Chapter 15

GENERAL EQUATION OF THE SECOND DEGREE

: TRACING OF CURVES

348. Particular cases of Conic Sections. The general definition of a Conic Section in Art. 196 was that it is the locus of a point P which moves so that its distance from a given point S is in a constant ratio to its perpendicular distance PM from a given straight line ZK.

When S does not lie on the straight line ZK, we have found that the locus is an ellipse, a parabola, or a hyperbola according as the eccentricity e is $< = $ or > 1.

The Circle is a sub-case of the Ellipse. For the equation of Art. 139 is the same as the equation (6) of Art. 247 when $b^2 = a^2$, *i.e.* when $e = 0$. In this case $CS = 0$, and $SZ = \frac{a}{e} - ae = \infty$. The Circle is therefore a Conic Section, whose eccentricity is zero, and whose directrix is at an infinite distance.

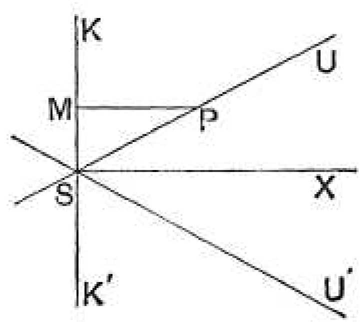
Next, let S lie on the straight line ZK, so that S and Z coincide.

In this case, since

$$SP = e \, . \, PM,$$

we have

$$\sin PSM = \frac{PM}{SP} = \frac{1}{e}.$$



If $e > 1$, then P lies on one or other of the two straight lines SU and SU' inclined to KK' at an angle

$$\sin^{-1}\left(\frac{1}{e}\right).$$

If $e = 1$, then PSM is a right angle, and the locus becomes two coincident straight lines coinciding with SX.

If $e < 1$, the $\angle\, PSM$ is imaginary, and the locus consists of two imaginary straight lines.

If, again, both KK' and S be at infinity and S be on KK', the lines SU and SU' of the previous figure will be two straight lines meeting at infinity, *i.e.* will be two parallel straight lines.

Finally, it may happen that the axes of an ellipse may both be zero, so that it reduces to a point.

Under the head of a conic section we must therefore include:

(1) An Ellipse (including a circle and a point).

(2) A Parabola.

(3) A Hyperbola.

(4) Two straight lines, real or imaginary, intersecting, coincident, or parallel.

349. *To shew that the general equation of the second degree*

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \ldots\ldots\ldots (1)$$

always represents a conic section.

Let the axes of coordinates be turned through an angle θ, so that, as in Art. 129, we substitute for x and y the quantities $x \cos\theta - y \sin\theta$ and $x \sin\theta + y \cos\theta$ respectively.

The equation (1) then becomes

$$a(x\cos\theta - y\sin\theta)^2 + 2h(x\cos\theta - y\sin\theta)(x\sin\theta + y\cos\theta)$$
$$+ b(x\sin\theta + y\cos\theta)^2 + 2g(x\cos\theta - y\sin\theta)$$
$$+ 2f(x\sin\theta + y\cos\theta) + c = 0,$$

i.e. $$x^2(a\cos^2\theta + 2h\cos\theta\sin\theta + b\sin^2\theta)$$
$$+ 2xy\{h(\cos^2\theta - \sin^2\theta) - (a - b)\cos\theta\sin\theta\}$$
$$+ y^2(a\sin^2\theta - 2h\cos\theta\sin\theta + b\cos^2\theta) + 2x(g\cos\theta + f\sin\theta)$$
$$+ 2y(f\cos\theta - g\sin\theta) + c = 0 \ldots\ldots\ldots\ldots (2).$$

Now choose the angle θ so that the coefficient of xy in this equation may vanish,

i.e. so that $\quad h(\cos^2\theta - \sin^2\theta) = (a-b)\sin\theta\cos\theta,$

i.e. $$2h\cos 2\theta = (a-b)\sin 2\theta,$$

i.e. so that $$\tan 2\theta = \frac{2h}{a-b}.$$

Whatever be the values of a, b, and h, there is always a value of θ satisfying this equation and such that it lies between $-45°$ and $+45°$. The values of $\sin\theta$ and $\cos\theta$ are therefore known.

On substituting their values in (2), let it become

$$Ax^2 + By^2 + 2Gx + 2Fy + c = 0 \ldots\ldots\ldots\ldots (3).$$

First, let neither A nor B be zero.

The equation (3) may then be written in the form

$$A\left(x + \frac{G}{A}\right)^2 + B\left(y + \frac{F}{B}\right)^2 = \frac{G^2}{A} + \frac{F^2}{B} - c.$$

Transform the origin to the point $\left(-\frac{G}{A}, \ -\frac{F}{B}\right)$.

The equation becomes

$$Ax^2 + By^2 = \frac{G^2}{A} + \frac{F^2}{B} - c = K \text{ (say)} \ldots\ldots\ldots (4),$$

i.e. $$\frac{x^2}{\frac{K}{A}} + \frac{y^2}{\frac{K}{B}} = 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5).$$

If $\frac{K}{A}$ and $\frac{K}{B}$ be both positive, the equation represents an ellipse. (Art. 247.)

If $\frac{K}{A}$ and $\frac{K}{B}$ be one positive and the other negative, it represents a hyperbola (Art. 295). If they be both negative, the locus is an imaginary ellipse.

If K be zero, then (4) represents two straight lines, which are real or imaginary according as A and B have opposite or the same signs.

Secondly, let either A or B be zero, and let it be A.

Then (3) can be written in the form

$$B\left(y+\frac{F}{B}\right)^2+2G\left[x+\frac{c}{2G}-\frac{F^2}{2BG}\right]=0.$$

Transform the origin to the point whose coordinates are

$$\left(-\frac{c}{2G}+\frac{F^2}{2BG},\quad -\frac{F}{B}\right).$$

This equation then becomes

$$By^2+2Gx=0,$$

i.e.

$$y^2=-\frac{2G}{B}x,$$

which represents a parabola. (Art. 197.)

If, in addition to A being zero, we also have G zero, the equation (3) becomes

$$By^2+2Fy+c=0,$$

i.e.

$$y+\frac{F}{B}=\pm\sqrt{\frac{F^2}{B^2}-\frac{c}{B}},$$

and this represents two parallel straight lines, real or imaginary.

Thus in every case the general equation represents one of the conic sections enumerated in Art. 348.

350. Centre of a Conic Section. Def. The centre of a conic section is a point such that all chords of the conic which pass through it are bisected there.

When the equation to the conic is in the form

$$ax^2+2hxy+by^2+c=0 \quad\ldots\ldots\ldots\ldots\ldots (1),$$

the origin is the centre.

For let (x', y') be *any* point on (1), so that we have

$$ax'^2+2hx'y'+by'^2+c=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

This equation may be written in the form

$$a(-x')^2+2h(-x')(-y')+b(-y')^2+c=0,$$

and hence shews that the point $(-x', -y')$ also lies on (1).

But the points (x', y') and $(-x', -y')$ lie on the same straight line through the origin, and are at equal distances from the origin.

The chord of the conic which passes through the origin and any point (x', y') of the curve is therefore bisected at the origin.

The origin is therefore the centre.

351. When the equation to the conic is given in the form

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \text{.........}(1),$$

the origin is the centre only when both f and g are zero.

For, if the origin be the centre, then corresponding to *each* point (x', y') on (1), there must be also a point $(-x', -y')$ lying on the curve.

Hence we must have

$$ax'^2 + 2hx'y' + by'^2 + 2gx' + 2fy' + c = 0 \text{......} (2),$$

and $$ax'^2 + 2hx'y' + by'^2 - 2gx' - 2fy' + c = 0 \text{......} (3).$$

Subtracting (3) from (2), we have

$$gx' + fy' = 0.$$

This relation is to be true for *all* the points (x', y') which lie on the curve (1). But this can only be the case when $g = 0$ and $f = 0$.

