (0, 0) satisfy the inequality $3x + 4y < 12$. So, the region represented by $3x + 4y < 12$ is region containing the origin as shown in the figure.

(ii) Circle

The general equation of a circle is

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

$\therefore$ The second degree equation in x, y such that coeff. of x^2 = coeff. of y^2 and there is no term containing xy; it always represents a circle. To draw a sketch of a circle, we write the equation instandard form $(x - h)^2 + (y - k)^2 = r^2$, whose centre is (h, k) and radius is r.

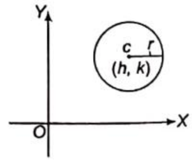


Figure 3.1

Remark

1. The inequality $(x - a)^2 + (y - b)^2 < r^2$ represents the interior of a circle.
2. The inequality $(x - a)^2 + (y - b)^2 > r^2$ represents the exterior of a circle (i.e. region lying outside the circle).

(iii) Parabola

It is the locus of points such that its distance from a fixed point is equal to its distance from a fixed straight line.

Taking the fixed straight line $x = -a, a > 0$ and fixed point $(a, 0)$, we get the equation of parabola $y^2 = 4ax$.

Steps to Sketch the Curve

(i) It passes through (0, 0).

(ii) It is symmetrical about axis of X.

(iii) No part of the curve lies on the negative side of axis of X.

(iv) Curve turns at (0, 0) which is called the vertex of the curve.

(v) The curve extends to infinity. It is not a closed curve.

(1) $y^2 = 4ax$ (Standard equation of parabola)

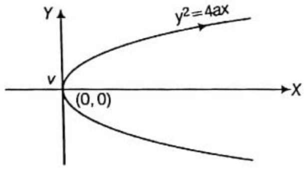


Figure 3.2

(2) $y^2 = 4a(x - h)$; where a and h are positive.

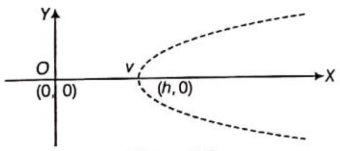


Figure 3.3

(3) $y^2 = 4a(x + h)$; where a and h are positive.

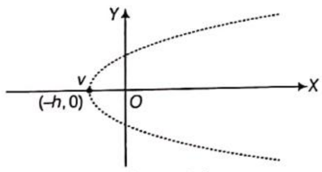


Figure 3.4

(4) $y^2 = -4ax$; $a > 0$

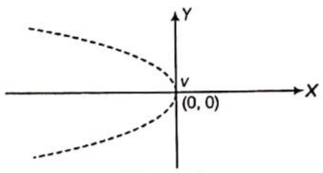


Figure 3.5

(5) $y^2 = -4a(x - h)$; $a, h > 0$

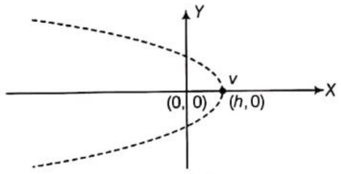


Figure 3.6

(6) $y^2 = -4a(x + h)$; $a, h > 0$

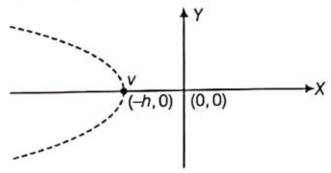


Figure 3.7

(7) $x^2 = 4ay$; $a > 0$

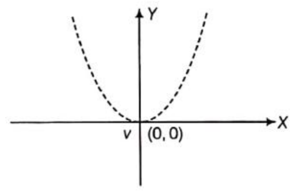


Figure 3.8

(8) $x^2 = 4a(y + k)$; $a > 0, k > 0$

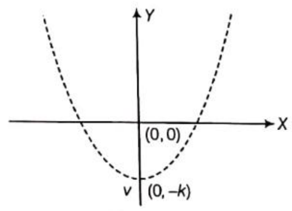


Figure 3.9

(9) $x^2 = 4a(y - k)$; $a, k > 0$

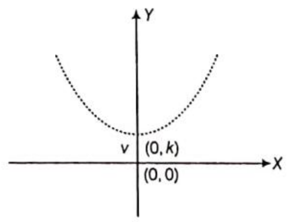


Figure 3.10

(10) $x^2 = -4ay;\ a > 0$

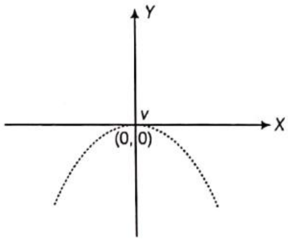


Figure 3.11

(11) $x^2 = -4a(y + k);\ a, k > 0$

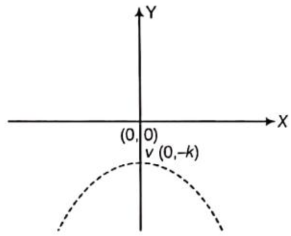


Figure 3.12

(12) $x^2 = -4a(y - k);\ a, k > 0$

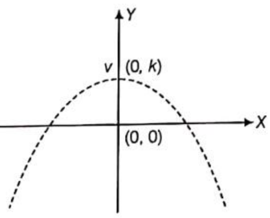


Figure 3.13

(iv) Ellipse

Basics of Ellipse

Definition 1. An ellipse is the locus of a moving point such that the ratio of its distance from a fixed point to its distance from a fixed line is a constant less than unity. This constant is termed the eccentricity of the ellipse. The fixed point is the focus while the fixed line is the directrix.

The symmetrical nature of the ellipse ensures that there will be two foci and two directrices.

Definition 2. An ellipse is the locus of a moving point such that the sum of the its distances from two fixed points is constant. The two fixed points are the two foci of the ellipse. To plot the ellipse, we can use the peg-and-thread method described earlier.

Stantard Equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

	If $a > b$	**If $a < b$**
Vertices	$(a, 0)$ and $(-a, 0)$	$(0, b)$ and $(0, -b)$
Foci	$(ae, 0)$ and $(-ae, 0)$	$(0, be)$ and $(0, -be)$
Major axis	$2a$ (along x-axis)	$2b$ (along y-axis)
Minor axis	$2b$ (along y-axis)	$2a$ (along x-axis)
Directrices	$x = \frac{a}{e}$ and $x = -\frac{a}{e}$	$y = \frac{b}{e}$ and $y = -\frac{b}{e}$
Eccentricity e	$\sqrt{1 - \frac{b^2}{a^2}}$	$\sqrt{1 - \frac{a^2}{b^2}}$
Latus-rectum	$\frac{2b^2}{a}$	$\frac{2a^2}{b}$
Focal distances of (x, y)	$a \pm ex$	$b \pm ey$

And lastly, if the equation of the ellipse is

$$\frac{(x - \alpha)^2}{a^2} + \frac{(y - \beta)^2}{b^2} = 1$$

instead of the usual standard form, we can use the transformation $X \to x - \alpha$ and $Y \to y - \beta$ (basically a translation of the axes so the axes so that the origin of the new system coincides with (α, β). The equation then becomes

$$\frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1$$

We can now work on this form, use all the standard formulae that we'd like to and obtain whatever it is that we wish to obtain. The final result (in the x-y system) is obtained using the reverse transformation $x \to X + \alpha$ and $y \to Y + \beta$.

(1) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1;\ a > 0, b > 0$ (Standard equation of the ellipse)

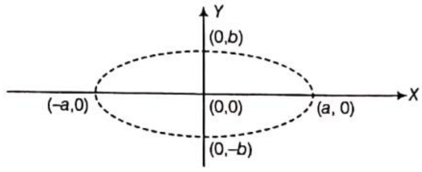


Figure 3.14

(2) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$; $b>a>0$ (Conjugate ellipse)

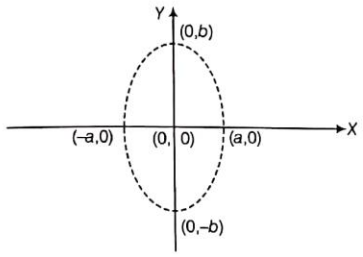


Figure 3.15

(3) $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$; $a>0, b>0$ and $a>b$

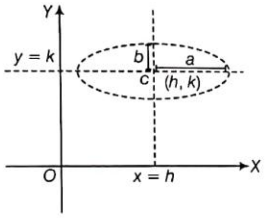


Figure 3.16

(4) $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$, where $a<b$

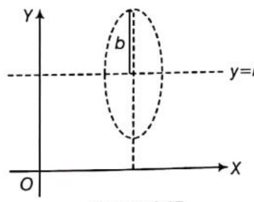


Figure 3.17

(v) Hyperbola

A hyperbola is the locus of a moving point such that the **difference** of its distances from two fixed points is always constant. The two fixed point are called the foci of the hyperbola. Constrast this with the definition of the ellipse where we had the sum of focal distances (instead of difference) as constant. As in the case of the ellipse, we have

Focal distance of $P(x, y)$

$$d_1 = e(PF) = e\left(x-\frac{a}{e}\right) = ex - a$$

$$d_2 = e(PF') = e\left(x+\frac{a}{e}\right) = ex + a$$

Latus-Rectum

The chord(s) of the hyperbola passing through the focus F (or F') and perpendicular to the transverse axis. The length of the latus-rectum can be evaluated by substituting $x=\pm ae$ in the equation for the hyperbola :

$$\frac{x^2}{a^2}-\frac{y^2}{a^2(e^2-1)}=1$$

$$\Rightarrow \quad e^2-\frac{y^2}{a^2(e^2-1)}=1$$

$$\Rightarrow \quad y^2 = a^2(e^2-1)^2 = \frac{a^2\cdot b^4}{a^4}=\frac{b^4}{a^2}$$

$$\Rightarrow \quad y=\pm\frac{b^2}{a}.$$

Thus, the length of the latus-rectum is $\frac{2b^2}{a}$.

We discussed in the unit of Ellipose that an ellipse with centre at (α, β) instead of the origin and the major and minor axis parallel to the coordinate axes will have the equation

$$\frac{(x-\alpha)^2}{a^2}+\frac{(y-\beta)}{b^2}=1 \text{ or } \frac{X^2}{a^2}+\frac{Y^2}{b^2}=1$$

where $X\to x-\alpha$ and $Y\to y-\beta$.

The same holds true for a hyperbola. Any hyperbola with centre at (α, β) and the transverse and conjugate axis parallel to the coordinate axes will have the form

$$\frac{(x-\alpha)^2}{a^2}-\frac{(y-\beta)^2}{b^2}=1$$

or

$$\frac{X^2}{a^2}-\frac{Y^2}{b^2}=1$$

where $X\to x-\alpha$ and $Y\to y-\beta$.

We can, using the definition of a hyperbola, write the equation of any hyperbola with an arbitrary focus and directrix, but we will rarely have the occassion to use it.

(1) $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ (Standard equation of hyperbola)

Figure 3.18

(2) $\dfrac{y^2}{b^2} - \dfrac{x^2}{a^2} = 1$ (Conjugate hyperbola)

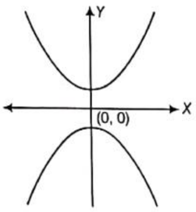


Figure 3.19

(3) $xy = c^2$ (Rectangular hyperbola)

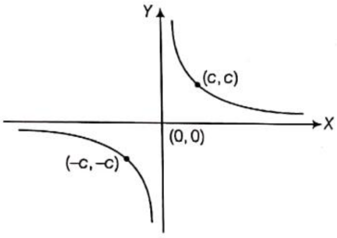


Figure 3.20

(4) $\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = 1;\ a > b > 0$

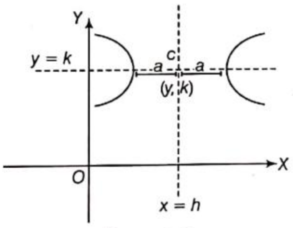


Figure 3.21

(5) $\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = -1$

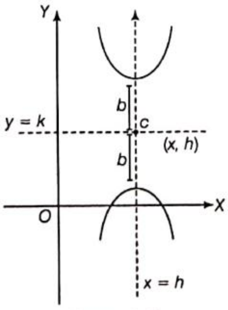


Figure 3.22

Some More Curves which Occur Frequently in Mathematics in Standard Forms

1.

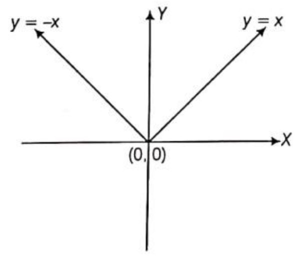


Figure 3.23

Modulus function, $y = |x|$

2.

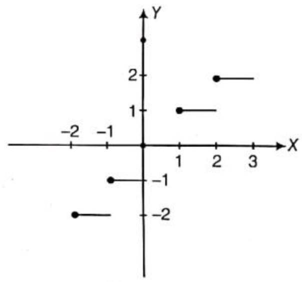


Figure 3.24

Greatest integer function $y = [x]$

3.

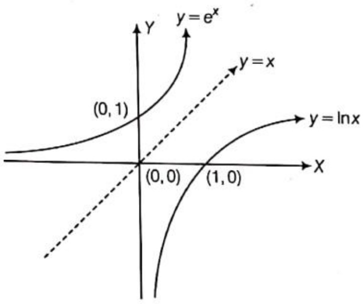


Figure 3.25

4.

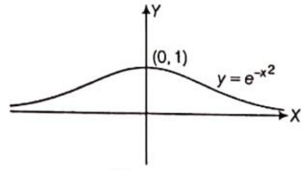


Figure 3.26

5.

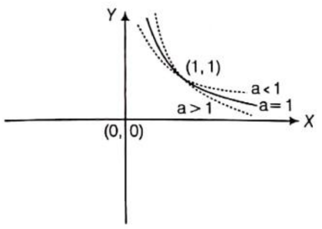


Figure 3.27

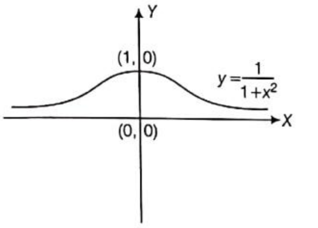


Figure 3.28

$$y=\frac{1}{x^{\alpha}}, \alpha>0$$

7. The Astroid

Its cartesian equation is $x^{2/3}+y^{2/3}=a^{2/3}$

Its parametric equation is $x=a\cos^3 t, y=a\sin^3 t$

and it could be plotted as

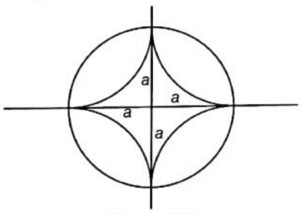


Figure 3.29

Curve Sketching

For the evaluation of area of bounded regions it is very essential to know the rough sketch of the curves. The following points are very useful to draw a rough sketch of a curve.

(i) Symmetry

(a) **Symmetry about X-axis** If all powers of y in the equation of the given curve are even, then it is symmetric about X-axis, i.e. the shape of the curve above X-axis is exactly identical to its shape below X-axis. **e.g.** $y^2=4ax$ is symmetric about X-axis.

(b) **Symmetry about Y-axis** If all powers of x in the equation of the given curve are even, then it is symmetric about Y-axis. **e.g.** $x^2=4ay$ is symmetric about Y-axis.

(c) **Symmetry in opposite quadrants** If by putting $-x$ for x and $-y$ for y, the equation of curve remains same, then it is symmetric in opposite quadrants.

e.g. $xy=c^2$, $x^2+y^2=a^2$ are symmetric in opposite quadrants.

(d) **Symmetric about the line $y=x$** If the equation of a given curve remains unaltered by interchanging x and y, then it is symmetric about the line $y=x$ which passes through the origin and makes an angle of 45° with positive direction of X-axis.

(ii) Origin and Tangents at the Origin

See whether the curve passes through origin or not. If the point (0, 0) satisfies the equation of the curve, then it passes through the origin and in such a case to find the equations of the tangents at the origin, equate the lowest degree term to zero. **e.g.** $y^2=4ax$ passes through the origin. The lowest degree term in this equation is $4ax$. Equating $4ax$ to zero, we get $x=0$.

So, $x=0$ i.e. Y-axis is tangent at the origin to $y^2=4ax$.

(iii) Points of Intersection of Curve with the Coordinate Axes

By putting $y=0$ in the equation of the given curve, find points where the curve crosses the X-axis. Similarly, by putting $x=0$ in the equation of the given curve we can find points where the curve crosses the Y-axis.

e.g. To find the points where the curve $xy^2=4a^2(2a-x)$ meets X-axis, we put $y=0$ in the equation which gives $4a^2(2a-x)=0$ or $x=2a$. So the curve $xy^2=4a^2(2a-x)$, meets X-axis at $(2a, 0)$. This curve does not intersect Y-axis, because by putting $x=0$ in the equation of the given curve get an absurd result.

(iv) Regions where the Curve Does Not Exist

Determine the regions in which the curve does not exist. For this, find the value of y in terms of x from the equation of the curve and find the value of x for which y is imaginary. Similarly, find the value of x in terms of y and determine the values of y for which x is imaginary. The curve does not exist for these values of x and y.

e.g. The values of y obtained from $y^2 = 4ax$ are imaginary for negative values of x. So, the curve does not exist on the left side of Y-axis. Similarly, the curve $a^2y^2 = x^2(a-x)$ does not exist for $x > a$ as the values of y are imaginary for $x > a$.

(v) Special Points

Find the points at which $\frac{dy}{dx} = 0$. At these points the tangent to the curve is parallel to X-axis.

Find the points at which $\frac{dx}{dy} = 0$. At these points the tangents to the curve is parallel to Y-axis.

(vi) Sign of dy/dx and Points of Maxima and Minima

Find the interval in which $\frac{dy}{dx} > 0$. In this interval, the function is monotonically increasing, find the interval in which $\frac{dy}{dx} < 0$. In this interval, the function is monotonically decreasing.

Put $\frac{dy}{dx} = 0$ and check the sign of $\frac{d^2y}{dx^2}$ at the points so obtained to find the points of maxima and minima.

Keeping the above facts in mind and plotting some points on the curve one can easily have a rough sketch of the curve. Following examples will clear the procedure.

Example 2 Sketch the curve $y = x^3$.

Sol. We observe the following points about the given curve

(i) The equation of the curve remains unchanged, if x is replaced by $-x$ and y by $-y$. So, it is symmetric in opposite quadrants. Consequently, the shape of the curve is similar in the first and the third quadrants.

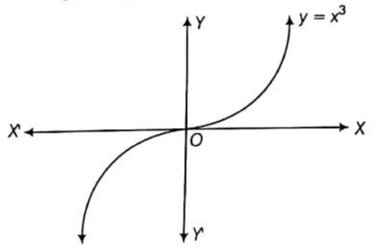
(ii) The curve passes through origin. Equating lowest degree term y to zero, we get $y = 0$ i.e. X-axis is the tangent at the origin.

(iii) Putting $y = 0$ in the equation of the curve, we get $x = 0$. Similarly, when $x = 0$, we get $y = 0$. So, the curve meets the coordinate axes at (0, 0) only.

(iv) $y = x^3 \Rightarrow \frac{dy}{dx} = 3x^2, \frac{d^2y}{dx^2} = 6x$ and $\frac{d^3y}{dx^3} = 6$

Clearly, $\frac{dy}{dx} = 0 = \frac{d^2y}{dx^2}$ at the origin but $\frac{d^3y}{dx^3} \neq 0$.

So, the origin is a point of inflexion.



(v) As x increases from 0 to ∞, y also increases from 0 to ∞. Keeping all the above points in mind, we obtain a sketch of the curve as shown in figure.

Example 3 Sketch the curve $y = x^3 - 4x$.

Sol. We note the following points about the curve

(i) The equation of the curve remains same, if x is replaced by $(-x)$ and y by $(-y)$, so it is symmetric in opposite quadrants.

Consequently, the curve in the first quadrant is identical to the curve in third quadrant and the curve in second quadrant is similar to the curve in fourth quadrant.

(ii) The curve passes through the origin. Equating the lowest degree term $y + 4x$ to zero, we get $y + 4x = 0$ or $y = -4x$. So, $y = -4x$ is tangent to the curve at the origin.

(iii) Putting $y = 0$ in the equation of the curve, we obtain $x^3 - 4x = 0 \Rightarrow x = 0, \pm 2$. So, the curve meets X-axis at (0, 0), (2, 0), (−2, 0).

Putting $x = 0$ in the equation of the curve, we get $y = 0$. So, the curve meets Y-axis at (0, 0) only.

(iv) $y = x^3 - 4x \Rightarrow \frac{dy}{dx} = 3x^2 - 4$

Now, $\frac{dy}{dx} > 0 \Rightarrow 3x^2 - 4 > 0$

$\Rightarrow \left(x - \frac{2}{\sqrt{3}}\right)\left(x + \frac{2}{\sqrt{3}}\right) > 0$

$\Rightarrow x < -\frac{2}{\sqrt{3}}$ or $x > \frac{2}{\sqrt{3}}$ (using number line rule)

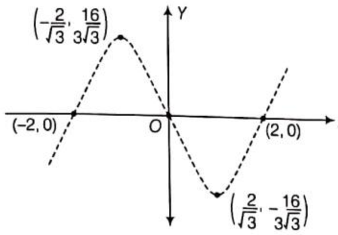
and $\frac{dy}{dx} < 0 \Rightarrow -\frac{2}{\sqrt{3}} < x < \frac{2}{\sqrt{3}}$

So, the curve is decreasing in the interval $(-2/\sqrt{3}, 2/\sqrt{3})$ and increasing for $x > \frac{2}{\sqrt{3}}$ or $x < -\frac{2}{\sqrt{3}}$.

$x = -\frac{2}{\sqrt{3}}$ is a point of local maximum and $x = \frac{2}{\sqrt{3}}$ is point of local minimum.

When $x = \frac{2}{\sqrt{3}}$, then $y = -\frac{16}{3\sqrt{3}}$

When $x = -\frac{2}{\sqrt{3}}$, then $y = \frac{16}{3\sqrt{3}}$



Keeping above points in mind, we sketch the curve as shown in figure.

| Example 4 Sketch the curve $y = (x-1)(x-2)(x-3)$.

Sol. We note the following points about the given curve

(i) The curve does not have any type of symmetry about the coordinate axes and also in opposite quadrants.

(ii) The curve does not pass through the origin.

(iii) Putting $y = 0$ in the equation of the curve, we get $(x-1)(x-2)(x-3) = 0 \Rightarrow x = 1, 2, 3$. So, the curve meets X-axis at (1, 0), (2, 0) and (3, 0).

Putting $x = 0$ in the equation of the curve, we get $y = -6$. So, the curve crosses Y-axis at $(0, -6)$.

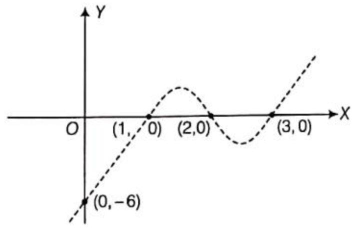
We observe that

$x < 1 \Rightarrow y < 0$

$1 < x < 2 \Rightarrow y > 0$

$2 < x < 3 \Rightarrow y < 0$

and $x > 3 \Rightarrow y > 0$



Clearly, y decreases as x decreases for all $x < 1$ and y increases as x increases for $x > 3$.

Keeping all the above points in mind, we sketch the curve as shown in figure.

| Example 5 Sketch the graph for $y = x^2 - x$.

Sol. We note the following points about the curve

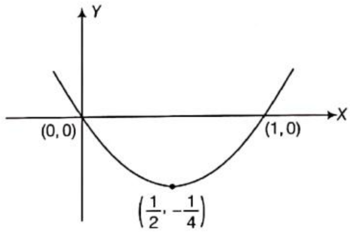
(i) The curve does not have any kind of symmetry.

(ii) The curve passes through the origin and the tangent at the origin is obtained by equating the lowest degree term to zero.

The lowest degree term is $x + y$. Equating it to zero, we get $x + y = 0$ as the equation of tangent at the origin.

(iii) Putting $y = 0$ in the equation of curve, we get $x^2 - x = 0 \Rightarrow x = 0, 1$. So, the curve crosses X-axis at (0, 0) and (1, 0).

Putting $x = 0$ in the equation of the curve, we obtain $y = 0$. So, the curve meets Y-axis at (0, 0) only.



(iv) $y = x^2 - x \Rightarrow \frac{dy}{dx} = 2x - 1$ and $\frac{d^2y}{dx^2} = 2$

Now, $\frac{dy}{dx} = 0 \Rightarrow x = \frac{1}{2}$

At $x = \frac{1}{2}, \frac{d^2y}{dx^2} > 0$

So, $x = \frac{1}{2}$ is point of local minima.

(v) $\frac{dy}{dx} > 0 \Rightarrow 2x - 1 > 0 \Rightarrow x > \frac{1}{2}$

So, the curve increases for all $x > \frac{1}{2}$ and decreases for all $x < \frac{1}{2}$. Keeping above points in mind, we sketch the curve as shown in figure.

| Example 6 Sketch the curve $y = \sin 2x$.

Sol. We note the following points about the curve

(i) The equation of the curve remains unchanged, if x is replaced by $(-x)$ and y by $(-y)$, so it is symmetric in opposite quadrants. Consequently, the shape of the curve is similar in opposite quadrants.

(ii) The curve passes through origin.

(iii) Putting $x = 0$ in the equation of the curve, we get $y = 0$. So, the curve crosses the Y-axis at (0, 0) only.

Putting $y = 0$ in the equation of the curve, we get

$$\sin 2x = 0 \Rightarrow 2x = n\pi, \; n \in Z$$

$$\Rightarrow \quad x = \frac{n\pi}{2}, n \in Z$$

So, the curve cuts the X-axis at the points

$\ldots, (-\pi, 0), (-\pi/2, 0), (0, 0), (\pi/2, 0), (\pi, 0), \ldots$

(iv) $y = \sin 2x \Rightarrow \frac{dy}{dx} = 2\cos 2x$ and $\frac{d^2y}{dx^2} = -4\sin 2x$

Now, $\frac{dy}{dx} = 0 \Rightarrow \cos 2x = 0$

$\Rightarrow \quad 2x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \dots$

$\Rightarrow \quad x = \pm\frac{\pi}{4}, \pm\frac{3\pi}{4}, \dots$

Clearly, $\frac{d^2y}{dx^2} < 0$ at $x = \frac{\pi}{4}, \frac{5\pi}{4}, \dots$

and at $x = -\frac{3\pi}{4}, -\frac{7\pi}{4}, \dots$

and $\frac{d^2y}{dx^2} > 0$ at $x = \frac{3\pi}{4}, \frac{7\pi}{4}, \dots$

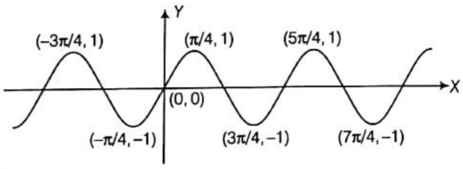
and at $x = -\frac{\pi}{4}, -\frac{5\pi}{4}, \dots$

So, the points $x = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}, \dots$

and $x = -\frac{3\pi}{4}, -\frac{7\pi}{4}, \dots$ are points of local maximum and local maximum values at these points are 1.

Similarly, $x = -\frac{\pi}{4}, -\frac{5\pi}{4}, \dots$ and $x = \frac{3\pi}{4}, \frac{7\pi}{4}, \dots$ are points of local minimum and local minimum value at these points is (-1).

(v) $\sin 2(x + \pi) = \sin 2x$ for all x. So, the periodicity of the function is π. This means that the pattern of the curve repeats at intervals of length π.



Thus, keeping in mind, we sketch the curve as shown in figure.

| Example 7 Sketch the curve $y = \sin^2 x$.

Sol. We note the following points about the curve

(i) The equation of the curve remains same, if x is replaced by $(-x)$. So, the curve is symmetric about Y-axis, i.e. the curve on the left side of Y-axis is identical to the curve on its right side.

(ii) The curve meets the coordinate axes at the same points where $y = \sin x$ meets them.

(iii) $y = \sin^2 x \Rightarrow \frac{dy}{dx} = \sin 2x$ and $\frac{d^2y}{dx^2} = 2\cos 2x$

Now, $\frac{dy}{dx} = 0 \Rightarrow \sin 2x = 0 \Rightarrow 2x = n\pi, n \in Z$

$\Rightarrow x = \frac{n\pi}{2}, \ n \in Z \Rightarrow x = \pm\pi/2, \pm\pi, \pm 3\pi/2, \pm 2\pi, \dots$

Clearly, $\frac{d^2y}{dx^2} < 0$ at $x = \pm\pi/2, \pm 3\pi/2, \pm 5\pi/2, \dots$

and $\frac{d^2y}{dx^2} > 0$ at $x = \pm\pi, \pm 2\pi, \pm 3\pi \dots$

So, $x = \pm\pi/2, \pm 3\pi/2, \pm 5\pi/2, \dots$ are the points of local maximum and local maximum value at these points is 1. Points $x = \pm\pi, \pm 2\pi, \pm 3\pi, \dots$ are points of local minimum and the local minimum value at these points is 0.

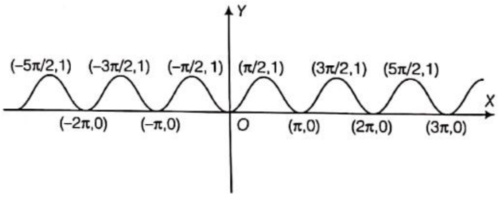
(iv) $y = \sin^2 x \Rightarrow \frac{dy}{dx} = \sin 2x$

Clearly, $\frac{dy}{dx} > 0$ when $0 < x < \frac{\pi}{2}$

and $\frac{dy}{dx} < 0$ when $\frac{\pi}{2} < x < \pi$.

So, the given curve is increasing in the interval $[0, \pi/2]$ and decreasing in $[\pi/2, \pi]$.

(v) $\sin^2(\pi + x) = \sin^2 x$ for all x. So, the periodicity of the function is π. This means that the shape of the curve repeats at the interval of length π.



Keeping the above facts in mind, we sketch the curve as shown in figure.

Asymptotes

The straight line AB is called the asymptote of curve $y = f(x)$, if the distance MK from M a point on the curve $y = f(x)$ to the straight line AB tends to zero as M recedes infinity.

In other words, the straight line AB meets the curve $y = f(x)$ at infinity (K is a point on AB). Thus,

1. If $f(x) \to \pm\infty$ for $x \to a$, then the straight line $x = a$ is the asymptote of the curve $y = f(x)$.

2. If in the right hand member of the equation of the curve $y = f(x)$ it is possible to single out a linear part so that the remaining part tends to zero as $x \to \pm\infty$, i.e. if $y = f(x) = Kx + b + g(x)$ and $g(x) \to 0$ for $x \to \pm\infty$, then the straight line $y = Kx + b$ is the asymptote of the curve.

3. If there exist finite limits $\lim_{x \to \pm\infty} \frac{f(x)}{x} = K$ and $\lim_{x \to \pm\infty} [f(x) - Kx] = b$, then the straight line $y = Kx + b$ is the asymptote of the curve.

Methods to Sketch Curves

While constructing the graphs of functions, it is expedient to follow the procedure given below

(1) Find the domain of definition of the function.

(2) Determine the odd-even nature of the function.

(3) Find the period of the function if its periodic.

(4) Find the asymptotes of the function.

(5) Check the behaviour of the function for $x \to 0\pm$

(6) Find the values of x, if possible for which $f(x) \to 0$.

(7) The interval of increase and decrease of the function in its range. Hence, determine the greatest and the least values of the function if any.

Remark

(5), (6) and (7) gives the points where the function cuts the coordinate axes.

| Example 8 Construct the graph for $f(x) = \frac{x^2 - 1}{x^2 + 1}$.

Sol. Here, $f(x) = \frac{x^2 - 1}{x^2 + 1} = 1 - \frac{2}{x^2 + 1}$

(1) The function $f(x)$ is well defined for all real x.

$\Rightarrow$ Domain of $f(x) \in R$.

(2) $f(-x) = f(x)$, so it is an even function.

(3) Since, algebraic $\to$ non-periodic function.

$f(x) \to 1$ for $x \to \pm\infty$

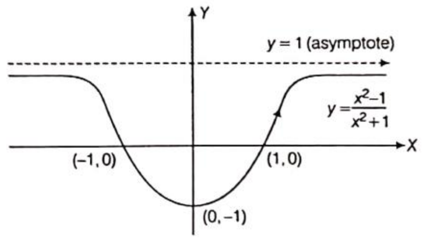
and $f(x) \to -1$ for $x \to 0\pm$

It may be observed that $f(x) < 1$ for any $x \in R$ and consequently its graph lies below the line $y = 1$ which is asymptote to the graph of the given function.

Again, $\frac{2}{x^2 + 1}$ decreases for $(0, \infty)$ and increases for $(-\infty, 0)$, thus $f(x)$ increases for $(0, \infty)$ and decreases for $(-\infty, 0)$ in its range.

(4) The greatest value $\to 1$ for $x \to \pm\infty$ and the least value is -1 for $x = 0$.

Thus, its graph is as shown in figure



| Example 9 Construct the graph for $f(x) = x + \frac{1}{x}$.

Sol. The function is defined for all x except for $x = 0$.

It is an odd function for $x \neq 0$.

It is not a periodic function.

For $x \to 0+$, $f(x) \to +\infty$; for $x \to 0-$, $f(x) \to -\infty$

For $x \to -\infty$, $f(x) \to -\infty$; for $x \to \infty$, $f(x) \to \infty$

$\therefore \quad \lim_{x \to \pm\infty} (f(x) - x) = 0$

$\therefore$ The straight lines $x = 0$ and $y = x$ are the asymptotes of the graph of the given function.

Now, consider $f(x_2) - f(x_1)$ (for $x_2 > x_1$)

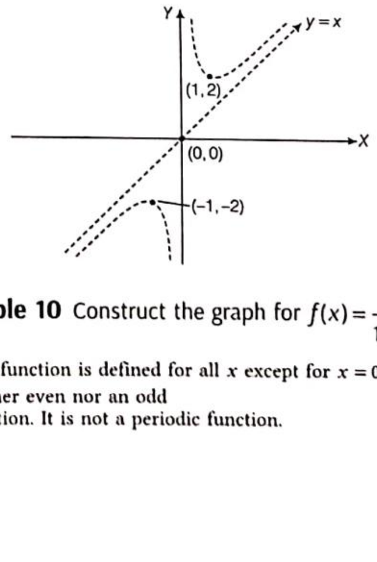
$$= (x_2 - x_1) + \frac{1}{x_2} - \frac{1}{x_1}$$

$$= (x_2 - x_1)\left[1 - \frac{1}{x_1 x_2}\right] < 0 \text{ for } x_1\ x_2 \in (0, 1]$$

and it is > 0 for $x_1\ x_2 \in [1, \infty)$.

Thus, $f(x)$ increases for $x \in [1, \infty)$ and decreases for $x \in (0, 1]$.

Thus, the least value of the function is at $x = 1$ which is $f(1) = 2$. Thus, its graph can be drawn as



| Example 10 Construct the graph for $f(x) = \frac{1}{1 + e^{1/x}}$.

Sol. The function is defined for all x except for $x = 0$. It is neither even nor an odd function. It is not a periodic function.

For $x \to 0+$ $f(x) \to 0$; for $x \to 0-$, $f(x) \to 1$

For $x \to \infty$ $f(x) \to \frac{1}{2}$; for $x \to -\infty$, $f(x) \to \frac{1}{2}$

$\therefore$ $\lim_{x \to \pm\infty} f(x) = \frac{1}{2}$

$\therefore$ The straight line $y = \frac{1}{2}$ is asymptote of the graph of the given function.

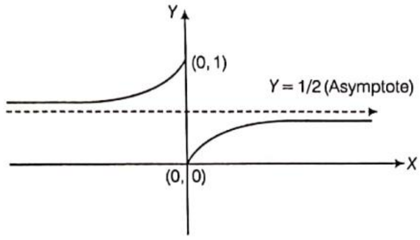
As x increases from $(0, \infty)$, $\frac{1}{x}$ decreases from $(0, \infty)$ and $e^{1/x}$ decreases from $(0, \infty)$. Thus, $(1 + e^{1/x})$ decreases from $(2, \infty)$.

$\therefore$ $f(x)$ increases from $\left(0, \frac{1}{2}\right)$ for $x \in (0, \infty)$.

Similarly, $f(x)$ increases from $(1/2, 1)$ for $x \to (-\infty, 0)$.

i.e. $f(x)$ is an increasing function except for $x = 0$.

Thus, its graph can be drawn as shown in figure



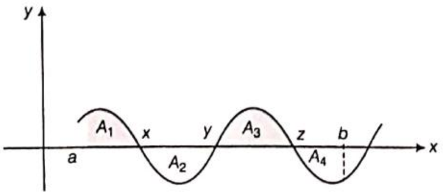
Areas of Curves

(1) Suppose that $f(x) < 0$ on some interval $[a, b]$. Then, the area under the curve $y = f(x)$ from $x = a$ to $x = b$ will be negative in sign, i.e.

$$\int_a^b f(x)\,dx \le 0$$

This is obvious once you consider how the definite integral was arrived at in the first place; as a limit of the sum of the n rectangles ($n \to \infty$). Thus, if $f(x) < 0$ in some interval then the area of the rectangles in that interval will also be negative.

This property means that for example, if $f(x)$ has the following form



then $\int_a^b f(x)\,dx$ will equal $A_1 - A_2 + A_3 - A_4$ and not $A_1 + A_2 + A_3 + A_4$.

If we need to evaluate $A_1 + A_2 + A_3 + A_4$ (the magnitude of the bounded area), we will have to calculate

$$\int_a^x f(x)\,dx + \left|\int_x^y f(x)\,dx\right| \int_y^z f(x)\,dx + \left|\int_z^b f(x)\,dx\right|$$

From this, it should also be obvious that

$$\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$$

(2) The area under the curve $y = f(x)$ from $x = a$ to $x = b$ is equal in magnitude but opposite in sign to the area under the same curve from $x = b$ to $x = a$, i.e.

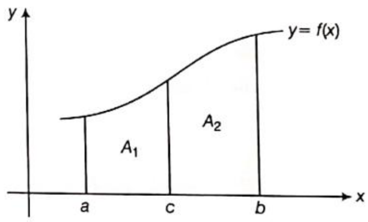
$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

This property is obvious if you consider the Newton-Leibnitz formula. If $g(x)$ is the anti-derivative of $x(f)$, then $\int_a^b f(x)\,dx$ is $g(b) - g(a)$ while $\int_a^b f(x)\,dx$ is $g(a) - g(b)$

(3) The area under the curve $y = f(x)$ from $x = a$ to $x = b$ can be written as the sum of the area under the curve from $x = a$ to $x = c$ and from $x = c$ to $x = b$, that is

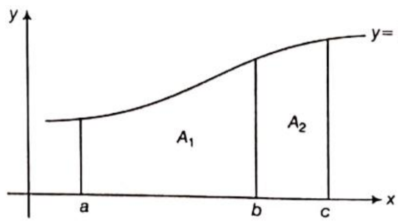
$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Let us consider an example of this. Let $c \in (a, b)$



It is clear that the area under the curve from $x = a$ to $x = b$, A is $A_1 + A_2$.

Note that c need not lie between a and b for this relation to hold true. Suppose that $c > b$.



Observe that $A = \int_a^b f(x)\,dx = (A + A_1) - A_1$

$$= \int_a^c f(x)\,dx - \int_b^c f(x)\,dx$$

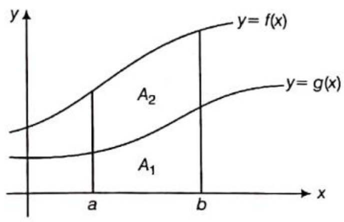
$$= \int_a^c f(x)\,dx + \int_a^b f(x)\,dx$$

Analytically, this relation can be proved easily using the Newton Leibnitz's formula.