352. *To obtain the coordinates of the centre of the conic given by the general equation, and to obtain the equation to the curve referred to axes through the centre parallel to the original axes.*

Transform the origin to the point $(\bar{x}, \bar{y})$, so that for x and y we have to substitute $x + \bar{x}$ and $y + \bar{y}$. The equation then becomes

$$a(x + \bar{x})^2 + 2h(x + \bar{x})(y + \bar{y}) + b(y + \bar{y})^2 + 2g(x + \bar{x}) + 2f(y + \bar{y}) + c = 0,$$

i.e. $$ax^2 + 2hxy + by^2 + 2x(a\bar{x} + h\bar{y} + g) + 2y(h\bar{x} + b\bar{y} + f) + a\bar{x}^2 + 2h\bar{x}\bar{y} + b\bar{y}^2 + 2g\bar{x} + 2f\bar{y} + c = 0 \text{.........} (2).$$

If the point $(\bar{x}, \bar{y})$ be the centre of the conic section, the coefficients of x and y in the equation (2) must vanish, so that we have

$$a\bar{x} + h\bar{y} + g = 0 \quad\text{.....................}(3),$$

and

$$h\bar{x} + b\bar{y} + f = 0 \quad\text{.....................}(4).$$

Solving (3) and (4), we have, in general,

$$\bar{x} = \frac{fh - bg}{ab - h^2}, \text{ and } \bar{y} = \frac{gh - af}{ab - h^2} \quad\text{..........}(5).$$

With these values the constant term in (2)

$$= a\bar{x}^2 + 2h\bar{x}\bar{y} + b\bar{y}^2 + 2g\bar{x} + 2f\bar{y} + c$$

$$= \bar{x}(a\bar{x} + h\bar{y} + g) + \bar{y}(h\bar{x} + b\bar{y} + f) + g\bar{x} + f\bar{y} + c$$

$$= \mathbf{g\bar{x} + f\bar{y} + c} \quad\text{.......................................}(6),$$

by equations (3) and (4),

$$= \frac{abc + 2fgh - af^2 - bg^2 - ch^2}{ab - h^2}, \text{ by equations (5)},$$

$$= \frac{\Delta}{ab - h^2},$$

where Δ is the discriminant of the given general equation (Art. 118).

The equation (2) can therefore be written in the form

$$ax^2 + 2hxy + by^2 + \frac{\Delta}{ab - h^2} = 0.$$

This is the required equation referred to the new axes through the centre.

Ex. *Find the centre of the conic section*

$$2x^2 - 5xy - 3y^2 - x - 4y + 6 = 0,$$

and its equation when transformed to the centre.

The centre is given by the equations $2\bar{x} - \frac{5}{2}\bar{y} - \frac{1}{2} = 0$, and $-\frac{5}{2}\bar{x} - 3\bar{y} - 2 = 0$, so that $\bar{x} = -\frac{2}{7}$, and $\bar{y} = -\frac{3}{7}$.

The equation referred to the centre is then

$$2x^2 - 5xy - 3y^2 + c' = 0,$$

where $c' = -\frac{1}{2}\cdot\bar{x} - 2\cdot\bar{y} + 6 = \frac{1}{7} + \frac{6}{7} + 6 = 7.$ (Art. 352.)

The required equation is thus

$$2x^2 - 5xy - 3y^2 + 7 = 0.$$

353. Sometimes the equations (3) and (4) of the last article do not give suitable values for $\bar{x}$ and $\bar{y}$.

For, if $ab - h^2$ be zero, the values of $\bar{x}$ and $\bar{y}$ in (5) are both infinite. When $ab - h^2$ is zero, the conic section is a parabola.

The centre of a parabola is therefore at infinity.

Again, if $\frac{a}{h} = \frac{h}{b} = \frac{g}{f}$, the result (5) of the last article is of the form $\frac{0}{0}$ and the equations (3) and (4) reduce to the same equation, viz.,

$$a\bar{x} + h\bar{y} + g = 0.$$

We then have only one equation to determine the centre, and there is therefore an infinite number of centres all lying on the straight line

$$ax + hy + g = 0.$$

In this case the conic section consists of a pair of parallel straight lines, both parallel to the line of centres.

354. The student who is acquainted with the Differential Calculus will observe, from equations (3) and (4) of Art. 352, that the coordinates of the centre satisfy the equations that are obtained by differentiating, with regard to x and y, the original equation of the conic section.

It will also be observed that the coefficients of $\bar{x}$, $\bar{y}$, and unity in the equations (3), (4), and (6) of Art. 352 are the quantities (in the order in which they occur) which make up the determinant of Art. 118.

This determinant being easy to write down, the student may thence recollect the equations for the centre and the value of c.

The reason why this relation holds will appear from the next article.

355. Ex. *Find the condition that the general equation of the second degree may represent two straight lines.*

The centre $(\bar{x}, \bar{y})$ of the conic is given by

$$a\bar{x} + h\bar{y} + g = 0 \quad \ldots\ldots\ldots\ldots (1),$$

and

$$h\bar{x} + b\bar{y} + f = 0 \quad \ldots\ldots\ldots\ldots (2).$$

Also, if it be transformed to the centre as origin, the equation becomes

$$ax^2+2hxy+by^2+c'=0\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

where $c'=g\bar{x}+f\bar{y}+c.$

Now the equation (3) represents two straight lines if c' be zero, *i.e.* if

$$g\bar{x}+f\bar{y}+c=0\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

The equation therefore represents two straight lines if the relations (1), (2), and (4) be simultaneously true.

Eliminating the quantities $\bar{x}$ and $\bar{y}$ from these equations, we have, by Art. 12,

$$\begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix}=0.$$

This is the condition found in Art. 118.

356. *To find the equation to the asymptotes of the conic section given by the general equation of the second degree.*

Let the equation be

$$ax^2+2hxy+by^2+2gx+2fy+c=0\ldots\ldots\ldots(1).$$

Since the equation to the asymptotes has been shewn to differ from the equation to the curve only in its constant term, the required equation must be

$$ax^2+2hxy+by^2+2gx+2fy+c+\lambda=0\ldots\ldots(2).$$

Also (2) is to be a pair of straight lines.

Hence

$$ab\,(c+\lambda)+2fgh-af^2-bg^2-(c+\lambda)\,h^2=0. \quad \text{(Art. 116.)}$$

Therefore $\lambda=-\dfrac{abc+2fgh-af^2-bg^2-ch^2}{ab-h^2}=-\dfrac{\Delta}{ab-h^2}.$

The required equation to the asymptotes is therefore

$$ax^2+2hxy+by^2+2gx+2fy+c-\frac{\Delta}{ab-h^2}=0\ldots(2).$$

Cor. Since the equation to the hyperbola, which is conjugate to a given hyperbola, differs as much from the equation to the common asymptotes as the original equation does, it follows that the equation to the hyperbola, which is conjugate to the hyperbola (1), is

$$ax^2+2hxy+by^2+2gx+2fy+c-2\,\frac{\Delta}{ab-h^2}=0.$$

357. *To determine by an examination of the general equation what kind of conic section it represents.*

[On applying the method of Art. 313 to the ellipse and parabola, it would be found that the asymptotes of the ellipse are imaginary, and that a parabola only has one asymptote, which is at an infinite distance and perpendicular to its axis.]

The straight lines $ax^2 + 2hxy + by^2 = 0$ ....(1)

are parallel to the lines (2) of the last article, and hence represent straight lines parallel to the asymptotes.

Now the equation (1) represents real, coincident, or imaginary straight lines according as h^2 is $> = $ or $< ab$, *i.e.* the asymptotes are real, coincident, or imaginary, according as $h^2 > = $ or $< ab$, *i.e.* the conic section is a hyperbola, parabola, or ellipse, according as $h^2 > = $ or $< ab$.

Again, the lines (1) are at right angles, *i.e.* the curve is a rectangular hyperbola, if $a + b = 0$.

Also, by Art. 143, the general equation represents a circle if $a = b$, and $h = 0$.

Finally, by Art. 116, the equation represents a pair of straight lines if $\Delta = 0$; also these straight lines are parallel if the terms of the second degree form a perfect square, *i.e.* if

$$h^2 = ab.$$

358. The results for the general equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

are collected in the following table, the axes of coordinates being rectangular.

Curve.	**Condition.**
Ellipse.	$h^2 < ab$.
Parabola.	$h^2 = ab$.
Hyperbola.	$h^2 > ab$.
Circle.	$a = b$, and $h = 0$.
Rectangular hyperbola.	$a + b = 0$.
Two straight lines, real or imaginary.	$\Delta = 0$, *i.e.* $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$.
Two parallel straight lines.	$\Delta = 0$, and $h^2 = ab$.

If the axes of coordinates be oblique, the lines (1) of Art. 356 are at right angles if $a+b-2h\cos\omega=0$ (Art. 93); so that the conic section is a rectangular hyperbola if $a+b-2h\cos\omega=0$.

Also, by Art. 175, the conic section is a circle if $b=a$ and

$$h=a\cos\omega.$$

The conditions for the other cases in the previous article are the same for both oblique and rectangular axes.

EXAMPLES XL

What conics do the following equations represent? When possible, find their centres, and also their equations referred to the centre.

1. $12x^2-23xy+10y^2-25x+26y=14$.

2. $13x^2-18xy+37y^2+2x+14y-2=0$.

3. $y^2-2\sqrt{3}xy+3x^2+6x-4y+5=0$.

4. $2x^2-72xy+23y^2-4x-28y-48=0$.

5. $6x^2-5xy-6y^2+14x+5y+4=0$.

6. $3x^2-8xy-3y^2+10x-13y+8=0$.

Find the asymptotes of the following hyperbolas and also the equations to their conjugate hyperbolas.

7. $8x^2+10xy-3y^2-2x+4y=2$. 8. $y^2-xy-2x^2-5y+x-6=0$.

9. $55x^2-120xy+20y^2+64x-48y=0$.

10. $19x^2+24xy+y^2-22x-6y=0$.

11. If $(\bar{x}, \bar{y})$ be the centre of the conic section

$$f(x, y)\equiv ax^2+2hxy+by^2+2gx+2fy+c=0,$$

prove that the equation to the asymptotes is $f(x, y)=f(\bar{x}, \bar{y})$.

If t be a variable quantity, find the locus of the point (x, y) when

12. $x=a\left(t+\frac{1}{t}\right)$ and $y=a\left(t-\frac{1}{t}\right)$.

13. $x=at+bt^2$ and $y=bt+at^2$.

14. $x=1+t+t^2$ and $y=1-t+t^2$.

If θ be a variable angle, find the locus of the point (x, y) when

15. $x=a\tan(\theta+\alpha)$ and $y=b\tan(\theta+\beta)$.

16. $x=a\cos(\theta+\alpha)$ and $y=b\cos(\theta+\beta)$.

What are represented by the equations

17. $(x-y)^2+(x-a)^2=0$. 18. $xy+a^2=a(x+y)$.

19. $x^3 - y^3 = (y - a)(x^2 - y^2)$.

20. $x^3 + y^3 - xy(x + y) + a^2(y - x) = 0$.

21. $(x^2 - a^2)^2 - y^4 = 0$.

22. $x^3 + y^3 + (x + y)(xy - ax - ay) = 0$.

23. $x^2 + xy + y^2 = 0$.

24. $(r\cos\theta - a)(r - a\cos\theta) = 0$.

25. $r\sin^2\theta = 2a\cos\theta$.

26. $r + \frac{1}{r} = 3\cos\theta + \sin\theta$.

27. $\frac{1}{r} = 1 + \cos\theta + \sqrt{3}\sin\theta$.

28. $r(4 - 3\sin^2\theta) = 8a\cos\theta$.

ANSWERS

1. A hyperbola; $(2, 1)$; $c' = -26$.

2. An ellipse; $(-\frac{1}{4}, -\frac{1}{4})$; $c' = -4$.

3. A parabola.

4. A hyperbola; $(-\frac{11}{25}, -\frac{2}{25})$; $c' = -46$.

5. Two straight lines; $(-\frac{11}{13}, \frac{19}{13})$; $c' = 0$.

6. A hyperbola; $(-\frac{41}{25}, \frac{1}{50})$; $c' = -\frac{33}{100}$.

7. $(2x + 3y - 1)(4x - y + 1) = 0$; $8x^2 + 10xy - 3y^2 - 2x + 4y = 0$.

8. $(y + x - 2)(y - 2x - 3) = 0$; $y^2 - xy - 2x^2 - 5y + x + 18 = 0$.

9. $(11x - 2y + 4)(5x - 10y + 4) = 0$;
$55x^2 - 120xy + 20y^2 + 64x - 48y + 32 = 0$.

10. $19x^2 + 24xy + y^2 - 22x - 6y + 4 = 0$;
$19x^2 + 24xy + y^2 - 22x - 6y + 8 = 0$.

12. $x^2 - y^2 = 4a^2$.

13. $(ax - by)^2 = (a^2 - b^2)(ay - bx)$.

14. $(x - y)^2 - 2(x + y) + 4 = 0$.

15. $(xy + ab)\tan(\alpha - \beta) = bx - ay$.

16. $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 2\frac{xy}{ab}\cos(\alpha - \beta) = \sin^2(\alpha - \beta)$.

17. A point.

18. Two straight lines.

19. A straight line and a parabola.

20. A straight line and a rectangular hyperbola.

21. A circle and a rectangular hyperbola.

22. A straight line and a circle.

23. Two imaginary straight lines.

24. A circle and a straight line.

25. A parabola.

26. A circle.

27. A hyperbola.

28. An ellipse.

SOLUTIONS/HINTS

1. The centre is given by
$12x - \frac{23}{2}y - \frac{25}{2} = 0$, and $-\frac{23}{2}x + 10y + 13 = 0$,

whence $x = 2,\ y = 1.$

$$\therefore\ c' = 2\left(-\tfrac{25}{2}\right) + 13 - 14 = -26. \quad \text{[Art. 352.]}$$

Since $\left(\frac{23}{2}\right)^2 - 10 \times 12 > 0$, the conic is a hyperbola.

2. The centre is given by

$$13x - 9y + 1 = 0, \text{ and } -9x + 37y + 7 = 0,$$

whence $x = -\frac{1}{4},\ y = -\frac{1}{4}.$

$$\therefore\ c' = -\tfrac{1}{4} + 7\left(-\tfrac{1}{4}\right) - 2 = -4.$$

Since $9^2 - 13 \times 37 < 0$, the conic is an ellipse.

3. Since $y^2 - 2\sqrt{3}xy + 3x^2$ is a perfect square, and $\Delta \neq 0$, the equation represents a parabola.

4. The centre is given by

$$2x - 36y - 2 = 0, \text{ and } -36x + 23y - 14 = 0.$$

$$\therefore\ x = -\tfrac{11}{25},\ y = -\tfrac{2}{25}.$$

$$\therefore\ c' = -2\left(-\tfrac{11}{25}\right) - 14\left(-\tfrac{2}{25}\right) - 48 = -46.$$

Since $36^2 - 2 \times 23 > 0$, the conic is a hyperbola.

5. The centre is given by

$$6x - \tfrac{5}{2}y + 7 = 0, \text{ and } -\tfrac{5}{2}x - 6y + \tfrac{5}{2} = 0,$$

whence $x = -\frac{11}{13},\ y = \frac{10}{13}.$

$$\therefore\ c' = -\tfrac{11}{13} \cdot 7 + \tfrac{5}{13} \cdot 5 + 4 = 0.$$

$\therefore$ the equation represents two straight lines [Art. 120].

6. The centre is given by

$$3x - 4y + 5 = 0, \text{ and } 4x + 3y + \tfrac{13}{2} = 0,$$

whence $x = -\frac{41}{25},\ y = \frac{1}{50}.$

$$\therefore\ c' = -\tfrac{41}{25} \cdot 5 + \tfrac{1}{50}\left(-\tfrac{13}{2}\right) + 8 = -\tfrac{33}{100}.$$

Since $a + b = 0$, the conic is a rectangular hyperbola.

7. Here

$$\frac{\Delta}{ab - h^2} = \frac{8 \cdot 3 \cdot 2 - 10 \cdot 2 \cdot 1 - 8 \cdot 4 + 3 \cdot 1 + 2 \cdot 25}{-24 - 25} = -1.$$

Hence the equation to the asymptotes is

$$8x^2 + 10xy - 3y^2 - 2x + 4y - 1 = 0,$$

and that to the conjugate hyperbola is

$$8x^2 + 10xy - 3y^2 - 2x + 4y = 0.$$

8. Here $\dfrac{\Delta}{ab-h^2} = \dfrac{2 \cdot 6 + \frac{5}{4} + 2 \cdot \frac{25}{4} - \frac{1}{4} + 6 \cdot \frac{1}{4}}{-2 - \frac{1}{4}} = -12.$

Hence the equation to the asymptotes is

$$y^2 - xy - 2x^2 - 5y + x + 6 = 0,$$

and that to the conjugate hyperbola is

$$y^2 - xy - 2x^2 - 5y + x + 18 = 0.$$

9. Here

$$\frac{\Delta}{ab-h^2} = \frac{2 \cdot 60 \cdot 32 \cdot 24 - 55 \cdot 24^2 - 20 \cdot 32^2}{55 \cdot 20 - 60^2} = -16.$$

Hence the equation to the asymptotes is

$$55x^2 - 120xy + 20y^2 + 64x - 48y + 16 = 0,$$

and that to the conjugate hyperbola is

$$55x^2 - 120xy + 20y^2 + 64x - 48y + 32 = 0.$$

10. Here $\dfrac{\Delta}{ab-h^2} = \dfrac{2 \cdot 12 \cdot 11 \cdot 3 - 19 \cdot 3^2 - 11^2}{19 - 12^2} = -4.$

Hence, etc.