(4) Let $f(x) > g(x)$ on the interval $[a, b]$. Then,

$$\int_a^b f(x)\,dx > \int_a^b g(x)\,dx$$

This is because the curve of $f(x)$ lies above the curve of $g(x)$, or equivalently, the curve of $f(x) - g(x)$ lies above the x-axis for $[a, b]$



This is an example where $f(x) > g(x) > 0$.

$$\int_a^b f(x)\,dx = A_1 + A_2$$

while $\int_a^b g(x)\,dx = A_1$

Similarly, if $f(x) < g(x)$ on the interval $[a, b]$, then

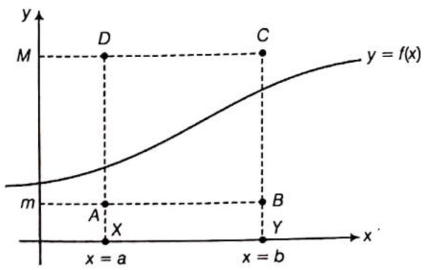
$$\int_a^b f(x)\,dx < \int_a^b g(x)\,dx$$

(5) For the interval $[a, b]$, suppose $m < f(x) < M$.

That is, m is a lower-bound for $f(x)$ while M is an upper bound.

Then $m(b-a) < \int_a^b f(x)\,dx < M(b-a)$

This is obvious once we consider the figure below :



Observe that area (rect $AXYB$) $< \int_a^b f(x)\,dx <$ area (rect $DXYC$)

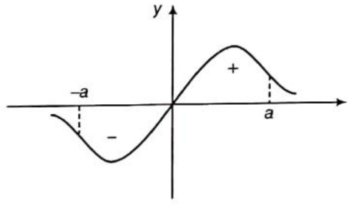
(6) Let us consider the integral of $f_1(x) + f_2(x)$ from $x = a$ to $x = b$. To evaluate the area under $f_1(x) + f_2(x)$, we can separately evaluate the area under $f_1(x)$ and the area under $f_2(x)$ and add the two area (algebraically).

Thus, $\int_a^b (f_1(x) + f_2(x))\,dx = \int_a^b f_1(x)\,dx + \int_a^b f_2(x)\,dx$

Now consider the integral of $kf(x)$ from $x = a$ to $x = b$. To evaluate the area under $kf(x)$, we can first evaluate the area under $f(x)$ and then multiply it by k, that is :

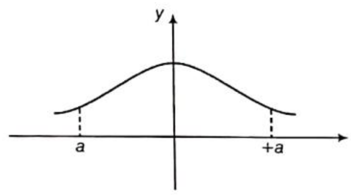
$$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$$

(7) Consider the odd function $f(x)$, i.e. $f(x) = -f(-x)$. This measn that the graph of $f(x)$ is symmetric about the origin.

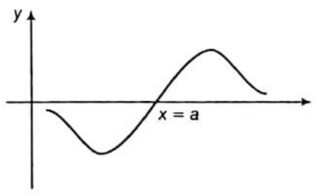


From the figure, it should be obvious that $\int_{-a}^a f(x)\,dx = 0$, because the area on the left side and that on the right algebraically add to 0.

Similarly, if $f(x)$ was even, i.e. $f(x) = f(-x)$



$\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx$ because the graph is symmetrical about the y-axis. If you recall the discussion in the unit on functions, a function can also be even or odd about any arbitrary point $x = a$. Let us suppose that $f(x)$ is odd about $x = a$, i.e. $f(x) = -f(2a - x)$

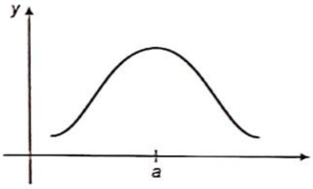


The points x and $2a - x$ lie equidistant from $x = a$ at either sides of it.

Suppose for example, that we need to calculate $\int_a^{2a} f(x)\,dx$.

It is obvious that this will be 0, since we are considering equal variation on either side of $x = a$, the area from $x = 0$ to $x = a$ and the area from $x = a$ to $x = 2a$ will add algebraically to 0.

Similarly, if $f(x)$ is even about $x = a$, i.e. $f(x) = f(2a - x)$

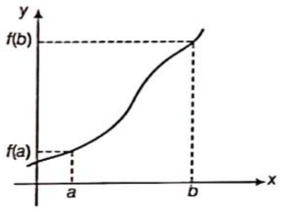


then we have, for example

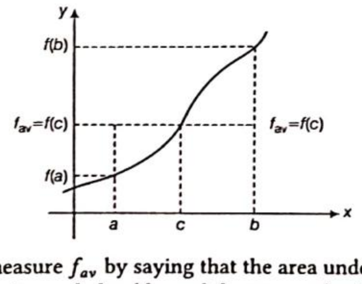
$$\int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx$$

From the discussion, you will get a general idea as to how to approach such issues regarding even/odd functions.

(8) Let us consider a function $f(x)$ on $[a, b]$



We want to somehow define the "average" value that $f(x)$ takes on the interval $[a, b]$. What would be an appropriate way to define such an average? Let f_{av} be the average value that we are seeking. Let it be such that it is obtained at some $x = c \in [a, b]$



We can measure f_{av} by saying that the area under $f(x)$ from $x = a$ to $x = b$ should equal the area under the average value from $x = a$ to $x = b$. This seems to be the only logical way to define the average (and this is how it is actually defined!).

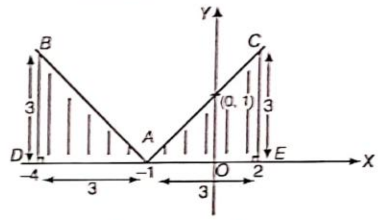
Thus $f_{av}(b - a) = \int_a^b f(x)\,dx \Rightarrow f_{av} = \frac{1}{b-a}\int_a^b f(x)\,dx$

This value is attained for at least one $c \in (a, b)$ (under the constraint that f is continuous, of course).

| Example 11 Sketch the graph $y = |x + 1|$. Evaluate $\int_{-4}^{2} |x+1|\,dx$. What does the value of this integral represents on the graph.

Sol. Here, $y = |x + 1| = \begin{cases} (x+1), & \text{if } x \ge -1 \\ -(x+1), & \text{if } x \le -1 \end{cases}$

which can be shown as



$$\therefore \int_{-4}^{2} |x+1|\,dx = \int_{-4}^{-1} |x+1|\,dx + \int_{-1}^{2} |x+1|\,dx$$

$$= \int_{-4}^{-1} -(x+1)\,dx + \int_{-1}^{2} |x+1|\,dx + \int_{-1}^{2} (x+1)\,dx$$

$$= -\left[\frac{x^2}{2} + x\right]_{-4}^{-1} + \left[\frac{x^2}{2} + x\right]_{-1}^{2} = 9$$

Representation of the value 9 of integral on graph.

$\therefore \int_{-4}^{2} |x+1|\,dx = 9$ represents the area bounded by the curve $y = |x + 1|$. X-axis and the lines $x = -4$ and $x = 2$, i.e. if is equal to the sum of the ares of ΔABD and ACE,

i.e. $\frac{1}{2}(3)(3) + \frac{1}{3}(3)(3) = \frac{9}{2} + \frac{9}{2} = 9$

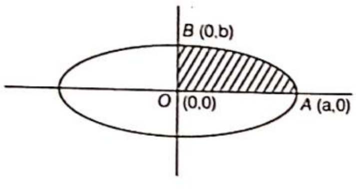
($\because$ area of triangle $= \frac{1}{2} \times$ base $\times$ height)

| Example 12 Find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Sol. Using the symmetry of the figure; required area is given by

$A = 4$ (area $OABO$)

$= 4\int_0^a y\,dx$, where $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



$\therefore \quad \dfrac{y^2}{b^2} = 1 - \dfrac{x^2}{a^2}$

$\Rightarrow \quad y^2 = \dfrac{b^2}{a^2}(a^2 - x^2)$

$\Rightarrow \quad y = \pm \dfrac{b}{a}\sqrt{a^2 - x^2}$

In the first quadrant,

$y = \dfrac{b}{a}\sqrt{a^2 - x^2}$

$A = 4\int_0^a \dfrac{b}{a}\sqrt{a^2 - x^2}\, dx = 4\dfrac{b}{a}\int_0^a \sqrt{a^2 - x^2}\, dx$

$= 4\dfrac{b}{a}\left\{ \dfrac{x}{2}\sqrt{a^2 - x^2} + \dfrac{a^2}{2}\sin^{-1}\dfrac{x}{a} \right\}_0^a$

$= 4\dfrac{b}{a}\left[\left\{0 + \dfrac{a^2}{2}\sin^{-1}\dfrac{a}{a}\right\} - \{0+0\}\right] = \dfrac{4b}{a}\cdot\dfrac{a^2}{2}\sin^{-1}(1)$

$= 2ab\left(\dfrac{\pi}{2}\right)$

$A = \pi ab$ sq units

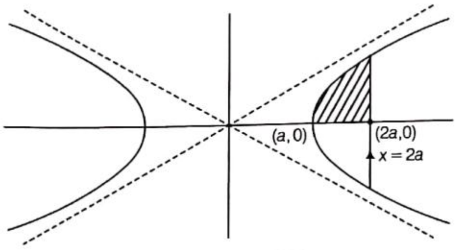
| Example 13 Find the area bounded by the hyperbola $x^2 - y^2 = a^2$ between the straight lines $x = a$ and $x = 2a$.

Sol. We use the symmetry of figure.

Required area, $A = 2\int_a^{2a} y\, dx$, where $x^2 - y^2 = a^2$

i.e. $\quad x^2 - a^2 = y^2$

$\therefore \quad y = \pm\sqrt{x^2 - a^2}$



In the first quadrant; $y = +\sqrt{x^2 - a^2}$

$\therefore\ A = 2\int_a^{2a} \sqrt{x^2 - a^2}\, dx$

$= 2\left[\dfrac{x}{2}\sqrt{x^2 - a^2} - \dfrac{a^2}{2}\log(x + \sqrt{x^2 - a^2})\right]_a^{2a}$

$= 2\left[\left\{a\sqrt{4a^2 - a^2} - \dfrac{a^2}{2}\log(2a + \sqrt{4a^2 - a^2})\right\} - \left\{0 - \dfrac{a^2}{2}\log a\right\}\right]$

$= 2\left[a\sqrt{3a^2} - \dfrac{a^2}{2}\log(2a + a\sqrt{3}) + \dfrac{a^2}{2}\log a\right]$

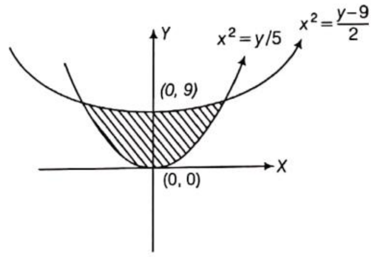
$= 2\left[a^2\sqrt{3} - \dfrac{a^2}{2}\log\left(\dfrac{2a + a\sqrt{3}}{a}\right)\right]$

$= 2a^2\sqrt{3} - a^2\log(2 + \sqrt{3})$ sq units

| Example 14 Find the area common to the parabola $5x^2 - y = 0$ and $2x^2 - y + 9 = 0$.

Sol. Given curves are $\quad y = 5x^2 \quad$ ...(i)

and $\quad y = 2x^2 + 9 \quad$ ...(ii)



Remark

In such examples, figure is the most essential thing. Without figure it just becomes difficult to judge whether y_1 to be subtracted from y_2 or otherwise.

Let us solve Eqs. (i) and (ii) simultaneously,

$\therefore \quad 5x^2 = 2x^2 + 9$

$\Rightarrow \quad 3x^2 = 9 \Rightarrow x^2 = 3$

$\therefore \quad x = -\sqrt{3}$

or $\quad x = \sqrt{3}$

In the usual notations, the required area is given by

$$A = \int_{-\sqrt{3}}^{\sqrt{3}} (y_1 - y_2)\, dx$$

We have to find which curve is above and which is below w.r.t. X-axis in order to decide y_1 and y_2.

Take any point between $x = -\sqrt{3}$ and $x = \sqrt{3}$

Let us take $x = 0$, which lies between

$x = -\sqrt{3}$ and $x = \sqrt{3}$

When $x = 0$ from Eq. (i) $y = 0$

When $x = 0$ from Eq. (ii) $y = 9$

Now, $\quad 9 > 0$

$\therefore$ Parabola Eq. (ii) is above parabola Eq. (i) between

$x = -\sqrt{3}$

and $\quad x = \sqrt{3}$.

$\therefore$ y of the curve (ii) is to be taken as y_1 and y of the curve (i) is to be taken as y_2.

$$\therefore \quad \text{Area}(A) = \int_{-\sqrt{3}}^{\sqrt{3}} \{(2x^2+9) - 5x^2\}\, dx$$

$$= \int_{-\sqrt{3}}^{\sqrt{3}} (9 - 3x^2)\, dx$$

$$= 2\int_0^{\sqrt{3}} (9 - 3x^2)\, dx$$

$$= 2\,[9x - x^3]_0^{\sqrt{3}}$$

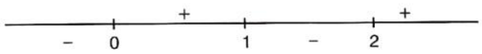
$$= 2\,[9\sqrt{3} - 3\sqrt{3}]$$

$$\text{Area} = 12\sqrt{3} \text{ sq units}$$

Example 15 Find the area enclosed by $y = x(x-1)(x-2)$ and X-axis.

Sol. The given curve is $y = x(x-1)(x-2)$. It passes through (0, 0), (1, 0) and (2, 0).

The sign scheme for $y = x(x-1)(x-2)$ is as shown in figure.



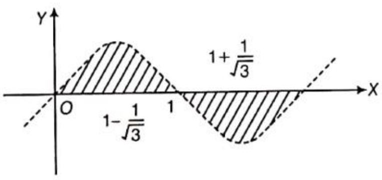
From the sign scheme it is clear that the curve is + ve when $0 < x < 1$ or $x > 2$, hence in these regions the curve lies above X-axis while in the rest regions the curve lies below X-axis.

Remark

Sometimes the discussion of monotonicity of function helps us in sketching polynomials. In the present case, $\frac{dy}{dx} = 3x^2 - 6x + 2$

When $\frac{dy}{dx} = 0$, then; $x = 1 \pm \frac{1}{\sqrt{3}}$. Sign scheme for $\frac{dy}{dx}$ is

Thus, it is clear that the curve increases in $\left(-\infty, 1 - \frac{1}{\sqrt{3}}\right)$, decreases in $\left(1 - \frac{1}{\sqrt{3}}, 1 + \frac{1}{\sqrt{3}}\right)$ and again increases in $\left(1 + \frac{1}{\sqrt{3}}, \infty\right)$. Therefore, the graph of the curve is as below



Hence, required area

$$= \int_0^1 x(x-1)(x-2)\, dx + \left| \int_1^2 x(x-1)(x-2)\, dx \right|$$

$$= \int_0^1 (x^3 - 3x^2 + 2x)\, dx + \left| \int_1^2 (x^3 - 3x^2 + 2x)\, dx \right|$$

$$= \left(\frac{x^4}{4} - x^3 + x^2 \right)_0^1 + \left| \left(\frac{x^4}{4} - x^3 + x^2 \right)_1^2 \right|$$

$$= \left(\frac{1}{4} - 1 + 1 \right) + \left| (4 - 8 + 4) - \left(\frac{1}{4} - 1 + 1 \right) \right|$$

$$= \frac{1}{4} + \frac{1}{4} = \frac{1}{2} \text{ sq unit}$$

Example 16 Find the area between the curves $y = 2x^4 - x^2$, the X-axis and the ordinates of two minima of the curve.

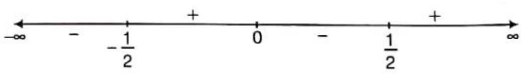
Sol. The given curve is $y = 2x^4 - x^2$.

When $y = 0$, then $x = 0, 0, \pm \frac{1}{\sqrt{2}}$

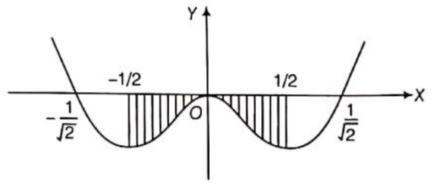
The sign scheme is as shown below

Therefore, it is clear that the curve cuts the X-axis at $x = -\frac{1}{\sqrt{2}}$, 0 and $\frac{1}{\sqrt{2}}$.

The curve is –ve in $\left(-\frac{1}{\sqrt{2}}, 0\right)$ and $\left(0, \frac{1}{\sqrt{2}}\right)$ while positive in the rest. Now, $\frac{dy}{dx} = 8x^3 - 2x$. The sign scheme for $\frac{dy}{dx}$ is as below



i.e. The curve decreases in $(-\infty, -1/2)$ and $(0, 1/2)$ and increases in the rest of portions. Also, the function possess minimum at $x = -1/2$ and 1/2 while maximum at $x = 0$. Therefore, the graph of the curve is as shown below



$$\therefore \text{ Required area} = 2\int_0^{1/2} |2x^4 - x^2|\, dx$$

$$= 2\int_0^{1/2} -(2x^4 - x^2)\, dx = -2\left[2\frac{x^5}{5} - \frac{x^3}{3} \right]_0^{1/2}$$

$$= \frac{7}{120} \text{ sq unit}$$

Exercise for Session 1

1. Draw a rough sketch of $y = \sin 2x$ and determine the area enclosed by the curve, X-axis and the lines $x = \pi/4$ and $x = 3\pi/4$.

2. Find the area under the curve $y = (x^2 + 2)^2 + 2x$ between the ordinates $x = 0$ and $x = 2$.

3. Find the area of the region bounded by the curve $y = 2x - x^2$ and the X-axis.

4. Find the area bounded by the curve $y^2 = 2y - x$ and the Y-axis.

5. Find the area bounded by the curve $y = 4 - x^2$ and the line $y = 0$ and $y = 3$.

6. Find the area bounded by $x = at^2$ and $y = 2at$ between the ordinates corresponding to $t = 1$ and $t = 2$.

7. Find the area of the parabola $y^2 = 4ax$ and the latusrectum.

8. Find the area bounded by $y = 1 + 2\sin^2 x$, X-axis, $x = 0$ and $x = \pi$.

9. Sketch the graph of $y = \sqrt{x} + 1$ in $[0, 4]$ and determine the area of the region enclosed by the curve, the axis of X and the lines $x = 0$, $x = 4$.

10. Find the area of the region bounded by the curve $xy - 3x - 2y - 10 = 0$, X-axis and the lines $x = 3$, $x = 4$.

Session 2

Area Bounded by Two or More Curves

Area Bounded by Two or More Curves

Area bounded by the curves $y = f(x)$, $y = g(x)$ and the lines $x = a$ and $x = b$.

Let the curves $y = f(x)$ and $y = g(x)$ be represented by AB and CD, respectively. We assume that the two curves do not intersect each other in the interval $[a, b]$.

Thus, shaded area = Area of curvilinear trapezoid $APQB$ − Area of curvilinear trapezoid $CPQD$

$$= \int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \int_a^b \{f(x) - g(x)\}\,dx$$

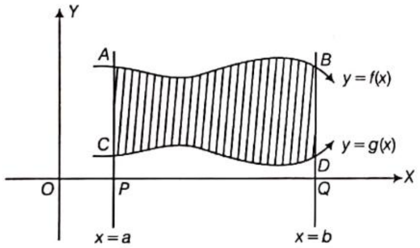


Figure 3.32

Now, consider the case when $f(x)$ and $g(x)$ intersect each other in the interval $[a, b]$.

First of all we should find the intersection point of $y = f(x)$ and $y = g(x)$. For that we solve $f(x) = g(x)$. Let the root is $x = c$. (We consider only one intersection point to illustrate the phenomenon).

Thus, required (shaded) area

$$= \int_a^c \{f(x) - g(x)\}dx + \int_c^b \{g(x) - f(x)\}\,dx$$

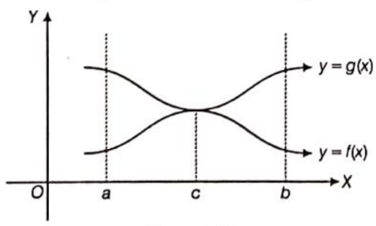


Figure 3.33

If confusion arises in such case evaluate $\int_a^b |f(x) - g(x)|\,dx$ which gives the required area.

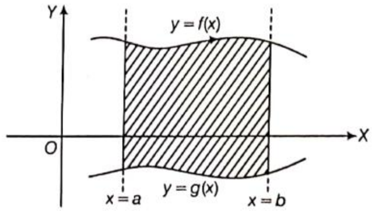


Figure 3.34

Area between two curves $y = f(x)$, $y = g(x)$ and the lines $x = a$ and $x = b$ is always given by $\int_a^b \{f(x) - g(x)\}\,dx$ provided $f(x) > g(x)$ in $[a, b]$; the position of the graph is immaterial. As shown in Fig. 3.34, Fig. 3.35, Fig. 3.36.

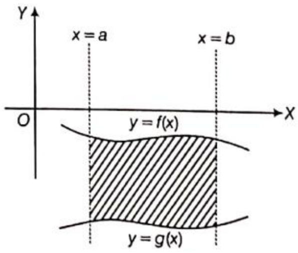


Figure 3.35

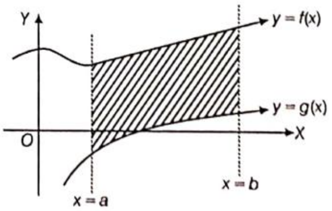
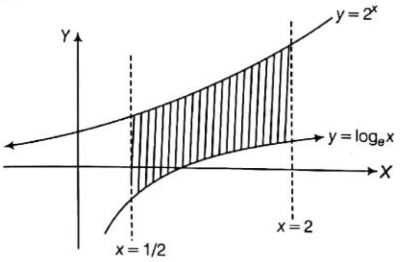


Figure 3.36

| Example 17 Sketch the curves and identify the region bounded by $x = 1/2$, $x = 2$, $y = \log_e x$ and $y = 2^x$. Find the area of this region. *[IIT JEE 1991]*

Sol. The required area is the shaded portion in the following figure.



In the region $\frac{1}{2} \le x \le 2$; the curve $y = 2^x$ lies above as compared to $y = \log_e x$.

Hence, required area $= \int_{1/2}^{2} (2^x - \log x)\, dx$

$$= \left[\frac{2^x}{\log 2} - (x \log x - x) \right]_{1/2}^{2}$$

$$= \left(\frac{4 - \sqrt{2}}{\log 2} - \frac{5}{2} \log 2 + \frac{3}{2} \right) \text{ sq units}$$

Example 18 Find the area given by $x + y \le 6, x^2 + y^2 \le 6y$ and $y^2 \le 8x$.

Sol. Let us consider the curves

$$P \equiv y^2 - 8x = 0 \qquad \text{...(i)}$$

$$C \equiv x^2 + y^2 = 6y$$

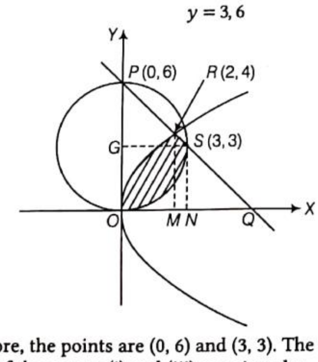
i.e. $$x^2 + (y - 3)^2 - 9 = 0 \qquad \text{...(ii)}$$

and $$S \equiv x + y - 6 = 0 \qquad \text{...(iii)}$$

The intersection points of the curves (ii) and (iii) are given by

$$(6 - y)^2 + y^2 - 6y = 0$$

i.e. $$y = 3, 6$$



Therefore, the points are (0, 6) and (3, 3). The intersection points of the curves (i) and (iii) are given by

$$y^2 = 8(6 - y), \quad \text{i.e. } y = 4, -12$$

Therefore, the point of intersection in 1st quadrant is (2, 4).

Now, we know that

$C \le 0$ denotes the region, inside the circle $C = 0$.

$P \le 0$ denotes the region, inside the parabola $P = 0$.

$S \le 0$ denotes the region, which is negative side of the line $S = 0$.

$\therefore$ Required area = Area of curvilinear $\Delta OMRO$ + Area of trapezium $MNSR$ − Area of curvilinear $\Delta ONSO$

$$= \int_0^2 \sqrt{8x}\, dx + \frac{1}{2}(MR + NS) \cdot MN$$

$$- (\text{Area of square } ONSG - \text{Area of sector } OSGO)$$

$$= \int_0^2 \sqrt{8x}\, dx + \frac{1}{2}(4 + 3) \cdot 1 - \left(3^2 - \frac{\pi \cdot 3^2}{4}\right)$$

$$= \left(\frac{9\pi}{4} - \frac{1}{6}\right) \text{ sq units}$$

Example 19 Find the area of the region $\{(x, y) : 0 \le y \le x^2 + 1, 0 \le y \le x + 1, 0 \le x \le 2\}$.

Sol. Let $R = \{(x, y) : 0 \le y \le x^2 + 1, 0 \le y \le x + 1, 0 \le x \le 2\}$

$$= \{(x, y) : 0 \le y \le x^2 + 1\} \cap \{(x, y) : 0 \le y \le x + 1\} \cap \{(x, y) : 0 \le x \le 2\}$$

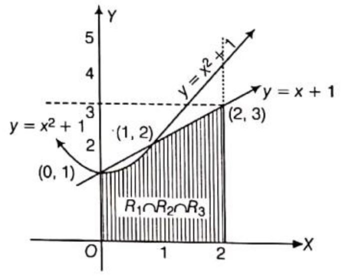
$$= R_1 \cap R_2 \cap R_3$$

where, $R_1 = \{(x, y) : 0 \le y \le x^2 + 1\}$

$R_2 = \{(x, y) : 0 \le y \le x + 1\}$

and $R_3 = \{(x, y) : 0 \le x \le 2\}$

Thus, the sketch of R_1, R_2 and R_3 are



From the above figure,

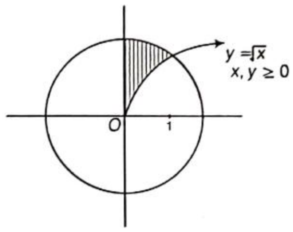
$$\text{Required area} = \int_0^1 (x^2 + 1)\, dx + \int_1^2 (x + 1)\, dx$$

$$= \left(\frac{x^3}{3} + x\right)_0^1 + \left(\frac{x^2}{2} + x\right)_1^2 = \frac{23}{6} \text{ sq units}$$

Example 20 The area common to the region determined by $y \ge \sqrt{x}$ and $x^2 + y^2 < 2$ has the value

(a) π sq units
(b) $(2\pi - 1)$ sq units
(c) $\left(\frac{\pi}{4} - \frac{1}{6}\right)$ sq units
(d) None of these

Sol. The region formed by $y \geq \sqrt{x}$ is the outer region of the parabola $y^2 = x$, when $y \geq 0$ and $x \geq 0$ and $x^2 + y^2 < 2$ is the region inner to circle $x^2 + y^2 = 2$ shown as in figure.



Now, to find the point of intersection put $y^2 = x$ in $x^2 + y^2 = 2$.

$$\Rightarrow \quad x^2 + x - 2 = 0$$

$$\Rightarrow \quad (x+2)(x-1) = 0$$

$$\Rightarrow \quad x = 1, \text{ as } x \geq 0$$

$$\therefore \text{ Required area} = \int_0^1 (\sqrt{2 - x^2} - \sqrt{x})\, dx$$

$$= \left[\frac{x\sqrt{2-x^2}}{2} + \sin^{-1}\frac{x}{\sqrt{2}} - \frac{2}{\sqrt{3}}x^{3/2}\right]_0^1$$

$$= \frac{1}{2} + \frac{\pi}{4} - \frac{2}{3} = \left(\frac{\pi}{4} - \frac{1}{6}\right) \text{ sq units}$$

Hence, (c) is the correct answer.

| Example 21 Find the area of the region enclosed by the curve $5x^2 + 6xy + 2y^2 + 7x + 6y + 6 = 0$.

Sol. Comparing $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, we get $a = 5, b = 2, h = 3, g = 7/2, f = 3$ and $c = 6$

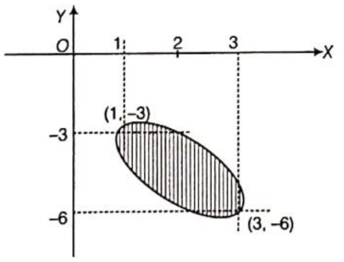
$$\Rightarrow \quad h^2 - ab = -1 < 0$$

So, the above equation represents an ellipse.

$$\therefore \quad 2y^2 + 6(1+x)y + (5x^2 + 7x + 6) = 0$$

$$\Rightarrow \quad y = \frac{-3(1+x) \pm \sqrt{(3-x)(x-1)}}{2}$$

Clearly, the values of y are real for all $x \in [1, 3]$. Thus, the graph is as shown below



Thus, required area

$$= \left|\int_1^3 \left(\frac{-3(1+x) - \sqrt{(3-x)(x-1)}}{2}\right) - \left(\frac{-3(1+x) + \sqrt{(3-x)(x-1)}}{2}\right) dx\right|$$

$$= \left|-\int_1^3 \sqrt{(3-x)(x-1)}\, dx\right| = \left|\int_1^3 \sqrt{1^2 - (x-2)^2}\, dx\right|$$

$$= \left|-\left\{\frac{1}{2}(x-2)\sqrt{-x^2 + 4x - 3} + \frac{1}{2}\sin^{-1}\left(\frac{x-2}{1}\right)\right\}_1^3\right|$$

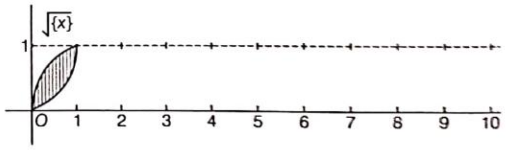
$$= \frac{\pi}{2} \text{ sq units}$$

| Example 22 If $f(x) = \begin{cases} \sqrt{\{x\}}, & x \notin Z \\ 1, & x \in Z \end{cases}$ and $g(x) = \{x\}^2$ (where, {.} denotes fractional part of x), then the area bounded by $f(x)$ and $g(x)$ for $x \in [0, 10]$ is

(a) $\frac{5}{3}$ sq units (b) 5 sq units

(c) $\frac{10}{3}$ sq units (d) None of these

Sol. As, $f(x) = \begin{cases} \sqrt{\{x\}}, & x \notin z \\ 1, & x \in z \end{cases}$ and $g(x) = \{x\}^2$, where both $f(x)$ and $g(x)$ are periodic with period '1' shown as



$$\text{Thus, required area} = 10\int_0^1 [\sqrt{\{x\}} - \{x\}^2]\, dx$$

$$= 10\int_0^1 [(x)^{1/2} - x^2]\, dx$$

$$= 10\left[\frac{x^{3/2}}{3/2} - \frac{x^3}{3}\right]_0^1$$

$$= 10\left(\frac{2}{3} - \frac{1}{3}\right) = \frac{10}{3} \text{ sq units}$$

Hence, (c) is the correct answer.

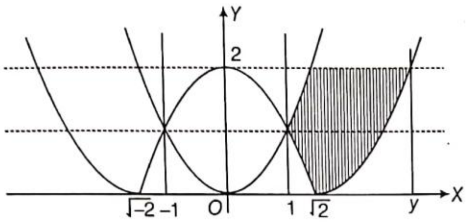
| Example 23 Find the area of the region bounded by the curves $y = x^2, y = |2 - x^2|$ and $y = 2$, which lies to the right of the line $x = 1$. *[IIT JEE 2002]*

Sol. The region bounded by given curves on the right side of $x = 1$ is shown as

$$\text{Required area} = \int_1^{\sqrt{2}} \{x^2 - (2 - x^2)\}\, dx + \int_{\sqrt{2}}^2 \{4 - x^2\}\, dx$$

$$= \int_1^{\sqrt{2}} (2x^2 - 2)\, dx + \int_{\sqrt{2}}^{2} (4 - x^2)\, dx$$

$$= \left(2 \cdot \frac{x^3}{3} - 2x\right)_1^{\sqrt{2}} + \left(4x - \frac{x^3}{3}\right)_{\sqrt{2}}^{2}$$



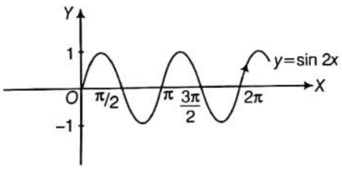
$$= \left(\frac{4\sqrt{2}}{3} - 2\sqrt{2}\right) - \left(\frac{2}{3} - 2\right) + \left(8 - \frac{8}{3}\right) - \left(4\sqrt{2} - \frac{2\sqrt{2}}{3}\right)$$

$$= \left(-4\sqrt{2} + \frac{20}{3}\right) = \left(\frac{20 - 12\sqrt{2}}{3}\right) \text{ sq units}$$

| Example 24 The area enclosed by the curve $|y| = \sin 2x$, when $x \in [0, 2\pi]$ is

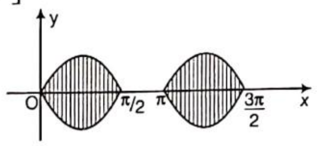
(a) 1 sq unit (b) 2 sq units
(c) 3 sq units (d) 4 sq units

Sol. As, we know $y = \sin 2x$ could be plotted as



Thus, $|y| = \sin 2x$ is whenever positive, y can have both positive and negative values, i.e. the curve is symmetric about the axes.

$\sin 2x$ is positive only in $0 \le x \le \frac{\pi}{2}$ and $\pi \le x \le \frac{3\pi}{2}$. Thus, the curve consists of two loops one in $\left[0, \frac{\pi}{2}\right]$ and another in $\left[\pi, \frac{3\pi}{2}\right]$.



Thus, required area $= 4 \int_0^{\pi/2} (\sin 2x)\, dx$

$$= 4\left(-\frac{\cos 2x}{2}\right)_0^{\pi/2} = -2(\cos \pi - \cos 0)$$

$$= -2(-1 - 1) = 4 \text{ sq units}$$

Hence, (d) is the correct answer.

| Example 25 Let $f(x) = x^2, g(x) = \cos x$ and $\alpha, \beta\, (\alpha < \beta)$ be the roots of the equation $18x^2 - 9\pi x + \pi^2 = 0$. Then, the area bounded by the curves $y = fog(x)$, the ordinates $x = \alpha, x = \beta$ and the X-axis is

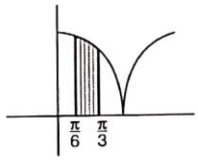
(a) $\frac{1}{2}(\pi - 3)$ sq units (b) $\frac{\pi}{3}$ sq units
(c) $\frac{\pi}{4}$ sq units (d) $\frac{\pi}{12}$ sq units

Sol. Here, $y = fog(x) = f\{g(x)\} = (\cos x)^2 = \cos^2 x$

Also, $18x^2 - 9\pi x + \pi^2 = 0$

$\Rightarrow (3x - \pi)(6x - \pi) = 0$

$\Rightarrow x = \frac{\pi}{6}, \frac{\pi}{3}$ (as α, β)



$\therefore$ Required area of curve

$$= \int_{\pi/6}^{\pi/3} \cos^2 x\, dx = \frac{1}{2}\int_{\pi/6}^{\pi/3} (1 + \cos 2x)\, dx$$

$$= \frac{1}{2}\left\{x + \frac{\sin 2x}{2}\right\}_{\pi/6}^{\pi/3} = \frac{1}{2}\left\{\left(\frac{\pi}{3} - \frac{\pi}{6}\right) + \frac{1}{2}\left(\sin\frac{2\pi}{3} - \sin\frac{2\pi}{6}\right)\right\}$$

$$= \frac{1}{2}\left\{\frac{\pi}{6} + \frac{1}{2}\left(\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\right)\right\} = \frac{\pi}{12}$$

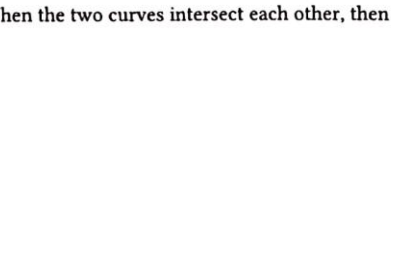
Hence, (d) is the correct answer.

| Example 26 Find the area bounded by the curves $x^2 + y^2 = 25, 4y = |4 - x^2|$ and $x = 0$ above the X-axis.

Sol. The 1st curve is a circle of radius 5 with centre at (0, 0).

The 2nd curve is $y = \left|\frac{4 - x^2}{4}\right| = \left|1 - \frac{x^2}{4}\right|$

which can be traced easily by graph transformation.



When the two curves intersect each other, then

$$x^2 + \left(1 - \frac{x^2}{4}\right)^2 = 25 \Rightarrow x = \pm 4$$

Hence, required area $= 2\int_0^4 \left(\sqrt{25 - x^2} - \left|1 - \frac{x^2}{4}\right|\right) dx$

$$= 2\left[\int_0^4 \sqrt{25 - x^2}\,dx - \int_0^2 \left(1 - \frac{x^2}{4}\right)dx + \int_2^4 \left(1 - \frac{x^2}{4}\right)dx\right]$$

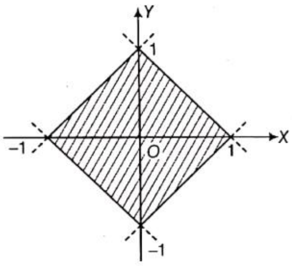
$$= 2\left[6 + \frac{25}{2}\sin^{-1}\left(\frac{4}{5}\right) - \frac{4}{3} - \frac{8}{3}\right] = \left\{25\sin^{-1}\left(\frac{4}{5}\right) + 4\right\}$$

| Example 27 Find the area enclosed by $|x| + |y| = 1$.

Sol. From the given equation, we have

$$|y| = 1 - |x| \geq 0 \qquad [\because |y| \geq 0]$$

$$\Rightarrow \qquad -1 \leq x \leq 1$$



Therefore, the curve exists for $x \in [-1, 1]$ only

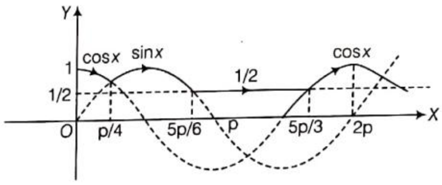
and for $-1 \leq x \leq 1$; $y = \pm(1 - |x|)$ i.e. $y = \begin{cases} |x| - 1 \\ -(|x| - 1) \end{cases}$

Thus, the required graph is as given in figure.

$\therefore$ Required area $= (\sqrt{2})^2 = 2$ sq units

| Example 28 Let $f(x) = \max\left\{\sin x, \cos x, \frac{1}{2}\right\}$, then determine the area of region bounded by the curves $y = f(x)$, X-axis, Y-axis and $x = 2\pi$.