11. Let $f(x, y) + \lambda = 0$ be the equation to the asymptotes. Since they pass through $(\bar{x}, \bar{y})$,

$$\therefore f(\bar{x}, \bar{y}) + \lambda = 0. \quad \therefore \lambda = -f(\bar{x}, \bar{y})$$

and the required equation is $f(x, y) = f(\bar{x}, \bar{y})$.

12. Square and subtract; $\therefore x^2 - y^2 = 4a^2$.

13. We have $ax - by = (a^2 - b^2)\,t$, and

$$ay - bx = (a^2 - b^2)\,t^2.$$

$$\therefore (ax - by)^2 = (a^2 - b^2)(ay - bx).$$

14. We have $\qquad x - y = 2t,$

and $\qquad 2x + 2y = 4 + 4t^2 = 4 + (x - y)^2.$

15. $\theta + \alpha = \tan^{-1}\frac{x}{a}$, and $\theta + \beta = \tan^{-1}\frac{y}{b}$.

$$\therefore\ \alpha - \beta = \tan^{-1}\frac{x}{a} - \tan^{-1}\frac{y}{b}.$$

$$\therefore\ \tan(\alpha - \beta) = \frac{\frac{x}{a} - \frac{y}{b}}{1 + \frac{xy}{ab}},\ \text{etc.}$$

16. $\theta + \alpha = \cos^{-1}\frac{x}{a}$, and $\theta + \beta = \cos^{-1}\frac{y}{b}$.

$$\therefore\ \alpha - \beta = \cos^{-1}\frac{x}{a} - \cos^{-1}\frac{y}{b}.$$

$$\therefore\ \cos(\alpha - \beta) = \frac{x}{a}\frac{y}{b} + \sqrt{\left(1 - \frac{x^2}{a^2}\right)\left(1 - \frac{y^2}{b^2}\right)}.$$

$$\therefore\ 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{x^2y^2}{a^2b^2} = \frac{x^2y^2}{a^2b^2} + \cos^2(\alpha - \beta) - \frac{2xy}{ab}\cos(\alpha - \beta).$$

$$\therefore\ \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2xy}{ab}\cos(\alpha - \beta) = \sin^2(\alpha - \beta).$$

17. We must have both $x = y$ and $x = a$, simultaneously; therefore the equation represents the point (a, a).

18. The equation is equivalent to $(x - a)(y - a) = 0$, and therefore represents the two lines $x = a$, and $y = a$.

19. The equation is equivalent to

$$(x - y)(x^2 + ax + ay) = 0,$$

and therefore represents the line $x - y = 0$ and the parabola

$$x^2 + ax + ay = 0.$$

20. The equation is

$$x^3 + y^3 - xy(x + y) + a^2(y - x) = 0,$$

i.e.
$$(x + y)(x - y)^2 + a^2(y - x) = 0,$$

i.e.
$$(x - y)(x^2 - y^2 - a^2) = 0.$$

Hence the equation represents the line $x - y = 0$ and the rectangular hyperbola $x^2 - y^2 = a^2$.

21. The equation is equivalent to

$$(x^2 - a^2 + y^2)(x^2 - a^2 - y^2) = 0,$$

and thus represents a circle and a rectangular hyperbola.

22. The equation is equivalent to

$$(x + y)(x^2 + y^2 - ax - ay) = 0,$$

and thus represents a straight line and a circle.

23. Two imaginary straight lines through the origin. [Art. 120.]

24. A circle and a straight line. [Arts. 88 and 172.]

25. In Cartesians, the equation is $y^2 = 2ax$, *i.e.* a parabola.

26. In Cartesians, the equation is $x^2 + y^2 + 1 = 3x + y$, *i.e.* a circle.

27. In Cartesians, the equation is

$x^2 + y^2 = (x + \sqrt{3}y - 1)^2$, *i.e.* a hyperbola. [Art. 358.]

28. In Cartesians, the equation is

$$4(x^2 + y^2) - 3y^2 = 8ax, \text{ or } 4x^2 + y^2 = 8ax.$$

This represents an ellipse. [Art. 358.]

359. *To trace the parabola given by the general equation of the second degree*

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \dots\dots\dots (1),$$

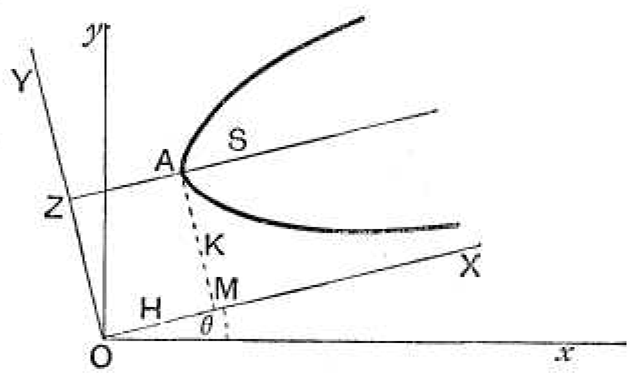
and to find its latus rectum.

First Method. Since the curve is a parabola we have $h^2 = ab$, so that the terms of the second degree form a perfect square.

Put then $a = \alpha^2$ and $b = \beta^2$, so that $h = \alpha\beta$, and the equation (1) becomes

$$(\alpha x + \beta y)^2 + 2gx + 2fy + c = 0 \dots\dots\dots\dots (2).$$

Let the direction of the axes be changed so that the straight line $\alpha x + \beta y = 0$, *i.e.* $y = -\frac{\alpha}{\beta} x$, may be the new axis of X.



We have therefore to turn the axes through an angle θ such that $\tan\theta = -\frac{\alpha}{\beta}$, and therefore

$$\sin\theta = -\frac{\alpha}{\sqrt{\alpha^2 + \beta^2}} \text{ and } \cos\theta = \frac{\beta}{\sqrt{\alpha^2 + \beta^2}}.$$

For x we have to substitute

$$X\cos\theta - Y\sin\theta,\ \textit{i.e.}\ \frac{\beta X + \alpha Y}{\sqrt{\alpha^2+\beta^2}},$$

and for y the quantity

$$X\sin\theta + Y\cos\theta,\ \textit{i.e.}\ \frac{-\alpha X + \beta Y}{\sqrt{\alpha^2+\beta^2}}. \quad \text{(Art. 129.)}$$

For $\alpha x + \beta y$ we therefore substitute $Y\sqrt{(\alpha^2+\beta^2)}$.

The equation (2) then becomes

$$Y^2(\alpha^2+\beta^2) + \frac{2}{\sqrt{\alpha^2+\beta^2}}[g(\beta X + \alpha Y) + f(\beta Y - \alpha X)] + c = 0,$$

i.e.
$$Y^2 + 2Y\frac{\alpha g + \beta f}{(\alpha^2+\beta^2)^{\frac{3}{2}}} = 2X\frac{\alpha f - \beta g}{(\alpha^2+\beta^2)^{\frac{3}{2}}} - \frac{c}{\alpha^2+\beta^2},$$

i.e.
$$(Y-K)^2 = 2\frac{\alpha f - \beta g}{(\alpha^2+\beta^2)^{\frac{3}{2}}}[X-H] \ \ldots\ldots\ldots \ (3),$$

where
$$K = -\frac{\alpha g + \beta f}{(\alpha^2+\beta^2)^{\frac{3}{2}}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (4),$$

and
$$-2\frac{\alpha f - \beta g}{(\alpha^2+\beta^2)^{\frac{3}{2}}} \times H = K^2 - \frac{c}{\alpha^2+\beta^2},$$

i.e.
$$H = \frac{\sqrt{\alpha^2+\beta^2}}{2(\alpha f - \beta g)}\left[c - \frac{(\alpha g + \beta f)^2}{(\alpha^2+\beta^2)^2}\right] \ldots\ldots\ldots (5).$$

The equation (3) represents a parabola whose latus rectum is $2\frac{\alpha f - \beta g}{(\alpha^2+\beta^2)^{\frac{3}{2}}}$, whose axis is parallel to the new axis of X, and whose vertex referred to the new axes is the point (H, K).

360. *Equation of the axis, and coordinates of the vertex, referred to the original axes.*

Since the axis of the curve is parallel to the new axis of X, it makes an angle θ with the old axis of x, and hence the perpendicular on it from the origin makes an angle $90° + \theta$.

Also the length of this perpendicular is K.

The equation to the axis of the parabola is therefore

$$x\cos(90^\circ+\theta)+y\sin(90^\circ+\theta)=K,$$

i.e. $$-x\sin\theta+y\cos\theta=K,$$

i.e. $$\alpha x+\beta y=K\sqrt{\alpha^2+\beta^2}=-\frac{\alpha g+\beta f}{\alpha^2+\beta^2}\;\ldots\ldots\ldots(6).$$

Again, the vertex is the point in which the axis (6) meets the curve (2).

We have therefore to solve (6) and (2), *i.e.* (6) and

$$\frac{(\alpha g+\beta f)^2}{(\alpha^2+\beta^2)^2}+2gx+2fy+c=0\;\ldots\ldots\ldots\ldots\ldots(7).$$

The solution of (6) and (7) therefore gives the required coordinates of the vertex.

361. It was proved in Art. 224 that if PV be a diameter of the parabola and QV the ordinate to it drawn through any point Q of the curve, so that QV is parallel to the tangent at P, and if θ be the angle between the diameter PV and the tangent at P, then

$$QV^2=4a\operatorname{cosec}^2\theta\,.\,PV\;\ldots\ldots\ldots\ldots\ldots(1).$$

If QL be perpendicular to PV and QL' be perpendicular to the tangent at P, we have

$$QL=QV\sin\theta,\ \text{ and }\ QL'=PV\sin\theta,$$

so that (1) is $$QL^2=4a\operatorname{cosec}\theta\,.\,QL'.$$

Hence the square of the perpendicular distance of any point Q on the parabola from any diameter varies as the perpendicular distance of Q from the tangent at the end of the diameter.

Hence, if $Ax+By+C=0$ be the equation of any diameter and $A'x+B'y+C'=0$ be the equation of the tangent at its end, the equation to the parabola is

$$(Ax+By+C)^2=\lambda(A'x+B'y+C')\;\ldots\ldots\ldots(2),$$

where λ is some constant.

Conversely, if the equation to a parabola can be reduced to the form (2), then

$$Ax+By+C=0\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

is a diameter of the parabola and the axis of the parabola is parallel to (3).

We shall apply this property in the following article.

362. *To trace the parabola given by the general equation of the second degree*

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad \text{.........(1)}.$$

Second Method. Since the curve is a parabola, the terms of the second degree must form a perfect square and $h^2 = ab$.

Put then $a = \alpha^2$ and $b = \beta^2$, so that $h = \alpha\beta$, and the equation (1) becomes

$$(\alpha x + \beta y)^2 = -(2gx + 2fy + c) \quad \text{...........(2)}.$$

As in the last article the straight line $\alpha x + \beta y = 0$ is a diameter, and the axis of the parabola is therefore parallel to it, and so its equation is of the form

$$\alpha x + \beta y + \lambda = 0 \quad \text{...................(3)}.$$

The equation (2) may therefore be written

$$(\alpha x + \beta y + \lambda)^2 = -(2gx + 2fy + c) + \lambda^2 + 2\lambda(\alpha x + \beta y)$$
$$= 2x(\lambda\alpha - g) + 2y(\beta\lambda - f) + \lambda^2 - c \quad \text{.......(4)}.$$

Choose λ so that the straight lines

$$\alpha x + \beta y + \lambda = 0 \quad \text{...................(5)}$$

and $$2x(\lambda\alpha - g) + 2y(\beta\lambda - f) + \lambda^2 - c = 0 \quad \text{.......(6)}$$

are at right angles, *i.e.* so that

$$\alpha(\lambda\alpha - g) + \beta(\beta\lambda - f) = 0,$$

i.e. so that $$\lambda = \frac{\beta f + \alpha g}{\alpha^2 + \beta^2} \quad \text{.....................(7)}.$$

The lines (5) and (6) are now, by the last article, a diameter and a tangent at its extremity; also, since they are at right angles, they must be the axis and the tangent at the vertex.

The equation (4) may now, by (7), be written

$$\{ax+\beta y+\lambda\}^2=\frac{2(af-\beta g)}{a^2+\beta^2}[\beta x-ay+\mu],$$

where $$\mu=\frac{a^2+\beta^2}{2(af-\beta g)}(\lambda^2-c),$$

i e. $$\left\{\frac{ax+\beta y+\lambda}{\sqrt{a^2+\beta^2}}\right\}^2=\frac{2(af-\beta g)}{(a^2+\beta^2)^{\frac{3}{2}}}\cdot\frac{\beta x-ay+\mu}{\sqrt{a^2+\beta^2}},$$

i.e. $$PN^2=\frac{2(af-\beta g)}{(a^2+\beta^2)^{\frac{3}{2}}}.AN,$$

where PN is the perpendicular from any point P of the curve on the axis, and A is the vertex.

Hence the axis and tangent at the vertex are the lines (5) and (6), where λ has the value (7), and the latus rectum

$$=2\frac{af-\beta g}{(a^2+\beta^2)^{\frac{3}{2}}}.$$

363. Ex. *Trace the parabola*

$$9x^2-24xy+16y^2-18x-101y+19=0.$$

The equation is

$$(3x-4y)^2-18x-101y+19=0 \qquad (1).$$

First Method. Take $3x-4y=0$ as the new axis of x, *i.e.* turn the axes through an angle θ, where $\tan\theta=\frac{3}{4}$, and therefore $\sin\theta=\frac{3}{5}$ and $\cos\theta=\frac{4}{5}$.

For x we therefore substitute $X\cos\theta-Y\sin\theta$, *i.e.* $\frac{4X-3Y}{5}$; for y we put $X\sin\theta+Y\cos\theta$, *i.e.* $\frac{3X+4Y}{5}$, and hence for $3x-4y$ the quantity $-5Y$.

The equation (1) therefore becomes

$$25Y^2-\tfrac{1}{5}[72X-54Y]-\tfrac{1}{5}[303X+404Y]+19=0,$$

i.e. $$25Y^2-75X-70Y+19=0 \qquad (2).$$

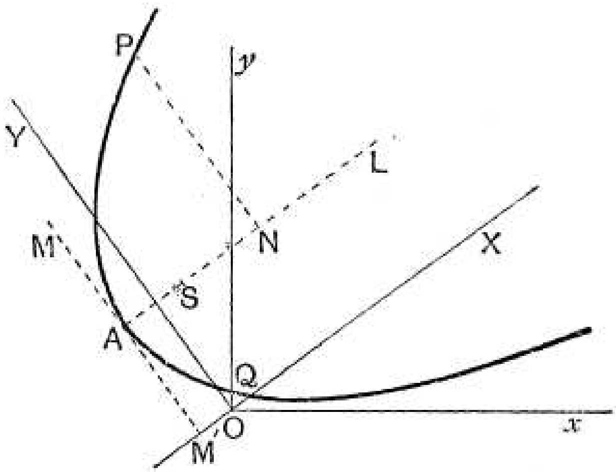
This is the equation to the curve referred to the axes OX and OY.

But (2) can be written in the form

$$Y^2-\frac{14Y}{5}=3X-\tfrac{19}{25},$$

i.e. $$(Y-\tfrac{7}{5})^2=3X-\tfrac{19}{25}+\tfrac{49}{25}=3(X+\tfrac{2}{5}).$$

Take a point A whose coordinates referred to OX and OY are $-\frac{2}{5}$ and $\frac{7}{5}$, and draw AL and AM parallel to OX and OY respectively.



Referred to AL and AM the equation to the parabola is $Y^2 = 3X$. It is therefore a parabola, whose vertex is A, whose latus rectum is 3, and whose axis is AL.