Sol. We have, $f(x) = \max\left\{\sin x, \cos x, \frac{1}{2}\right\}$. Graphically, $f(x)$ could be drawn as



Here, the graph is plotted between 0 to 2π and between the points of intersection the maximum portion is included, thus the shaded part is required area

	Interval	Value of $f(x)$
i.e. for	$0 \leq x < \pi/4$	$\cos x$
for	$\pi/4 \leq x < 5\pi/6$	$\sin x$
for	$5\pi/6 \leq x < 5\pi/3$	$1/2$
for	$5\pi/3 \leq x < 2\pi$	$\cos x$

Hence, required area

$$I = \int_0^{\pi/4} \cos x\,dx + \int_{\pi/4}^{5\pi/6} \sin x\,dx + \int_{5\pi/6}^{5\pi/3} \frac{1}{2}\,dx + \int_{5\pi/3}^{2\pi} \cos x\,dx$$

$$= (\sin x)_0^{\pi/4} - (\cos x)_{\pi/4}^{5\pi/6} + \frac{1}{2}(x)_{5\pi/6}^{5\pi/3} + (\sin x)_{5\pi/3}^{2\pi}$$

$$= \left(\frac{1}{\sqrt{2}} - 0\right) - \left(-\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}\right) + \frac{1}{2}\left(\frac{5\pi}{3} - \frac{5\pi}{6}\right) + \left(0 + \frac{\sqrt{3}}{2}\right)$$

$$= \left(\frac{5\pi}{12} + \sqrt{2} + \sqrt{3}\right) \text{ sq units}$$

Exercise for Session 2

1. The area of the region bounded by $y^2 = 2x + 1$ and $x - y - 1 = 0$ is

(a) 2/3 (b) 4/3 (c) 8/3 (d) 16/3

2. The area bounded by the curve $y = 2x - x^2$ and the straight line $y = x$ is given by

(a) 9/2 (b) 43/6 (c) 35/6 (d) None of these

3. The area bounded by the curve $y = x|x|$, X-axis and the ordinates $x = -1$, $x = 1$ is given by

(a) 0 (b) 1/3 (c) 2/3 (d) None of these

4. Area of the region bounded by the curves $y = 2^x$, $y = 2x - x^2$, $x = 0$ and $x = 2$ is given by

(a) $\frac{3}{\log 2} - \frac{4}{3}$ (b) $\frac{3}{\log 2} + \frac{4}{3}$ (c) $3\log 2 - \frac{4}{3}$ (d) None of these

5. The area of the figure bounded by the cuves $y = e^x$, $y = e^{-x}$ and the straight line $x = 1$ is

(a) $e + \frac{1}{e}$ (b) $e - \frac{1}{e}$ (c) $e + \frac{1}{e} - 2$ (d) None of these

6. Area of the region bounded by the curve $y^2 = 4x$, Y-axis and the line $y = 3$ is

(a) 2 (b) 9/4 (c) $6\sqrt{3}$ (d) None of these

7. The area of the figure bounded by $y = \sin x$, $y = \cos x$ is the first quadrant is

(a) $2(\sqrt{2}-1)$ (b) $\sqrt{3}+1$ (c) $2(\sqrt{3}-1)$ (d) None of these

8. The area bounded by the curves $y = xe^x$, $y = xe^{-x}$ and the line $x = 1$ is

(a) $\frac{2}{e}$ (b) $1-\frac{2}{e}$ (c) $\frac{1}{e}$ (d) $1-\frac{1}{e}$

9. The areas of the figure into which the curve $y^2 = 6x$ divides the circle $x^2 + y^2 = 16$ are in the ratio

(a) $\frac{2}{3}$ (b) $\frac{4\pi-\sqrt{3}}{8\pi+\sqrt{3}}$ (c) $\frac{4\pi+\sqrt{3}}{8\pi-\sqrt{3}}$ (d) None of these

10. The area bounded by the Y-axis, $y = \cos x$ and $y = \sin x$, $0 \geq x \leq \pi/2$ is

(a) $2(\sqrt{2}-1)$ (b) $\sqrt{2}-1$ (c) $(\sqrt{2}+1)$ (d) $\sqrt{2}$

11. The area bounded by the curve $y = \frac{3}{|x|}$ and $y + |2-x| = 2$ is

(a) $\frac{4-\log 27}{3}$ (b) $2-\log 3$ (c) $2+\log 3$ (d) None of these

12. The area bounded by the curves $y = x^2 + 2$ and $y = 2|x| - \cos + x$ is

(a) 2/3 (b) 8/3 (c) 4/3 (d) 1/3

13. The are bounded by the curve $y^2 = 4x$ and the circle $x^2 + y^2 - 2x - 3 = 0$ is

(a) $2\pi + \frac{8}{3}$ (b) $4\pi + \frac{8}{3}$ (c) $\pi + \frac{8}{3}$ (d) $\pi - \frac{8}{3}$

14. A point P moves inside a triangle formed by $A(0,0)$, $B\left(1, \frac{1}{\sqrt{3}}\right)$, $C(2,0)$ such that $\min\{PA, PB, PC\} = 1$, then the area bounded by the curve traced by P, is

(a) $3\sqrt{3} - \frac{3\pi}{2}$ (b) $\sqrt{3} + \frac{\pi}{2}$ (c) $\sqrt{3} - \frac{\pi}{2}$ (d) $3\sqrt{3} + \frac{3\pi}{2}$

15. The graph of $y^2 + 2xy + 40|x| = 400$ divides the plane into regions. The area of the bounded region is

(a) 400 (b) 800 (c) 600 (c) None of these

16. The area of the region defined by $||x| - |y|| \leq 1$ and $x^2 + y^2 \leq 1$ in the xy plane is

(a) π (b) 2π (c) 3π (d) 1

17. The area of the region defined by $1 \leq |x-2| + |y+1| \leq 2$ is

(a) 2 (b) 4 (c) 6 (d) None of these

18. The area of the region enclosed by the curve $|y| = -(1-|x|)^2 + 5$, is

(a) $\frac{8}{3}(7+5\sqrt{5})$ sq units (b) $\frac{2}{3}(7+5\sqrt{5})$ sq units (c) $\frac{2}{3}(5\sqrt{5}-7)$ sq units (d) None of these

19. The area bounded by the curve $f(x) = ||\tan x + \cot x| - |\tan x - \cot x||$ between the lines $x = 0$, $x = \frac{\pi}{2}$ and the X-axis is

(a) $\log 4$ (b) $\log\sqrt{2}$ (c) $2\log 2$ (d) $\sqrt{2}\log 2$

20. If $f(x) = \max\left\{\sin x, \cos x, \frac{1}{2}\right\}$, then the area of the region bounded by the curves $y = f(x)$, X-axis, Y-axis and $x = \frac{5\pi}{3}$ is

(a) $\left(\sqrt{2} - \sqrt{3} + \frac{5\pi}{12}\right)$ sq units (b) $\left(\sqrt{2} + \sqrt{3} + \frac{5\pi}{2}\right)$ sq units

(c) $\left(\sqrt{2} + \sqrt{3} + \frac{5\pi}{2}\right)$ sq units (d) None of these

JEE Type Solved Examples :
Single Option Correct Type Questions

• Ex. 1 *If A denotes the area bounded by* $f(x)=\left|\frac{\sin x+\cos x}{x}\right|$, *X-axis,* $x=\pi$ *and* $x=3\pi$, *then*

(a) $1< A< 2$ (b) $0< A< 2$
(c) $2< A< 3$ (d) None of these

Sol. $\frac{2\sqrt{2}}{2\pi}<\int_{\pi}^{2\pi}\frac{|\sin x+\cos x|}{x}\,dx<\frac{2\sqrt{2}}{2\pi}$

$$\left[\because \pi<x<2\pi \Rightarrow \frac{1}{2\pi}<\frac{1}{x}<\frac{1}{\pi}\right] \quad \text{...(i)}$$

$$\frac{2\sqrt{2}}{3\pi}<\int_{2\pi}^{3\pi}\frac{|\sin x+\cos x|}{x}\,dx<\frac{2\sqrt{2}}{2\pi} \quad \text{...(ii)}$$

On adding Eqs. (i) and (ii), we get

$$\frac{5\sqrt{2}}{3\pi}<A<\frac{3\sqrt{2}}{\pi}$$

$\Rightarrow$ $0.75<A<1.3$

Hence, (b) is the correct answer.

• Ex. 2 *If* $f(x)\geq 0, \forall x\in(0,2)$ *and* $y=f(x)$ *makes positive intercepts of 2 and 1 unit on X and Y-axes respectively and encloses an area of 3/4 unit with axes, then* $\int_0^2 xf'(x)\,dx$ *is*

(a) $\frac{3}{4}$ (b) 1 (c) $\frac{5}{4}$ (d) $-\frac{3}{4}$

Sol. $I=xf(x)|_0^2-\int_0^2 f(x)\,dx=0-\frac{3}{4}=-\frac{3}{4}$

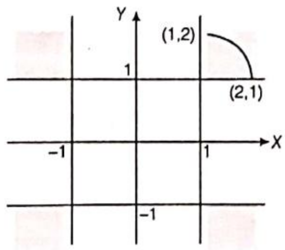
Hence, (d) is the correct answer.

• Ex. 3 *The area of the region included between the regions satisfying* $\min(|x|,|y|)\geq 1$ *and* $x^2+y^2\leq 5$ *is*

(a) $\frac{5}{2}\left(\sin^{-1}\frac{2}{\sqrt{5}}-\sin^{-1}\frac{1}{\sqrt{5}}\right)-4$ (b) $10\left(\sin^{-1}\frac{2}{\sqrt{5}}-\sin^{-1}\frac{1}{\sqrt{5}}\right)-4$

(c) $\frac{2}{5}\left(\sin^{-1}\frac{2}{\sqrt{5}}-\sin^{-1}\frac{1}{\sqrt{5}}\right)-4$ (d) $15\left(\sin^{-1}\frac{2}{\sqrt{5}}-\sin^{-1}\frac{1}{\sqrt{5}}\right)-4$

Sol. Shaded region depicts $\min(|x|,|y|)\geq 1$



Required area $=4\int_1^2(\sqrt{5-x^2}-1)\,dx$

$$=10\left(\sin^{-1}\frac{2}{\sqrt{5}}-\sin^{-1}\frac{1}{\sqrt{5}}\right)-4$$

Hence, (b) is the correct answer.

• Ex. 4 *The area of the region between the curves* $y=\sqrt{\frac{1+\sin x}{\cos x}}$ *and* $y=\sqrt{\frac{1-\sin x}{\cos x}}$ *and bounded by the lines* $x=0$ *and* $x=\frac{\pi}{4}$ *is* ***[IIT JEE 2003]***

(a) $\int_0^{\sqrt{2}-1}\frac{t}{(1+t^2)\sqrt{1-t^2}}\,dt$

(b) $\int_0^{\sqrt{2}-1}\frac{4t}{(1+t^2)\sqrt{1-t^2}}\,dt$

(c) $\int_0^{\sqrt{2}+1}\frac{4t}{(1+t^2)\sqrt{1-t^2}}\,dt$

(d) $\int_0^{\sqrt{2}+1}\frac{t}{(1+t^2)\sqrt{1-t^2}}\,dt$

Sol. Required area $=\int_0^{\pi/4}\left(\sqrt{\frac{1+\sin x}{\cos x}}-\sqrt{\frac{1-\sin x}{\cos x}}\right)dx$

$$\left(\because \frac{1+\sin x}{\cos x}>\frac{1-\sin x}{\cos x}>0\right)$$

$$=\int_0^{\pi/4}\left(\sqrt{\frac{1+\dfrac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}}{\dfrac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}}}-\sqrt{\frac{1-\dfrac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}}{\dfrac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}}}\right)dx$$

$$=\int_0^{\pi/4}\left(\sqrt{\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}}-\sqrt{\frac{1-\tan\frac{x}{2}}{1+\tan\frac{x}{2}}}\right)dx$$

$$=\int_0^{\pi/4}\frac{1+\tan\frac{x}{2}-1+\tan\frac{x}{2}}{\sqrt{1-\tan^2\frac{x}{2}}}\,dx=\int_0^{\pi/4}\frac{2\tan\frac{x}{2}}{\sqrt{1-\tan^2\frac{x}{2}}}\,dx$$

Put $\tan\frac{x}{2}=t$

$\therefore$ Area $=\int_0^{\sqrt{2}-1}\frac{4t}{(1+t^2)\sqrt{1-t^2}}\,dt$

Hence, (b) is the correct answer.

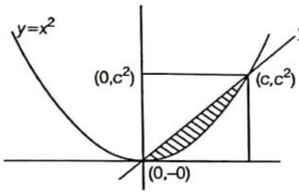
JEE Type Solved Examples :
More than One Correct Option Type Questions

• **Ex. 5** *Let T be the triangle with vertices $(0,0),(0,c^2)$ and (c,c^2) and let R be the region between $y=cx$ and $y=x^2$, where $c>0$, then*

(a) Area $(R)=\dfrac{c^3}{6}$ (b) Area of $R=\dfrac{c^3}{3}$

(c) $\lim\limits_{c\to 0^+}\dfrac{\text{Area }(T)}{\text{Area }(R)}=3$ (d) $\lim\limits_{c\to 0^+}\dfrac{\text{Area }(T)}{\text{Area }(R)}=\dfrac{3}{2}$

Sol. Area $(T)=\dfrac{c\cdot c^2}{2}=\dfrac{c^3}{2}$



$$\text{Area }(R)=\frac{c^3}{2}-\int_0^c x^2\,dx=\frac{c^3}{2}-\frac{c^3}{3}=\frac{c^3}{6}$$

$$\therefore \quad \lim_{c\to 0^+}\frac{\text{Area}(T)}{\text{Area }(R)}=\lim_{c\to 0^+}\frac{c^3}{2}\cdot\frac{6}{c^3}=3$$

Hence, (a) and (c) are the correct answers.

• **Ex. 6** *Suppose f is defined from $R\to[-1,1]$ as $f(x)=\dfrac{x^2-1}{x^2+1}$ where R is the set of real number. Then, the statement which does not hold is*

(a) f is many-one onto

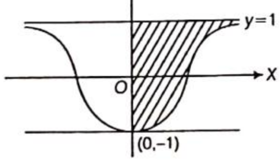
(b) f increases for $x>0$ and decreases for $x<0$

(c) minimum value is not attained even though f is bounded

(d) the area included by the curve $y=f(x)$ and the line $y=1$ is π sq units

Sol. $y=f(x)=\dfrac{x^2-1}{x^2+1}=1-\dfrac{2}{x^2+1}$

$f'(x)=\dfrac{4x}{(x^2+1)^2}$ $x>0$, f is increasing and $x<0$ f is decreasing.



$\Rightarrow$ (b) is true; range is $[-1,1)$ $\Rightarrow$ into $\Rightarrow$ (a) is false; minimum value occurs at $x=0$ and $f(0)=-1$ $\Rightarrow$ (c) is false.

$$A=2\int_0^\infty\left(1-\frac{x^2-1}{x^2+1}\right)dx=4\int_0^\infty\frac{dx}{x^2+1}$$

$$=[4\cdot\tan^{-1}x]_0^\infty=4\cdot\frac{\pi}{2}=2\pi \Rightarrow \text{(d) is false.}$$

Hence, (a), (c) and (d) are the correct answers.

• **Ex. 7** *Consider* $f(x)=\begin{cases}\cos x, & 0\le x<\dfrac{\pi}{2}\\ \left(\dfrac{\pi}{2}-x\right)^2, & \dfrac{\pi}{2}\le x<\pi\end{cases}$ *such that f is periodic with period π, then*

(a) the range of f is $\left[0,\dfrac{\pi^2}{4}\right)$

(b) f is continuous for all real x, but not differentiable for some real x

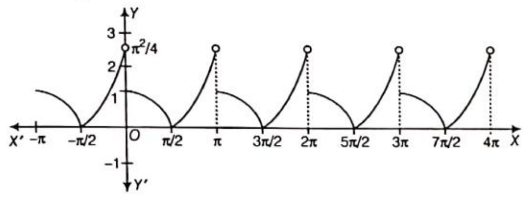
(c) f is continuous for all real x

(d) the area bounded by $y=f(x)$ and the X-axis from $x=-n\pi$ to $x=n\pi$ is $2n\left(1+\dfrac{\pi^2}{24}\right)$ for a given $n\in N$

Sol. Given, $f(x)=\begin{cases}\cos x, & 0\le x<\dfrac{\pi}{2}\\ \left(\dfrac{\pi}{2}-x\right)^2, & \dfrac{\pi}{2}\le x<\pi\end{cases}$ and f is periodic with period π. Let us draw the graph of $y=f(x)$

From the graph, the range of the function is $\left[0,\dfrac{\pi^2}{4}\right)$.

It is discontinuous at $x=n\pi$, $n\in I$. It is not differentiable at $x=\dfrac{n\pi}{2}$, $n\in I$.



Area bounded by $y=f(x)$ and the X-axis from $-n\pi$ to $n\pi$ for $n\in N$

$$=2n\int_0^\pi f(x)\,dx=2n\left[\int_0^{\pi/2}\cos x\,dx+\int_{\pi/2}^{\pi}\left(\frac{\pi}{2}-x\right)^2dx\right]$$

$$=2n\left(1+\frac{\pi^3}{24}\right)$$

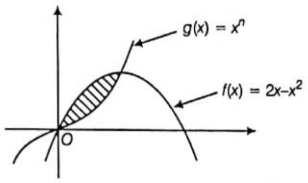
Hence, (a) and (d) are the correct answers.

● **Ex. 8** *Consider the functions $f(x)$ and $g(x)$, both defined from $R \to R$ and are defined as $f(x) = 2x - x^2$ and $g(x) = x^n$ where $n \in N$. If the area between $f(x)$ and $g(x)$ is 1/2, then n is a divisor of*

(a) 12 (b) 15 (c) 20 (d) 30

Sol. Solving, $f(x) = 2x - x^2$ and $g(x) = x^n$ we have

$2x - x^2 = x^n \Rightarrow x = 0$ and $x = 1$



$$A = \int_0^1 (2x - x^2 - x^n)\, dx = \left[x^2 - \frac{x^3}{3} - \frac{x^{n+1}}{n+1}\right]_0^1$$

$$= 1 - \frac{1}{3} - \frac{1}{n+1} = \frac{2}{3} - \frac{1}{n+1}$$

Since, $\frac{2}{3} - \frac{1}{n+1} = \frac{1}{2} \Rightarrow \frac{2}{3} - \frac{1}{2} = \frac{1}{n+1}$

$\Rightarrow \frac{4-3}{6} = \frac{1}{n+1} \Rightarrow n + 1 = 6$

$\Rightarrow n = 5$

Thus, n is a divisor of 15, 20, 30.

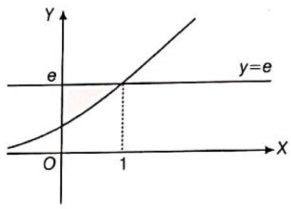
Hence, (b), (c) and (d) are the correct answers.

● **Ex. 9** *Area of the region bounded by the curve $y = e^x$ and lines $x = 0$ and $y = e$ is* **[IIT JEE 2009]**

(a) $e - 1$ (b) $\int_1^e \ln(e + 1 - y)\, dy$

(c) $e - \int_0^1 e^x\, dx$ (d) $\int_1^e \ln y\, dy$

Sol. Shaded area $= e - \left(\int_0^1 e^x\, dx\right) = 1$



Also, $\int_1^e \ln(e + 1 - y)\, dy$

Put $e + 1 - y = t$

$\Rightarrow -dy = dt$

$= \int_e^1 \ln t(-dt) = \int_1^e \ln t\, dt$

$= \int_1^e \ln y\, dy = 1$

Hence, (b), (c) and (d) are the correct answers.

JEE Type Solved Examples : Passage Based Questions

Passage I

(Q. Nos. 10 to 12)

Consider the function $f(x) = x^3 - 8x^2 + 20x - 13$.

● **Ex. 10** *Number of positive integers x for which $f(x)$ is a prime number, is*

(a) 1 (b) 2
(c) 3 (d) 4

Sol. $f(x) = (x - 1)(x^2 - 7x + 13)$ for $f(x)$ to be prime atleast one of the factors must be prime.

Therefore, $x - 1 = 1$

$\Rightarrow x = 2$

or $x^2 - 7x + 13 = 1$

$\Rightarrow x^2 - 7x + 12 = 0$

$\Rightarrow x = 3$ or 4

$\Rightarrow x = 2, 3, 4$

Hence, (c) is the correct answer.

● **Ex. 11** *The function $f(x)$ defined for $R \to R$*

(a) is one-one onto
(b) is many-one onto
(c) has 3 real roots
(d) is such that $f(x_1) \cdot f(x_2) < 0$ where x_1 and x_2 are the roots of $f'(x) = 0$

Sol. $f(x)$ is many-one as it increases and decreases, also range of $f(x) \in R \Rightarrow$ many-one onto.

Hence, (b) is the correct answer.

● **Ex. 12** *Area enclosed by $y = f(x)$ and the coordinate axes is*

(a) 65/12 (b) 13/12
(c) 71/12 (d) None of these

Sol. $A = \left|\int_0^1 f(x)\, dx\right| = -\int_0^1 (x^3 - 8x^2 + 20x - 13)\, dx = \frac{65}{12}$

Hence, (a) is the correct answer.

Passage II

(Q. Nos. 13 to 15)

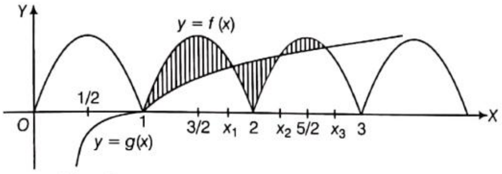
Let $h(x) = f(x) - g(x)$, where $f(x) = \sin^4 \pi x$ and $g(x) = \ln x$. Let $x_0, x_1, x_2, \ldots, x_{n+1}$ be the roots of $f(x) = g(x)$ in increasing order.

• **Ex. 13** *The absolute area enclosed by $y = f(x)$ and $y = g(x)$ is given by*

(a) $\sum_{r=0}^{n} \int_{x_r}^{x_{r+1}} (-1)^r h(x)\, dx$ (b) $\sum_{r=0}^{n} \int_{x_r}^{x_{r+1}} (-1)^{r+1} h(x)\, dx$

(c) $2\sum_{r=0}^{n} \int_{x_r}^{x_{r+1}} (-1)^r h(x)\, dx$ (d) $\frac{1}{2}\sum_{r=0}^{n} \int_{x_r}^{x_{r+1}} (-1)^{r+1} h(x)\, dx$

Sol.



Hence, (a) is the correct answer.

• **Ex. 14** *In the above question, the value of n is*

(a) 1 (b) 2 (c) 3 (d) 4

Sol. $x_{n+1} = x_3 \Rightarrow n = 2.$

Hence, (b) is the correct answer.

• **Ex. 15** *The whole area bounded by $y = f(x)$, $y = g(x)$ and $x = 0$ is*

(a) $\frac{11}{8}$ (b) $\frac{8}{3}$ (c) 2 (d) $\frac{13}{3}$

Sol. Required area $= \int_0^1 \sin^4 \pi x\, dx - \int_0^1 \ln x\, dx = \frac{11}{8}$

Hence, (a) is the correct answer.

Passage III

(Q. Nos. 16 to 18)

Consider the function defined implicitly by the equation $y^3 - 3y + x = 0$ on various intervals in the real line. If $x \in (-\infty, -2) \cup (2, \infty)$, the equation implicitly defines a unique real-valued differentiable function $y = f(x)$. If $x \in (-2, 2)$, the equation implicitly defines a unique real-valued differentiable function $y = g(x)$ satisfying $g(0) = 0$. **[IIT JEE 2008]**

• **Ex. 16** *If $f(-10\sqrt{2}) = 2\sqrt{2}$, then $f''(-10\sqrt{2})$ is equal to*

(a) $\frac{4\sqrt{2}}{7^3 3^2}$ (b) $-\frac{4\sqrt{2}}{7^3 3^2}$ (c) $\frac{4\sqrt{2}}{7^3 3}$ (d) $-\frac{4\sqrt{2}}{7^3 3}$

Sol. ∴ $y^3 - 3y + x = 0$

On differentiating, we get $3y^2 y' - 3y' - 1 = 0$

$$\Rightarrow \quad y' = \frac{1}{3(1 - y^2)}$$

$$\Rightarrow \quad y'(-10\sqrt{2}) = \frac{1}{3\{1 - (2\sqrt{2})^2\}} \quad \ldots\text{(i)}$$

$$\Rightarrow \quad y'(-10\sqrt{2}) = \frac{1}{3\{1 - (2\sqrt{2})^2\}} = \frac{1}{3\,(1 - 8)} = -\frac{1}{21}$$

Again differentiating Eq. (i), we get

$$y'' = \frac{6yy'^2}{3(1 - y^2)}$$

$$y''(-10\sqrt{2}) = \frac{6 \cdot 2\sqrt{2} \cdot \left(\frac{1}{21}\right)^2}{3(1 - 8)} = -\frac{4\sqrt{2}}{7^3 \cdot 3^2}$$

Hence, (b) is the correct answer.

• **Ex. 17** *The area of the region bounded by the curve $y = f(x)$, the X-axis and the line $x = a$ and $x = b$, where $-\infty < a < b < -2$ is*

(a) $\int_a^b \frac{x}{3[\{f(x)\}^2 - 1]} dx + bf(b) - af(a)$

(b) $-\int_a^b \frac{x}{3[\{f(x)\}^2 - 1]} dx - bf(b) + af(a)$

(c) $\int_a^b \frac{x}{3[\{f(x)\}^2 - 1]} dx - bf(b) + af(a)$

(d) $-\int_a^b \frac{x}{3[\{f(x)\}^2 - 1]} dx - bf(b) + af(a)$

Sol. Required area $= \int_a^b f(x)\, dx = [xf(x)]_a^b - \int_a^b xf'(x)\, dx$

$$= bf(b) - af(a) + \int_a^b \frac{x}{3[\{f(x)\}^2 - 1]} dx$$

Hence, (a) is the correct answer.

• **Ex. 18** *$\int_{-1}^{1} g'(x)\, dx$ is equal to*

(a) $2g(-1)$ (b) 0 (c) $-2g(1)$ (d) $2g(1)$

Sol. $I = \int_{-1}^{1} g'(x)\, dx = [g(x)]_{-1}^{1} = g(1) - g(-1)$

Since, $y^3 - 3y + x = 0$ ...(i)

and $y = g(x)$

Since, $\{g(x)\}^3 - 3g(x) + x = 0$ [by Eq. (i)]

At $x = 1$, $\{g(1)\}^3 - 3g(1) + 1 = 0$ ...(ii)

At $x = -1$, $\{g(-1)\}^3 - 3g(-1) - 1 = 0$ ...(iii)

On adding Eqs. (i) and (ii), we get

$$\{g(1)\}^3 + \{g(-1)\}^3 - 3\{g(1) + g(-1)\} = 0$$

$$\{g(1) + g(-1)\}\{g(1)^2 + g(-1)^2 - g(1)g(-1) - 3\} = 0$$

$\Rightarrow$ $g(1) + g(-1) = 0$, $g(1) = -g(-1)$

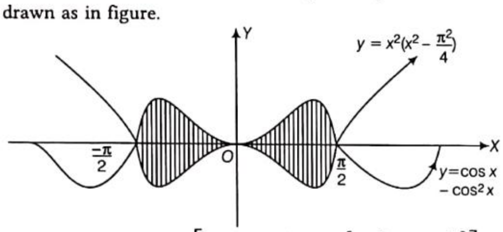
$\Rightarrow$ $I = g(1) - g(-1) = g(1) - (-g(1)) = 2g(1)$

Hence, (d) is the correct answer.

Subjective Type Questions

● ***Ex. 19*** *Find the total area bounded by the curves* $y = \cos x - \cos^2 x$ *and* $y = x^2\left(x^2 - \dfrac{\pi^2}{4}\right)$.

Sol. Here, $y = \cos x - \cos^2 x$ and $y = x^2\left(x^2 - \dfrac{\pi^2}{4}\right)$ could be drawn as in figure.



Thus, the area $= 2\int_0^{\pi/2}\left[(\cos x - \cos^2 x) - \left\{x^2\left(x^2 - \dfrac{\pi^2}{4}\right)\right\}\right]dx$

$= 2\int_0^{\pi/2}\left(\cos x - \cos^2 x - x^4 + \dfrac{\pi^2}{4}x^2\right)dx$

$= 2\left[\sin x - \dfrac{x}{2} - \dfrac{\sin 2x}{4} - \dfrac{x^5}{5} + \dfrac{\pi^2 x^3}{12}\right]_0^{\pi/2}$

$= \left(2 - \dfrac{\pi}{2} + \dfrac{\pi^5}{120}\right)$

● ***Ex. 20*** *A curve* $y = f(x)$ *passes through the point* $P(1, 1)$*, the normal to the curve at P is* $a(y-1) + (x-1) = 0$. *If the slope of the tangent at any point on the curve is proportional to the ordinate of that point, determine the equation of the curve. Also obtain the area bounded by the Y-axis, the curve and the normal to the curve at P.*

Sol. Here, slope of the normal at $P(x, y)$.

$\Rightarrow$ Slope of the line $a(y-1) + (x-1) = 0$ is $-\dfrac{1}{a}$

$\therefore$ Slope of the tangent at $P = a$,

$$\left(\frac{dy}{dx}\right)_P = a \quad \text{...(i)}$$

It is given that the slope of the tangent at any point on the curve $y = f(x)$ is proportional to the ordinate of the point.

$\therefore \quad \dfrac{dy}{dx} \propto y \Rightarrow \dfrac{dy}{dx} = \lambda y$

$\Rightarrow \quad \left(\dfrac{dy}{dx}\right)_{(1,1)} = \lambda \Rightarrow a = \lambda$

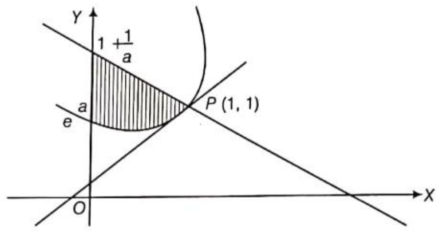
$\therefore \quad \dfrac{dy}{dx} = ay \quad \Rightarrow \quad \dfrac{dy}{y} = a\,dx$

$\Rightarrow \quad \log y = ax + \log c$

$\Rightarrow \quad y = ce^{ax}$, which passes through $P(1, 1)$

$\therefore \quad c = e^{-a}$

$\Rightarrow$ Curve is $\quad y = e^{-a}e^{ax} \Rightarrow y = e^{a(x-1)}$



$\therefore$Required area $= \dfrac{1}{a}\int_{e^{-a}}^{1}(\log y + a)\,dy + \int_1^{1+1/a}\{(1+a) - ay\}\,dy$

$= \dfrac{1}{a}\{y(\log y - 1) + ay\}_{e^{-a}}^{1} + \left\{(1+a)y - \dfrac{a}{2}y^2\right\}_1^{1+\frac{1}{a}}$

$= \dfrac{1}{a}\{(-1+a) - e^{-a}(-a-1) - ae^{-a}\}$

$+ \left[(1+a)\left(1 + \dfrac{1}{a} - 1\right) - \dfrac{a}{2}\left\{\left(1 + \dfrac{1}{a}\right)^2 - 1^2\right\}\right]$

$= \dfrac{1}{a}[-1 + a + ae^{-a} + e^{-a} - ae^{-a}] + \left[\dfrac{1+a}{a} - \dfrac{a}{2}\left(2 + \dfrac{1}{a}\right)\dfrac{1}{a}\right]$

$= \dfrac{1}{a}(-1 + a + e^{-a}) + \left(\dfrac{1}{a} + 1 - 1 - \dfrac{1}{2a}\right) = \left(1 + \dfrac{e^{-a}}{a} - \dfrac{1}{2a}\right)$ sq units

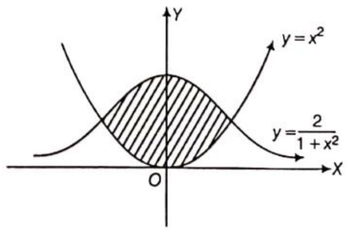
● ***Ex. 21*** *Sketch the region bounded by the curves* $y = x^2$ *and* $y = \dfrac{2}{1+x^2}$. *Find the area.*

Sol. For intersection point, $x^2 = \dfrac{2}{1+x^2}$

i.e. $\quad x^4 + x^2 - 2 = 0$

i.e. $\quad (x^2 + 2)(x^2 - 1) = 0$

i.e. $\quad x = \pm 1$



Hence, required area $= 2\int_0^1 \left(\frac{2}{1+x^2} - x^2\right) dx$

$$= 2\left[2\tan^{-1} x - \frac{x^3}{3}\right]_0^1 = 2\left(\frac{\pi}{2} - \frac{1}{3}\right)$$

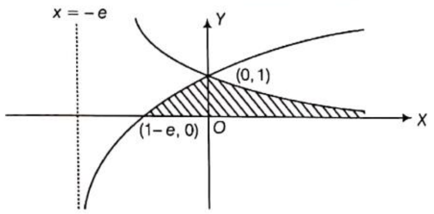
Ex. 22 *Find the area enclosed between the curves*

$$y = \log(x+e);\ x = \log_e\left(\frac{1}{y}\right) \text{ and X-axis.}$$

Sol. The given curves are $y = \log(x+e)$ and

$$x = \log_e\left(\frac{1}{y}\right) \Rightarrow \frac{1}{y} = e^x \Rightarrow y = e^{-x}$$

Using graph transformation we can sketch the curves.



Hence, required area $= \int_{1-e}^{0} \log(x+e)\,dx + \int_0^{\infty} e^{-x}\,dx$

$$= \int_1^e \log(t)\,dt + \int_0^{\infty} e^{-x}\,dx$$

(putting $x + e = t$)

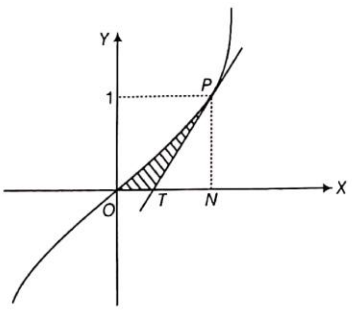
$$= [t\log t - t]_1^e - [e^{-x}]_0^{\infty} = 1 + 1 = 2$$

Ex. 23 *Find the area of the region bounded by the curve $c: y = \tan x$, tangent drawn to c at $x = \pi/4$ and the X-axis.*

Sol. The given curve is $y = \tan x$

$$\therefore \quad \frac{dy}{dx} = \sec^2 x$$

$$\left(\frac{dy}{dx}\right)_{x=\pi/4} = \sec^2\frac{\pi}{4} = 2$$



Also, at $x = \frac{\pi}{4}$; $y = 1$

$\therefore$ The equation of the tangent to the curve at the point $\left(\frac{\pi}{4}, 1\right)$ is $\quad y - 1 = 2\left(x - \frac{\pi}{4}\right)$

when $\quad y = 0;\ x = \frac{\pi}{4} - \frac{1}{2} = OT$

Now, the required area = Area of curvilinear ΔOPN − Area of ΔPTN

$$= \int_0^{\pi/4} (\tan x)\,dx - \frac{1}{2}\cdot NT\cdot PN$$

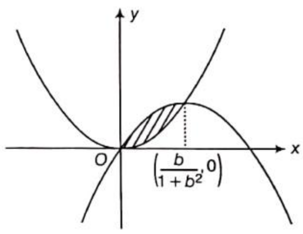
$$= [\log(\sec x)]_0^{\pi/4} - \frac{1}{2}\left(\frac{\pi}{4} - \frac{\pi}{4} + \frac{1}{2}\right)\cdot 1 = \frac{1}{2}\left(\log 2 - \frac{1}{2}\right)$$

Ex. 24 *Find all the possible values of $b > 0$, so that the area of the bounded region enclosed between the parabolas $y = x - bx^2$ and $y = \frac{x^2}{b}$ is maximum.*

Sol. Eliminating y from $y = \frac{x^2}{b}$ and $y = x - bx^2$, we get

$$x^2 = bx - b^2x^2$$

$$\Rightarrow \quad x = 0, \frac{b}{1+b^2}$$



Thus, the area enclosed between the parabolas,

$$A = \int_0^{b/(1+b^2)} \left(x - bx^2 - \frac{x^2}{b}\right) dx$$

$$= \int_0^{b/1+b^2} \left\{x - x^2\left(\frac{1+b^2}{b}\right)\right\} dx$$

$$= \left(\frac{x^2}{2} - \frac{x^3}{3}\cdot\frac{1+b^2}{b}\right)_0^{b/1+b^2} = \frac{1}{6}\cdot\frac{b^2}{(1+b^2)^2}$$

For maximum value of A, $\frac{dA}{db} = 0$

But $\frac{dA}{db} = \frac{1}{6}\cdot\frac{(1+b^2)^2\cdot 2b - b^2\cdot 2(1+b^2)\cdot 2b}{(1+b^2)^4} = \frac{1}{3}\cdot\frac{b(1-b^2)}{(1+b^2)^3}$

Hence, $\frac{dA}{db} = 0$ gives $b = -1, 0, 1$ since $b > 0$

Therefore, we consider only $b = 1$

Sign scheme for $\frac{dA}{db}$ around $b = 1$ is as below

0 ← + — 1 — − → ∞

From sign scheme it is clear that A is maximum.

Ex. 25 *Let C_1 and C_2 be the graphs of the function $y = x^2$ and $y = 2x, 0 \le x \le 1$, respectively. Let C_3 be the graph of a function $y = f(x); 0 \le x \le 1, f(0) = 0$. For a point P on C_1, let the lines through P parallel to the axes, meets C_2 and C_3 at Q and R respectively. If for every position of P on (C_1), the areas of the shaded region OPQ and ORP are equal, determine the function $f(x)$.* **[IIT JEE 1998]**

Sol. On the curve C_1, i.e. $y = x^2$.

Let P be (α, α^2). So, ordinate of point Q on C_2 is also α^2.

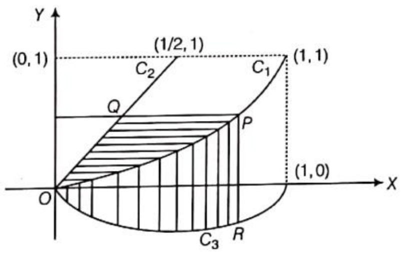
Now, C_2 ($y = 2x$) the abscissae of Q is given by $x = \frac{y}{2} = \frac{\alpha^2}{2}$.

$\therefore$ Q is $\left(\frac{\alpha^2}{2}, \alpha\right)$ and R on C_3 is $\{\alpha, f(x)\}$.

Now, area of $\Delta OPQ = \int_0^{\alpha^2} (x_1 - x_2)\, dy = \int_0^{\alpha^2} \left(\sqrt{y} - \frac{y}{2}\right) dy$

$$= \frac{2}{3}\alpha^3 - \frac{\alpha^4}{4} \quad \text{...(i)}$$

Again, area of $\Delta ORP = \int_0^{\alpha} (y_1 - y_2) dx = \int_0^{\alpha^2} \{x^2 - f(x)\}\, dx$ ...(ii)



Thus, from Eqs. (i) and (ii), we get

$$\frac{2\alpha^3}{3} - \frac{\alpha^4}{4} = \int_0^{\alpha} \{x^2 - f(x)\}\, dx$$

Differentiating both the sides w.r.t. α, we get

$$2\alpha^2 - \alpha^3 = \alpha^2 - f(\alpha)$$

$$\Rightarrow \quad f(\alpha) = \alpha^3 - \alpha^2 \Rightarrow f(x) = x^3 - x^2$$

Ex. 26 *Find the area of the region bounded by the curves $y = ex \log x$ and $y = \frac{\log x}{ex}$.* **[IIT JEE 1990]**

Sol. Both the curves are defined for $x > 0$. Both are positive when $x > 1$ and negative when $0 < x < 1$.

We know that, $\lim_{x \to 0^+} \log x \to -\infty$

Therefore, $\lim_{x \to 0^+} \frac{\log x}{ex} \to -\infty$

Thus, Y-axis is asymptote of second curve.

and $\lim_{x \to 0^+} ex \log x$ [(0) $(-\infty)$ form]

$$= \lim_{x \to 0^+} \frac{e \log x}{1/x} \qquad \left(-\frac{\infty}{\infty} \text{ form}\right)$$

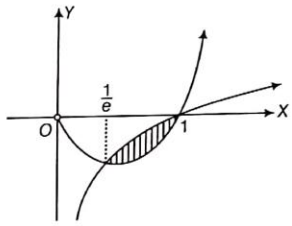
$$= \lim_{x \to 0^+} \frac{e(1/x)}{(-1/x^2)} = 0 \quad \text{(using L'Hospital's rule)}$$

Thus, the first curve starts from (0, 0) but does not include (0, 0). Now, the given curves intersect therefore

$$ex \log x = \frac{\log x}{ex}$$

i.e. $(e^2x^2 - 1) \log x = 0$

i.e. $x = 1, \frac{1}{e}$ $\qquad (\because x > 0)$



Therefore, using the above results figure could be drawn as

$\therefore$ Required area $= \int_{1/e}^{1} \left(\frac{\log x}{ex} - ex \log x\right) dx$

$$= \frac{1}{e}\left[\frac{(\log x)^2}{2}\right]_{1/e}^{1} - e\left[\frac{x^2}{4}(2\log x - 1)\right]_{1/e}^{1} = \frac{e^2 - 5}{4e}$$

Ex. 27 *Let A_n be the area bounded by the curve $y = (\tan x)^n$ and the lines $x = 0, y = 0$ and $x = \frac{\pi}{4}$. Prove that for $n > 2$; $A_n + A_{n-2} = \frac{1}{n-1}$ and deduce that $\frac{1}{2n+2} < A_n < \frac{1}{2n-2}$.*

Sol. First part We have, $A_n = \int_0^{\pi/4} (\tan x)^n\, dx$

Hence, $A_{n-2} = \int_0^{\pi/4} (\tan x)^{n-2}\, dx$

$\therefore$ $A_n + A_{n-2} = \int_0^{\pi/4} (\tan x)^{n-2} (\tan^2 x + 1)\, dx$

$$= \int_0^{\pi/4} (\tan x)^{n-2} \cdot \sec^2 x\, dx$$

Let $\tan x = t$, so that $\sec^2 x\, dx = dt$

$\therefore$ $A_n + A_{n-2} = \int_0^1 t^{n-2}\, dt = \left(\frac{t^{n-1}}{n-1}\right)_0^1 = \frac{1}{n-1}$ ...(i)

Second part

Since, $0 \le x \le \pi/4 \therefore 0 \le \tan x \le 1$

$\Rightarrow$ $\tan^{n+2} x < \tan^n x < \tan^{n-2} x$

$\Rightarrow$ $\int_0^{\pi/4} \tan^{n+2} x\, dx < \int_0^{\pi/4} \tan^n x\, dx < \int_0^{\pi/4} \tan^{n-2} x\, dx$

$\Rightarrow$ $A_{n+2} < A_n < A_{n-2} \Rightarrow A_n + A_{n+2} < 2A_n < A_n + A_{n-2}$

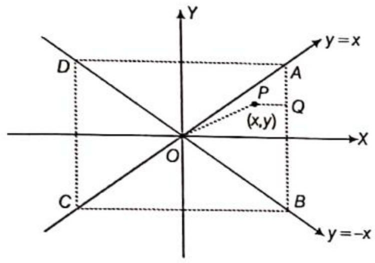
$\therefore \qquad \dfrac{1}{n+1} < 2A_n < \dfrac{1}{n-1}$

$\Rightarrow \qquad \dfrac{1}{2(n+1)} < A_n < \dfrac{1}{2(n-1)}$ [using Eq. (i)]

• Ex. 28 *Consider a square with vertices at (1, 1) (−1, 1), (−1, −1) and (1, −1). Let S be the region consisting of all points inside the square which are nearer to the origin than to any edge. Sketch the region S and find its area.* **[IIT JEE 1995]**

Sol. For the points lying in the ΔOAB the edge AB, i.e. $x = 1$ is the closest edge. Therefore, if the distance of a point $P(x, y)$ (lying in the ΔOAB) from origin is less than that of its distance from the edge $x = 1$ it will fall in the region S.

$\therefore \qquad OP \le PQ$



$\Rightarrow \qquad \sqrt{x^2 + y^2} \le 1 - x$

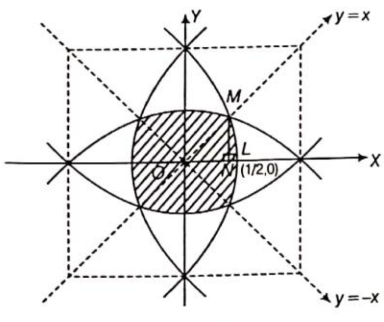
$\Rightarrow \qquad x^2 + y^2 \le x^2 - 2x + 1$

$\Rightarrow \qquad y^2 \le 1 - 2x$

Similarly, for points lying in the ΔOAD the side $y = 1$ is the closest side and therefore the region S is determined by

$$x^2 \le 1 - 2y$$

Since, the edges are symmetric about the origin. Hence, by the above inequality and by symmetry, the required area will be the shaded portion in the figure given below



Now, when the curves $y^2 = 1 - 2x$ and $y = x$ intersect each other, then

$$x^2 = 1 - 2x \Rightarrow x^2 + 2x - 1 = 0$$

$\Rightarrow \qquad x = \sqrt{2} - 1, -\sqrt{2} - 1$

Hence, the intersection points in the first quadrant is $(\sqrt{2} - 1, \sqrt{2} - 1)$.

$\therefore$ Required area = 8 [Area of curvilinear ΔOLM]

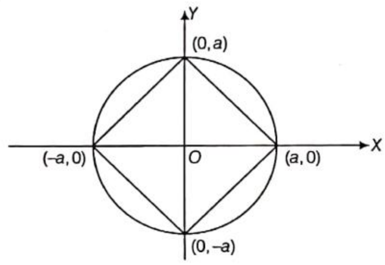
$$= 8\left[\frac{1}{2}(\sqrt{2}-1)(\sqrt{2}-1) + \int_{\sqrt{2}-1}^{1/2} \sqrt{1-2x}\, dx\right]$$

$$= 8\left[\frac{1}{2}(3 - 2\sqrt{2}) + \left(\frac{2(1-2x)^{3/2}}{3(-2)}\right)_{\sqrt{2}-1}^{1/2}\right]$$

$$= \frac{4}{3}(4\sqrt{2} - 5)$$

• Ex. 29 *Sketch the region included between the curves $x^2 + y^2 = a^2$ and $\sqrt{|x|} + \sqrt{|y|} = \sqrt{a}$ $(a > 0)$ and find its area.*

Sol. The graphs $|x| + |y| = a$ and $|x|^2 + |y|^2 = a^2$ are as shown in figure.



From the figure it can be concluded that when powers of $|x|$ and $|y|$ both is reduced to half the straight lines get stretched inside taking the shape as above.

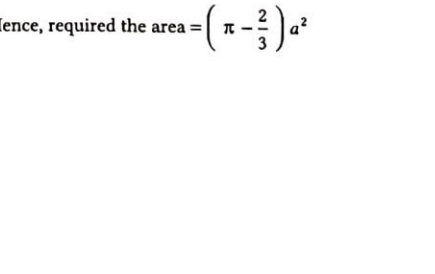
Thus, required area = 4 [shaded area in the first quadrant]

$$= 4\left[\frac{\pi a^2}{4} - \int_0^a (\sqrt{a} - \sqrt{x})^2\, dx\right]$$

(since in 1st quadrant $x, y > 0$), hence

$$\sqrt{|x|} + \sqrt{|y|} = \sqrt{a} \Rightarrow \sqrt{x} + \sqrt{y} = \sqrt{a}$$

$\Rightarrow \qquad y = (\sqrt{a} - \sqrt{x})^2$



Hence, required the area $= \left(\pi - \dfrac{2}{3}\right)a^2$

• Ex. 30 *Show that the area included between the parabolas $y^2 = 4a(x+a)$ and $y^2 = 4b(b-x)$ is $\frac{8}{3}(a+b)\sqrt{ab}$.*

Sol. Given parabolas are

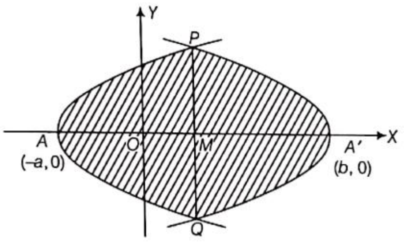
$$y^2 = 4a(x+a) \quad \text{...(i)}$$

and $$y^2 = 4b(b-x) \quad \text{...(ii)}$$

Solving Eqs. (i) and (ii), we get

$$x = (b-a) \quad \text{and} \quad y = \pm 2\sqrt{ab}$$

$\therefore$ P and Q are $(b-a, 2\sqrt{ab})$ and $(b-a, -2\sqrt{ab})$ respectively, and the points A and A' are $(-a, 0)$ and $(b, 0)$, respectively.



Now, required area = Area $APA'QA$ = 2 Area $APA'A$

$$= 2\,[\text{Area } APMA + \text{Area } MPA'M]$$

$$= 2\left[\int_{-a}^{b-a} 2\sqrt{a(a+x)}\,dx + \int_{b-a}^{b} 2\sqrt{b(b-x)}\,dx\right]$$

$$= 4\sqrt{a}\int_{-a}^{b-a}\sqrt{a+x}\,dx + 4\sqrt{b}\int_{b-a}^{b}\sqrt{b-x}\,dx$$

$$= 4\sqrt{a}\left[\frac{2}{3}(a+x)^{3/2}\right]_{-a}^{b-a} + 4\sqrt{b}\left[-\frac{2}{3}(b-x)^{3/2}\right]_{b-a}^{b}$$

$$= \frac{8}{3}\sqrt{a}\,(b)^{3/2} + \frac{8\sqrt{b}}{3}(a)^{3/2} = \frac{8}{3}\sqrt{ab}(a+b) \text{ sq units}$$

• Ex. 31 *Determine the area of the figure bounded by two branches of the curve $(y-x)^2 = x^3$ and the straight line $x = 1$.*

Sol. Given curves are $(y-x)^2 = x^3$

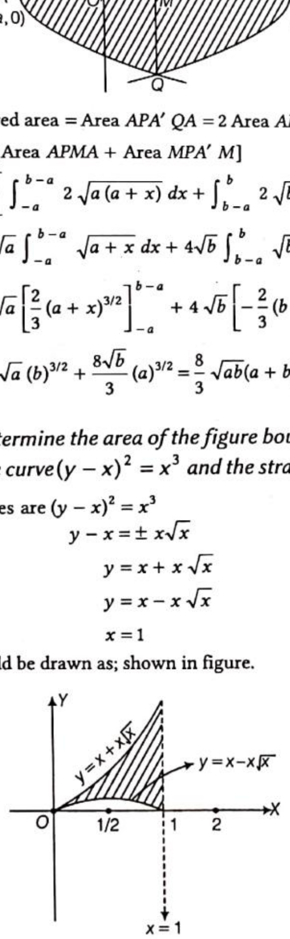
$$y - x = \pm x\sqrt{x}$$

$$y = x + x\sqrt{x} \quad \text{...(i)}$$

$$y = x - x\sqrt{x} \quad \text{...(ii)}$$

and $$x = 1 \quad \text{...(iii)}$$

Which could be drawn as; shown in figure.



Hence, required area

$$= \int_0^1 \{(x + x\sqrt{x}) - (x - x\sqrt{x})\}\,dx$$

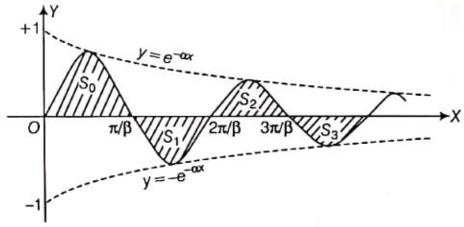
$$= \int_0^1 (2x\sqrt{x})\,dx$$

$$= 2\int_0^1 x^{3/2}\,dx = \frac{4}{5} \text{ sq unit}$$

• Ex. 32 *Prove that the areas $S_0, S_1, S_2, \ldots$ bounded by the X-axis and half-waves of the curve $y = e^{-\alpha x}\sin\beta x$, $x\,|\,0$. form a geometric progression with the common ratio $g = e^{-\pi\alpha/\beta}$.*

Sol. The curve $y = e^{-\alpha x}\sin\beta x$ intersects the positive X-axis at the points where $y = 0$.

$$\therefore \quad e^{-\alpha x}\sin\beta x = 0$$



$$\Rightarrow \quad \sin\beta x = 0 \Rightarrow x_n = \frac{n\pi}{\beta}, \; n = 0, 1, 2, \ldots$$

The function $y = e^{-\alpha x}\sin\beta x$ is positive in the interval (x_{2K}, x_{2K+1}) and negative in (x_{2K+1}, x_{2K+2}), i.e. the sign of the function in the interval (x_n, x_{n+1}), therefore

$$S_n = \left|\int_{n\pi/\beta}^{(n+1)\pi/\beta} e^{-\alpha x}\sin\beta x\,dx\right|$$

$$= \left|\left\{\frac{(-1)^{n+1}e^{-\alpha x}}{\alpha^2+\beta^2}(\alpha\sin\beta x + \beta\cos\beta x)\right\}_{n\pi/\beta}^{(n+1)\pi/\beta}\right|$$

$$= \frac{\beta e^{-n\pi\alpha/\beta}}{(\alpha^2+\beta^2)}\{1 + e^{-\pi\alpha/\beta}\}$$

$$\text{Hence, } g = \frac{S_{n+1}}{S_n} = \frac{\dfrac{\beta e^{-(n+1)\pi\alpha/\beta}\{1+e^{-\pi\alpha/\beta}\}}{(\alpha^2+\beta^2)}}{\dfrac{\beta e^{-n\pi\alpha/\beta}\{1+e^{-\pi\alpha/\beta}\}}{\alpha^2+\beta^2}} = e^{-\pi\alpha/\beta}$$

which completes the proof.

• Ex. 33 *Let $b \neq 0$ and for $j = 0, 1, 2, \ldots, n$. Let S_j be the area of the region bounded by Y-axis and the curve $x \cdot e^{ay} = \sin by$, $\frac{j\pi}{b} \leq y \leq \frac{(j+1)\pi}{b}$. Show that $S_0, S_1, S_2, \ldots, S_n$ are in geometric progression. Also, find their sum for $a = -1$ and $b = \pi$.*

Sol. Here, $S_j = \left|\int_{j\pi/b}^{(j+1)\pi/b} x\,dy\right| = \left|\int_{j\pi/b}^{(j+1)\pi/b} e^{-ay}\sin by\,dy\right|$

$$= \left| \left\{ \frac{e^{-ay}}{a^2+b^2}(-a\sin by - b\cos by) \right\}_{j\pi/b}^{(j+1)\pi/b} \right|$$

$$= \left| \frac{e^{-a(j+1)\pi/b}}{a^2+b^2} \times (-b)(-1)^{j+1} - \frac{e^{-aj\pi/b}}{a^2+b^2}(-b)(-1)^j \right|$$

$$= |b| \left\{ \frac{e^{-a(j+1)\pi/b}}{a^2+b^2} + \frac{e^{-aj\pi/b}}{a^2+b^2} \right\}$$

$$= |b| \frac{e^{-aj\pi/b}}{a^2+b^2} \{e^{-a\pi/b}+1\}; \ \{j = 0, 1, 2, \dots, n\}$$

Now, $\dfrac{S_{j+1}}{S_j} = |b| \dfrac{e^{-a(j+1)\pi/b}}{a^2+b^2}\{e^{-a\pi/b}+1\} \,\Big|\, |b| \dfrac{e^{-aj\pi/b}}{a^2+b^2}\{e^{-a\pi/b}+1\}$

$= e^{-a\pi/b}$ for all $j = 0, 1, 2, \dots, n$

Hence, $S_0, S_1, S_2, \dots, S_n$ are in GP with common ratio $e^{-a\pi/b}$.

For $a = -1$ and $b = \pi$, we have

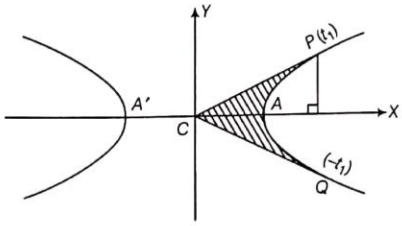
$$S_j = \frac{e^j\,\pi\,(e+1)}{\pi^2+1}; \ j = 0, 1, 2, \dots, n$$

$$\therefore \quad \sum_{j=0}^{n} S_j = \sum_{j=0}^{n} \frac{e^{j\pi}(e+1)}{\pi^2+1} = \frac{(e+1)\pi}{\pi+1}\left\{\frac{e^{n+1}-1}{e-1}\right\}$$

● ***Ex. 34*** *For any real t, $x = 2 + \dfrac{e^t + e^{-t}}{2}$, $y = 2 + \dfrac{e^t - e^{-t}}{2}$ is a point on the hyperbola $x^2 - y^2 - 4x + 4y - 1 = 0$. Find the area bounded by the hyperbola and the lines joining the centre to the points corresponding to t_1 and $-t_1$.*

Sol. The points $x = 2 + \dfrac{e^t + e^{-t}}{2}$, $y = 2 + \dfrac{e^t - e^{-t}}{2}$ is on the curve

$(x-2)^2 - (y-2)^2 = 1$ or $x^2 - y^2 - 4x + 4y - 1 = 0$



Put $(x-2) = x, (y-2) = y$

$$\therefore \quad x^2 - y^2 = 1 \text{ and } x = \frac{e^t + e^{-t}}{2}, y = \frac{e^t - e^{-t}}{2}$$

We have to find the area of the region bounded by the curve $x^2 - y^2 = 1$ and the lines joining the centre $x = 0, y = 0$ to the point (t_1) and $(-t_1)$.

$$\therefore \text{Required area} = 2\left[\text{Area of } \Delta PCN - \int_1^{\frac{e^{t_1}+e^{-t_1}}{2}} y\,dx\right]$$

$$= 2\left[\frac{1}{2}\left(\frac{e^{t_1}+e^{-t_1}}{2}\right)\left(\frac{e^{t_1}-e^{-t_1}}{2}\right) - \int_1^{t_1} y\frac{dx}{dt}\,dt\right]$$

$$= 2\left[\left(\frac{e^{2t_1}-e^{-2t_1}}{8}\right) - \int_0^{t_1}\left(\frac{e^t-e^{-t}}{2}\right)^2 dt\right]$$

$$= \frac{e^{2t_1}-e^{-2t_1}}{4} - \frac{1}{2}\int_0^{t_1}(e^{2t}+e^{-2t}-2)\,dt$$

$$= \frac{e^{2t_1}-e^{-2t_1}}{4} - \frac{1}{2}\left[\frac{e^{2t_1}}{2} - \frac{e^{-2t_1}}{2} - 2t_1\right]_0^{t_1}$$

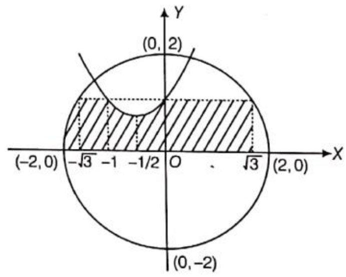
$$= \frac{e^{2t_1}-e^{-2t_1}}{4} - \frac{1}{2}\left[\frac{e^{2t_1}}{2} - \frac{e^{-2t_1}}{2} - 2t_1\right] = t_1$$

● ***Ex. 35*** *Find the area enclosed by circle $x^2 + y^2 = 4$, parabola $y = x^2 + x + 1$, the curve $y = \left[\sin^2\dfrac{x}{4} + \cos\dfrac{x}{4}\right]$ and X-axis (where, [.] is the greatest integer function).*

Sol. $\because \quad y = \left[\sin^2\dfrac{x}{4} + \cos\dfrac{x}{4}\right]$

$\because \quad 1 < \sin^2\dfrac{x}{4} + \cos\dfrac{x}{4} < 2 \qquad$ for $x \in (-2, 2]$

$\therefore \quad y = \left[\sin^2\dfrac{x}{4} + \cos\dfrac{x}{4}\right] = 1$



Now, we have to find out the area enclosed by the circle $x^2 + y^2 = 4$, parabola $\left(y - \dfrac{3}{4}\right) = \left(x + \dfrac{1}{2}\right)^2$, line $y = 1$ and X-axis. Required area is shaded area in the figure.

Hence, required area

$$= \sqrt{3}\times 1 + (\sqrt{3}-1)\times 1 + \int_{-1}^{0}(x^2 + x + 1)\,dx + 2\int_{\sqrt{3}}^{2}(\sqrt{4-x^2})\,dx$$

$$= (2\sqrt{3}-1) + \left[\frac{x^3}{3} + \frac{x^2}{2} + x\right]_{-1}^{0} + 2\left[\frac{x}{2}\sqrt{4-x^2} + 2\sin^{-1}\left(\frac{x}{2}\right)\right]_{\sqrt{3}}^{2}$$

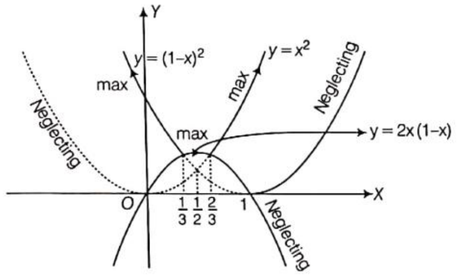
$$= (2\sqrt{3}-1) + \left[0 - \left(-\frac{1}{3} + \frac{1}{2} - 1\right)\right] + 2\left[(0+\pi) - \left(\frac{\sqrt{3}}{2} + \frac{2\pi}{3}\right)\right]$$

$$= (2\sqrt{3}-1) + \frac{5}{6} + \frac{2\pi}{3} - \sqrt{3} = \left(\frac{2\pi}{3} + \sqrt{3} - \frac{1}{6}\right) \text{ sq units}$$

● ***Ex. 36*** *Let $f(x) = \max\{x^2, (1-x)^2, 2x(1-x)\}$, where $0 \le x \le 1$. Determine the area of the region bounded by the curves $y = f(x)$, X-axis, $x = 0$ and $x = 1$.* **[IIT JEE 1997]**

Sol. We have, $f(x) = \max\{x^2, (1-x^2), 2x(1-x)\}$

Graphically it could be shown as;



For figure it is clear that maximum graph (i.e. above max graph is considered and others are neglected).

$\therefore$ For $x \in \left[0, \frac{1}{3}\right]$, $x^2 \le 2x(1-x) \le (1-x)^2$

For $x \in \left[\frac{1}{3}, \frac{1}{2}\right]$, $x^2 \le (1-x)^2 \le 2x(1-x)$

For $x \in \left[\frac{1}{2}, \frac{2}{3}\right]$, $(1-x)^2 \le x^2 \le 2x(1-x)$

For $x \in \left[\frac{2}{3}, 1\right]$, $(1-x)^2 \le 2x(1-x) \le x^2$

Hence, $f(x)$ can be written as

$$f(x) = \begin{cases} (1-x)^2, & \text{for } 0 \le x \le 1/3 \\ 2x(1-x), & \text{for } 1/3 \le x \le 2/3 \\ x^2, & \text{for } 2/3 \le x \le 1 \end{cases}$$

Hence, the area bounded by the curve $y = f(x)$; X-axis and the lines $x = 0$ and $x = 1$ is given by

$$= \int_0^{1/3} (1-x)^2\,dx + \int_{1/3}^{2/3} 2x(1-x)\,dx + \int_{2/3}^{1} (x^2)\,dx$$

$$= \frac{17}{27} \text{ sq unit}$$

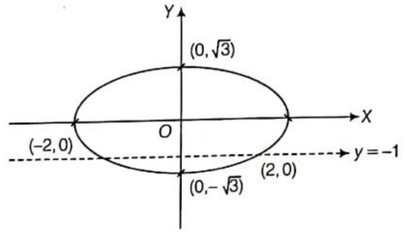
● ***Ex. 37*** *Find the ratio in which the curve, $y = [-0.01x^4 - 0.02x^2]$ [where, [.] denotes the greatest integer function) divides the ellipse $3x^2 + 4y^2 = 12$.*

Sol. Here, $y = [-0.01x^4 - 0.02x^2]$

i.e. $y = -1$, when $-2 < x < 2$

$y = -1$ cut the ellipse $3x^2 + 4y^2 = 12$

At $x^2 = \frac{8}{3}$ or $x = \pm\frac{2\sqrt{2}}{\sqrt{3}}$



$$\text{Required area} = \int_{-2\sqrt{6}/3}^{2\sqrt{6}/3} \left\{\sqrt{\frac{12-3x^2}{4}} - 1\right\} dx$$

$$= \frac{\sqrt{3}}{2}\int_{-2\sqrt{6}/3}^{2\sqrt{6}/3} \sqrt{4-x^2}\,dx - \int_{-2\sqrt{6}/3}^{2\sqrt{6}/3} 1\,dx$$

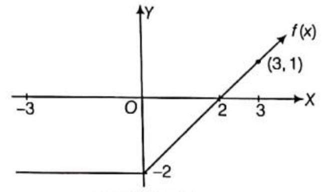
$$= 2\sin^{-1}\sqrt{\frac{2}{3}} + \frac{2}{3} - \frac{4}{3}\sqrt{6}$$

● ***Ex. 38*** *Let* $f(x) = \begin{cases} -2, & -3 \le x \le 0 \\ x-2, & 0 < x \le 3 \end{cases}$, *where $g(x) = \min\{f(|x|) + |f(x)|, f(|x|) - |f(x)|\}$. Find the area bounded by the curve $g(x)$ and the X-axis between the ordinates at $x = 3$ and $x = -3$.*

Sol. Here, $f(|x|) = \begin{cases} -x-2, & -3 \le x \le 0 \\ x-2, & 0 < x \le 3 \end{cases}$

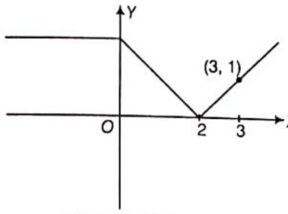
$$|f(x)| = \begin{cases} 2, & -3 \le x \le 0 \\ -x+2, & 0 < x \le 2 \\ x-2, & 2 < x \le 3 \end{cases}$$

$$\therefore\ f(|x|) - |f(x)| = \begin{cases} -x-4, & -3 \le x \le 0 \\ 2x-4, & 0 < x \le 2 \\ 0, & 2 < x \le 3 \end{cases}$$

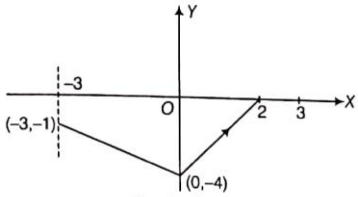


Graph of $f(x)$

Graph of $f(|x|)$



Graph of $|f(x)|$



Graph of $g(x)$

Since, $|f(x)|$ is always positive.

$$g(x) = f(|x|) - |f(x)|$$

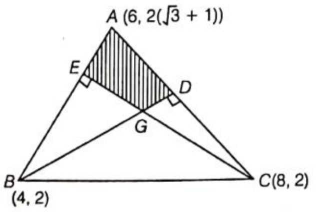
where the graphs could be drawn as shown in above figures.

From the graph, required area

$$= \frac{1}{2}(1+4)\times 3 + \left(\frac{1}{2}\times 2\times 4\right) + 0 = \frac{23}{2} \text{ sq units}$$

Ex. 39 *Let ABC be a triangle with vertices $A \equiv (6, 2(\sqrt{3}+1))$, $B \equiv (4, 2)$ and $C \equiv (8, 2)$. Let R be the region consisting of all those points P inside Δ ABC which satisfy* $d(P, BC) \geq \max\{d(P, AB), d(P, AC)\}$, where $d(P, L)$ denotes the distance of the point P from the line L. Sketch the region R and find its area.

Sol. It is easy to see that ABC is an equilateral triangle with side of length 4. BD and CE are angle bisectors of angle B and C, respectively. Any point inside the $\Delta\, AEC$ is nearer to AC than BC and any point inside the $\Delta\, BDA$ is nearer to AB than BC. So any point inside the quadrilateral $AEGC$ will satisfy the given condition. Hence, shaded region is the required region, whose area is to be found, shown as in figure



Thus, required area = 2 Area of $\Delta EAG = 2 \times \frac{1}{2} AE \times EG$

$$= \frac{1}{2} AB \times \frac{1}{3} CE = \frac{1}{6}\times 4 \times \sqrt{4^2 - 2^2}$$

$$= \frac{4\sqrt{3}}{3} \text{ sq units}$$

Ex. 40 *Let $O(0,0)$, $A(2,0)$ and $B\left(1, \frac{1}{\sqrt{3}}\right)$ be the vertices of a triangle. Let R be the region consisting of all those points P inside Δ OAB which satisfy $d(P, OA) \leq \min\{d(P, OB), d(P, AB)\}$, when 'd' denotes the distance from the point to the corresponding line. Sketch the region R and find its area.* **[IIT JEE 1997]**

Sol. Let the coordinate of P be (x, y).

Equation of line $OA \equiv y = 0$

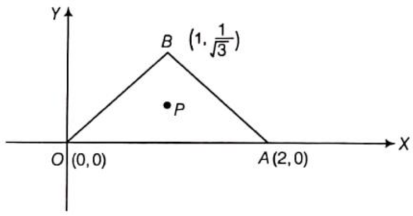
Equation of line $OB \equiv \sqrt{3}y = x$

Equation of line $AB = \sqrt{3}y = 2 - x$

$d(P, OA)$ = Distance of P from line $OA = y$

$d(P, OB)$ = Distance of P from line $OB = \dfrac{|\sqrt{3}y - x|}{2}$

$d(P, AB)$ = Distance of P from line $AB = \dfrac{|\sqrt{3}y + x - 2|}{2}$



Given, $d(P, OA) \leq \min\{d(P, OB), d(P, AB)\}$

$$y \leq \min\left\{\frac{|\sqrt{3}y - x|}{2}, \frac{|\sqrt{3}y + x - 2|}{2}\right\}$$

$$\Rightarrow \quad y \leq \frac{|\sqrt{3}y - x|}{2} \quad \text{...(i)}$$

and $$y \leq \frac{|\sqrt{3}y + x - 2|}{2} \quad \text{...(ii)}$$

Case I If $y \leq \dfrac{|\sqrt{3}y - x|}{2}$

$\therefore \quad y \leq \dfrac{x - \sqrt{3}y}{2}$, i.e. $x > \sqrt{3}y \qquad (\because \sqrt{3}y - x < 0)$

$\Rightarrow \quad (2+\sqrt{3})\, y \leq x$

$\Rightarrow \quad y \leq (2-\sqrt{3})\, x$

$\Rightarrow \quad y \leq x \tan 15° \quad$ ...(iii)

($\because y = x \tan 15°$ is an acute angle bisector of $\angle AOB$)

Case II If $y \leq \dfrac{|\sqrt{3}y + x - 2|}{2}$

$\Rightarrow \quad 2y \leq 2 - x - \sqrt{3}y \qquad$ (i.e. $\sqrt{3}y + x - 2 < 0$)

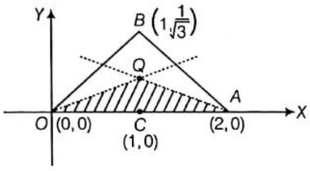
$\Rightarrow \quad (2+\sqrt{3})\, y \leq 2 - x$

$\Rightarrow \quad y \leq -(2-\sqrt{3})(x-2)$

$\Rightarrow \quad y \leq -(\tan 15°)(x-2) \quad$ ...(iv)

[$\because y = (x-2)\tan 15°$ is an acute angle bisector of CA]

From Eqs. (iii) and (iv), P moves inside the triangle as shown in figure.



As $\angle QOB = \angle OBQ = 15°$, ΔOQB is an isosceles triangle

$\Rightarrow$ $OC = AC =$ unit

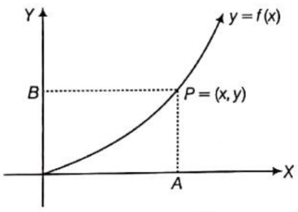
Area of shaded region = area of $\Delta OQA = \frac{1}{2}$ (base) $\times$ (height)

$$= \frac{1}{2}(2)(1 \tan 15°) = \tan 15°$$

$$= (2 - \sqrt{3}) \text{ sq units}$$

• Ex. 41 *A curve $y = f(x)$ passes through the origin and lies entirely in the first quadrant. Through any point $P(x, y)$ on the curve, lines are drawn parallel to the coordinate axes. If the curve divides the area formed by these lines and coordinate axes in $m:n$, find $f(x)$.*

Sol. Area of $(OAPB) = xy$, Area of $(OAPO) = \int_0^x f(t)\, dt$



Therefore, area of $(OBPO) = xy - \int_0^x f(t)\, dt$

According to the given condition,

$$\frac{xy - \int_0^x f(t)\, dt}{\int_0^x f(t)\, dt} = \frac{m}{n}$$

$$\Rightarrow \quad n\, xy = (m + n) \int_0^x f(t)\, dt$$

Differentiating w.r.t. x, we get

$$n\left(x\frac{dy}{dx} + y\right) = (m + n) f(x) = (m + n)\, y, \text{ as } y = f(x)$$

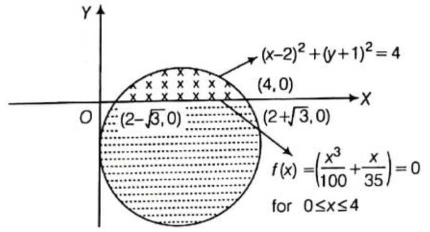
$$\frac{m}{n} \cdot \frac{dx}{x} = \frac{dy}{y} \Rightarrow \frac{m}{n} \cdot (\log x) = \log y - \log c, \text{ where } c \text{ is a constant.}$$

$$\Rightarrow \quad y = cx^{m/n}$$

• Ex. 42 *Find the ratio of the areas in which the curve $y = \left[\frac{x^3}{100} + \frac{x}{35}\right]$ divides the circle $x^2 + y^2 - 4x + 2y + 1 = 0$ (where, [.] denotes the greatest integer function).*

Sol. We have, $x^2 + y^2 - 4x + 2y + 1 = 0$

or $\quad (x - 2)^2 + (y + 1)^2 = 4 \quad$ ...(i)



Now, for $0 \le x \le 4$,

$$0 \le \frac{x^3}{100} + \frac{x}{35} < 1 \Rightarrow \left[\frac{x^3}{100} + \frac{x}{35}\right] = 0$$

So, we have to find out the ratio in which X-axis divides the circle (i).

Now, at X-axis, $\quad y = 0$

So, $\quad (x - 2)^2 = 3$

So, it cuts the X-axis at $(2 - \sqrt{3}, 0)$ and $(2 + \sqrt{3}, 0)$.

Therefore, required area, $A = \int_{2-\sqrt{3}}^{2+\sqrt{3}} \left(\sqrt{4 - (x - 2)^2} - 1\right) dx$

$$= \frac{4\pi - 3\sqrt{3}}{3}$$

$\therefore \quad$ Required ratio $= \frac{A}{4\pi - A} = \frac{4\pi - 3\sqrt{3}}{8\pi + 3\sqrt{3}}$

• Ex. 43 *Area bounded by the line $y = x$, curve $y = f(x)$, $(f(x) > x, \forall\, x > 1)$ and the lines $x = 1$, $x = t$ is $(t + \sqrt{1 + t^2}) - (1 + \sqrt{2})$ for all $t > 1$. Find $f(x)$.*

Sol. The area bounded by $y = f(x)$ and $y = x$ between the lines $x = 1$ and $x = t$ is $\int_{+1}^{t} (f(x) - x)\, dx$. But it is equal to $(t + \sqrt{1 + t^2}) - (1 + \sqrt{2})$.

So, $\quad \int_1^t (f(x) - x)\, dx = (t + \sqrt{1 + t^2}) - (1 + \sqrt{2})$

Differentiating both the sides w.r.t. t, we get

$$f(t) - t = 1 + \frac{t}{\sqrt{1 + t^2}} \Rightarrow f(t) = 1 + t + \frac{t}{\sqrt{1 + t^2}}$$

or $\quad f(x) = 1 + x + \frac{x}{\sqrt{1 + x^2}}$

• Ex. 44 *The area bounded by the curve $y = f(x)$, X-axis and ordinates $x = 1$ and $x = b$ is $(b - 1)\sin(3b + 4)$, find $f(x)$.*

Sol. We know that the area bounded by the curve $y = f(x)$, X-axis and the ordinates $x = 1$ and $x = b$ is $\int_1^b f(x)\, dx$.

From the question; $\int_1^b f(x)\, dx = (b - 1)\sin(3b + 4)$

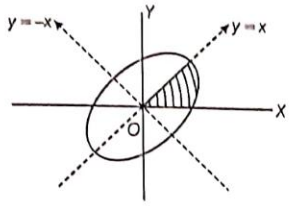
Differentiating w.r.t. b, we get

$$f(b) \cdot 1 = 3(b - 1)\cos(3b + 4) + \sin(3b + 4)$$

$$\Rightarrow \quad f(x) = 3(x - 1)\cos(3x + 4) + \sin(3x + 4)$$

• Ex. 45 *Find the area of region enclosed by the curve* $\frac{(x-y)^2}{a^2}+\frac{(x+y)^2}{b^2}=2\ (a>b)$, the line $y=x$ and the positive X-axis.

Sol. The given curve $\frac{(x-y)^2}{a^2}+\frac{(x+y)^2}{b^2}=2$ is an ellipse major and minor axes are $x-y=0$ and $x+y=0$, respectively. The required area is shown with shaded region.