Second Method. The equation (1) can be written

$$(3x - 4y + \lambda)^2 = (6\lambda + 18)\, x + y\,(101 - 8\lambda) + \lambda^2 - 19 \;\ldots\ldots\; (3).$$

Choose λ so that the straight lines

$$3x - 4y + \lambda = 0$$

and $$(6\lambda + 18)\, x + y\,(101 - 8\lambda) + \lambda^2 - 19 = 0$$

may be at right angles.

Hence λ is given by

$$3\,(6\lambda + 18) - 4\,(101 - 8\lambda) = 0 \text{ (Art. 69)},$$

and therefore $\lambda = 7$.

The equation (3) then becomes

$$(3x - 4y + 7)^2 = 15\,(4x + 3y + 2),$$

i.e. $$\left(\frac{3x - 4y + 7}{\sqrt{25}}\right)^2 = 3\,.\,\frac{4x + 3y + 2}{\sqrt{25}} \;\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

Let AL be the straight line

$$3x - 4y + 7 = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (5),$$

and AM the straight line $4x + 3y + 2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$.

These are at right angles.

If P be any point on the parabola and PN be perpendicular to AL, the equation (4) gives $PN^2 = 3\,.\,AN$.

Hence, as in the first method, we have the parabola.

The vertex is found by solving (5) and (6) and is therefore the point $\left(-\frac{29}{25}, \frac{22}{25}\right)$.

In drawing curves it is often advisable, as a verification, to find whether they cut the original axes of coordinates.

Thus the points in which the given parabola cuts the axis of x are found by putting $y=0$ in the original equation. The resulting equation is $9x^2-18x+19=0$, which has imaginary roots.

The parabola does not therefore meet Ox.

Similarly it meets Oy in points given by $16y^2-101y+19=0$, the roots of which are nearly $6\frac{1}{3}$ and $\frac{3}{16}$.

The values of OQ and OQ' should therefore be nearly $\frac{3}{16}$ and $6\frac{1}{3}$.

364. *To find the direction and magnitude of the axes of the central conic section*

$$ax^2+2hxy+by^2=1\quad\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

First Method. We know that, when the equation to a central conic section has no term containing xy and the axes are rectangular, the axes of coordinates are the axes of the curve.

Now in Art. 349 we shewed that, to get rid of the term involving xy, we must turn the axes through an angle θ given by

$$\tan 2\theta=\frac{2h}{a-b}\quad\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

The axes of the curve are therefore inclined to the axes of coordinates at an angle θ given by (2).

Now (2) can be written

$$\frac{2\tan\theta}{1-\tan^2\theta}=\frac{2h}{a-b}=\frac{1}{\lambda}\text{ (say)},$$

$$\therefore\ \tan^2\theta+2\lambda\tan\theta-1=0\quad\ldots\ldots\ldots\ldots(3).$$

This, being a quadratic equation, gives two values for θ, which differ by a right angle, since the product of the two values of $\tan\theta$ is -1. Let these values be θ_1 and θ_2, which are therefore the inclinations of the required axes of the curve to the axis of x.

Again, in polar coordinates, equation (1) may be written

$$r^2(a\cos^2\theta+2h\cos\theta\sin\theta+b\sin^2\theta)=1=\cos^2\theta+\sin^2\theta,$$

i.e.

$$r^2=\frac{\cos^2\theta+\sin^2\theta}{a\cos^2\theta+2h\cos\theta\sin\theta+b\sin^2\theta}=\frac{1+\tan^2\theta}{a+2h\tan\theta+b\tan^2\theta}\quad\ldots\ldots\ldots\ldots(4).$$

If in (4) we substitute either value of $\tan\theta$ derived from (3) we obtain the length of the corresponding semi-axis.

The directions and magnitudes of the axes are therefore both found.

Second Method. The directions of the axes of the conic are, as in the first method, given by

$$\tan 2\theta = \frac{2h}{a-b}.$$

When referred to the axes of the conic section as the axes of coordinates, let the equation become

$$\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots(5).$$

Since the equation (1) has become equation (5) by a change of axes without a change of origin, we have, by Art. 135,

$$\frac{1}{\alpha^2} + \frac{1}{\beta^2} = a + b \quad \ldots\ldots\ldots\ldots\ldots\ldots(6),$$

and

$$\frac{1}{\alpha^2\beta^2} = ab - h^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots(7).$$

These two equations easily determine the semi-axes α and β. [For if from the square of (6) we subtract four times equation (7) we have $\left(\frac{1}{\alpha^2} - \frac{1}{\beta^2}\right)^2$, and hence $\frac{1}{\alpha^2} - \frac{1}{\beta^2}$; hence by (6) we get $\frac{1}{\alpha^2}$ and $\frac{1}{\beta^2}$.]

The difficulty of this method lies in the fact that we cannot always easily determine to which direction for an axis the value α belongs and to which the value β.

If the original axes be inclined at an angle ω, the equations (6) and (7) are, by Art. 137,

$$\frac{1}{\alpha^2} + \frac{1}{\beta^2} = \frac{a + b - 2h\cos\omega}{\sin^2\omega},$$

and

$$\frac{1}{\alpha^2\beta^2} = \frac{ab - h^2}{\sin^2\omega}.$$

Cor. 1. The reciprocals of the squares of the semi-axes are, by (6) and (7), the roots of the equation

$$Z^2 - (a+b)Z + ab - h^2 = 0.$$

Cor. 2. From equation (4) we have

$$\text{Area of an ellipse} = \pi\alpha\beta = \frac{\pi}{\sqrt{ab-h^2}}.$$

365. Ex. 1. *Trace the curve*

$$14x^2 - 4xy + 11y^2 - 44x - 58y + 71 = 0 \quad\ldots\ldots\ldots\ldots(1).$$

Since $(-2)^2 - 14 . 11$ is negative, the curve is an ellipse. [Art. 358.]

By Art. 352 the centre $(\bar{x}, \bar{y})$ of the curve is given by the equations

$$14\bar{x} - 2\bar{y} - 22 = 0, \text{ and } -2\bar{x} + 11\bar{y} - 29 = 0.$$

Hence $\bar{x} = 2$, and $\bar{y} = 3$.

The equation referred to parallel axes through the centre is

therefore $$14x^2 - 4xy + 11y^2 + c' = 0,$$

where $$c' = -22\bar{x} - 29\bar{y} + 71 = -60,$$

so that the equation is

$$14x^2 - 4xy + 11y^2 = 60 \quad\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

The directions of the axes are given by

$$\tan 2\theta = \frac{2h}{a-b} = \frac{-4}{14-11} = -\tfrac{4}{3},$$

so that $$\frac{2\tan\theta}{1-\tan^2\theta} = -\tfrac{4}{3},$$

and hence $$2\tan^2\theta - 3\tan\theta - 2 = 0.$$

Therefore $\tan\theta_1 = 2$, and $\tan\theta_2 = -\tfrac{1}{2}$.

Referred to polar coordinates the equation (2) is

$$r^2(14\cos^2\theta - 4\cos\theta\sin\theta + 11\sin^2\theta) = 60(\cos^2\theta + \sin^2\theta),$$

i.e. $$r^2 = 60\frac{1+\tan^2\theta}{14 - 4\tan\theta + 11\tan^2\theta}.$$

When $\tan\theta_1 = 2$, $r_1{}^2 = 60 \times \dfrac{1+4}{14-8+44} = 6.$

When $\tan\theta_2 = -\tfrac{1}{2}$, $r_2{}^2 = 60 \times \dfrac{1+\frac{1}{4}}{14+2+\frac{11}{4}} = 4.$

The lengths of the semi-axes are therefore $\sqrt{6}$ and 2.

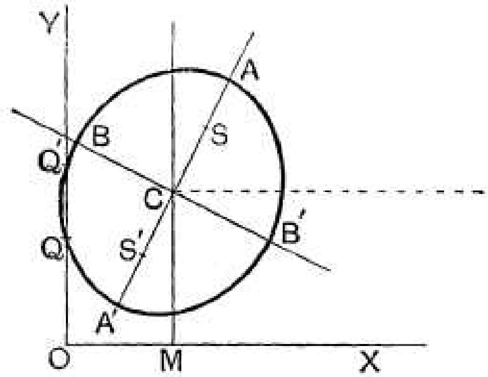
Hence, to draw the curve, take the point C, whose coordinates are (2, 3).

Through it draw $A'CA$ inclined at an angle $\tan^{-1} 2$ to the axis of x and mark off

$$A'C = CA = \sqrt{6}.$$

Draw BCB' at right angles to ACA' and take $B'C = CB = 2$.