Instead of directly solving the problem we can solve equivalent problem with equivalent ellipse whose axes are $x=0$ and $y=0$. The equivalent region is shown as $(OA'B'O)$ where the equation of ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

$\therefore$ Required area = Area $(\Delta OA'A'' + A'A''B')$

where, $A'=\left(\frac{ab}{\sqrt{a^2+b^2}},\frac{ab}{\sqrt{a^2+b^2}}\right)$

$$\text{Area } \Delta OA'A''=\frac{1}{2}\times\frac{ab}{\sqrt{a^2+b^2}}\times\frac{ab}{\sqrt{a^2+b^2}}=\frac{1}{2}\left(\frac{a^2b^2}{a^2+b^2}\right) \quad ...(i)$$

$$\text{Area of } A'B'A''=\int_{\frac{ab}{\sqrt{a^2+b^2}}}^{a} b\sqrt{\left(1-\frac{x^2}{a^2}\right)}\,dx$$

$$=\frac{b}{a}\cdot\int_{\frac{ab}{\sqrt{a^2+b^2}}}^{a}\sqrt{a^2-x^2}\,dx$$

$$=\frac{b}{a}\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_{ab/\sqrt{a^2+b^2}}^{a}$$

$$=\frac{b}{a}\left[0+\frac{a^2}{2}\cdot\frac{\pi}{2}-\frac{ab}{2\sqrt{a^2+b^2}}\cdot\sqrt{\left(a^2-\frac{a^2b^2}{a^2+b^2}\right)}-\frac{a^2}{2}\sin^{-1}\frac{b}{\sqrt{a^2+b^2}}\right]$$

$$=\frac{b}{a}\left[\frac{\pi a^2}{4}-\frac{a^2}{2}\sin^{-1}\left(\frac{b}{\sqrt{a^2+b^2}}\right)-\frac{a^3b}{2(a^2+b^2)}\right]$$

$$=\frac{\pi ab}{4}-\frac{ab}{2}\sin^{-1}\left(\frac{b}{\sqrt{a^2+b^2}}\right)-\frac{a^2b^2}{2(a^2+b^2)} \quad ...(ii)$$

Hence, required area = Sum of Eqs. (i) and (ii)

$$=\frac{\pi ab}{4}-\frac{ab}{2}\sin^{-1}\left(\frac{b}{\sqrt{a^2+b^2}}\right)$$

• Ex. 46 *Let $f(x)$ be a function which satisfy the equation $f(xy)=f(x)+f(y)$ for all $x>0, y>0$ such that $f'(1)=2$. Find the area of the region bounded by the curves $y=f(x), y=|x^3-6x^2+11x-6|$ and $x=0$.*

Sol. Take $x=y=1\Rightarrow f(1)=0$

Now, $y=\frac{1}{x}$

$$\Rightarrow \quad 0=f\left(x\cdot\frac{1}{x}\right)=f(x)+f\left(\frac{1}{x}\right)\Rightarrow f\left(\frac{1}{x}\right)=-f(x)$$

$$\therefore f\left(\frac{x}{y}\right)=f(x)+f\left(\frac{1}{y}\right)=f(x)-f(y) \quad ...(i)$$

Now, $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$$=\lim_{h\to 0}\frac{f\left(\frac{x+h}{x}\right)}{h} \quad \text{[using Eq. (i)]}$$

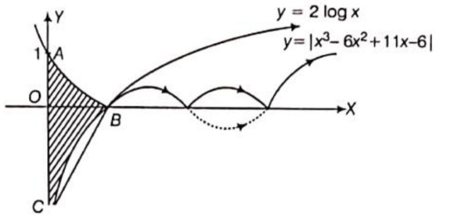
$$=\lim_{h\to 0}\frac{f\left(1+\frac{h}{x}\right)-f(1)}{\frac{h}{x}\cdot x}=\frac{f'(1)}{x} \quad [\because f(1)=0]$$

$$f'(x)=\frac{2}{x}$$

$\Rightarrow \quad f(x)=2\log x+c \quad$ [since, $f(1)=0\Rightarrow c=0$]

$\Rightarrow \quad f(x)=2\log x$

Thus, $f(x)=2\log x$ and $y=|x^3-6x^2+11x-6|$ could be plotted as



Hence, required area

$$=\int_0^1 (x^3-6x^2+11x-6)\,dx+\int_{-\infty}^{0} e^{y/2}\,dy$$

$$=\left(\frac{x^4}{4}-\frac{6x^3}{3}+\frac{11x^2}{2}-6x\right)_0^1+2\,(e^{y/2})_{-\infty}^{0}$$

$$=\left(\frac{1}{4}-2+\frac{11}{2}-6\right)-(0)+2\,(e^0-e^{-\infty})$$

$$=\frac{1}{4}+\frac{11}{2}-8+2=-\frac{1}{4}\text{ sq unit}$$

• Ex. 47 *Find the area of the region which contains all the points satisfying condition $|x-2y|+|x+2y|\le 8$ and $xy\ge 2$.*

Sol. The line $y=\pm\frac{x}{2}$ divide the xy plane in four parts

Region I $2y-x\le 0$ and $2y+x\ge 0$

So that, $|x-2y|+|x+2y|\le 8$

$\Rightarrow \quad (x-2y)+(x+2y)\le 8 \Rightarrow 0\le x\le 4$

Region II $2y - x | 0$ and $2y + x \geq 0$

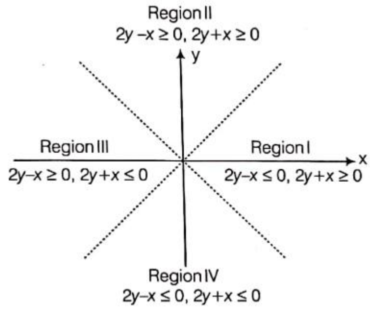
So that, $|x - 2y| + |x + 2y| \leq 8$

$\Rightarrow \quad -(x - 2y) + (x + 2y) \leq 8 \Rightarrow 0 \leq y \leq 2$

Region III $2y + x \leq 0, 2y - x \geq 0$

So that, $|x - 2y| + |x + 2y| \leq 8$

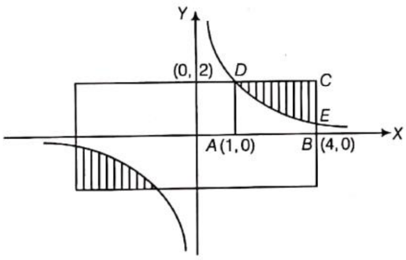
$\Rightarrow \quad -(x - 2y) - (x + 2y) \leq 8 \Rightarrow -4 \leq x \leq 0$



Region IV $2y + x \leq 0, 2y - x \leq 0$

So that, $|x - 2y| + |x + 2y| \leq 8 \Rightarrow -2 \leq y \leq 0$

Here, all the points lie in the rectangle.



Also, the hyperbola $xy = 2$ meets the sides of the rectangle at the points $(1, 2)$ and $(4, 1/2)$ in the 1st quadrant graphically.

Hence, required area

$= 2$ (Area of rectangle $ABCD$ – Area of $ABEDA$)

$$= 2\left(3 \times 2 - \int_1^4 \frac{2}{x}\, dx\right) = 2(6 - 2\log 4) \text{ sq units}$$

• Ex. 48 *Consider the function*

$$f(x) = \begin{cases} x - [x] - \dfrac{1}{2}, & \text{if } x \notin I \\ 0, & \text{if } x \in I \end{cases}$$

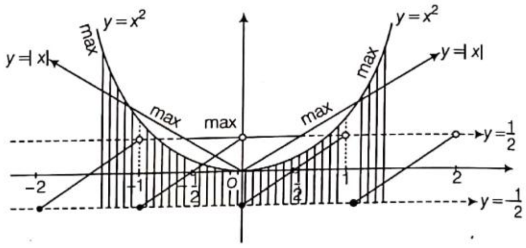
, where [.] denotes the greatest integer function and I is the set of integers. If $g(x) = \max\{x^2, f(x), |x|\}, -2 \leq x \leq 2$, then find the area bounded by $g(x)$ when $-2 \leq x \leq 2$.

Sol. Here, $f(x) = \begin{cases} x - [x] - \dfrac{1}{2}, & \text{if } x \notin I \\ 0, & \text{if } x \in I \end{cases}$

$$= \begin{cases} \{x\} - \dfrac{1}{2}, & \text{if } x \notin I \\ 0, & \text{if } x \in I \end{cases}$$

Thus, $g(x) = \max\{x^2, f(x), |x|\}$

which could be graphically expressed as



Clearly, $g(x) = \begin{cases} x^2, & -2 \leq x \leq -1 \\ -x, & -1 \leq x \leq -1/4 \\ x + \dfrac{1}{2}, & -1/4 \leq x \leq 0 \\ x, & 0 \leq x \leq 1 \\ x^2, & 1 \leq x \leq 2 \end{cases}$

Hence, required area $= \int_{-2}^{2} g(x)\, dx$

$$= \int_{-2}^{-1} (x^2)\, dx + \int_{-1}^{-1/4} (-x)\, dx + \int_{-1/4}^{0} \left(x + \frac{1}{2}\right) dx + \int_0^1 x\, dx + \int_1^2 x^2\, dx$$

$$= \left(\frac{x^3}{3}\right)_{-2}^{-1} - \left(\frac{x^2}{2}\right)_{-1}^{-1/4} + \left(\frac{x^2}{2} + \frac{1}{2}x\right)_{-1/4}^{0} + \left(\frac{x^2}{2}\right)_0^1 + \left(\frac{x^3}{3}\right)_1^2 = \frac{275}{48} \text{ sq units}$$

• Ex. 49 *Find the area of the region bounded by $y = f(x), y = |g(x)|$ and the lines $x = 0, x = 2$, where f, g are continuous functions satisfying*

$$f(x + y) = f(x) + f(y) - 8xy, \forall x, y \in R$$

and $g(x + y) = g(x) + g(y) + 3xy(x + y), \forall x, y \in R$

Also, $f'(0) = 8$ and $g'(0) = -4$.

Sol. Here, $f(x + y) = f(x) + f(y) - 8xy$

Replacing $x, y \to 0$, we get $f(0) = 0$

Now, $f'(x) = \lim_{h \to 0} \dfrac{f(x + h) - f(x)}{h} = \lim_{y \to 0} \dfrac{f(x + y) - f(x)}{y}$

$$= \lim_{y \to 0} \frac{f(x) + f(y) - 8xy - f(x)}{y}$$

$$= \lim_{y \to 0} \left\{\frac{f(y)}{y} - \frac{8xy}{y}\right\}$$

$$= \lim_{y \to 0} \left(\frac{f'(y)}{1}\right) - 8x \text{ (using L'Hospital's rule)}$$

$= f'(0) - 8x = 8 - 8x$ [given, $f'(0) = 8$]

$\Rightarrow \quad f'(x) = 8 - 8x$

Integrating both the sides, we get

$$f(x) = 8x - 4x^2 + c$$

As $\quad f(0) = 0 \Rightarrow c = 0$

$\Rightarrow \quad f(x) = 8x - 4x^2 \quad$ ...(i)

Also, $\quad g(x + y) = g(x) + g(y) + 3xy(x + y)$

Replacing $x, y \Rightarrow 0$, we get $g(0) = 0$

Now, $\quad g'(x) = \lim_{y \to 0} \dfrac{g(x+y) - g(x)}{y}$

$$= \lim_{y \to 0} \frac{g(x) + g(y) + 3x^2 y + 3xy^2 - g(x)}{y}$$

$$= \lim_{y \to 0} \left[\frac{g(y)}{y} + \frac{y(3x^2 + 3xy)}{y} \right]$$

$$= \frac{g'(0)}{1} + 3x^2 = -4 + 3x^2$$

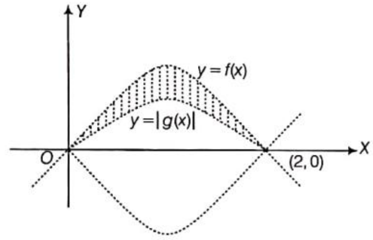
$\therefore \quad g'(x) = -4 + 3x^2$

$\Rightarrow \quad g(x) = x^3 - 4x \quad$ [as $g(0) = 0$] ...(ii)

Points where $f(x)$ and $g(x)$ meets, we have

$$f(x) = g(x) \quad \text{or} \quad 8x - 4x^2 = x^3 - 4x$$

$\Rightarrow \quad x = 0, 2, -6$



Now, $\quad |g(x)| = \begin{cases} x^3 - 4x, & x \in [-2, 0] \cup (2, \infty) \\ 4x - x^3, & x \in [-\infty, -2] \cup (0, 2) \end{cases}$

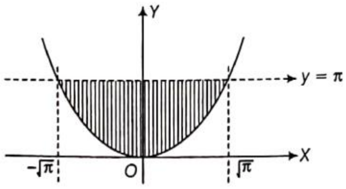
$\therefore$ Area bounded by $y = f(x)$ and $y = |g(x)|$ between $x = 0$ to $x = 2$

$$= \int_0^2 \{(8x - 4x^2) - (4x - x^3)\}\, dx$$

$$= \int_0^2 (x^3 - 4x^2 + 4x)\, dx = \frac{4}{3} \text{ sq units}$$

• Ex. 50 *Find the area of the region bounded by the curve $y = x^2$ and $y = \sec^{-1}[-\sin^2 x]$, where [.] denotes the greatest integer function.*

Sol. As we know, $[-\sin^2 x] = 0$ or -1. But $\sec^{-1}(0)$ is not defined.



$\Rightarrow \quad \sec^{-1}[-\sin^2 x] = \sec^{-1}(-1) = \pi$

Thus, to find the area bounded between

$$y = x^2 \quad \text{and} \quad y = \pi$$

i.e. when $x^2 = \pi$ or $(x = -\sqrt{\pi}$ to $x = \sqrt{\pi})$

$\therefore$ Required area $= \int_{-\sqrt{\pi}}^{\sqrt{\pi}} (\pi - x^2)\, dx = \left(\pi x - \dfrac{x^3}{3}\right)_{-\sqrt{\pi}}^{\sqrt{\pi}}$

$$= \pi(\sqrt{\pi} + \sqrt{\pi}) - \frac{1}{3}(\pi\sqrt{\pi} + \pi\sqrt{\pi}) = \frac{4\pi}{3}\sqrt{\pi}$$

• Ex. 51 *Sketch the graph of $\cos^{-1}(4x^3 - 3x)$ and find the area enclosed between $y = 0$, $y = f(x)$ and $x \geq -1/2$.*

Sol. Here, $\quad f(x) = \cos^{-1}(4x^3 - 3x)$

Let $\quad x = \cos\theta \quad$ and $\quad 0 \leq \theta \leq \pi$

$\Rightarrow \quad f(x) = \cos^{-1}(4\cos^3\theta - 3\cos\theta)$

$$= \cos^{-1}(\cos 3\theta);\ 0 \leq 3\theta \leq 3\pi$$

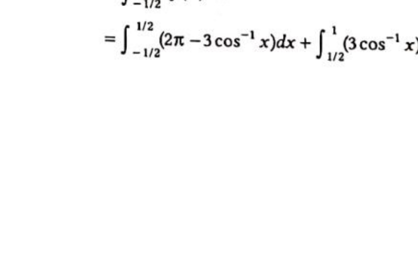
$\Rightarrow \quad f(x) = \begin{cases} 3\theta, & 0 \leq 3\theta \leq \pi \\ 2\pi - 3\theta, & \pi < 3\theta \leq 2\pi \\ 3\theta - 2\pi, & 2\pi < 3\theta \leq 3\pi \end{cases}$

$$= \begin{cases} 3\cos^{-1} x, & 1/2 \leq x \leq 1 \\ 2\pi - 3\cos^{-1} x, & -1/2 \leq x < 1 \\ 3\cos^{-1} x - 2\pi, & -1 \leq x < -1/2 \end{cases}$$

$\therefore \quad f'(x) = \begin{cases} -3/\sqrt{1 - x^2}, & 1/2 < x < 1 \\ 3/\sqrt{1 - x^2}, & -1/2 < x < 1/2 \\ -3/\sqrt{1 - x^2}, & -1 < x < -1/2 \end{cases}$

and $\quad f''(x) = \begin{cases} \dfrac{-3x}{(1 - x^2)^{3/2}}, & 1/2 < x < 1 \\ \dfrac{3x}{(1 - x^2)^{3/2}}, & -1/2 < x < 1/2 \\ \dfrac{-3x}{(1 - x^2)^{3/2}}, & -1 < x < -1/2 \end{cases}$

Thus, the graph for $f(x) = \cos^{-1}(4x^3 - 3x)$ is



Thus, required area $= \int_{-1/2}^{1} f(x)\, dx$

$$= \int_{-1/2}^{1/2} (2\pi - 3\cos^{-1} x)\,dx + \int_{1/2}^{1} (3\cos^{-1} x)\,dx$$

$$= 2\pi - 3\int_{-1/2}^{1/2}\left(\frac{\pi}{2} - \sin^{-1} x\right) dx + 3\int_{1/2}^{1} (\cos^{-1} x)\, dx$$

$$= \frac{\pi}{2} + 3\int_{1/2}^{1} \cos^{-1} x\, dx \qquad \left[\text{as} \int_{-1/2}^{1/2} \sin^{-1} x\, dx = 0\right]$$

$$= \frac{\pi}{2} + 3\left\{(x\cos^{-1} x)_{1/2}^{1} + \int_{1/2}^{1} \frac{x}{\sqrt{1-x^2}}\, dx\right\}$$

On solving, we get $= \frac{3\sqrt{3}}{2}$ sq units

● ***Ex. 52*** *Consider two curves $y^2 = 4a(x - \lambda)$ and $x^2 = 4a(y - \lambda)$, where $a > 0$ and λ is a parameter. Show that*

(i) there is a single positive value of λ for which the two curves have exactly one point of intersection in the 1st quadrant find it.

(ii) there are infinitely many negative values of λ for which the two curves have exactly one point of intersection in the 1st quadrant.

(iii) if $\lambda = -a$, then find the area of the bounded by the two curves and the axes in the 1st quadrant.

Sol. The two curves are inverse of each other. Hence, the two curves always meet along the line $y = x$.

Consider, $\quad y^2 = 4a(x - \lambda)$ and put $x = y$

$\Rightarrow \quad y^2 - 4ay + 4a\lambda = 0$

$\Rightarrow \quad y = \frac{4a \pm 4\sqrt{a^2 - a\lambda}}{2} = 2a \pm \sqrt{a^2 - a\lambda}$

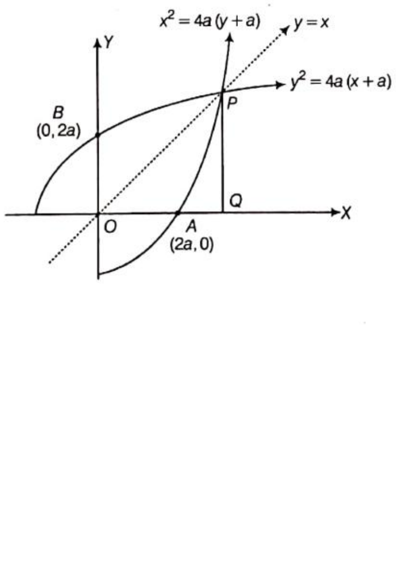
Since, y is real $\Rightarrow a^2 - a\lambda \mid 0$ or $\lambda \le a$

(i) If $0 < \lambda < a$, then there are two distinct values of y and both $2(a + \sqrt{a^2 - a\lambda})$ and $2(a - \sqrt{a^2 - a\lambda})$ are positive, i.e. both points lie in the first quadrant.

If $\lambda = a$, then $y = 2a$ only, i.e. only one point of intersection $(2a, 2a)$.

Hence, there is exactly one point of intersection in 1st quadrant for $\lambda = a$. It is infact the points of tangency of the two curves.

(ii) If $\lambda < 0$, then $y = 2(a + \sqrt{a^2 - a\lambda}) > 0$ and $y = 2(a - \sqrt{a^2 - a\lambda}) < 0$. i.e. the only point of intersection is in the first quadrant, the other in the 3rd quadrant. Hence, there are infinitely many such values.



(iii) For $\lambda = -a$, we have $y^2 = 4a(x + a)$

$$x^2 = 4a(y + a)$$

The point of intersection in the 1st quadrant $P = ((2 + 2\sqrt{2})a, (2 + 2\sqrt{2})a)$

Required area = 2 (area of ΔOPQ – area $\overline{APQA}$)

of $\Delta OPQ = \frac{1}{2}\cdot(2 + 2\sqrt{2})^2 a^2 = 2(1 + \sqrt{2})^2 a^2$

Area of $APQA = \int_{2a}^{(2+2\sqrt{2})a} y\, dx = \int_{2a}^{(2+2\sqrt{2})a} \left(\frac{x^2}{4a} - a\right) dx$

$$= \frac{1}{4a}\left[\frac{x^3}{3}\right]_{2a}^{(2+2\sqrt{2})a} - a(x)_{2a}^{(2+2\sqrt{2})a}$$

$$= \frac{1}{12a}[(2 + 2\sqrt{2})^3 \cdot a^3 - 8a^3] - 2\sqrt{2}\, a^2$$

$$= \left(\frac{4\sqrt{2} + 12}{3}\right) a^2$$

$$\therefore \text{ Required area} = 2\left(2(1 + \sqrt{2})^2 + \frac{4\sqrt{2} + 12}{3}\right) a^2$$

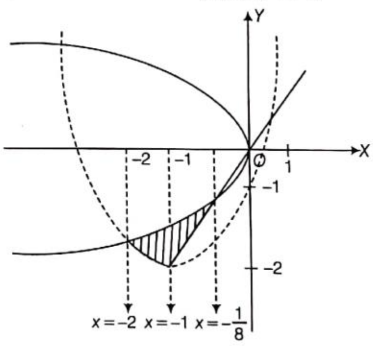
$$= \frac{4}{3}(15 + 8\sqrt{2})\, a^2$$

● ***Ex. 53*** *Let $f(x)$ be continuous function given by*

$$f(x) = \begin{cases} 2x, & |x| \le 1 \\ x^2 + ax + b, & |x| > 1 \end{cases}$$

Find the area of the region in the third quadrant bounded by the curves $x = -2y^2$ and $y = f(x)$ lying on the left of the line $8x + 1 = 0$. ***[IIT JEE 1999]***

Sol. Given, a continuous function $f(x)$, given by



$$f(x) = \begin{cases} 2x, & |x| \le 1 \\ x^2 + ax + b, & |x| > 1 \end{cases}$$

i.e. $$f(x) = \begin{cases} x^2 + ax + b, & -\infty < x < -1 \\ 2x, & -1 \le x \le 1 \\ x^2 + ax + b, & 1 < x < \infty \end{cases}$$

$\because$ f is continuous.

$\therefore$ It is continuous at $x = -1$ and $x = 1$.

$\therefore \quad (-1)^2 + a(-1) + b = 2(-1)$

and $\quad (1)^2 + a(1) + b = 2(1)$

$\Rightarrow \quad 1 - a + b = -2$

and $\quad 1 + a + b = 2$

$\Rightarrow \quad -a + b = -3$

and $\quad a + b = 1$

$\Rightarrow \quad b = -1$ and $a = 2$

$$\therefore \quad f(x) = \begin{cases} x^2 + 2x - 1, & \text{when } -\infty < x < -1 \\ 2x, & \text{when } -1 \le x \le 1 \end{cases}$$

Now, $\quad y = x^2 + 2x - 1 = (x+1)^2 - 2$

or $\quad (y+2) = (x+1)^2$

We need the area of the region in third quadrant bounded by the curves $a = -2y^2$, $y = f(x)$ lying on the left of the line $8x + 1 = 0$.

$\Rightarrow \quad (y+2) = (x+1)^2$

Cuts the Y-axis at $(0, -1)$ and the X-axis at $(-1-\sqrt{2}, 0)$ and $(-1+\sqrt{2}, 0)$.

When $x = 1$ and $y = 2$

Solving $x = -2y^2$ and $(y+2) = (x+1)^2$, we get

$$(y+2) = (-2y^2 + 1)^2$$

$\Rightarrow \quad 4y^4 - 4y^2 + 1 - y - 2 = 0$

$\Rightarrow \quad 4y^4 - 4y^2 - y - 1 = 0$

For $\quad y = -1,\ 4y^4 - 4y^2 - y - 1 = 0$

At $\quad y = -1,\ x = -2$

$$\therefore \quad \text{Required area} = \int_{-2}^{-1} \left\{ -\sqrt{-\frac{x}{2}} - (x+1)^2 + 2 \right\} dx + \int_{-1}^{-1/8} \left\{ -\sqrt{-\frac{x}{2}} - 2x \right\} dx$$

$$= \left\{ \frac{(-x/2)^{3/2}}{\frac{3}{2}(1/2)} - \frac{(x+1)^3}{3} + 2x \right\}_{-2}^{-1} + \left\{ \frac{(-x/2)^{3/2}}{\frac{3}{2}(1/2)} - \frac{2x^2}{2} \right\}_{-1}^{-1/8}$$

$$= \left\{ \frac{4}{3}\left(-\frac{x}{2}\right)^{3/2} - \frac{(x+1)^3}{3} + 2x \right\}_{-2}^{-1} + \left\{ \frac{4}{3}\left(-\frac{x}{2}\right)^{3/2} - x^2 \right\}_{-1}^{-1/8}$$

$$= \left\{ \frac{4}{3}\left(\frac{1}{2}\right)^{3/2} - 0 - 2 \right\} - \left\{ \frac{4}{3}(1)^{3/2} - \frac{(-1)^3}{3} - 4 \right\} - \left\{ \frac{4}{3}\left(\frac{1}{2}\right)^{3/2} - (-1)^2 \right\} + \left\{ \frac{4}{3}\left(\frac{1}{16}\right)^{3/2} - \frac{1}{64} \right\}$$

$$= \left\{ \frac{4}{3\cdot 2\sqrt{2}} - 2 - \frac{4}{3} - \frac{1}{3} + 4 - \frac{4}{3\cdot 2\sqrt{2}} + 1 + \frac{4}{3(64)} - \frac{1}{64} \right\}$$

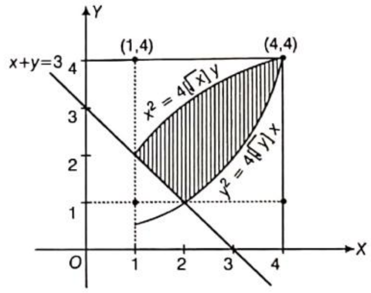
$$= \frac{4}{3} + \frac{1}{3(64)} = \frac{257}{192} \text{ sq units}$$

Ex. 54 *Let $[x]$ denotes the greatest integer function. Draw a rough sketch of the portions of the curves $x^2 = 4[\sqrt{x}]\,y$ and $y^2 = 4[\sqrt{y}]\,x$ that lie within the square $\{(x, y) \mid 1 \le x \le 4, 1 \le y \le 4\}$. Find the area of the part of the square that is enclosed by the two curves and the line $x + y = 3$.*

Sol. We have, $\quad 1 \le x \le 4$ and $1 \le y \le 4$

$\Rightarrow \quad 1 \le \sqrt{x} \le 2$ and $1 \le \sqrt{y} \le 2$

$\Rightarrow [\sqrt{x}] = 1$ and $[\sqrt{y}] = 1$ for all (x, y) lying with in the square.



Thus, $\quad x^2 = 4[\sqrt{x}]\,y$ and $y^2 = 4[\sqrt{y}]\,x$

$\Rightarrow \quad x^2 = 4y$ and $y^2 = 4x$ when $1 \le x, y \le 4$ which could be plotted as;

Thus, required area

$$= \int_1^2 (2\sqrt{x} - 3 + x)\,dx + \int_2^4 \left(2\sqrt{x} - \frac{x^2}{4}\right) dx$$

$$= \left(\frac{4}{3}x^{3/2} - 3x + \frac{x^2}{2}\right)_1^2 + \left(\frac{4}{3}x^{3/2} - \frac{x^3}{12}\right)_2^4$$

$$= \left(\frac{8\sqrt{2}}{3} - 6 + 2\right) - \left(\frac{4}{3} - 3 + \frac{1}{2}\right) + \left(\frac{32}{3} - \frac{64}{12}\right) - \left(\frac{8\sqrt{2}}{3} - \frac{8}{12}\right)$$

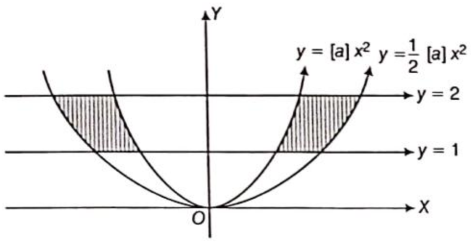
$$= \frac{19}{6} \text{ sq units}$$

Ex. 55 *Find all the values of the parameter a ($a \ge 1$) for which the area of the figure bounded by pair of straight lines $y^2 - 3y + 2 = 0$ and the curves $y = [a]\,x^2$, $y = \frac{1}{2}[a]\,x^2$ is greatest, where $[.]$ denotes the greatest integer function.*

Sol. The curves $y = [a]\,x^2$ and $y = \frac{1}{2}[a]\,x^2$ represent parabolas which are symmetric about Y-axis.

The equation $y^2 - 3y + 2 = 0$ gives a pair of straight lines $y = 1$, $y = 2$ which are parallel to X-axis.

Thus, the area bounded is shown as



From the above figure,

Required area $= 2\int_{y=1}^{y=2} (x_2 - x_1)\, dy$

$$= 2\int_1^2 \left(\sqrt{\frac{2y}{[a]}} - \sqrt{\frac{y}{[a]}} \right) dy$$

$$= \frac{2(\sqrt{2}-1)}{[a]} \int_1^2 \sqrt{y}\, dy = \frac{2(\sqrt{2}-1)}{[a]} \frac{2}{3} (y^{3/2})_1^2$$

$$= \frac{4}{3} \frac{(\sqrt{2}-1)}{[a]} \cdot (2^{3/2} - 1)$$

$$= \frac{4}{3} \frac{(5 - \sqrt{2} - 2\sqrt{2})}{[a]} = \frac{4}{3} \frac{(5 - 3\sqrt{2})}{[a]}$$

∴ Area is the greatest when $[a]$ is least, i.e. 1.

∴ Area is the greatest when $[a] = 1$

Hence, $a \in [1, 2)$

• **Ex. 56** *Find the area in the first quadrant bounded by $[x] + [y] = n$, where $n \in N$ and $y = i$ (where, $i \in N\ \forall\ i \le n+1$), $[\cdot]$ denotes the greatest integer less than or equal to x.*

Sol. As we know, $[x] + [y] = n \Rightarrow [x] = n - [y]$

When $y = 0$, $[x] = n \Rightarrow n \le x < n + 1$

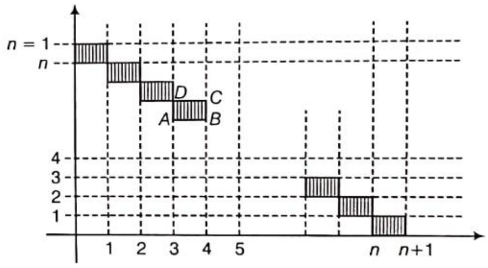
When $y = 1$, $[x] = n - 1 \Rightarrow n - 1 \le x < n - 2$

When $y = 2$, $[x] = n - 2$

$\Rightarrow$ $n - 2 \le x < n - 3$. ... and so on.

When $y = n$, $[x] = 0 \Rightarrow 0 \le x < 1$

which could be shown as



From the above figure,

$= (n + 1)$ area of square $ABCD = (n + 1) \cdot 1$

Required area $= (n + 1)$ sq units

Area of Bounded Regions Exercise 1 :
Single Option Correct Type Questions

1. A Point $P(x, y)$ moves such that $[x+y+1]=[x]$. (where $[\cdot]$ denotes greatest integer function) and $x \in (0, 2)$, then the area represented by all the possible positions of P, is

(a) $\sqrt{2}$ (b) $2\sqrt{2}$
(c) $4\sqrt{2}$ (d) 2

2. If $f:[-1,1] \to \left[-\frac{1}{2}, \frac{1}{2}\right]$, $f(x)=\frac{x}{1+x^2}$. The area bounded by $y=f^{-1}(x)$, X-axis, $x=\frac{1}{2}$, $x=-\frac{1}{2}$ is

(a) $\frac{1}{2}\log e$ (b) $\log\left(\frac{e}{2}\right)$
(c) $\frac{1}{2}\log\frac{e}{3}$ (d) $\frac{1}{2}\log\left(\frac{e}{2}\right)$

3. If the length of latusrectum of ellipse
$E_1: 4(x+y-1)^2+2(x-y+3)^2=8$
and $E_2: \frac{x^2}{p}+\frac{y^2}{p^2}=1, (0<p<1)$ are equal, then area of ellipse E_2, is

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{\sqrt{2}}$
(c) $\frac{\pi}{2\sqrt{2}}$ (d) $\frac{\pi}{4}$

4. The area of bounded by the curve
$4|x-2017^{2017}|+5|y-2017^{2017}| \le 20$, is

(a) 60 (b) 50
(c) 40 (d) 30

5. If the area bounded by the corve $y=x^2+1$, $y=x$ and the pair of lines $x^2+y^2+2xy-4x-4y+3=0$ is K units, then the area of the region bounded by the curve $y=x^2+1$, $y=\sqrt{x-1}$ and the pair of lines $(x+y-1)(x+y-3)=0$, is

(a) K (b) $2K$
(c) $\frac{K}{2}$ (d) None of these

6. Suppose $y=f(x)$ and $y=g(x)$ are two functions whose graphs intersect at the three points (0, 4), (2, 2) and (4, 0) with $f(x)>g(x)$ for $0<x<2$ and $f(x)<g(x)$ for $2<x<4$. If $\int_0^4 [f(x)-g(x)]\,dx=10$ and $\int_2^4 [g(x)-f(x)]\,dx=5$, then the area between two curves for $0<x<2$, is

(a) 5 (b) 10
(c) 15 (d) 20

7. Let 'a' be a positive constant number. Consider two curves $C_1: y=e^x$, $C_2: y=e^{a-x}$. Let S be the area of the part surrounding by C_1, C_2 and the Y-axis, then $\lim_{a\to 0}\frac{S}{a^2}$ equals

(a) 4 (b) 1/2
(c) 0 (d) 1/4

8. 3 points $O(0,0)$, $P(a, a^2)$, $Q(-b, b^2)$ $(a>0, b>0)$ are on the parabola $y=x^2$. Let S_1 be the area bounded by the line PQ and the parabola and let S_2 be the area of the ΔOPQ, the minimum value of S_1/S_2 is

(a) 4/3 (b) 5/3
(c) 2 (d) 7/3

9. Area enclosed by the graph of the function $y=\ln^2 x-1$ lying in the 4th quadrant is

(a) $\frac{2}{e}$ (b) $\frac{4}{e}$
(c) $2\left(e+\frac{1}{e}\right)$ (d) $4\left(e-\frac{1}{e}\right)$

10. The area bounded by $y=2-|2-x|$ and $y=\frac{3}{|x|}$ is

(a) $\frac{4+3\ln 3}{2}$ (b) $\frac{19}{8}-3\ln 2$
(c) $\frac{3}{2}+\ln 3$ (d) $\frac{1}{2}+\ln 3$

11. Suppose $g(x)=2x+1$ and $h(x)=4x^2+4x+5$ and $h(x)=(fog)(x)$. The area enclosed by the graph of the function $y=f(x)$ and the pair of tangents drawn to it from the origin, is

(a) 8/3 (b) 16/3
(c) 32/3 (d) None of these

12. The area bounded by the curves $y=-\sqrt{-x}$ and $x=-\sqrt{-y}$ where $x, y \le 0$

(a) cannot be determined
(b) is 1/3
(c) is 2/3
(d) is same as that of the figure bounded by the curves $y=\sqrt{-x}$; $x\le 0$ and $x=\sqrt{-y}$; $y\le 0$

13. $y=f(x)$ is a function which satisfies
(i) $f(0)=0$ (ii) $f''(x)=f'(x)$ and (iii) $f'(0)=1$
Then, the area bounded by the graph of $y=f(x)$, the lines $x=0$, $x-1=0$ and $y+1=0$, is

(a) e (b) $e-2$
(c) $e-1$ (d) $e+1$

14. Area of the region enclosed between the curves $x = y^2 - 1$ and $x = |y|\sqrt{1-y^2}$ is

(a) 1 (b) 4/3
(c) 2/3 (d) 2

15. The area bounded by the curve $y = xe^{-x}$; $xy = 0$ and $x = c$ where c is the x-coordinate of the curve's inflection point, is

(a) $1-3e^{-2}$ (b) $1-2e^{-2}$
(c) $1-e^{-2}$ (d) 1

16. If $(a, 0)$; $a > 0$ is the point where the curve $y = \sin 2x - \sqrt{3}\sin x$ cuts the X-axis first, A is the area bounded by this part of the curve, the origin and the positive X-axis, then

(a) $4A + 8\cos a = 7$ (b) $4A + 8\sin a = 7$
(c) $4A - 8\sin a = 7$ (d) $4A - 8\cos a = 7$

17. The curve $y = ax^2 + bx + c$ passes through the point $(1, 2)$ and its tangent at origin is the line $y = x$. The area bounded by the curve, the ordinate of the curve at minima and the tangent line is

(a) $\frac{1}{24}$ (b) $\frac{1}{12}$ (c) $\frac{1}{8}$ (d) $\frac{1}{6}$

18. A function $y = f(x)$ satisfies the differential equation $\frac{dy}{dx} - y = \cos x - \sin x$, with initial condition that y is bounded when $x \to \infty$. The area enclosed by $y = f(x)$, $y = \cos x$ and the Y-axis in the 1st quadrant

(a) $\sqrt{2}-1$ (b) $\sqrt{2}$
(c) 1 (d) $1/\sqrt{2}$

19. If the area bounded between X-axis and the graph of $y = 6x - 3x^2$ between the ordinates $x = 1$ and $x = a$ is 19 sq units, then 'a' can take the value

(a) 4 or -2
(b) two values are in $(2, 3)$ and one in $(-1, 0)$
(c) two values one in $(3, 4)$ and one in $(-2, -1)$
(d) None of the above

20. Area bounded by $y = f^{-1}(x)$ and tangent and normal drawn to it at the points with abscissae π and 2π, where $f(x) = \sin x - x$ is

(a) $\frac{\pi^2}{2} - 1$ (b) $\frac{\pi^2}{2} - 2$
(c) $\frac{\pi^2}{2} - 4$ (d) $\frac{\pi^2}{2}$

21. If $f(x) = x - 1$ and $g(x) = |f(|x|) - 2|$, then the area bounded by $y = g(x)$ and the curve $x^2 - 4y + 8 = 0$ is equal to

(a) $\frac{4}{3}(4\sqrt{2}-5)$ (b) $\frac{4}{3}(4\sqrt{2}-3)$
(c) $\frac{8}{3}(4\sqrt{2}-3)$ (d) $\frac{8}{3}(4\sqrt{2}-5)$

22. Let $S = \left\{(x, y): \frac{y(3x-1)}{x(3x-2)} < 0\right\}$,
$S' = \{(x, y) \in A \times B: -1 \le A \le 1, -1 \le B \le 1\}$,
then the area of the region enclosed by all points in $S \cap S'$ is

(a) 1 (b) 2
(c) 3 (d) 4

23. The area of the region bounded between the curves $y = e||x|\ln|x||$, $x^2 + y^2 - 2(|x| + |y|) + 1 \ge 0$ and X-axis where $|x| \le 1$, if α is the x-coordinate of the point of intersection of curves in 1st quadrant, is

(a) $4\left[\int_0^\alpha ex\ln x\,dx + \int_\alpha^1 (1 - \sqrt{1-(x-1)^2})\,dx\right]$

(b) $4\left[\int_0^\alpha ex\ln x\,dx + \int_1^\alpha (1 - \sqrt{1-(x-1)^2})\,dx\right]$

(c) $4\left[-\int_0^\alpha ex\ln x\,dx + \int_\alpha^1 (1 - \sqrt{1-(x-1)^2})\,dx\right]$

(d) $2\left[\int_0^\alpha ex\ln x\,dx + \int_1^\alpha (1 - \sqrt{1-(x-1)^2})\,dx\right]$

24. A point P lying inside the curve $y = \sqrt{2ax - x^2}$ is moving such that its shortest distance from the curve at any position is greater than its distance from X-axis. The point P enclose a region whose area is equal to

(a) $\frac{\pi a^2}{2}$ (b) $\frac{a^2}{3}$
(c) $\frac{2a^2}{3}$ (d) $\left(\frac{3\pi - 4}{6}\right)a^2$

Area of Bounded Regions Exercise 2 : More than One Option Correct Type Questions

25. The triangle formed by the normal to the curve $f(x) = x^2 - ax + 2a$ at the point (2, 4) and the coordinate axes lies in second quadrant, if its area is 2 sq units, then a can be
(a) 2
(b) 17/4
(c) 5
(d) 19/4

26. Let f and g be continuous function on $a \le x \le b$ and set $p(x) = \max\{f(x),\ g(x)\}$ and $q(x) = \min\{f(x),\ g(x)\}$, then the area bounded by the curves $y = p(x)$, $y = q(x)$ and the ordinates $x = a$ and $x = b$ is given by
(a) $\int_a^b |f(x) - g(x)|\, dx$
(b) $\int_a^b |p(x) - q(x)|\, dx$
(c) $\int_a^b \{f(x) - g(x)\}\, dx$
(d) $\int_a^b \{p(x) - q(x)\}\, dx$

27. The area bounded by the parabola $y = x^2 - 7x + 10$ and X-axis equals
(a) area bounded by $y = -x^2 + 7x - 10$ and X-axis
(b) 1/6 sq units
(c) 5/6 sq units
(d) 9/2 sq units

28. Area bounded by the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$ is equal to
(a) 6π sq units
(b) 3π sq units
(c) 12π sq units
(d) area bounded by the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$

29. There is a curve in which the length of the perpendicular from the origin to tangent at any point is equal to abscissa of that point. Then,
(a) $x^2 + y^2 = 2$ is one such curve
(b) $y^2 = 4x$ is one such curve
(c) $x^2 + y^2 = 2cx$ (c parameters) are such curves
(d) there are no such curves

Area of Bounded Regions Exercise 3 : Statement I and II Type Questions

- **Direction** (Q. No. 30-34) *For the following questions, choose the correct answers from the codes (a), (b), (c) and (d) defined as follows :*
  - (a) Statement I is true, Statement II is also true; Statement II is the correct explanation of Statement I
  - (b) Statement I is true, Statement II is also true; Statement II is not the correct explanation of Statement I
  - (c) Statement I is true, Statement II is false
  - (d) Statement I is false, Statement II is true

30. Statement I The area of the curve $y = \sin^2 x$ from 0 to π will be more than that of the curve $y = \sin x$ from 0 to π.

Statement II $x^2 > x$, if $x > 1$.

31. Statement I The area bounded by the curves $y = x^2 - 3$ and $y = kx + 2$ is least if $k = 0$.

Statement II The area bounded by the curves $y = x^2 - 3$ and $y = kx + 2$ is $\sqrt{k^2 + 20}$.

32. Statement I The area of region bounded parabola $y^2 = 4x$ and $x^2 = 4y$ is $\frac{32}{3}$ sq units.

Statement II The area of region bounded by parabola $y^2 = 4ax$ and $x^2 = 4by$ is $\frac{16}{3}ab$.

33. Statement I The area by region $|x + y| + |x - y| \le 2$ is 8 sq units.

Statement II Area enclosed by region $|x + y| + |x - y| \le 2$ is symmetric about X-axis.

34. Statement I Area bounded by $y = x(x - 1)$ and $y = x(1 - x)$ is $\frac{1}{3}$.

Statement II Area bounded by $y = f(x)$ and $y = g(x)$ is $\left| \int_a^b (f(x) - g(x))\, dx \right|$ is true when $f(x)$ and $g(x)$ lies above X-axis. (Where a and b are intersection of $y = f(x)$ and $y = g(x)$).

Area of Bounded Regions Exercise 4 : Passage Based Questions

Passage I

(Q. Nos. 35 to 37)

Let $f(x)=\dfrac{ax^2+bx+c}{x^2+1}$ *such that* $y=-2$ *is an asymptote of the curve* $y=f(x)$*. The curve* $y=f(x)$ *is symmetric about Y-axis and its maximum values is 4. Let* $h(x)=f(x)-g(x)$*, where* $f(x)=\sin^4 \pi x$ *and* $g(x)=\log_e x$*. Let* $x_0, x_1, x_2, \ldots. x_{n+1}$ *be the roots of* $f(x)=g(x)$ *in increasing order.*

35. Then, the absolute area enclosed by $y=f(x)$ and $y=g(x)$ is given by

(a) $\sum_{r=0}^{n} \int_{x_r}^{x_{r+1}} (-1)^r \cdot h(x)dx$

(b) $\sum_{r=0}^{n} \int_{x_r}^{x_{r+1}} (-1)^{r+1} \cdot h(x)dx$

(c) $2\sum_{r=0}^{n} \int_{x_r}^{x_{r+1}} (-1)^r \cdot h(x)dx$

(d) $\frac{1}{2}\cdot\sum_{r=0}^{n} \int_{x_r}^{x_{r+1}} (-1)^{r+1} h(x)dx$

36. In above inquestion the value of n, is

(a) 1 (b) 2
(c) 3 (d) 4

37. The whole area bounded by $y=f(x), y=g(x)$ $x=0$ is

(a) 11/8 (b) 8/3
(c) 2 (d) 13/3

Passage II

(Q. Nos. 38 to 40)

Consider the function $f:(-\infty,\infty)\to(-\infty,\infty)$ *defined by*
$f(x)=\dfrac{x^2-ax+1}{x^2+ax+1}, 0<a<2.$

38. Which of the following is true?

(a) $(2+a)^2 f''(1)+(2-a)^2 f''(-1)=0$
(b) $(2-a)^2 f''(1)-(2+a)^2 f''(-1)=0$
(c) $f'(1)f'(-1)=(2-a)^2$
(d) $f'(1)f'(-1)=-(2+a)^2$

39. Which of the following is true?

(a) $f(x)$ is decreasing on $(-1,1)$ and has a local minimum at $x=1$
(b) $f(x)$ is increasing on $(-1,1)$ and has a local maximum at $x=1$
(c) $f(x)$ is increasing on $(-1,1)$ but has neither a local maximum nor a local minimum at $x=1$
(d) $f(x)$ is decreasing on $(-1,1)$ but has neither a local maximum nor a local minimum at $x=1$

40. Let $g(x)=\int_0^{e^x} \dfrac{f'(t)}{1+t^2}dt$. Which of the following is true?

(a) $g'(x)$ is positive on $(-\infty,0)$ and negative on $(0,\infty)$
(b) $g'(x)$ is negative on $(-\infty,0)$ and positive on $(0,\infty)$
(c) $g'(x)$ change sign on both $(-\infty,0)$ and $(0,\infty)$
(d) $g'(x)$ does not change sign on $(-\infty,\infty)$

Passage III

(Q. Nos. 41 to 43)

Computing areas with parametrically represented boundaries : If the boundary of a figure is represented by parametric equations i.e. $x=x(t)$, $y=y(t)$*, then the area of the figure is evaluated by one of the three formulas*

$$S=-\int_\alpha^\beta y(t)\cdot x'(t)\,dt$$

$$S=\int_\alpha^\beta x(t)\cdot y'(t)\,dt$$

$$S=\frac{1}{2}\int_\alpha^\beta (xy'-yx')\,dt$$

where α *and* β *are the values of the parameter '*t*' corresponding, respectively to the beginning and the end of the traversal of the curve corresponding to increasing '*t*'.*

41. The area enclosed by the asteroid $\left(\dfrac{x}{a}\right)^{2/3}+\left(\dfrac{y}{a}\right)^{2/3}=1$ is

(a) $\frac{3}{4}a^2\pi$ (b) $\frac{3}{18}\pi a^2$
(c) $\frac{3}{8}\pi a^2$ (d) $\frac{3}{4}a\pi$

42. The area of the region bounded by an arc of the cycloid $x=a(t-\sin t), y=a(1-\cos t)$ and the X-axis is

(a) $6\pi a^2$ (b) $3\pi a^2$
(c) $4\pi a^2$ (d) None of these

43. Area of the loop described as $x=\frac{t}{3}(6-t), y=\frac{t^2}{8}(6-t)$ is

(a) $\frac{27}{5}$ (b) $\frac{24}{5}$
(c) $\frac{27}{6}$ (d) $\frac{21}{5}$

Area of Bounded Regions Exercise 5 : Matching Type Questions

44. Match the statements of Column I with values of Column II.

	Column I		Column II
(A)	The area bounded by the curve $y = x + \sin x$ and its inverse function between the ordinates $x = 0$ to $x = 2\pi$ is $4s$. Then, the value of s is	(p)	2
(B)	The area bounded by $y = xe^{\|x\|}$ and lies $\|x\| = 1$, $y = 0$ is	(q)	1
(C)	The area bounded by the curves $y^2 = x^3$ and $\|y\| = 2x$ is	(r)	$\frac{16}{5}$
(D)	The smaller are included between the curves $\sqrt{\|x\|} + \sqrt{\|y\|} = 1$ and $\|x\| + \|y\| = 1$ is	(s)	$\frac{1}{3}$

45. Match the following :

	Column I		Column II
(A)	Area enclosed by $y = \|x\|$, $\|x\| = 1$ and $y = 0$ is	(p)	2
(B)	Area enclosed by the curve $y = \sin x$, $x = 0, x = \pi$ and $y = 0$ is	(q)	4
(C)	If the area of the region bounded by $x^2 \le y$ and $y \le x + 2$ is $\frac{k}{4}$, then k is equal to	(r)	27
(D)	Area of the quadrilateral formed by tangents at the ends of latusrectum of ellipse of ellipse $\frac{x^2}{9} + \frac{y^2}{5} = 1$ is	(s)	18

Area of Bounded Regions Exercise 6 : Single Integer Answer Type Questions

46. Consider $f(x) = x^2 - 3x + 2$. The area bounded by $|y| = |f(|x|)|, x \ge 1$ is A, then find the value of $3A + 2$.

47. The value of $c + 2$ for which the area of the figure bounded by the curve $y = 8x^2 - x^5$; the straight lines $x = 1$ and $x = c$ and X-axis is equal to $\frac{16}{3}$, is ....

48. The area bounded by $y = 2 - |2 - x|; y = \frac{3}{|x|}$ is $\frac{k - 3\ln 3}{2}$, then k is equal to ....

49. The area of the ΔABC, coordinates of whose vertices are $A(2, 0)$, $B(4, 5)$ and $C(6, 3)$ is ....

50. A point P moves in XY-plane in such a way that $[|x|] + [|y|] = 1$, where $[\cdot]$ denotes the greatest integer function. Area of the region representing all possible of the point P is equal to ....

51. Let $f : [0, 1] \to \left[0, \frac{1}{2}\right]$ be a function such that $f(x)$ is a polynomial of 2nd degree, satisfy the following condition :

(a) $f(0) = 0$

(b) has a maximum value of $\frac{1}{2}$ at $x = 1$.

If A is the area bounded by $y = f(x); y = f^{-1}(x)$ and the line $2x + 2y - 3 = 0$ in 1st quadrant, then the value of $24A$ is equal to ....

52. Let $f(x) = \min\left\{\sin^{-1} x, \cos^{-1} x, \frac{\pi}{6}\right\}, x \in [0, 1]$. If area bounded by $y = f(x)$ and X-axis, between the lines $x = 0$ and $x = 1$ is $\frac{a - X}{b(\sqrt{3} + 1)}$. Then, $(a - b)$ is ....

53. Let f be a real valued function satisfying $f\left(\frac{x}{y}\right) = f(x) - f(y)$ and $\lim_{x \to 0} \frac{f(1 + x)}{x} = 3$. Find the area bounded by the curve $y = f(x)$, the Y-axis and the line $y = 3$, where $x, y \in R^+$.

Area of Bounded Regions Exercise 7 :
Subjective Type Questions

54. Find a continuous function 'f', $(x^4 - 4x^2) \le f(x) \le (2x^2 - x^3)$ such that the area bounded by $y = f(x)$, $y = x^4 - 4x^2$, the Y-axis and the line $x = t, (0 \le t \le 2)$ is k times the area bounded by $y = f(x), y = 2x^2 - x^3$, Y-axis and line $x = t, (0 \le t \le 2)$.

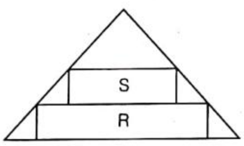
55. Let $f(t) = |t-1| - |t| + |t+1|, \forall t \in R$ and $g(x) = \max\{f(t) : x+1 \le t \le x+2\}; \forall x \in R$. Find $g(x)$ and the area bounded by the curve $y = g(x)$, the X-axis and the lines $x = -3/2$ and $x = 5$.

56. Let $f(x) = \text{minimum}\{e^x, 3/2, 1 + e^{-x}\}, 0 \le x \le 1$. Find the area bounded by $y = f(x)$, X-axis, Y-axis and the line $x = 1$.

57. Find the area bounded by $y = f(x)$ and the curve $y = \frac{2}{1+x^2}$, where f is a continuous function satisfying the conditions $f(x) \cdot f(y) = f(xy), \forall x, y \in R$ and $f'(1) = 2, f(1) = 1$.

58. Find out the area bounded by the curve $y = \int_{1/8}^{\sin^2 x} (\sin^{-1}\sqrt{t})\, dt + \int_{1/8}^{\cos^2 x} (\cos^{-1}\sqrt{t})\, dt \; (0 \le x \le \pi/2)$ and the curve satisfying the differential equation $y(x + y^3)\, dx = x(y^3 - x)\, dy$ passing through $(4, -2)$.

59. Let T be an acute triangle. Inscribe a pair R, S of rectangles in T as shown :

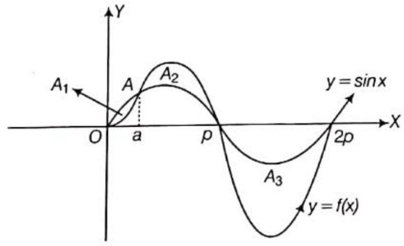


Let $A(x)$ denote the area of polygon X find the maximum value (or show that no maximum exists), of $\frac{A(R) + A(S)}{A(T)}$, where T ranges over all triangles and R, S over all rectangles as above.

60. Find the maximum area of the ellipse that can be inscribed in an isosceles triangles of area A and having one axis lying along the perpendicular from the vertex of the triangles to its base.

61. In the adjacent figure the graphs of two function $y = f(x)$ and $y = \sin x$ are given. $y = \sin x$ intersects, $y = f(x)$ at $A(a, f(a))$; $B(\pi, 0)$ and $C(2\pi, 0)$.

$A_i\ (i = 1, 2, 3)$ is the area bounded by the curves $y = f(x)$ and $y = \sin x$. between $x = 0$ and $x = a$; $i = 1$



between $x = a$ and $x = \pi$; $i = 2$ between $x = \pi$ and $x = 2\pi$; $i = 3$.

If $A_1 = 1 - \sin a + (a - 1)\cos a$, determine the function $f(x)$. Hence, determine a and A_1. Also, calculate A_2 and A_3.

62. Find the area of the region bounded by curve $y = 25^x + 16$ and the curve $y = b \cdot 5^x + 4$, whose tangent at the point $x = 1$ make an angle $\tan^{-1}(40 \ln 5)$ with the X-axis.

63. If the circles of the maximum area inscribed in the region bounded by the curves $y = x^2 - x - 3$ and $y = 3 + 2x - x^2$, then the area of region $y - x^2 + 2x + 3 \le 0$, $y + x^2 - 2x - 3 \le 0$ and $s \le 0$.

64. Find limit of the ratio of the area of the triangle formed by the origin and intersection points of the parabola $y = 4x^2$ and the line $y = a^2$, to the area between the parabola and the line as a approaches to zero.

65. Find the area of curve enclosed by : $|x + y| + |x - y| \le 4, |x| \le 1, y \ge \sqrt{x^2 - 2x + 1}$.

66. Calculate the area enclosed by the curve

$$4 \le x^2 + y^2 \le 2(|x| + |y|).$$

67. Find the area enclosed by the curve $[x] + [y] = 4$ in 1st quadrant (where [.] denotes greatest integer function).

68. Sketch the region and find the area bounded by the curves $|y + x| \le 1, |y - x| \le 1$ and $2x^2 + 2y^2 = 1$.

69. Find the area of the region bounded by the curve, $2^{|x|}\,|y| + 2^{|x|-1} \le 1$, with in the square formed by the lines $|x| \le 1/2, |y| \le 1/2$.

70. Find all the values of the parameter $a\ (a \le 1)$ for which the area of the figure bounded by the pair of straight lines $y^2 - 3y + 2 = 0$ and the curves $y = [a]x^2, y = \frac{1}{2}[a]x^2$ is the greatest, where [.] denotes greatest integer function.

71. If $f(x)$ is positive for all positive values of X and $f'(x) < 0, f''(x) > 0, \forall x \in R^+$, prove that

$$\frac{1}{2} f(1) + \int_1^n f(x)\, dx < \sum_{r=1}^{n} f(r) < \int_1^n f(x)\, dx + f(1).$$

Area of Bounded Regions Exercise 8 : Questions Asked in Previous 10 Years' Exams

(i) JEE Advanced & IIT-JEE

72. Area of the region $\{(x, y)\} \in R^2 : y \geq \sqrt{|x+3|}$, $5y \leq (x+9) \leq 15\}$ is equal to **[Single Correct Option 2016]**

(a) $\frac{1}{6}$ (b) $\frac{4}{3}$ (c) $\frac{3}{2}$ (d) $\frac{5}{3}$

73. Let $F(x) = \int_x^{x^2+\frac{\pi}{6}} 2\cos^2 t \, dt$ for all $x \in R$ and $f : \left[0, \frac{1}{2}\right] \to [0, \infty)$ be a continuous function. For $a \in \left[0, \frac{1}{2}\right]$, if $F'(a) + 2$ is the area of the region bounded by $x = 0$, $y = 0$, $y = f(x)$ and $x = a$, then $f(0)$ is **[Integer Answer Type 2015]**

74. The common tangents to the circle $x^2 + y^2 = 2$ and the parabola $y^2 = 8x$ touch the circle at the points P, Q and the parabola at the points R, S. Then, the area (in sq units) of the quadrilateral $PQRS$ is **[Single Correct Option 2014]**

(a) 3 (b) 6 (c) 9 (d) 15

75. The area enclosed by the curves $y = \sin x + \cos x$ and $y = |\cos x - \sin x|$ over the interval $\left[0, \frac{\pi}{2}\right]$ is **[Single Correct Option 2014]**

(a) $4(\sqrt{2}-1)$ (b) $2\sqrt{2}(\sqrt{2}-1)$

(c) $2(\sqrt{2}+1)$ (d) $2\sqrt{2}(\sqrt{2}+1)$

76. If S be the area of the region enclosed by $y = e^{-x^2}$, $y = 0$, $x = 0$ and $x = 1$. Then, **[More than One Option Correct 2012]**

(a) $S \geq \frac{1}{e}$ (b) $S \geq 1 - \frac{1}{e}$

(c) $S \leq \frac{1}{4}\left(1 + \frac{1}{\sqrt{e}}\right)$ (d) $S \leq \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{e}}\left(1 - \frac{1}{\sqrt{2}}\right)$

77. Let $f : [-1, 2] \to [0, \infty)$ be a continuous function such that $f(x) = f(1-x), \forall x \in [-1, 2]$. If $R_1 = \int_{-1}^{2} x f(x) \, dx$ and R_2 are the area of the region bounded by $y = f(x)$, $x = -1$, $x = 2$ and the X-axis. Then, **[Single Correct Option 2011]**

(a) $R_1 = 2R_2$ (b) $R_1 = 3R_2$ (c) $2R_1 = R_2$ (d) $3R_1 = R_2$

78. If the straight line $x = b$ divide the area enclosed by $y = (1-x)^2$, $y = 0$ and $x = 0$ into two parts $R_1 (0 \leq x \leq b)$ and $R_2 (b \leq x \leq 1)$ such that $R_1 - R_2 = \frac{1}{4}$. Then, b equals to **[Single Correct Option 2011]**

(a) $\frac{3}{4}$ (b) $\frac{1}{2}$ (c) $\frac{1}{3}$ (d) $\frac{1}{4}$

79. Area of the region bounded by the curve $y = e^x$ and lines $x = 0$ and $y = e$ is **[More than One Option Correct 2009]**

(a) $e - 1$ (b) $\int_1^e \ln(e + 1 - y) \, dy$

(c) $e - \int_0^1 e^x \, dx$ (d) $\int_1^e \ln y \, dy$

80. The area of the region between the curves $y = \sqrt{\frac{1 + \sin x}{\cos x}}$ and $y = \sqrt{\frac{1 - \sin x}{\cos x}}$ and bounded by the lines $x = 0$ and $x = \frac{\pi}{4}$ is **[Single Correct Option 2008]**

(a) $\int_0^{\sqrt{2}-1} \frac{t}{(1+t^2)\sqrt{1-t^2}} dt$ (b) $\int_0^{\sqrt{2}-1} \frac{4t}{(1+t^2)\sqrt{1-t^2}} dt$

(c) $\int_0^{\sqrt{2}+1} \frac{4t}{(1+t^2)\sqrt{1-t^2}} dt$ (d) $\int_0^{\sqrt{2}+1} \frac{t}{(1+t^2)\sqrt{1-t^2}} dt$

- **Directions** (Q. Nos. 81 to 83) Consider the functions defined implicity by the equation $y^3 - 3y + x = 0$ on various intervals in the real line. If $x \in (-\infty, -2) \cup (2, \infty)$, the equation implicitly defines a unique real-valued differentiable function $y = f(x)$. If $x \in (-2, 2)$, the equation implicitly defines a unique real-valued differentiable function $y = g(x)$, satisfying $g(0) = 0$. **[Passage Based Questions 2008]**

81. If $f(-10\sqrt{2}) = 2\sqrt{2}$, then $f''(-10\sqrt{2})$ is equal to

(a) $\frac{4\sqrt{2}}{7^3 3^2}$ (b) $-\frac{4\sqrt{2}}{7^3 3^2}$

(c) $\frac{4\sqrt{2}}{7^3 3}$ (d) $-\frac{4\sqrt{2}}{7^3 3}$

82. The area of the region bounded by the curve $y = f(x)$, the X-axis and the lines $x = a$ and $x = b$, where $-\infty < a < b < -2$, is

(a) $\int_a^b \frac{x}{3[\{f(x)\}^2 - 1]} dx + bf(b) - af(a)$

(b) $-\int_a^b \frac{x}{3[\{f(x)\}^2 - 1]} dx + bf(b) - af(a)$

(c) $\int_a^b \frac{x}{3[\{f(x)\}^2 - 1]} dx - bf(b) + af(a)$

(d) $-\int_a^b \frac{x}{3[\{f(x)\}^2 - 1]} dx - bf(b) + af(a)$

83. $\int_{-1}^{1} g'(x) \, dx$ is equal to

(a) $2g(-1)$ (b) 0 (c) $-2g(1)$ (d) $2g(1)$

(ii) JEE Main & AIEEE

84. The area (in sq. units) of the region $\{(x, y): x \geq 0, x + y \leq 3, x^2 \leq 4y\}$ and $y \leq 1 + \sqrt{x}\}$ is **[2017 JEE Main]**

(a) $\frac{5}{2}$ (b) $\frac{59}{12}$ (c) $\frac{3}{2}$ (d) $\frac{7}{3}$

85. The area (in sq units) of the region $\{(x, y): y^2 \geq 2x$ and $x^2 + y^2 \leq 4x, x \geq 0, y \geq 0\}$ is **[2016 JEE Main]**

(a) $\pi - \frac{4}{3}$ (b) $\pi - \frac{8}{3}$

(c) $\pi - \frac{4\sqrt{2}}{3}$ (d) $\frac{\pi}{2} - \frac{2\sqrt{2}}{3}$

86. The area (in sq units) of the region described by $\{(x, y): y^2 \leq 2x$ and $y \geq 4x - 1\}$ is **[2015 JEE Main]**

(a) $\frac{7}{32}$ (b) $\frac{5}{64}$

(c) $\frac{15}{64}$ (d) $\frac{9}{32}$

87. The area (in sq units) of the quadrilateral formed by the tangents at the end points of the latusrectum to the ellipse $\frac{x^2}{9} + \frac{y^2}{5} = 1$ is **[2015 JEE Main]**

(a) $\frac{27}{4}$ (b) 18

(c) $\frac{27}{2}$ (d) 27

88. The area of the region described by $A = \{(x, y): x^2 + y^2 \leq 1$ and $y^2 \leq 1 - x\}$ is **[2014 JEE Main]**

(a) $\frac{\pi}{2} + \frac{4}{3}$ (b) $\frac{\pi}{2} - \frac{4}{3}$

(c) $\frac{\pi}{2} - \frac{2}{3}$ (d) $\frac{\pi}{2} + \frac{2}{3}$

89. The area (in sq units) bounded by the curves $y = \sqrt{x}$, $2y - x + 3 = 0$, X-axis and lying in the first quadrant is **[2013 JEE Main]**

(a) 9 (b) 36

(c) 18 (d) $\frac{27}{4}$

90. The area bounded between the parabolas $x^2 = \frac{y}{4}$ and $x^2 = 9y$ and the straight line $y = 2$ is **[2012 AIEEE]**

(a) $20\sqrt{2}$ (b) $\frac{10\sqrt{2}}{3}$

(c) $\frac{20\sqrt{2}}{3}$ (d) $10\sqrt{2}$

91. The area of the region enclosed by the curves $y = x$, $x = e$, $y = \frac{1}{x}$ and the positive X-axis is **[2011 AIEEE]**

(a) 1 sq unit (b) $\frac{3}{2}$ sq units

(c) $\frac{5}{2}$ sq units (d) $\frac{1}{2}$ sq unit

92. The area bounded by the curves $y = \cos x$ and $y = \sin x$ between the ordinates $x = 0$ and $x = \frac{3\pi}{2}$ is **[2010 AIEEE]**

(a) $(4\sqrt{2} - 2)$ sq units (b) $(4\sqrt{2} + 2)$ sq units

(c) $(4\sqrt{2} - 1)$ sq units (d) $(4\sqrt{2} + 1)$ sq units

93. The area of the region bounded by the parabola $(y - 2)^2 = x - 1$, the tangent to the parabola at the point $(2, 3)$ and the X-axis is **[2009 AIEEE]**

(a) 6 sq units (b) 9 sq units

(c) 12 sq units (d) 3 sq units

94. The area of the plane region bounded by the curves $x + 2y^2 = 0$ and $x + 3y^2 = 1$ is equal to **[2008 AIEEE]**

(a) $\frac{5}{3}$ sq units (b) $\frac{1}{3}$ sq unit

(c) $\frac{2}{3}$ sq unit (d) $\frac{4}{3}$ sq units

Answers

Exercise for Session 1

1. 1 sq unit
2. $\frac{436}{15}$ sq units
3. $\frac{4}{3}$ sq units
4. $\frac{4}{3}$ sq units
5. $\frac{28}{3}$ sq units
6. $\frac{56}{3}a^2$ sq units
7. $\frac{8}{3}a^2$ sq units
8. 2z sq units
9. $\frac{28}{3}$ sq units
10. 3+16 log2 sq units

Exercise for Session 2

1. (d)	**2.** (a)	**3.** (c)	**4.** (d)	**5.** (a)
6. (b)	**7.** (a)	**8.** (a)	**9.** (c)	**10.** (b)
11. (d)	**12.** (b)	**13.** (a)	**14.** (c)	**15.** (b)
16. (a)	**17.** (c)	**18.** (a)	**19.** (a)	**20.** (b)

Chapter Exercises

1. (d)	**2.** (b)	**3.** (b)	**4.** (c)	**5.** (b)
6. (c)	**7.** (d)	**8.** (a)	**9.** (b)	**10.** (b)
11. (b)	**12.** (b)	**13.** (c)	**14.** (d)	**15.** (a)
16. (a)	**17.** (a)	**18.** (a)	**19.** (c)	**20.** (b)
21. (a)	**22.** (b)	**23.** (d)	**24.** (c)	**25.** (b, c)
26. (a, b, d)		**27.** (a, d)	**28.** (a, d)	**29.** (a, c)
30. (d)	**31.** (c)	**32.** (d)	**33.** (b)	**34.** (c)
35. (a)	**36.** (b)	**37.** (a)	**38.** (a)	**39.** (a)
40. (b)	**41.** (c)	**42.** (b)	**43.** (a)	

44. (A) → (p); (B) → (p); (C) → (r); (D) → (s)

45. (A) → (p); (B) → (p); (C) → (r); (D) → (r)

46. (7)	**47.** (1)	**48.** (4)	**49.** (7)
50. (8)	**51.** (5)	**52.** (3)	

53. 3 e sq units

54. $f(x) = \frac{1}{k+1}[x^4 - kx^3 + (2k-4)x^2]$

55. $g(x) = \begin{cases} -x-1, & x \le -5/2 \\ 4+x, & -5/2 < x \le -2 \\ 2, & -2 < x \le -1 \\ 1-x, & -1 < x \le -1/2 \\ 1+x, & x > -1/2 \end{cases}$

and area = $\frac{101}{4}$ sq units

56. $\left[2 + \log\left(\frac{4}{3\sqrt{3}}\right) - \frac{1}{e}\right]$ sq units

57. $\left(\pi - \frac{2}{3}\right)$ sq units

58. $\frac{1}{8}\left(\frac{3\pi}{16}\right)^4$ sq units

59. Required maximum ratio = $\frac{2}{3}$

60. $(A)_{max} = \frac{\sqrt{3}\,\pi A}{9}$ sq units

61. $A_1 = 1 - \sin 1, A_2 = \pi - 1 - \sin 1, A_3 = 3\pi - 2$

62. $4 \log_5\left(\frac{e^4}{27}\right)$ sq units

63. $4\left(\frac{16}{3} - 4\pi\right)$ sq units

64. $\frac{3}{2}$

65. 2 sq units

66. 8 sq units

67. 5 sq units

68. $\left(2 - \frac{\pi}{2}\right)$ sq units

69. $\left[\frac{4}{\log 2}(1 - 2^{-1/2}) - 1\right]$ sq units

70. $a \in [1, 2)$

72. (c)	**73.** (3)	**74.** (d)	**75.** (b)	
76. (b,d)	**77.** (c)	**78.** (b)	**79.** (b,c,d)	**80.** (b)
81. (b)	**82.** (a)	**83.** (d)	**84.** (b)	**85.** (d)
86. (d)	**87.** (a)	**88.** (a)	**89.** (c)	**90.** (b)
91. (a)	**92.** (b)	**93.** (d)		

Solutions

1. Here, $[x+y]=[x]-1$

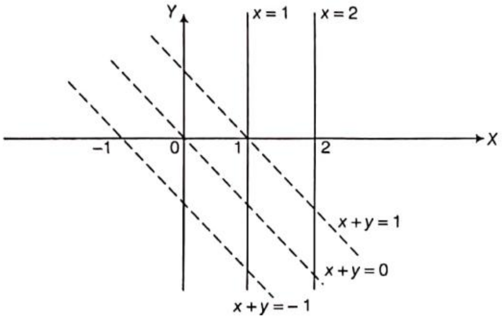
when $x \in [0,1) \Rightarrow [x+y]=-1$

$$-1 \le x+y < 0 \qquad \text{...(i)}$$

when $x \in [1,2) \Rightarrow [x+y]=0$

$\therefore$ $0 \le x+y < 1$ ...(ii)

which can be shown, as



$\therefore$ Required area = 2

2. Required area $= 2\int_0^{1/2} f^{-1}(x)\,dx$

Let, $f^{-1}(x)=t \Rightarrow x=f(t)$

$dx = f'(t)dt$

$\therefore$ $A = 2\int_0^1 (t\cdot f'(t))dt$

$$= 2\left[\left(t\cdot f\;(t)\right)_0^1 - \int_0^1 1\cdot f(t)dt\right]$$

$$= 2\left[f(1) - \int_0^1 \frac{t}{1+t^2}\,dt\right]$$

$$= 2\left[f(1) - \frac{1}{2}\left(\log(1+t^2)\right)_0^1\right]$$

$$= 2\left[f(1) - \frac{1}{2}\log 2\right]$$

$$= 2\left[\frac{1}{2} - \frac{1}{2}\log 2\right] = 1 - \log 2 = \log\left(\frac{e}{2}\right)$$

3. Here, $E_1 : \dfrac{\left(\dfrac{x+y-1}{\sqrt{2}}\right)^2}{12} + \dfrac{\left(\dfrac{x-y+3}{\sqrt{2}}\right)^2}{(\sqrt{2})^2} = 1$

Length of latusrectum $= 2\dfrac{a^2}{b} = \dfrac{2}{\sqrt{2}} = \sqrt{2}$

Now, $\dfrac{2p^2}{\sqrt{p}} = \sqrt{2} \Rightarrow p^{3/2} = 2^{-1/2}$

$\Rightarrow$ $P = 2^{-1/3}$

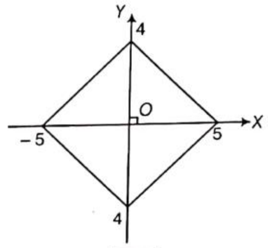
$\therefore$ $E_2 : \dfrac{x^2}{2^{-1/3}} + \dfrac{y^2}{4^{-1/3}} = 1$

Area of ellipse E_2, is

$$\pi\cdot\sqrt{p}\cdot p = \pi\, p^{3/2} = \frac{\pi}{\sqrt{2}}$$

4. Area of bounded region by $4|x-2017^{2017}| + 5|y-2017^{2017}| \le 20$, is same as area of the region bounded by $4|x|+5|y| \le 20$.

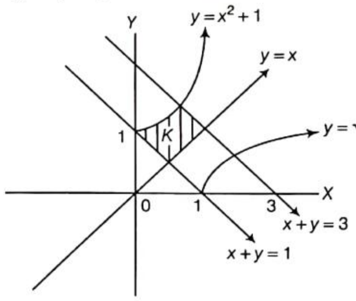
$\Rightarrow$ $4 \times \dfrac{1}{2} \times 4 \times 5 = 40$



5. Here, $y = x^2+1$ and $y = \sqrt{x-1}$ are inverse of each other.

The shaded area is given K units

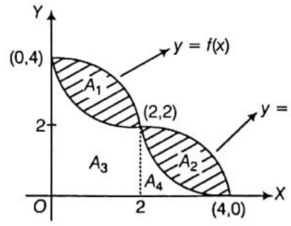
$\Rightarrow$ Area of the region bounded by $y = x^2+1$, $y=\sqrt{x-1}$ and $(x+y-1)(x+y-3)=0$, is $2K$ units.



6. Given, $\int_0^4 f(x)\,dx - \int_0^4 g(x)\,dx = 10$

$$(A_1+A_3+A_4)-(A_2+A_3+A_4) = 10$$

$$A_1 - A_2 = 10 \qquad \text{...(i)}$$



Again, $\int_2^4 g(x)\,dx - \int_2^4 f(x)\,dx = 5$

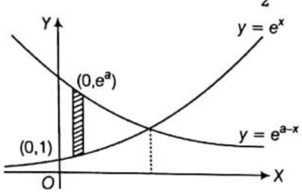
$$(A_2+A_4) - A_4 = 5$$

$$A_2 = 5 \qquad \text{...(ii)}$$

Adding Eqs. (i) and (ii),

$$A_1 = 15$$

7. Solving, $e^x = e^{a-x}$, we get $e^{2x} = e^a \Rightarrow x = \frac{a}{2}$



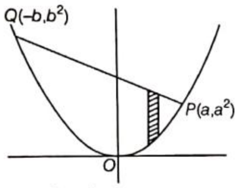
$$S = \int_0^{a/2} (e^a \cdot e^{-x} - e^x)\, dx = [-(e^a \cdot e^{-x} + e^x)]_0^{a/2}$$

$$= (e^a + 1) - (e^{a/2} + e^{a/2}) = e^a - 2e^{a/2} + 1 = (e^{a/2} - 1)^2$$

$$\therefore \quad \frac{S}{a^2} = \left(\frac{e^{a/2} - 1}{a}\right)^2 = \frac{1}{4}\left(\frac{e^{a/2} - 1}{a/2}\right)^2$$

$$\therefore \quad \lim_{a \to 0} \frac{S}{a^2} = \frac{1}{4}$$

8. $m_{PQ} = \dfrac{a^2 - b^2}{a + b} = a - b$ equation of PQ



$$y - a^2 = \frac{a^2 - b^2}{a + b}(x - a) \quad \text{or} \quad y - a^2 = (a - b)(x - a)$$

$$y = a^2 + x(a - b) - a^2 + ab$$

$$y = (a - b)x + ab$$

$$\therefore \quad S_1 = \int_{-b}^{a} ((a - b)x + ab - x^2)\, dx$$

which simplifies to $\dfrac{(a + b)^3}{6}$. ...(i)

Also, $S_2 = \dfrac{1}{2}\begin{vmatrix} a & a^2 & 1 \\ -b & b^2 & 1 \\ 0 & 0 & 1 \end{vmatrix} = \dfrac{1}{2}[ab^2 + a^2b] = \dfrac{1}{2}ab\,(a + b)$ ...(ii)

$$\therefore \quad \frac{S_1}{S_2} = \frac{(a + b)^3}{6} \cdot \frac{2}{ab\,(a + b)} = \frac{(a + b)^2}{3ab} = \frac{1}{3}\left[\frac{a}{b} + \frac{b}{a} + 2\right]$$

$$\therefore \quad \left.\frac{S_1}{S_2}\right|_{\min} = \frac{4}{3}$$

9. $y = \ln^2 x - 1 \Rightarrow y' = \dfrac{2 \ln x}{x} = 0 \Rightarrow x = 1$

$x > 1$, y increasing and $0 < x < 1$, y is decreasing

$$A = \left| \int_{1/e}^{e} (\ln^2 x - 1)\, dx \right|$$

$$= \left| \int_{1/e}^{e} \ln^2 x\, dx - \int_{1/e}^{e} dx \right|$$

$$= \left| [x \ln^2 x]_{1/e}^{e} - 2\int_{1/e}^{e} \left(\frac{\ln x}{x}\right) \cdot x\, dx - \left(e - \frac{1}{e}\right) \right|$$

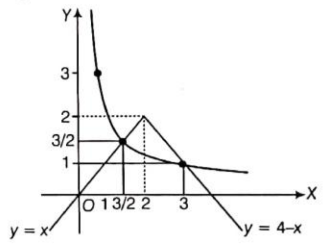
$$= \left| \left(e - \frac{1}{e}\right) - 2\int_{1/e}^{e} \left(\frac{\ln x}{x}\right) \cdot x\, dx - \left(e - \frac{1}{e}\right) \right|$$

$$= \left| -2\left[[x \ln x]_{1/e}^{e} - \int_{1/e}^{e} dx \right] \right|$$

$$= \left| -2\left[\left(e + \frac{1}{e}\right) - \left(e - \frac{1}{e}\right) \right] \right| = \left|\frac{4}{e}\right| = \frac{4}{e}$$

10. $y = \begin{cases} 2 - (2 - x), & \text{if } x \le 2 = x \\ 2 - (x - 2), & \text{if } x \ge 2 = 4 - x \end{cases} = \begin{cases} x, & \text{if } x \le 2 \\ 4 - x, & \text{if } x \ge 2 \end{cases}$

Also, $y = \begin{cases} \dfrac{3}{x}, & \text{if } x > 0 \\ -\dfrac{3}{x}, & \text{if } x < 0 \end{cases}$



$$A = \int_{3/2}^{2} \left(x - \frac{3}{x}\right) dx + \int_{2}^{3} \left((4 - x) - \frac{3}{x}\right) dx$$

$$= \left[\frac{x^2}{2} - 3 \ln x\right]_{3/2}^{2} + \left[4x - \frac{x^2}{2} - 3 \ln x\right]_{2}^{3}$$

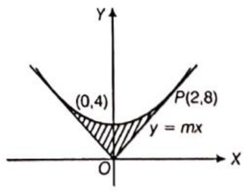
$$= \left[2 - 3 \ln 2 - \left(\frac{9}{8} - 3 \ln \frac{3}{2}\right)\right] + \left[12 - \frac{9}{2} - 3 \ln 3 - (8 - 2 - \ln 2)\right]$$

$$= \frac{7}{8} - 3 \ln 2 + 3 \ln 3 - 3 \ln 2 + \frac{3}{2} - 3 \ln 3 + 3 \ln 2 = \frac{19}{8} - 3 \ln 2$$

11. Given, $g(x) = 2x + 1$; $h(x) = (2x + 1)^2 + 4$

Now, $h(x) = f[g(x)]$

$$(2x + 1)^2 + 4 = f(2x + 1)$$



Let $\quad 2x+1=t \Rightarrow f(t)=t^2+4$

$\therefore \quad f(x)=x^2+4 \quad$ ...(i)

Solving, $\quad y=mx$ and $y=x^2+4$

$x^2-mx+4=0$

Put $D=0$; $\quad m^2=16 \Rightarrow m=\pm 4$

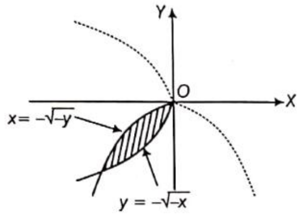
Tangents are $y=4x$ and $y=-4x$

$$A=2\int_0^2[(x^2+4)-4x]\,dx=2\int_0^2(x-2)^2dx$$

$$=\left[\frac{2}{3}(x-2)^3\right]_0^2=\frac{16}{3}\text{ sq units}$$

12. $y=-\sqrt{-x} \Rightarrow y^2=-x$, where x and y both negative

$x=-\sqrt{-y} \Rightarrow x^2=-y$, where x and y both negative

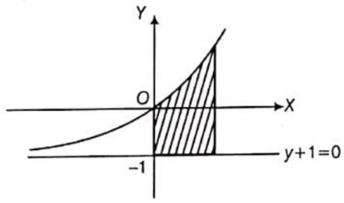


Since, $\quad A=\frac{16ab}{3}$ where, $a=b=\frac{1}{4}$

$\therefore \quad A=\frac{1}{3}$

13. $\frac{f''(x)}{f'(x)}=1.$

Integrating, $\ln f'(x)=x+C,\ f'(0)=1 \Rightarrow C=0$

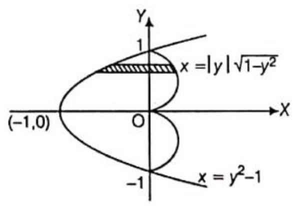


$f'(x)=e^x f(x)=e^x+k,\ f(0)=0 \Rightarrow k=-1$

$f(x)=e^x-1$

$$\text{Area}=\int_0^1(e^x-1+1)\,dx=[e^x]_0^1=e-1$$

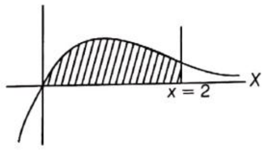
14. $A=2\int_0^1[y\sqrt{1-y^2}-(y^2-1)]\,dy=2$



15. $y=xe^{-x}$

$y'=e^{-x}-xe^{-x}=(1-x)e^{-x}$ increasing for $x<1$

$y''=-e^{x}-[e^{-x}-xe^{-x}]=e^{-x}[-1-1+x]=(x-2)e^{-x}$



For point of inflection $y''=0 \Rightarrow x=2$

$$A=\int_0^2 xe^{-x}dx=[-xe^{-x}]_0^2+\int_0^2 e^{-x}=(-2e^{-2})-(e^{-x})_0^2$$

$$=-2e^{-2}-(e^{-2}-1)=1-e^{-2}-2e^{-2}=1-3e^{-2}$$

16. $(a, 0)$ lies on the given curve

$\therefore \quad 0=\sin 2a-\sqrt{3}\sin a \Rightarrow \sin a=0$ or $\cos a=\sqrt{3}/2$

$\Rightarrow \quad a=\frac{\pi}{6}$ (as $a>0$ and the first point of intersection with positive X-axis)

and $A=\int_0^{\pi/6}(\sin 2x-\sqrt{3}\sin x)\,dx=\left(-\frac{\cos 2x}{2}+\sqrt{3}\cos x\right)_0^{\pi/6}$

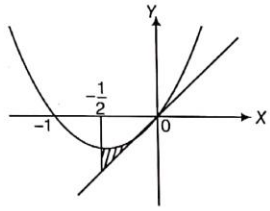
$$=\left(-\frac{1}{4}+\frac{3}{2}\right)-\left(-\frac{1}{2}+\sqrt{3}\right)=\frac{7}{4}-\sqrt{3}=\frac{7}{4}-2\cos a$$

$\Rightarrow \quad 4A+8\cos a=7$

17. $x=1;\ y=2$

$2=a+b+c \quad$ ...(i)

$x=0,\ y=0 \Rightarrow c=0 \Rightarrow a+b=2$



Now, $\left.\frac{dy}{dx}\right|_{(0,0)}=2a\,x+b=1$

$\therefore \quad b=1,\ a=1$

Hence, the curve is $y=x^2+x$

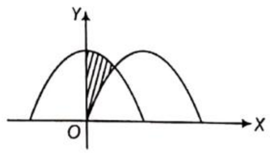
$$A=\int_{-\frac{1}{2}}^{0}(x^2+x-x)\,dx=\int_{-\frac{1}{2}}^{0}(x^2)\,dx=\frac{1}{24}\text{ sq units}$$

18. $\text{IF}=e^{-x}$

$\therefore\ ye^{-x}=\int e^{-x}(\cos x-\sin x)\,dx$. Put $-x=t$

$=-\int e^t(\cos t+\sin t)\,dt=-e^t\sin t+C$

$ye^{-x}=e^{-x}\sin x+C$



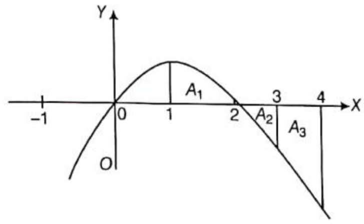
Since, y is bounded when $x\to\infty \Rightarrow C=0$

$\therefore \qquad y = \sin x$

$$\text{Area} = \int_0^{\pi/4} (\cos x - \sin x)\, dx = \sqrt{2} - 1$$

19. $I = \int (6x - 3x^2)\, dx = \dfrac{6x^2}{2} - \dfrac{3x^3}{3} = 3x^2 - x^3 = x^2(3 - x)$

$A_1 = I(2) - I(1) = 4 - 2 = 2$ units
$A_2 = I(2) - I(3) = 4 - 0 = 4$ units
$A_3 = I(3) - I(4) = 0 - (-16) = 16$ units



$\Rightarrow$ One value of a will lie in (3, 4).
Using symmetry, other will lie in (–2, –1).