The required ellipse has AA' and BB' as its axes.



It would be found, as a verification, that the curve does not meet the original axis of x, and that it meets the axis of y at distances from the origin equal to about 2 and $3\frac{1}{2}$ respectively.

Ex. 2. *Trace the curve*

$$x^2 - 3xy + y^2 + 10x - 10y + 21 = 0 \quad\text{..................(1)}.$$

Since $\left(\frac{-3}{2}\right)^2 - 1 . 1$ is positive, the curve is a hyperbola.

[Art. 358.]

The centre $(\bar{x}, \bar{y})$ is given by

$$\bar{x} - \frac{3}{2}\bar{y} + 5 = 0,$$

and

$$\frac{-3}{2}\bar{x} + \bar{y} - 5 = 0,$$

so that

$$\bar{x} = -2, \text{ and } \bar{y} = 2.$$

The equation to the curve, referred to parallel axes through the centre, is then

$$x^2 - 3xy + y^2 + 5(-2) - 5 \times 2 + 21 = 0,$$

i.e.

$$x^2 - 3xy + y^2 = -1 \quad\text{..........................(2)}.$$

The direction of the axes is given by

$$\tan 2\theta = \frac{2h}{a-b} = \frac{-3}{1-1} = \infty,$$

so that

$$2\theta = 90^\circ \text{ or } 270^\circ,$$

and hence

$$\theta_1 = 45^\circ \text{ and } \theta_2 = 135^\circ.$$

The equation (2) in polar coordinates is

$$r^2(\cos^2\theta - 3\cos\theta\sin\theta + \sin^2\theta) = -(\sin^2\theta + \cos^2\theta),$$

i.e.

$$r^2 = -\frac{1 + \tan^2\theta}{1 - 3\tan\theta + \tan^2\theta}.$$

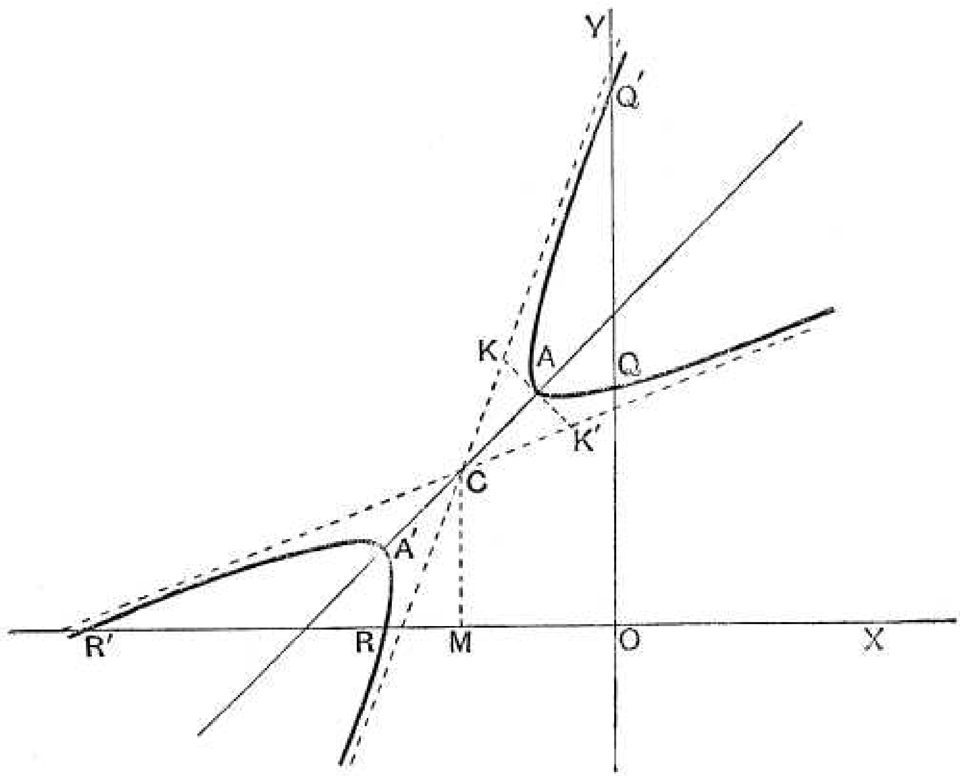
When $\theta_1 = 45°$, $r_1^2 = -\frac{2}{1-3+1} = 2$, so that $r_1 = \sqrt{2}$.

When $\theta_2 = 135°$, $r_2^2 = -\frac{2}{1+3+1} = \frac{-2}{5}$, so that $r_2 = \sqrt{\frac{-2}{5}}$.

To construct the curve take the point C whose coordinates are -2 and 2. Through C draw a straight line ACA' inclined at 45° to the axis of x and mark off $A'C = CA = \sqrt{2}$.

Also through A draw a straight line KAK' perpendicular to CA and take $AK = K'A = \sqrt{\frac{2}{5}}$. By Art. 315, CK and CK' are then the asymptotes.

The curve is therefore a hyperbola whose centre is C, whose transverse axis is $A'A$, and whose asymptotes are CK and CK'.



On putting $x = 0$ it will be found that the curve meets the axis of y where $y = 3$ or 7, and, on putting $y = 0$, that it meets the axis of x where $x = -3$ or -7.

Hence $OQ = 3$, $OQ' = 7$, $OR = 3$, and $OR' = 7$.

366. *To find the eccentricity of the central conic section*

$$ax^2 + 2hxy + by^2 = 1 \quad\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

First, let $h^2 - ab$ be negative, so that the curve is an ellipse, and let the equation to the ellipse, referred to its axes, be

$$\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}=1.$$

By the theory of Invariants (Art. 135) we have

$$\frac{1}{\alpha^2}+\frac{1}{\beta^2}=a+b\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

and

$$\frac{1}{\alpha^2\beta^2}=ab-h^2\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Also, if e be the eccentricity, we have, if α be $>\beta$,

$$e^2=\frac{\alpha^2-\beta^2}{\alpha^2}.$$

$$\therefore\ \frac{e^2}{2-e^2}=\frac{\alpha^2-\beta^2}{\alpha^2+\beta^2}.$$

But, from (2) and (3), we have

$$\alpha^2+\beta^2=\frac{a+b}{ab-h^2}\ \text{ and }\ \alpha^2\beta^2=\frac{1}{ab-h^2}.$$

Hence

$$\alpha^2-\beta^2=+\sqrt{(\alpha^2+\beta^2)^2-4\alpha^2\beta^2}=+\frac{\sqrt{(a-b)^2+4h^2}}{ab-h^2}.$$

$$\therefore\ \frac{e^2}{2-e^2}=+\frac{\sqrt{(a-b)^2+4h^2}}{a+b}\ldots\ldots\ldots\ldots(4).$$

This equation at once gives e^2.

Secondly, let h^2-ab be positive, so that the curve is a hyperbola, and let the equation referred to its principal axes be

$$\frac{x^2}{\alpha^2}-\frac{y^2}{\beta^2}=1,$$

so that in this case

$$\frac{1}{\alpha^2}-\frac{1}{\beta^2}=a+b,\ \text{and}\ -\frac{1}{\alpha^2\beta^2}=ab-h^2=-(h^2-ab).$$

Hence $\alpha^2-\beta^2=-\dfrac{a+b}{h^2-ab}$ and $\alpha^2\beta^2=\dfrac{1}{h^2-ab}$,

so that $\alpha^2+\beta^2=+\sqrt{(\alpha^2-\beta^2)^2+4\alpha^2\beta^2}=+\dfrac{\sqrt{(a-b)^2+4h^2}}{h^2-ab}$.

In this case, if e be the eccentricity, we have

$$e^2 = \frac{\alpha^2 + \beta^2}{\alpha^2},$$

i.e.
$$\frac{e^2}{2-e^2} = \frac{\alpha^2+\beta^2}{\alpha^2-\beta^2} = -\frac{\sqrt{(a-b)^2+4h^2}}{a+b} \quad \text{.........(5).}$$

This equation gives e^2.