20. Required area, $A = \int_{\pi}^{2\pi} ((\sin x - x) + 2\pi)\, dx$

$$= \frac{\pi^2}{2} - 2 \text{ sq units}$$

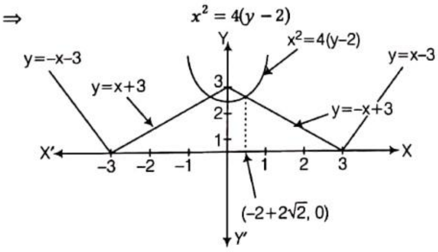
21. $g(x) = |\, f(|\,x\,|) - 2\,| = |\,|\,x\,| - 1 - 2\,| = |\,|\,x\,| - 3\,|$

$$= \begin{cases} (|x| - 3), & x < -3 \\ -(|x| - 3), & -3 \le x < 3 \\ |\,x\,| - 3 & x \ge 3 \end{cases} = \begin{cases} -x - 3, & x < -3 \\ -(-x - 3), & -3 \le x < 0 \\ -(x - 3), & 0 \le x < 3 \\ x - 3, & x \ge 3 \end{cases}$$

$$= \begin{cases} -x - 3, & x < -3 \\ x + 3, & -3 \le x < 0 \\ -x + 3, & 0 \le x < 3 \\ x - 3, & x \ge 3 \end{cases}$$

Now, $\qquad x^2 - 4y + 8 = 0$

$\Rightarrow \qquad x^2 = 4(y - 2)$



For point of intersection,

$$x^2 - 4(-x + 3) + 8 = 0$$

$\Rightarrow \qquad x^2 + 4x - 4 = 0$

$\Rightarrow \qquad x = \dfrac{-4 \pm \sqrt{16 + 16}}{2} = -2 \pm 2\sqrt{2}$

$\therefore$ Point of intersection is at $x = -2 + 2\sqrt{2}$

$\therefore$ Required area $= 2\int_0^{-2+2\sqrt{2}} \left[(-x + 3) - \left(\dfrac{x^2 + 8}{4}\right)\right] dx$

$$= \frac{2}{4}\int_0^{-2+2\sqrt{2}} (4 - 4x - x^2)\, dx = \frac{1}{2}\left[4x - 2x^2 - \frac{x^3}{3}\right]_0^{-2+2\sqrt{2}}$$

$$= \frac{1}{6}[12x - 6x^2 - x^3]_0^{-2+2\sqrt{2}}$$

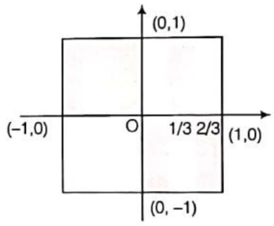
$$= \frac{1}{6}[12(-2 + 2\sqrt{2}) - 6(-2 + 2\sqrt{2})^2 - (-2 + 2\sqrt{2})^3]$$

$$= \frac{1}{6}[-24 + 24\sqrt{2} - 6(4 + 8 - 8\sqrt{2}) - (-8 + 16\sqrt{2}) + 24\sqrt{2} - 48]$$

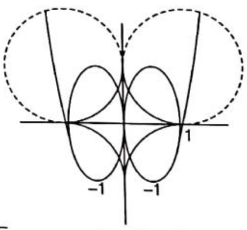
$$= \frac{1}{6}[-24 + 24\sqrt{2} - 72 + 48\sqrt{2} + 56 - 40\sqrt{2}]$$

$$= \frac{1}{6}[32\sqrt{2} - 40] = \frac{8}{6}(4\sqrt{2} - 5) = \frac{4}{3}(4\sqrt{2} - 5)$$

22. Shaded region represents $S \cap S'$ clearly area enclosed is 2 sq units.



23. Required area is $2\left[\int_0^{\alpha} ex \ln x\, dx + \int_1^{\alpha} (1 - \sqrt{1 - (x - 1)^2})\, dx\right]$

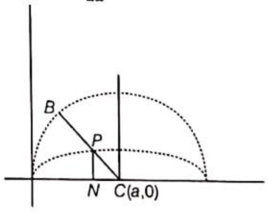


24. $y = \sqrt{2ax - x^2} \Rightarrow (x - a)^2 + y^2 = a^2$

Let $P(h, k)$ be a point, then $BP > PN$
For the bounded condition $BP = PN = k$

Now, $\qquad AP = a - k = \sqrt{(h - a)^2 + k^2}$

$\Rightarrow \qquad k = h - \dfrac{h^2}{2a}$



$\therefore$ Boundary of the region is $y = x - \dfrac{x^2}{2a}$

Required area $= 2\int_0^a \left(x - \dfrac{x^2}{2a}\right) dx = \dfrac{2a^2}{3}$

25. $f'(x) = 2x - a$. At (2, 4), $f'(x) = 4 - a$

Equation of normal at (2, 4) is $(y-4) = -\dfrac{1}{(4-a)}(x-2)$

Let point of intersection with X and Y-axes be A and B respectively, then

$$A \equiv (-4a+18, 0)\ B \equiv \left(0, \frac{4a-18}{a-4}\right)$$

Since, $a > \dfrac{9}{2}$ as

$\therefore$ Area of triangle $= \dfrac{1}{2}(4a-18)\dfrac{(4a-18)}{(a-4)} = 2$

$\Rightarrow$ $(4a-17)(a-5) = 0$

$\Rightarrow$ $a = 5$ or $\dfrac{17}{4}$

26. Max $\{f(x), g(x)\} = \dfrac{1}{2}[|f(x)+g(x)| + |f(x)-g(x)|]$

Min $\{f(x), g(x)\} = \dfrac{1}{2}[|f(x)+g(x)| - |f(x)-g(x)|]$

$\therefore$ Area $= \int_a^b$ [max $\{f(x), g(x)\}$ − min $\{f(x), g(x)\}$] dx

27. Area bounded by parabola $y = x^2 - 7x + 10$ and X-axis is given by

$$\int_2^5 |x^2 - 7x + 10|\, dx = \int_2^5 |-x^2 + 7x - 10|\, dx = \frac{9}{2} \text{ sq units}$$

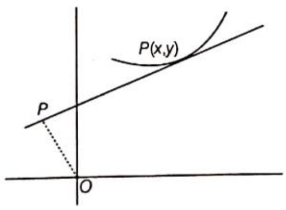
28. Area bounded by the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ will be the same as the area bounded by the ellipse $\dfrac{x^2}{b^2} + \dfrac{y^2}{a^2} = 1$ and πab

$\therefore$ Required area $= \pi(2)(3) = 6\pi$ sq units

29. $OP = x$

If slope is $\dfrac{dy}{dx}$, then equation of tangent is

$$Y - t = \frac{dy}{dx}(X - x)$$



Length of perpendicular from origin to this tangent is

$$x = \frac{Y - x\dfrac{dy}{dx}}{\sqrt{1 + \left(\dfrac{dy}{x}\right)^2}}$$

$$x^2\left\{1 + \left(\frac{dy}{dx}\right)^2\right\} = y^2 + x^2\left(\frac{dy}{dx}\right)^2 = 2xy\frac{dy}{dx}$$

$\Rightarrow$ $\dfrac{dy}{dx} = -\dfrac{x^2 - y^2}{2xy}$ [homogeneous form]

$$\frac{dy}{dx} = \frac{1-x}{y} = \frac{x - x^2}{xy} = \frac{\left[\dfrac{x^2+y^2}{2}\right] - x^2}{xy} = \frac{y^2 - x^2}{2xy}$$

Since, option (a) is true (equation of circle).

If option (a) is true (b) can't be (It is parabola).

If option (a) is true, option (c) is also true where $c = 1$.

30. $\therefore$ $\sin^2 x \le \sin x$, $\forall x \in (0, \pi)$

Therefore, area of $y = \sin^2 x$ will be lesser from area of $y = \sin x$.

Statement II is obviously true.

Hence, (d) is the correct answer.

31. Let the line $y = kx + 2$ cuts $y = x^2 - 3$ at $x = \alpha$ and $\alpha = \beta$, area bounded by the curves $= \int_\alpha^\beta (y_1 - y_2) = \int_\alpha^\beta \{(kx+2) - (x^2-3)\}\, dx$

$\Rightarrow$ $f(x) = \dfrac{(k^2+20)^{3/2}}{6}$

which clearly, shows the Statement II is false but $f(k)$ is least when $k = 0$.

Hence, (c) is the correct answer.

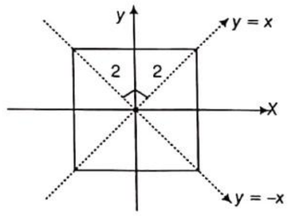
32. As of region bounded by parabola $y^2 = 4x$ and $x^2 = 4y$ is

$$\int_0^4 \left(2\sqrt{x} - \frac{x^2}{4}\right) dx = \left[\frac{4}{3}x^{3/2} - \frac{x^3}{12}\right]_0^4 = \frac{32}{3} - \frac{16}{3} = \frac{16}{3} \text{ sq units}$$

Hence, Statement I is false.

33. As the area enclosed by $|x| + |y| \le a$ is the area of square (i.e., $2a^2$).

$\therefore$ Area enclosed by $|x+y| + |x-y| \le 2$ is area of square shown as



$\therefore$ Area $= 4\left(\dfrac{1}{2} \times 2 \times 2\right) = 8$ sq units

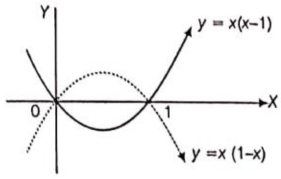
Also, the area enclosed by $|x+y| + |x-y| \le 2$ is symmetric about X-axis, Y-axis, $y = x$ and $y = -x$.

$\therefore$ Both the Statements are true but Statement II is not the correct explanation of Statement I.

34. $\left|\int_a^b \{f(x) - g(x)\}\, dx\right|$ is true for all quadrants.

$\therefore$ Statement II is false.

The area bounded by $y = x(x-1)$ and $y = x(1-x)$.



$\therefore$ Area enclosed $= 2\int_0^1 x(1-x)\,dx = 2\left[\frac{x^2}{2}-\frac{x^3}{3}\right]_0^1$

$$= 2\left(\frac{1}{2}-\frac{1}{3}\right) = 2\left(\frac{3-2}{6}\right) = \frac{1}{3}$$

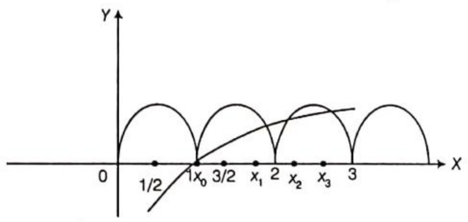
35. Since, absolute area

$$= \int_{x_0}^{x_1} h(x)\,dx + \int_{x_1}^{x_2} -h(x)\,dx + \int_{x_2}^{x_3} h(x)\,dx$$

$$= \sum_{r=0}^{n} \int_{x_r}^{x_{r+1}} (-1)^r \cdot h(x)\,dx$$

36. Also, $x_{n+1} = x_3 \Rightarrow n = 2$

37. Required area $= \int_0^1 \sin^4 \pi x\,dx - \int_0^1 \log_e x\,dx = \frac{11}{8}$



38. $f(x) = \frac{x^2-ax+1}{x^2+ax+1}$

For differentiation, better write $f(x)$ as

$$f(x) = 1 - \frac{2ax}{x^2+ax+1}$$

Now, on differentiation $f(x) = \frac{2a(x^2-1)}{(x^2+ax+1)^2}$ ...(i)

$$f'(1) = 0 = f'(-1)$$

Then, the options (b) and (d) are eliminated.

Again, for the differentiating Eq. (i) gives

$$(x^2+ax+1)^2 \cdot 2x - (x^2-1)$$

$$f''(x) = 2a \cdot \frac{2(x^2+ax+1)(2x+a)}{(x^2+ax+1)^4}$$

$$f''(-1) = -\frac{4a}{(2-a)^2},\ f''(1) = -\frac{4a}{(2+a)^2}$$

Combining both $f''(1)(2+a)^2 + f''(-1)(2-a)^2 = 0$

39. $f'(x) = 2a\frac{(x^2-1)}{(x^2+ax+1)^2} = 2a\frac{(x-1)(x+1)}{(x^2+ax+1)^2}$

If is easily seen that $f(x)$ decreases on $(-1,1)$ and has a local minimum at $x=1$, because the derivatives changes its sign from negative to positive.

40. $\because$ $g'(x) = \frac{f'(e^x)e^x}{1+e^{2x}}$

Now, $f'(e^x) = \frac{2a\cdot(e^{2x}-1)}{(e^{2x}+ae^x+1)^2}$

It is seen from the above that $f'(e^x)$ and so $g'(x)$ is positive on $(0,\infty)$ and negative on $(-\infty, 0)$.

41. Clearly, $x = a\sin^3 t, y = a\cos^3 t, (0 \le t \le 2\pi)$

$$S = \int_0^{2\pi} a\sin^3 t \cdot a \cdot 3\sin^2 t(-\sin t)\,dt$$

$$= -3a^2 \times \int_0^{\pi/2} \sin^4 t \cos^2 t\,dt$$

$$= -3a^2 \times \frac{\Gamma\frac{5}{2}\,\Gamma\frac{3}{2}}{2\,\Gamma\frac{4+2+2}{2}} = -3a^2\frac{\frac{3}{2}\times\frac{1}{2}\sqrt{\pi}\times\frac{1}{2}\sqrt{\pi}}{2\times3\times2\times1}$$

$$= -\frac{3}{8}\pi a^2 = \frac{3}{8}\pi a^2 \quad \text{[absolute value]}$$

42. $S = -\int_0^{2\pi} a(1-\cos t)a(1-\cos t)\,dt$

$$= -a^2\int_0^{2\pi}(1-2\cos t+\cos^2 t)\,dt$$

$$= -a^2\int_0^{2\pi}\left(1-2\cos t+\left(\frac{1+\cos 2t}{2}\right)\right)dt$$

$$= -\frac{a^2}{2}\int_0^{2\pi}[3-4\cos t+\cos 2t]_0^{2\pi}$$

$$= -3\pi a^2 = 3\pi a^2 \quad \text{[absolute value]}$$

43. $x = \frac{t}{3}(6-t) = \frac{1}{3}(6t-t^2),\ y = \frac{t^2}{8}(6-t) = \frac{1}{8}(6t^2-t^3)$

$\therefore$ Area $= \int_0^{6t}\frac{t}{3}(6-t)\cdot\frac{1}{8}(12t-13t^2)dt$

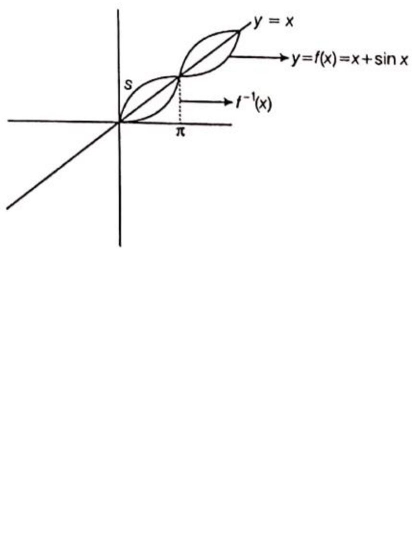
$$= \frac{1}{8}\int_0^6 (t^4+24t^2-10t^3)\,dt = \frac{1}{8}\left[\frac{t^5}{5}+8t^3-\frac{5}{2}t^4\right]_0^6$$

$$= \frac{1}{8}\left[\frac{7776}{5}+1728-3240\right] = \frac{216}{5\times8} = \frac{27}{5}\text{ sq units}$$

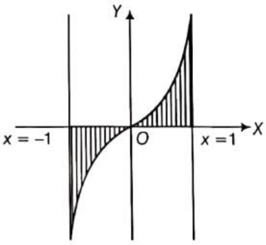
44. (A) Required area = 4s

$$s = \int_0^{\pi}(x+\sin x)\,dx - \int_0^{\pi} x\,dx$$

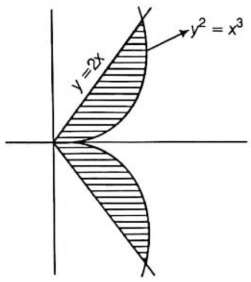
$$= \frac{\pi^2}{2} - \cos\pi + \cos 0 - \frac{\pi^2}{2} = 2\text{ sq units}$$



(B) Required area $= 2\int_0^1 xe^x\,dx = 2[xe^x - e^x]_0^1 = 2$



(C) $y^2 = x^3$ and $|y| = 2x$ both the curve are symmetric about Y-axis

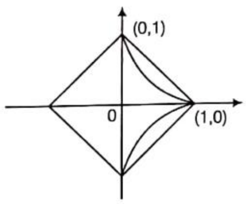


$$4x^2 = x^3 \Rightarrow x = 0, 4$$

Required area $= 2\int_0^4 (2x - x^{3/2})\,dx = \frac{16}{5}$

(D) $\sqrt{x} + \sqrt{|y|} = 1$

Above curve is symmetric about X-axis



$$\sqrt{|y|} = 1 - \sqrt{x} \text{ and } \sqrt{x} = 1 - \sqrt{|y|}$$

$\Rightarrow$ for $x > 0, y > 0, \sqrt{y} = 1 - \sqrt{x}$

$$\frac{1}{2\sqrt{y}}\frac{dy}{dx} = -\frac{1}{2\sqrt{x}}$$

$$\frac{dy}{dx} = -\sqrt{\frac{x}{y}}$$

$$\frac{dy}{dx} < 0$$

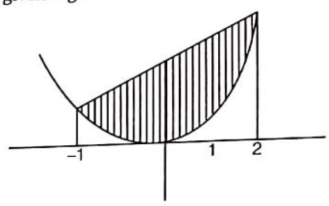
Function is decreasing required area $= \int_0^1 (2\sqrt{x} - 2x) = \frac{1}{3}$.

45. (A) The area = 2 unit

(B) Area enclosed $= \int_0^{\pi} \sin x\,dx = 2$

(C) The line $y = x + 2$ intersects $y = x^2$ at $x = -1$ and $x = 2$

The given region is shaded region area

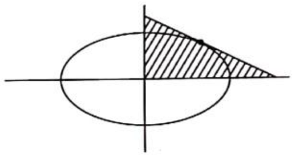


$$= \frac{15}{2} - \int_{-1}^{2} x^2\,dx = \frac{9}{2}$$

(D) Here, $a^2 = 9, b^2 = 5, b^2 = a^2(1 - e^2)$

$\Rightarrow \quad e^2 = \frac{4}{9} \Rightarrow = \frac{2}{3}$

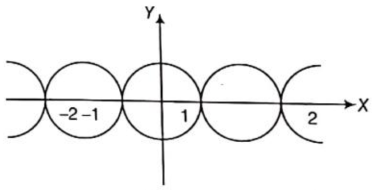
Equation of tangent at $\left(2, \frac{5}{3}\right)$ is $\frac{2x}{9} + \frac{y}{3} = 1$



x-intercept $= \frac{9}{2}$, y-intercept $= 3$

Area $= 4 \times \frac{9}{2} \times 3 \times \frac{1}{2} = 27$ sq units

46.



Area is given by $A = 2\int_0^1 (x^2 - 3x + 2)\,dx = \frac{5}{3}$

$\Rightarrow \quad 3A + 2 = 3 \cdot \frac{5}{3} + 2 = 7$

47. For $c < 1; \int_c^1 (8x^2 - x^5)\,dx = \frac{1}{3}$

$\Rightarrow \quad \frac{8}{3} - \frac{1}{6} - \frac{8c^3}{3} - \frac{c^6}{6} = \frac{16}{6}$

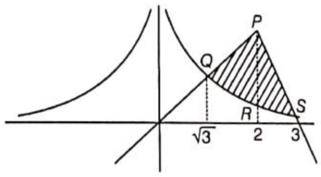
$\Rightarrow \quad c^3\left[-\frac{8}{3} + \frac{c^3}{6}\right] = \frac{16}{3} - \frac{8}{3} + \frac{1}{6} = \frac{17}{6}$

$\Rightarrow \quad c = -1$

Again, for $c \geq 1$ none of the values of c satisfy the required condition that

$$\int_1^c (8x^2 - x^5)\,dx = \frac{16}{3} \Rightarrow c + 2 = 1$$

48. $y = \begin{cases} x; & x < 2 \\ 4 - x; & x \ge 2 \end{cases}$ and $y = \begin{cases} 3/x; & x > 0 \\ -3/x & x < 2 \end{cases}$



Required area = $PQRSP$ = Area $PQRP$ + Area $PRSP$

$$= \left| \int_{\sqrt{3}}^{2} \left(x - \frac{3}{x} \right) dx \right| + \left| \int_{2}^{3} \left((4 - x) - \frac{3}{x} \right) dx \right|$$

$$= \frac{4 - 3 \ln 3}{2} \text{ sq units}$$

49. Equation of AB $y = \frac{5}{2}(x - 2)$

Equation of BC $y - 5 = \frac{3-5}{6-4}(x - 4) \Rightarrow y = -x + 9$

Equation of CA $y - 3 = \frac{0-3}{2-6}(x - 6) \Rightarrow y = \frac{3}{4}(x - 2)$

Required area

$$= \frac{5}{2}\int_{2}^{4}(x-2)\,dx + \int_{4}^{6} -(x-9)\,dx - \frac{3}{4}\int_{2}^{6}(x-2)\,dx$$

$$= \frac{5}{2}\left[\frac{(x-2)^2}{2}\right]_2^4 - \left[\frac{(x-9)^2}{2}\right]_4^6 - \frac{3}{4}\left[\frac{(x-2)^2}{2}\right]_2^6$$

$$= \frac{5}{2}[2^2 - 0] - \frac{1}{2}[(-3)^2 - (-5)^2] - \frac{3}{8}[4^2 - 0]$$

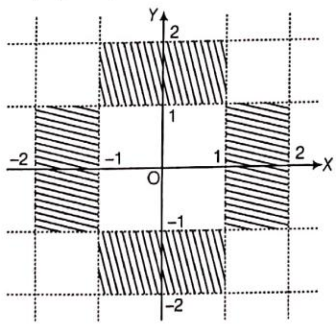
$$= \frac{5}{4} \times 4 - \frac{1}{2}[9 - 25] - \frac{3}{8}[16 - 0]$$

$$= 5 - \frac{1}{2}[-16] - \frac{3}{8} \times 16 = 5 + 8 - 6 = 7 \text{ sq units}$$

50. If $[|x|] = 1$ and $[|y|] = 0$, then $1 \le |x| < 2, 0 \le |y| < 1$

$\Rightarrow$ $x \in (-2, -1] \cup [1, 2), y \in (-1, 1)$

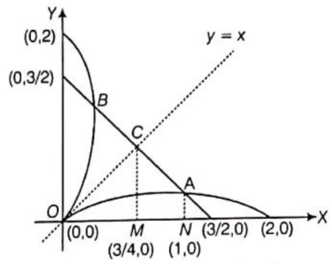
If $[|x|] = 0, [|y|] = 1$



Then, $x \in (-1, 1), y \in (-2, -1] \cup [1, 2]$

Area of required region $= 4(2 - 1)(1 - (-1)) = 8$ sq units

51. Clearly, $f(x) = \frac{2x - x^2}{2}$



Since, $2x + 2y = 3$; passes through $A\left(1, \frac{1}{2}\right)$ and $B\left(\frac{1}{2}, 1\right)$

so bounded area A

= Area OAB = 2 [Area OCM + Area $CMNA$ − Area ONA]

$$= 2\left[\frac{1}{2} \times \frac{3}{4} \times \frac{3}{4} + \frac{1}{2}\left(\frac{3}{4} + \frac{1}{2}\right) \times \frac{1}{4} - \frac{1}{2}\int_0^1 (2x - x^2)\,dx\right] = \frac{5}{24}$$

$\Rightarrow$ $24A = 5$

52. $f(x) = \min\left\{\sin^{-1} x, \cos^{-1} x, \frac{\pi}{6}\right\}, x \in [0, 1]$

$$\therefore \text{ Area} = \int_0^{1/2} \sin^{-1} x\, dx + \frac{1}{2}\left(\frac{\sqrt{3}}{2} - \frac{1}{2}\right) + \int_{\sqrt{3}/2}^{1} \cos^{-1} x\, dx$$

$$= (x \sin^{-1} x + \sqrt{1 - x^2})_0^{1/2} + \frac{\sqrt{3} - 1}{4} + (x \cos^{-1} x - \sqrt{1 - x^2})_{\sqrt{3}/2}^{1}$$

$$= \frac{\pi}{2}(1 - \sqrt{3}) + \frac{\sqrt{3} - 1}{2} + \frac{\sqrt{3} - 1}{2}$$

$$= (\sqrt{3} - 1)\left(\frac{3}{4} - \frac{\pi}{12}\right) = \frac{9 - \pi}{6(\sqrt{3} + 1)} = \frac{a - \pi}{b(\sqrt{3} + 1)}$$

$\therefore$ $a = 9, b = 6 \Rightarrow a - b = 3$

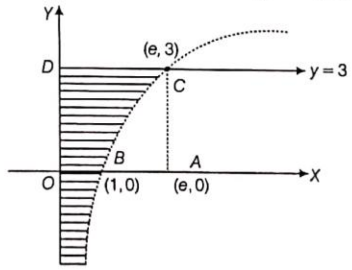
53. Given, $f\left(\frac{x}{y}\right) = f(x) - f(y)$ ...(i)

Putting, $x = y = 1$, $f(1) = 0$

Now, $f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{f\left(1 + \frac{h}{x}\right)}{h}$ [from Eq. (i)]

$$= \lim_{h \to 0} \frac{f\left(1 + \frac{h}{x}\right)}{\frac{h}{x} \cdot x}$$

$\Rightarrow$ $f'(x) = \frac{3}{x}$ $\left[\text{since, } \lim_{x \to 0} \frac{f(1 + x)}{x} = 3\right]$



$\Rightarrow \qquad f(x) = 3 \log x + c$

Putting $\qquad x = 1 \Rightarrow c = 0$

$\Rightarrow \qquad f(x) = 3 \log x = y \qquad$ [say]

$\therefore$ Required area $= \int_{-\infty}^{3} x \, dy = \int_{-\infty}^{3} e^{y/3} \, dy = 3 \, [e^{y/3}]_{-\infty}^{3}$

$= 3(e - 0) = 3e$ sq units

54. According to given conditions,

$$\int_0^t [f(x) - (x^4 - 4x^2)] \, dx = k \int_0^t [(2x^2 - x^3) - f(x)] \, dx$$

Differentiable both the sides w.r.t. t, we get

$$f(t) - (t^4 - 4t^2) = k((2t^2 - t^3) - f(t))$$

or $\qquad (1 + k) f(t) = 2kt^2 - kt^3 + t^4 - 4t^2.$

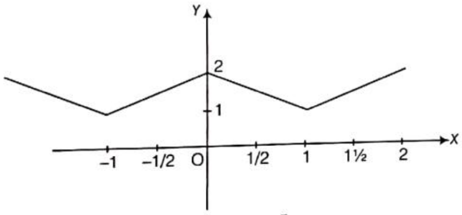
$$\Rightarrow \qquad f(t) = \frac{1}{k+1} \{t^4 - kt^3 + (2k - 4) t^2\}$$

Hence, required f is given by;

$$f(x) = \frac{1}{k+1} [x^4 - kx^3 + (2k - 4) x^2]$$

55. $f(t) = |t - 1| = |t| + |t + 1|$

$$\Rightarrow \qquad f(t) = \begin{cases} -t, & 1 \le -1 \\ 2 + t, & -1 < t \le 0 \\ 2 - t, & 0 < t \le 1 \\ t, & t > 1 \end{cases}$$



***Case* I** $\quad x + 2 \le -\frac{1}{2} \Rightarrow x \le -\frac{5}{2}$

$$g(x) = \max\{f(t) : x + 1 \le t \le x + 2\}$$
$$= f(x + 1) = -x - 1, \; x \le -5/2$$

***Case* II** $\quad -\frac{1}{2} < x + 2 \le 0 \Rightarrow -\frac{5}{2} < x \le -2$

$$g(x) = f(x + 2) = 4 + x, \; -\frac{5}{2} < x \le -2$$

***Case* III** $\quad 0 < x + 2 \le 1 \Rightarrow -2 < x \le -1$

$$g(x) = 2$$

***Case* IV** $\quad 1 < x + 2 \le 3/2 \Rightarrow -1 < x \le -1/2$

$$g(x) = f(x + 1) = 1 - x$$

***Case* V** $\quad x + 2 > 3/2 \Rightarrow x > -1/2$

$$g(x) = f(x + 2) = 2 + x$$

Hence, $\qquad g(x) = \begin{cases} -x - 1, & x \le -5/2 \\ 4 + x, & -5/2 < x \le -2 \\ 2, & -2 < x \le -1 \\ 1 - x, & -1 < x \le -1/2 \\ 2 + x, & x > -1/2 \end{cases}$

Now, required area $= \int_{-3/2}^{5} g(x) \, dx$

$$= \int_{-3/2}^{-1} (2) \, dx + \int_{-1}^{-1/2} (1 - x) \, dx + \int_{-1/2}^{5} (2 + x) \, dx$$

$$= 2\left(-1 + \frac{3}{2}\right) + \left(-\frac{1}{2} - \frac{1}{8}\right) - \left(-1 - \frac{1}{2}\right) + 2\left(5 + \frac{1}{2}\right) + \frac{1}{2}\left(25 - \frac{1}{4}\right)$$

$$= \frac{101}{4} \text{ sq units}$$

56. It is easy to see that,

$$f(x) = \begin{cases} e^x, & 0 \le x < \log(3/2) \\ 3/2, & \log(3/2) \le x < \log(2) \\ 1 + e^x, & \log(2) \le x \le 1 \end{cases}$$

Let A be the required area. Then,

$$A = \int_0^{(\log 3/2)} e^x \, dx + \int_{\log 3/2}^{\log 2} \frac{3}{2} \, dx + \int_{\log 2}^{1} (1 + e^{-x}) \, dx$$

$$= (e^x)_0^{\log 3/2} + \frac{3}{2} (x)_{\log 3/2}^{\log 2} + (x - e^{-x})_{\log 2}^{1}$$

$$= \left(\frac{3}{2} - 1\right) + \frac{3}{2}\left(\log 2 - \log \frac{3}{2}\right) + \left(1 - \frac{1}{e} - \log 2 + \frac{1}{2}\right)$$

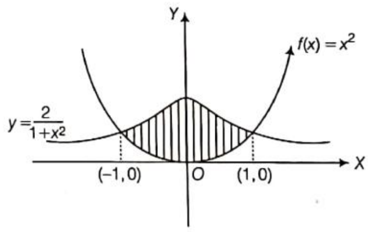
$$= \left[2 + \log\left(\frac{4}{3\sqrt{3}}\right) - \frac{1}{e}\right] \text{sq units}$$

57. $f'(x) = \lim_{h \to 0} \dfrac{f(x + h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{f(x(1 + h/x)) - f(x)}{h}$$

$$= \frac{f(x)}{x} \lim_{h \to 0} \frac{f(1 + h/x) - f(1)}{h/x} = \frac{f(x)}{x} \cdot f'(1)$$

$\therefore f'(x) = \dfrac{2f(x)}{x}$ or $\dfrac{f'(x)}{f(x)} = \dfrac{2}{x}$



Integrating both the sides, we get $f(x) = Cx^2$, since $f(1) = 1 \Rightarrow C = 1$.

So, $\qquad f(x) = x^2$

Now, $\qquad \dfrac{2}{1 + x^2} = x^2 \Rightarrow x^4 + x^2 - 2 = 0$

$\Rightarrow \qquad x^2 = 1 \Rightarrow x = \pm 1$

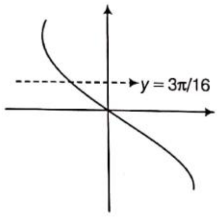
Required area $= 2\left[\int_0^1 \left(\dfrac{2}{1 + x^2} - x^2\right) dx\right]$

$$= 2\left[2 \tan^{-1} x - \frac{x^3}{3}\right]_0^1 = 2\left[\frac{\pi}{2} - \frac{1}{3}\right]$$

$$= \left(\pi - \frac{2}{3}\right) \text{sq units}$$

58. $y(x+y^3)\,dx = x(y^3-x)\,dy$

$\Rightarrow xy\,dx + y^2\,dx = xy^3\,dy - x^2\,dy$



$$\Rightarrow \quad xd(xy) = x^2y^3\left(\frac{1}{x}dy - \frac{y}{x^2}dx\right)$$

$$\Rightarrow \quad \frac{d(xy)}{(xy)^2} = \frac{y}{x}\cdot d\left(\frac{y}{x}\right) \Rightarrow -\frac{1}{xy} = \frac{1}{2}\left(\frac{y}{x}\right)^2 + C$$

At, $x = 4,\ y = -2$

So, $\quad \frac{1}{8} = \frac{1}{2}\left(-\frac{1}{2}\right)^2 + C \Rightarrow C = 0$

Hence, $\quad y^3 + 2x = 0$

So, $\quad f(x) = (-2x)^{1/3}$

The second equation given is

$$y = \int_{1/8}^{\sin^2 x} \sin^{-1}\sqrt{t}\,dt + \int_{1/8}^{\cos^2 x} \cos^{-1}\sqrt{t}\,dt$$

$\Rightarrow y' = x\cdot 2\sin x\cos x + x\cdot 2\cos x(-\sin x) = 0$

So, y is constant.

Put $\sin x = \cos x = \frac{1}{\sqrt{2}}$

Hence, $y = \int_{1/8}^{1/2}(\sin^{-1}\sqrt{t} + \cos^{-1}\sqrt{t})\,dt$

$$= \int_{1/8}^{1/2}\left(\frac{\pi}{2}\right)dt = \frac{\pi}{2}\cdot\frac{3}{8} = \frac{3\pi}{16}, \text{ and } g(x) = \frac{3\pi}{16}$$

So, we must find the area between $y = f(x),\ y = \frac{3\pi}{16}$

At $y = \frac{3\pi}{16};\ x = -\frac{1}{2}\left(\frac{3\pi}{16}\right)^3 = P$ (say)

Hence, $\quad \text{area} = \int_P^0 \left(\frac{3\pi}{16} + (2x)^{1/3}\right)dx$

$$= \left(\frac{3\pi}{16}x + 2^{1/3}\cdot\frac{x^{4/3}}{4/3}\right)_P^0 = \frac{1}{8}\left(\frac{3\pi}{16}\right)^4 \text{ sq units}$$

59. As in the figure $\frac{A(R)+A(S)}{A(T)} = \frac{ay+bz}{h\,x/2}$

where $h = a+b+c$, the altitude of T.

By similar triangles $\frac{x}{h} = \frac{y}{b+c} = \frac{z}{c}$,

$$\text{So, } \frac{A(R)+A(S)}{A(T)} = \frac{\frac{a(b+c)x}{h} + b\cdot\frac{cx}{h}}{hx/2}$$

$$= \frac{2}{h^2}(ab+ac+bc)$$

We need to maximize $(ab+bc+ca)$ subject to $a+b+c=h$. One way to do this is first to fix a, so $b+c = h-a$.

Then, $\quad (ab+bc+ac) = a(h-a) + bc$

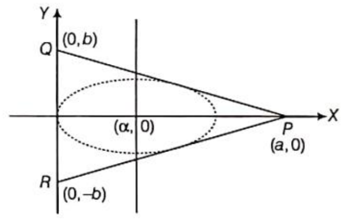
and bc is maximized when $b = c$. We now wish to maximize $2ab + b^2$ subject to $a + 2b = h$. This is a straight forward calculus problem giving $a = b = c = 1/3$. Hence, the maximum ratio is 2/3 (independent of T).

60. Consider a coordinate system with vertex P of the isosceles ΔPQR at $(a, 0)$ and Q and R at $(0, b)$ and $(0, -b)$ respectively.

$$A = \frac{1}{2}a\cdot 2b = ab \quad \text{...(i)}$$

Let the centre of ellipse be $(\alpha, 0)$ and the axes be of lengths, 2α and 2β.

So, the equation of ellipse is $\frac{(x-\alpha)^2}{\alpha^2} + \frac{(y)^2}{\beta^2} = 1$



Now, the line PQ is tangent to the ellipse. To apply condition of tangency, let us take a new system $x'y'$ whose origin is at $(\alpha, 0)$.

Then, $x = x' + \alpha$ and $y = y'$.

So, the ellipse becomes $\frac{x'^2}{\alpha^2} + \frac{y'^2}{\beta^2} = 1$

and the line PQ becomes $\frac{x'+\alpha}{a} + \frac{y'}{b} = 1$

which can be written as $y' = -\frac{b}{a}x' + b\left(1 - \frac{\alpha}{a}\right)$

So, $\quad b^2\left(1-\frac{\alpha}{a}\right)^2 = \alpha^2\left(-\frac{b}{a}\right)^2 + \beta^2$

$$\Rightarrow \quad \beta^2 = b^2\left(1 - \frac{2\alpha}{a}\right) \quad \text{...(ii)}$$

Now, area of ellipse $= \pi\alpha\beta \Rightarrow A^2 = \pi^2\alpha^2\beta^2$

Using Eq. (ii), $A^2 = \pi^2\alpha^2b^2\left(1 - \frac{2\alpha}{a}\right) = f(x)$ (say)

$$f'(x) = \pi^2 b^2\left(2\alpha - \frac{6\alpha^2}{a}\right)$$

$$f'(\alpha) = 0 \Rightarrow 2\alpha = \frac{6\alpha^2}{a} \Rightarrow \alpha = \frac{a}{3}\ (\alpha \neq 0)$$

Since, $f''(\alpha) = \pi^2 b^2 \left(2 - \frac{12\alpha}{a} \right) = -2\pi^2 b^2 = (-\text{ ve})$

(at $\alpha = a/3$)

So, $f\left(\frac{a}{3}\right) = \pi^2 \cdot \frac{a^2}{9} \cdot b^2 \left(1 - \frac{2}{3}\right) = \frac{\pi^2 A^2}{27}$ [using Eq. (i)]

Hence, $(A)_{\max} = \frac{\pi A}{3\sqrt{3}} = \frac{\sqrt{3}\pi A}{9}$ sq units

61. $A_1 = \int_0^a \{\sin x - f(x)\}\, dx = -[\cos x]_0^a - \int_0^a f(x)\, dx$

$= -(\cos a - 1) - \int_0^a f(x)\, dx = 1 - \sin a + (a-1)\cos a$ (given)

$\Rightarrow \; -\cos a - \int_0^a f(x)\, dx = -\sin a + (a-1)\cos a$

$\Rightarrow \; -\int_0^a f(x)\, dx = -\sin a + a\cos a$

$\Rightarrow \; \int_0^a f(x)\, dx = \sin a - a\cos a$

Differentiating w.r.t. a.

$f(a) = \cos a - (\cos a - a\sin a) = a\sin a$

$\therefore \; f(x) = x\sin x$

Now, $y = \sin x$ and $y = f(x)$ intersects at

$\Rightarrow \; a\sin a = \sin a \Rightarrow (a-1)\sin a = 0$

$\Rightarrow \; a = 1$ [as $\sin a = 0$]

Hence, $A_1 = 1 - \sin 1$

$A_2 = \int_1^{\pi} (x\sin x - \sin x)\, dx = \pi - 1 - \sin 1$

$A_3 = \int_{\pi}^{2\pi} |\, x\sin x - \sin x \,|\, dx = 3\pi - 2$

62. For $x = 1$, $y = b\cdot 5^x + 4 = 5b + 4$

and $\frac{dy}{dx} = b\cdot 5^x \log 5 \Rightarrow 5b\log 5 = 40\log 5 \Rightarrow b = 8$

The two curves intersects at points where

$$8\cdot 5^x + 4 = 25^x + 16$$

$\Rightarrow \; 5^{2x} - 8\cdot 5 + 12 = 0 \Rightarrow x = \log_5 \quad x = \log_5 6$

Hence, the area of the given region;

$= \int_{\log_5 2}^{\log_5 6} \{8\cdot 5^x + 4 - (25^x + 16)\}\, dx$

$= \int_{\log_5 2}^{\log_5 6} (8\cdot 5^x - 25^x - 12)\, dx$

$= \left[\frac{8\cdot 5^x}{\log_e 5} - 12x - \frac{25^x}{\log_e 25} \right]_{\log_5 2}^{\log_5 6}$

$= \frac{-5^{\log_5 36}}{\log_e 25} + \frac{8\cdot 5^{\log_5 6}}{\log_e 5} - 12(\log_5 6 - \log_5 2)$

$+ \frac{5^{\log_5 4}}{\log_e 25} - \frac{8\cdot 5^{\log_5 2}}{\log_e 5}$

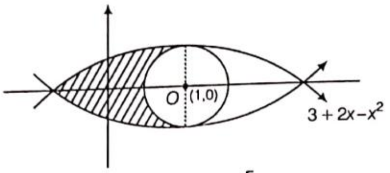
$= \frac{-36}{2\log_e 5} + \frac{48}{\log_e 5} - 12[\log_5 3] + \frac{4}{2(\log_e 5)} - \frac{16}{(\log_e 5)}$

$= \frac{16}{\log_e 5} - 12\log_5 3 = 4\log_5 e^4 - 4\log_5 25$

$= 4\log_5 \left[\frac{e^4}{27} \right]$ sq units

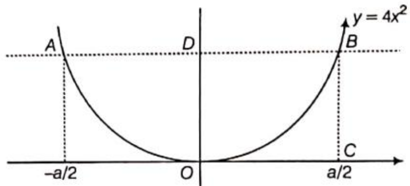
63. By the symmetry of the figure circle(s) of maximum area will have the end point of diameter at the vertex of the two parabola.

$\Rightarrow$ Radius of circle $= \frac{1}{2} \times AB = \frac{1}{2} \times 8 = 4$ units



So, the area of shaded region $= 4 \times \left[\int_1^3 (3 + 2x - x^2)\, dx - \frac{1}{4} \right.$ area of circle $= 4\left(\frac{16}{3} - 4\pi\right)$ sq units

64. Area of $\Delta AOB = 2 \times \left(\frac{1}{2} \times \frac{a}{2} \times a^2 \right) = \frac{a^3}{2}$

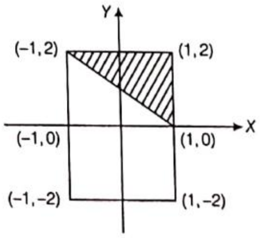


Area = Area $OCBD - \int_0^{a/2} 4x^2\, dx = \frac{a^3}{2} - \frac{a^3}{6} = \frac{a^3}{3}$

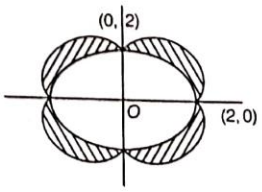
$= \lim_{x\to 0} \frac{\text{Area of triangle}}{\text{Area between the line and parabola}}$

$= \lim_{x\to 0} \frac{a^3/2}{a^3/3} = \frac{3}{2}$

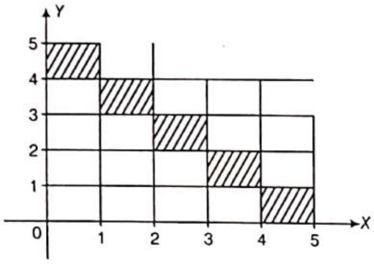
65. Required area $= \frac{1}{2} \times 2 \times 2 = 2$ sq units



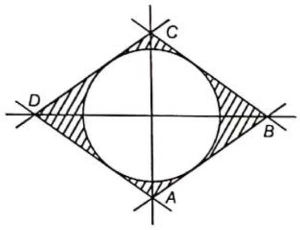
66. Required area $= 4 \times \left[\frac{\pi(\sqrt{2})^2}{2} - (\pi - 2) \right] = 8$ sq units



67. 5 sq units



68. Area of the square $ABCD = 2$ sq units



Area of the circle $= \pi \times \frac{1}{2} = \frac{\pi}{2}$ sq units.

Required area $= \left(2 - \frac{\pi}{2}\right)$ sq units

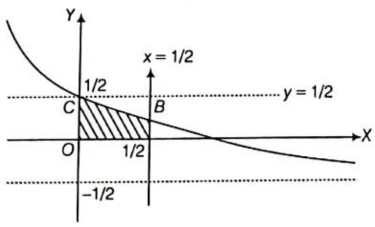
69. $$2^{|x|} \cdot |y| + 2^{|x|-1} \le 1 \quad \text{...(i)}$$

Clearly, this region is symmetrical about X and Y-axes.

Let $x < 0$, Eq. (i) gives,

$$2^x - y + 2^{x-1} \le 1$$

$$\Rightarrow \quad y \le \frac{1 - 2^{x-1}}{2^x} = 2^{-x} - \frac{1}{2}$$



Clearly, bounded region in the first quadrant is $OABC$. The required area is 4 times the area of the region $OABC$.

Required area $= 4\int_0^{1/2}\left(2^{-x} - \frac{1}{2}\right)dx = 4\left(-\frac{2^{-x}}{\ln 2} - \frac{x}{2}\right)_0^{1/2}$

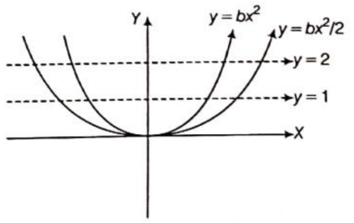
$= \left[\frac{4}{\ln 2}(1 - 2^{-1/2}) - 1\right]$ sq units

70. Pair of lines $y^2 - 3y + 2 = 0$

$\Rightarrow$ Lines are $y = 2, y = 1$.

Let $\quad [a] = b$

Now, curves are $y = bx^2$ and $y = \frac{b}{2}x^2$



Bounded area $= 2\left[\int_1^2 (x_2 - x_1)\,dy\right]$

$= 2\left[\int_1^2\left(\sqrt{\frac{2y}{b}} - \sqrt{\frac{y}{b}}\right)dy\right]$

$= 2\left[\sqrt{\frac{2}{b}} \cdot \frac{y^{3/2}}{3/2} - \frac{1}{\sqrt{b}} \cdot \frac{y^{3/2}}{3/2}\right]_1^2$

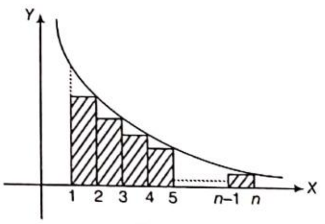
$= \frac{4}{3\sqrt{b}}((\sqrt{2} - 1)\,y^{3/2})_1^2 = \frac{4}{3\sqrt{b}}(\sqrt{2} - 1)(2\sqrt{2} - 1)$

Area will be maximum when $b = [a]$ is least.

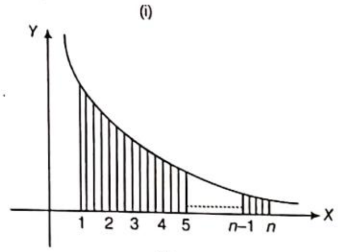
As $a \ge 2 \Rightarrow [a]_{\text{least}} = 1 \Rightarrow 1 \le a < 2$

71. Since, $f'(x) < 0 \Rightarrow f(x)$ is a decreasing function and also $f''(x) > 0 \Rightarrow f(x)$ is concave upwards.

Hence, the graph of the function $y = f(x)$ is as follows :



(i)



(ii)

Let S_L denotes the shaded area in figure (i).

$$\Rightarrow \quad S_L = f(2) + f(3) + \ldots + f(n) = \sum_{r=1}^{n} f(r) - f(1)$$

From the figure (i) it is clear that, $S_L < \int_1^n f(x)\,dx$

$$\Rightarrow \quad \sum_{r=1}^{n} f(r) - f(1) < \int_1^n f(x)\,dx$$

$$\Rightarrow \quad \sum_{r=1}^{n} f(r) < \int_1^n f(x)\,dx + f(1) \quad \text{...(i)}$$

Let S_u denotes the area of the shaded region in figure (ii).

$$\Rightarrow \quad S_u = \left(\frac{1}{2} f(1) + f(2)\right) + \frac{1}{2}(f(2) + f(3)) + \dots + \frac{1}{2}(f(n-1) + f(n))$$

$$= [(f(1) + f(2) + f(3)) + \dots + f(n-1) + f(n)] - \frac{1}{2}(f(1) + f(n))$$

$$= \sum_{r=1}^{n} f(r) - \frac{1}{2} f(1) - \frac{1}{2} f(n)$$

From figure (ii) it is clear that;

$$S_u > \int_1^n f(x)\,dx$$

$$\Rightarrow \sum_{r=1}^{n} f(r) - \frac{1}{2}(f(1) + f(n)) > \int_1^n f(x)\,dx$$

$$\Rightarrow \sum_{r=1}^{n} f(r) > \int_1^n f(x)\,dx + \frac{1}{2}(f(1) + f(n))$$

$$> \int_1^n f(x)\,dx + 4 \cdot \frac{1}{2} f(1) \quad \text{...(ii)}$$

From Eqs. (i) and (ii), we get

$$\frac{1}{2} f(1) + \int_1^n f(x)\,dx < \sum_{r=1}^{n} f(r) < \int_1^n f(x)\,dx + f(1).$$

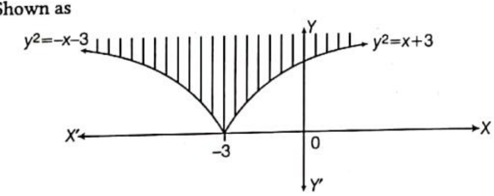
72. Here, $\{(x, y) \in R^2 : y \geq \sqrt{|x+3|}, 5y \leq (x+9) \leq 15\}$

$$\therefore \quad y \geq \sqrt{|x+3|}$$

$$\Rightarrow \quad y \geq \begin{cases} \sqrt{x+3}, \text{ when } x \geq -3 \\ \sqrt{-x-3}, \text{ when } x \leq -3 \end{cases}$$

or
$$y^2 \geq \begin{cases} x+3, \text{ when } x \geq -3 \\ -3-x, \text{ when } x \leq -3 \end{cases}$$

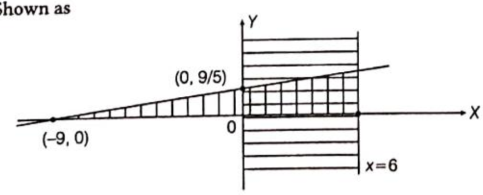
Shown as



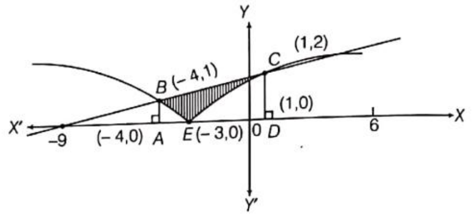
Also, $\quad 5y \leq (x+9) \leq 15$

$\Rightarrow \quad (x+9) \geq 5y$ and $x \leq 6$

Shown as



$\therefore \quad \{(x, y) \in R^2 : y \geq \sqrt{|x+3|}, 5y \leq (x+9) \leq 15\}$



$\therefore$ Required area = Area of trapezium $ABCD$ − Area of ABE under parabola − Area of CDE under parabola

$$= \frac{1}{2}(1+2)(5) - \int_{-4}^{-3} \sqrt{-(x+3)}\,dx - \int_{-3}^{1} \sqrt{(x+3)}\,dx$$

$$= \frac{15}{2} - \left[\frac{(-3-x)^{3/2}}{-\frac{3}{2}}\right]_{-4}^{-3} - \left[\frac{(x+3)^{3/2}}{\frac{3}{2}}\right]_{-3}^{1}$$

$$= \frac{15}{2} + \frac{2}{3}[0-1] - \frac{2}{3}[8-0] = \frac{15}{2} - \frac{2}{3} - \frac{16}{3} = \frac{15}{2} - \frac{18}{3} = \frac{3}{2}$$

73. Since, $F'(a) + 2$ is the area bounded by $x = 0, y = 0, y = f(x)$ and $x = a$.

$$\therefore \quad \int_0^a f(x)\,dx = F'(a) + 2$$

Using Newton-Leibnitz formula,

$$f(a) = F''(a) \text{ and } f(0) = F''(0) \quad \text{...(i)}$$

Given, $\quad F(x) = \int_x^{x^2 + \pi/6} 2\cos^2 t\,dt$

On differentiating,

$$F'(x) = 2\cos^2\left(x^2 + \frac{\pi}{6}\right) \cdot 2x - 2\cos^2 x \cdot 1$$

Again differentiating,

$$F''(x) = 4\left\{\cos^2\left(x^2+\frac{\pi}{6}\right) - 2x\cos\left(x^2+\frac{\pi}{6}\right)\sin\left(x^2+\frac{\pi}{6}\right)2x\right\} + \{4\cos x \cdot \sin x\}$$

$$= 4\left\{\cos^2\left(x^2+\frac{\pi}{6}\right) - 4x^2\cos\left(x^2+\frac{\pi}{6}\right)\sin\left(x^2+\frac{\pi}{6}\right)\right\} + 2\sin 2x$$

$$\therefore \quad F''(0) = 4\left\{\cos^2\left(\frac{\pi}{6}\right)\right\} = 3$$

$$\therefore \quad f(0) = 3$$

74. Let equation of tangent to parabola be $y = mx + \dfrac{2}{m}$

It also touches the circle $x^2 + y^2 = 2$.

$$\therefore \quad \left|\frac{2}{m\sqrt{1+m^2}}\right| = \sqrt{2}$$

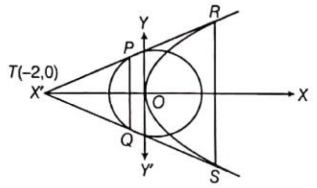
$$\Rightarrow \quad m^4 + m^2 = 2 \Rightarrow m^4 + m^2 - 2 = 0$$

$$\Rightarrow \quad (m^2 - 1)(m^2 + 2) = 0$$

$$\Rightarrow \quad m = \pm 1, m^2 = -2 \quad [\text{rejected } m^2 = -2]$$

So, tangents are $y = x + 2$, $y = -x - 2$.

They intersect at $(-2, 0)$.



Equation of chord PQ is $-2x = 2 \Rightarrow x = -1$

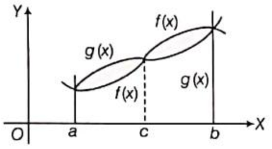
Equation of chord RS is $0 = 4(x-2) \Rightarrow x = 2$

$\therefore$ Coordinates of P, Q, R, S are

$$P(-1, 1), Q(-1, -1), R(2, 4), S(2, -4)$$

$\therefore$ Area of quadrilateral $= \dfrac{(2+8)\times 3}{2} = 15$ sq units

75. To find the bounded area between $y = f(x)$ and $y = g(x)$ between $x = a$ to $x = b$.



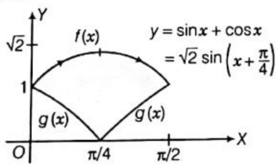
$\therefore$ Area bounded $= \int_a^c [g(x) - f(x)]dx + \int_c^b [f(x) - g(x)]dx$

$$= \int_a^b |f(x) - g(x)| dx$$

Here, $f(x) = y = \sin x + \cos x$, when $0 \le x \le \dfrac{\pi}{2}$

and $g(x) = y = |\cos x - \sin x| = \begin{cases} \cos x - \sin x, & 0 \le x \le \dfrac{\pi}{4} \\ \sin x - \cos x, & \dfrac{\pi}{4} \le x \le \dfrac{\pi}{2} \end{cases}$

could be shown as



$\therefore$ Area bounded $= \int_0^{\pi/4} \{(\sin x + \cos x) - (\cos x - \sin x)\} dx$

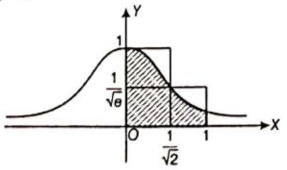
$$+ \int_{\pi/4}^{\pi/2} \{(\sin x + \cos x) - (\sin x - \cos x)\} dx$$

$$= \int_0^{\pi/4} 2\sin x\, dx + \int_{\pi/4}^{\pi/2} 2\cos x\, dx$$

$$= -2[\cos x]_0^{\pi/4} + 2[\sin x]_{\pi/4}^{\pi/2}$$

$$= 4 - 2\sqrt{2} = 2\sqrt{2}(\sqrt{2} - 1) \text{ sq units}$$

76. Graph for $y = e^{-x^2}$



Since, $x^2 \le x$ when $x \in [0, 1]$

$\Rightarrow \quad -x^2 \ge -x$ or $e^{-x^2} \ge e^{-x}$

$\therefore \quad \int_0^1 e^{-x^2} dx \ge \int_0^1 e^{-x} dx$

$\Rightarrow \quad S \ge -(e^{-x})_0^1 = 1 - \dfrac{1}{e}$ ...(i)

Also, $\int_0^1 e^{-x^2} dx \le$ Area of two rectangles

$$\le \left(1 \times \frac{1}{\sqrt{2}}\right) + \left(1 - \frac{1}{\sqrt{2}}\right) \times \frac{1}{\sqrt{e}}$$

$$\le \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{e}}\left(1 - \frac{1}{\sqrt{2}}\right) \quad \text{...(ii)}$$

$\therefore \quad \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{e}}\left(1 - \dfrac{1}{\sqrt{2}}\right) \ge S \ge 1 - \dfrac{1}{e}$ [from Eqs. (i) and (ii)]

77. $R_1 = \int_{-1}^{2} x\, f(x)\, dx$ ...(i)

Using $\int_a^b f(x)\, dx = \int_a^b f(a + b - x)\, dx$

$$R_1 = \int_{-1}^{2} (1 - x)\, f(1 - x)\, dx$$

$\therefore \quad R_1 = \int_{-1}^{2} (1 - x)\, f(x)\, dx$ ...(ii)

[$f(x) = f(1 - x)$, given]

Given, R_2 is area bounded by $f(x)$, $x = -1$ and $x = 2$.

$\therefore \quad R_2 = \int_{-1}^{2} f(x)\, dx$ ...(iii)

On adding Eqs. (i) and (ii), we get

$$2R_1 = \int_{-1}^{2} f(x)\, dx \quad \text{...(iv)}$$

From Eqs. (iii) and (iv), we get

$$2R_1 = R_2$$

78. Here, area between 0 to b is R_1 and b to 1 is R_2.

$\therefore \quad \int_0^b (1-x)^2\, dx - \int_b^1 (1-x)^2\, dx = \dfrac{1}{4}$

$\Rightarrow \quad \left[\dfrac{(1-x)^3}{-3}\right]_0^b - \left[\dfrac{(1-x)^3}{-3}\right]_b^1 = \dfrac{1}{4}$

$\Rightarrow -\dfrac{1}{3}[(1-b)^3 - 1] + \dfrac{1}{3}[0 - (1-b)^3] = \dfrac{1}{4}$

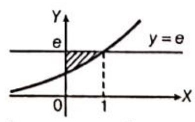
$\Rightarrow \quad -\dfrac{2}{3}(1-b)^3 = -\dfrac{1}{3} + \dfrac{1}{4} = -\dfrac{1}{12}$

$\Rightarrow \quad (1-b)^3 = \dfrac{1}{8}$

$\Rightarrow \quad (1-b) = \dfrac{1}{2} \Rightarrow b = \dfrac{1}{2}$

79. Shaded area $= e - \left(\int_0^1 e^x\, dx\right) = 1$

Also, $\int_1^e \ln(e + 1 - y)\, dy$ [put $e + 1 - y = t \Rightarrow -dy = dt$]



$$= \int_e^1 \ln t(-dt) = \int_1^e \ln t\, dt = \int_1^e \ln y\, dy = 1$$

80. Required area $= \int_0^{\pi/4}\left(\sqrt{\frac{1+\sin x}{\cos x}} - \sqrt{\frac{1-\sin x}{\cos x}}\right)dx$

$$\left[\because \frac{1+\sin x}{\cos x} > \frac{1-\sin x}{\cos x} > 0\right]$$

$$= \int_0^{\pi/4}\left(\sqrt{\frac{1+\dfrac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}}{\dfrac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}}} - \sqrt{\frac{1-\dfrac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}}{\dfrac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}}}\right)dx$$

$$= \int_0^{\pi/4}\left(\sqrt{\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}} - \sqrt{\frac{1-\tan\frac{x}{2}}{1+\tan\frac{x}{2}}}\right)dx$$

$$= \int_0^{\pi/4}\frac{1+\tan\frac{x}{2}-1+\tan\frac{x}{2}}{\sqrt{1-\tan^2\frac{x}{2}}}\,dx = \int_0^{\pi/4}\frac{2\tan\frac{x}{2}}{\sqrt{1-\tan^2\frac{x}{2}}}\,dx$$

Put $\tan\frac{x}{2} = t \Rightarrow \frac{1}{2}\sec^2\frac{x}{2}\,dx = dt$

$$= \int_0^{\tan\frac{\pi}{8}}\frac{4t\,dt}{(1+t^2)\sqrt{1-t^2}}$$

As $\int_0^{\sqrt{2}-1}\frac{4t\,dt}{(1+t^2)\sqrt{1-t^2}}$ $\left[\because \tan\frac{\pi}{8} = \sqrt{2}-1\right]$

81. Given, $y^3 - 3y + x = 0$

$\Rightarrow$ $3y^2\frac{dy}{dx} - 3\frac{dy}{dx} + 1 = 0$ ...(i)

$\Rightarrow 3y^2\left(\frac{d^2y}{dx^2}\right) + 6y\left(\frac{dy}{dx}\right)^2 - 3\frac{d^2y}{dx^2} = 0$ ...(ii)

At $x = -10\sqrt{2}$, $y = 2\sqrt{2}$

On substituting in Eq. (i) we get

$$3(2\sqrt{2})^2\cdot\frac{dy}{dx} - 3\cdot\frac{dy}{dx} + 1 = 0 \Rightarrow \frac{dy}{dx} = -\frac{1}{21}$$

Again, substituting in Eq. (ii), we get

$$3(2\sqrt{2})^2\frac{d^2y}{dx^2} + 6(2\sqrt{2})\cdot\left(-\frac{1}{21}\right)^2 - 3\cdot\frac{d^2y}{dx^2} = 0$$

$\Rightarrow$ $21\cdot\frac{d^2y}{dx^2} = -\frac{12\sqrt{2}}{(21)^2}$

$\Rightarrow$ $\frac{d^2y}{dx^2} = \frac{-12\sqrt{2}}{(21)^3} = \frac{-4\sqrt{2}}{7^3\cdot 3^2}$

82. Required area $= \int_a^b y\,dx = \int_a^b f(x)\,dx = [f(x)\cdot x]_a^b - \int_a^b f'(x)x\,dx$

$$= bf(b) - af(a) - \int_a^b f'(x)x\,dx$$

$$= bf(b) - af(a) + \int_a^b\frac{x\,dx}{3[\{f(x)\}^2-1]}$$

$$\left[\because f'(x) = \frac{dy}{dx} = \frac{-1}{3(y^2-1)} = \frac{-1}{3[\{f(x)\}^2-1]}\right]$$

83. Let $I = \int_{-1}^{1} g'(x)\,dx = [g(x)]_{-1}^{1} = g(1) - g(-1)$

Since, $y^3 - 3y + x = 0$ ...(i)

and $y = g(x)$

$\therefore$ $\{g(x)\}^3 - 3g(x) + x = 0$ [from Eq. (i)]

At $x = 1$, $\{g(1)\}^3 - 3g(1) + 1 = 0$ ...(ii)

At $x = -1$, $\{g(-1)\}^3 - 3g(-1) - 1 = 0$ ...(iii)

On adding Eqs. (i) and (ii), we get

$\{g(1)\}^3 + \{g(-1)\}^3 - 3\{g(1) + g(-1)\} = 0$

$\Rightarrow$ $[g(1) + g(-1)][\{g(1)\}^2 + \{g(-1)\}^2 - g(1)g(-1) - 3] = 0$

$\Rightarrow$ $g(1) + g(-1) = 0$

$\Rightarrow$ $g(1) = -g(-1)$

$\therefore$ $I = g(1) - g(-1) = g(1) - \{-g(1)\} = 2g(1)$

84. Given equations of curves are $y^2 = 2x$, ...(i)

which is a parabola with vertex (0, 0) and axis parallel to X-axis.

And $x^2 + y^2 = 4x$ ...(ii)

which is a circle with centre (2, 0) and radius = 2

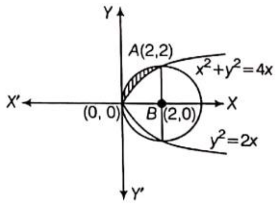
On substituting $y^2 = 2x$ in Eq. (ii), we get

$$x^2 + 2x = 4x \Rightarrow x^2 = 2x$$

$\Rightarrow$ $x = 0$ or $x = 2$

$\Rightarrow$ $y = 0$ or $y = \pm 2$ [using Eq. (i)]

Now, the required area is the area of shaded region, i.e.



Required area $= \frac{\text{Area of a circle}}{4} - \int_0^2\sqrt{2x}\,dx$

$$= \frac{\pi(2)^2}{4} - \sqrt{2}\int_0^2 x^{1/2}dx = \pi - \sqrt{2}\left[\frac{x^{3/2}}{3/2}\right]_0^2$$

$$= \pi - \frac{2\sqrt{2}}{3}[2\sqrt{2} - 0] = \left(\pi - \frac{8}{3}\right) \text{ sq units}$$

85. Given region is $\{(x, y) : y^2 \le 2x \text{ and } y \ge 4x - 1\}$

$y^2 \le 2x$ represents a region inside the parabola

$$y^2 = 2x \quad ...(i)$$

and $y \ge 4x - 1$ represents a region to the left of the line

$$y = 4x - 1 \quad ...(ii)$$

The point of intersection of the curve (i) and (ii) is

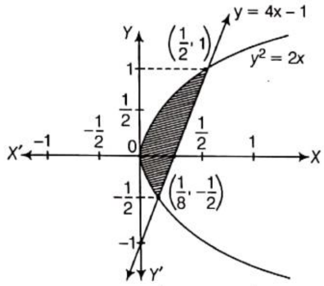
$$(4x-1)^2 = 2x$$

$\Rightarrow$ $16x^2 + 1 - 8x = 2x$

$\Rightarrow$ $16x^2 - 10x + 1 = 0$

$\Rightarrow$ $x = \frac{1}{2}, \frac{1}{8}$

$\therefore$ The points where these curves intersect, are $\left(\frac{1}{2}, 1\right)$ and $\left(\frac{1}{8}, -\frac{1}{2}\right)$.



Hence, required area $= \int_{-1/2}^{1}\left(\frac{y+1}{4} - \frac{y^2}{2}\right)dy$

$$= \frac{1}{4}\left(\frac{y^2}{2} + y\right)_{-1/2}^{1} - \frac{1}{6}(y^3)_{-1/2}^{1}$$

$$= \frac{1}{4}\left\{\left(\frac{1}{2} + 1\right) - \left(\frac{1}{8} - \frac{1}{2}\right)\right\} - \frac{1}{6}\left\{1 + \frac{1}{8}\right\}$$

$$= \frac{1}{4}\left\{\frac{3}{2} + \frac{3}{8}\right\} - \frac{1}{6}\left\{\frac{9}{8}\right\} = \frac{1}{4} \times \frac{15}{8} - \frac{3}{16} = \frac{9}{32}$$

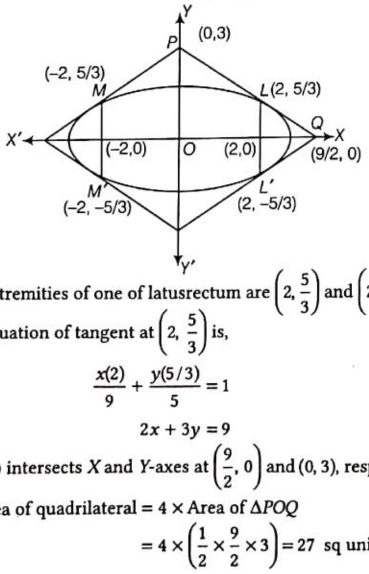
86. Given equation of ellipse is

$$\frac{x^2}{9} + \frac{y^2}{5} = 1 \qquad \text{...(i)}$$

$\therefore \qquad a^2 = 9, b^2 = 5 \Rightarrow a = 3, b = \sqrt{5}$

Now, $\qquad e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{5}{9}} = \frac{2}{3}$

Foci $= (\pm ae, 0) = (\pm 2, 0)$ and $\frac{b^2}{a} = \frac{5}{3}$



$\therefore$ Extremities of one of latusrectum are $\left(2, \frac{5}{3}\right)$ and $\left(2, \frac{-5}{3}\right)$.

$\therefore$ Equation of tangent at $\left(2, \frac{5}{3}\right)$ is,

$$\frac{x(2)}{9} + \frac{y(5/3)}{5} = 1$$

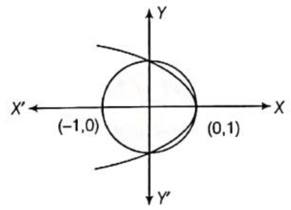
or $\qquad 2x + 3y = 9 \qquad$ ...(ii)

Eq.(ii) intersects X and Y-axes at $\left(\frac{9}{2}, 0\right)$ and $(0, 3)$, respectively.

$\therefore$ Area of quadrilateral $= 4 \times$ Area of ΔPOQ

$$= 4 \times \left(\frac{1}{2} \times \frac{9}{2} \times 3\right) = 27 \text{ sq units}$$

87. Given, $A = \{(x, y) : x^2 + y^2 \le 1 \text{ and } y^2 \le 1 - x\}$

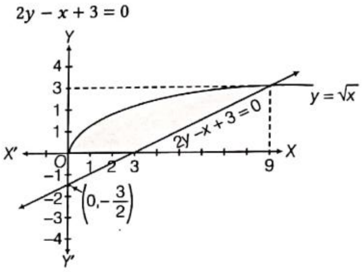


Required area $= \frac{1}{2}\pi r^2 + 2\int_0^1 (1 - y^2)dy = \frac{1}{2}\pi(1)^2 + 2\left(y - \frac{y^3}{3}\right)_0^1$

$$= \frac{\pi}{2} + \frac{4}{3}$$

88. Given curves are $y = \sqrt{x}$ $\qquad$ ...(i)

and $\qquad 2y - x + 3 = 0$ $\qquad$ ...(ii)



On solving Eqs. (i) and (ii), we get

$$2\sqrt{x} - (\sqrt{x})^2 + 3 = 0$$

$\Rightarrow \qquad (\sqrt{x})^2 - 2\sqrt{x} - 3 = 0$

$\Rightarrow \qquad (\sqrt{x} - 3)(\sqrt{x} + 1) = 0$

$\Rightarrow \qquad \sqrt{x} = 3 \quad [\because \sqrt{x} = -1 \text{ is not possible}]$

$\Rightarrow \qquad y = 3$

$\therefore$ Required area $= \int_0^3 (\text{line} - \text{curve})\, dy = \int_0^3 \{(2y + 3) - y^2\}\, dy$

$$= \left[y^2 + 3y - \frac{y^3}{3}\right]_0^3 = 9 + 9 - 9 = 9$$

89. **Given** Two parabolas $x^2 = \frac{y}{4}$ and $x^2 = 9y$

To find The area bounded between the parabolas and the straight line $y = 2$.

The required area is equal to the shaded region in the drawn figure.

The area of the shaded region (which can be very easily found by using integration) is twice the area shaded in first quadrant.

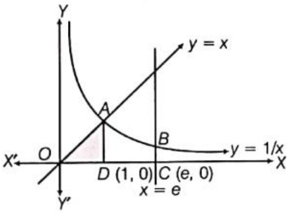
Required area $= 2\int_0^2\left(3\sqrt{y}-\frac{\sqrt{y}}{2}\right)dy = 2\int_0^2\left(\frac{5}{2}\sqrt{y}\right)dy$

$$= 5\left[\frac{y^{3/2}}{3/2}\right]_{y=0}^{y=2} = \frac{10}{3}(2^{3/2}-0) = \frac{20\sqrt{2}}{3}$$

90. Given, $y = x$, $x = e$ and $y = \frac{1}{x}$, $x \geq 0$

Since, $y = x$ and $x \geq 0 \Rightarrow y \geq 0$

$\therefore$ Area to be calculated in I quadrant shown as



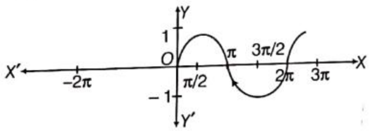
$\therefore$ Area = Area of ΔODA + Area of $DABCD$

$$= \frac{1}{2}(1\times 1) + \int_1^e \frac{1}{x}\,dx = \frac{1}{2} + (\log|x|)_1^e$$

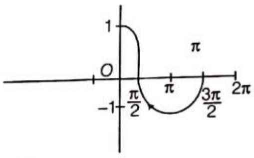
$$= \frac{1}{2} + \{\log|e| - \log 1\} \qquad [\because \log|e| = 1]$$

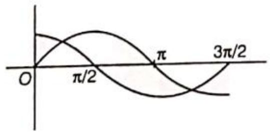
$$= \frac{1}{2} + 1 = \frac{3}{2} \text{ sq units}$$

91. Graph of $y = \sin x$ is



and graph of $y = \cos x$ is



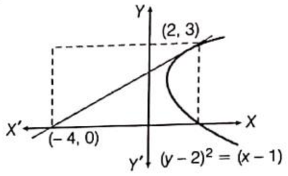


Required area $= \int_0^{\pi/4}(\cos x - \sin x)\,dx + \int_{\pi/4}^{5\pi/4}(\sin x - \cos x)dx + \int_{5\pi/4}^{3\pi/2}(\cos x - \sin x)dx$

$$= [\sin x + \cos x]_0^{\pi/4} + [-\cos x - \sin x]_{\pi/4}^{5\pi/4} + [\sin x + \cos x]_{5\pi/4}^{3\pi/2}$$

$$= (4\sqrt{2} - 2) \text{ sq units}$$

92. The equation of tangent at (2, 3) to the given parabola is $x = 2y - 4$



$\therefore$ Required area $= \int_0^3 \{(y-2)^2 + 1 - 2y + 4\}dy$

$$= \left[\frac{(y-2)^3}{3} - y^2 + 5y\right]_0^3$$

$$= \frac{1}{3} - 9 + 15 + \frac{8}{3} = 9 \text{ sq units}$$

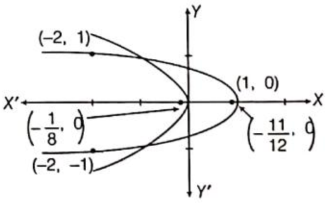
93. Given, equations of curves are $x + 3y^2 = 1$ ... (i)

and $x + 2y^2 = 0$ ... (ii)

On solving Eqs. (i) and (ii), we get

$$y = \pm 1 \text{ and } x = -2$$

$\therefore$ Required area $= \left|\int_{-1}^{1}(x_1 - x_2)dy\right|$



$$= \left|\int_{-1}^{1}(1 - 3y^2 + 2y^2)dy\right| = \left|\int_{-1}^{1}(1 - y^2)dy\right|$$

$$= \left|2\int_0^1(1-y^2)dy\right| = \left|2\left[y - \frac{y^3}{3}\right]_0^1\right| = \left|2\left(1 - \frac{1}{3}\right)\right| = \frac{4}{3} \text{ sq units}$$