In each case we see that e is a root of the equation

$$\left(\frac{e^2}{2-e^2}\right)^2 = \frac{(a-b)^2+4h^2}{(a+b)^2},$$

i.e. of the equation

$$e^4(ab-h^2) + \{(a-b)^2+4h^2\}(e^2-1) = 0.$$

367. *To obtain the foci of the central conic*

$$ax^2 + 2hxy + by^2 = 1.$$

Let the direction of the axes of the conic be obtained as in Art. 364, and let θ_1 be the inclination of the major axis in the case of the ellipse, and the transverse axis in the case of the hyperbola, to the axis of x.

Let r_1^2 be the square of the radius corresponding to θ_1, and let r_2^2 be the square of the radius corresponding to the perpendicular direction. [In the case of the hyperbola r_2^2 will be a negative quantity.]

The distance of the focus from the centre is $\sqrt{r_1^2 - r_2^2}$ (Arts. 247 and 295). One focus will therefore be the point

$$(\sqrt{r_1^2 - r_2^2}\cos\theta_1,\ \sqrt{r_1^2 - r_2^2}\sin\theta_1),$$

and the other will be

$$(-\sqrt{r_1^2 - r_2^2}\cos\theta_1,\ -\sqrt{r_1^2 - r_2^2}\sin\theta_1).$$

Ex. *Find the foci of the ellipse traced in Art.* 365.

Here $\tan\theta_1 = 2$, so that $\sin\theta_1 = \frac{2}{\sqrt{5}}$ and $\cos\theta_1 = \frac{1}{\sqrt{5}}$.

Also $r_1^2 = 6$, and $r_2^2 = 4$, so that $\sqrt{r_1^2 - r_2^2} = \sqrt{2}$.

The coordinates of the foci referred to axes through C are therefore

$$\left(\frac{\sqrt{2}}{\sqrt{5}},\ \frac{2\sqrt{2}}{\sqrt{5}}\right) \text{ and } \left(-\frac{\sqrt{2}}{\sqrt{5}},\ -\frac{2\sqrt{2}}{\sqrt{5}}\right).$$

Their coordinates referred to the original axes OX and OY are

$$\left(\bar{x} \pm \frac{\sqrt{2}}{\sqrt{5}},\ \bar{y} \pm \frac{2\sqrt{2}}{\sqrt{5}}\right),\ i.e. \left(2 \pm \frac{\sqrt{2}}{\sqrt{5}},\ 3 \pm \frac{2\sqrt{2}}{\sqrt{5}}\right).$$

368. The method of obtaining the coordinates of the focus of a parabola given by the general equation may be exemplified by taking the example of Art. 363.

Here it was shewn that the latus rectum is equal to 3, so that, if S be the focus, AS is $\frac{3}{4}$.

It was also shewn that the coordinates of A referred to OX and OY are $-\frac{2}{5}$ and $\frac{7}{5}$.

The coordinates of S referred to the same axes are

$$-\tfrac{2}{5}+\tfrac{3}{4} \text{ and } \tfrac{7}{5},\ i.e.\ \tfrac{7}{20} \text{ and } \tfrac{7}{5}.$$

Its coordinates referred to the original axes are therefore

$$\tfrac{7}{20}\cos\theta - \tfrac{7}{5}\sin\theta \text{ and } \tfrac{7}{20}\sin\theta + \tfrac{7}{5}\cos\theta,$$

i.e. $$\tfrac{7}{20}\cdot\tfrac{4}{5} - \tfrac{7}{5}\cdot\tfrac{3}{5} \text{ and } \tfrac{7}{20}\cdot\tfrac{3}{5} + \tfrac{7}{5}\cdot\tfrac{4}{5},$$

i.e. $$-\tfrac{14}{25} \text{ and } \tfrac{133}{100}.$$

In Art. 393 equations will be found to give the foci of any conic section directly, so that the conic need not first be traced.

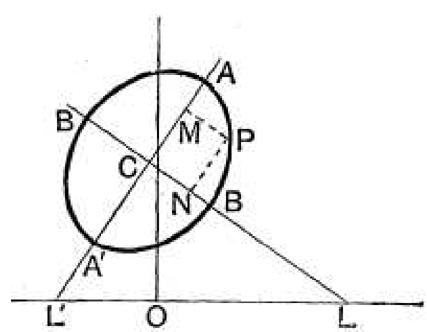
369. Ex. 1. *Trace the curve*

$$3(3x-2y+4)^2+2(2x+3y-5)^2=39 \quad \ldots\ldots\ldots\ldots(1).$$

The equation may be written

$$3\left(\frac{3x-2y+4}{\sqrt{13}}\right)^2+2\left(\frac{2x+3y-5}{\sqrt{13}}\right)^2=3 \quad \ldots\ldots\ldots\ldots(2).$$

Now the straight lines $3x-2y+4=0$ and $2x+3y-5=0$ are at right angles. Let them be CM and CN, intersecting in C which is the point $(-\frac{2}{13}, \frac{23}{13})$.



If P be any point on the curve and PM and PN the perpendiculars upon these lines, the lengths of PM and PN are

$$\frac{3x-2y+4}{\sqrt{13}} \text{ and } \frac{2x+3y-5}{\sqrt{13}}.$$

Hence equation (2) states that

$$3PM^2+2PN^2=3,$$

i.e. $$\frac{PM^2}{1}+\frac{PN^2}{\frac{3}{2}}=1.$$

The locus of P is therefore an ellipse whose semi-axes measured along CM and CN are $\sqrt{\frac{3}{2}}$ and 1 respectively.

Ex. 2. *What is represented by the equation*

$$(x^2-a^2)^2+(y^2-a^2)^2=a^4\,?$$

The equation may be written in the form

$$x^4+y^4-2a^2(x^2+y^2)+a^4=0,$$

i.e. $$(x^2+y^2)^2-2a^2(x^2+y^2)+a^4=2x^2y^2,$$

i.e. $$(x^2+y^2-a^2)^2-(\sqrt{2}xy)^2=0,$$

i.e. $$(x^2+\sqrt{2}xy+y^2-a^2)(x^2-\sqrt{2}xy+y^2-a^2)=0.$$

The locus therefore consists of the two ellipses

$$x^2+\sqrt{2}xy+y^2-a^2=0, \quad \text{and} \quad x^2-\sqrt{2}xy+y^2-a^2=0.$$

These ellipses are equal and their semi-axes would be found to be

$$a\sqrt{2+\sqrt{2}} \quad \text{and} \quad a\sqrt{2-\sqrt{2}}.$$

The major axis of the first is inclined at an angle of 135° to the axis of x, and that of the second at an angle of 45°.

EXAMPLES XLI

Trace the parabolas

1. $(x-4y)^2=51y$.
2. $(x-y)^2=x+y+1$.
3. $(5x-12y)^2=2ax+29ay+a^2$.
4. $(4x+3y+15)^2=5(3x-4y)$.
5. $16x^2+24xy+9y^2-5x-10y+1=0$.
6. $9x^2+24xy+16y^2-4y-x+7=0$.
7. $144x^2-120xy+25y^2+619x-272y+663=0$, and find its focus.
8. $16x^2-24xy+9y^2+32x+86y-39=0$.
9. $4x^2-4xy+y^2-12x+6y+9=0$.

Find the position and magnitude of the axes of the conics

10. $12x^2-12xy+7y^2=48$.
11. $3x^2+2xy+3y^2=8$.
12. $x^2-xy-6y^2=6$.

Trace the following central conics.

13. $x^2-2xy\cos 2a+y^2=2a^2$.
14. $x^2-2xy\operatorname{cosec} 2a+y^2=a^2$.
15. $xy=a(x+y)$.
16. $xy-y^2=a^2$.
17. $y^2-2xy+2x^2+2x-2y=0$.
18. $x^2+xy+y^2+x+y=1$.

19. $2x^2+3xy-2y^2-7x+y-2=0$.

20. $40x^2+36xy+25y^2-196x-122y+205=0$.

21. $9x^2-32xy+9y^2+60x+10y=64\frac{1}{2}$.

22. $x^2-xy+2y^2-2ax-6ay+7a^2=0$.

23. $10x^2-48xy-10y^2+38x+44y-5\frac{1}{2}=0$.

24. $4x^2+27xy+35y^2-14x-31y-6=0$.

25. $(3x-4y+a)(4x+3y+a)=a^2$.

26. $3(2x-3y+4)^2+2(3x+2y-5)^2=78$.

27. $2(3x-4y+5)^2-3(4x+3y-10)^2=150$.

Find the products of the semi-axes of the conics

28. $y^2-4xy+5x^2=2$. **29.** $4(3x+4y-7)^2+3(4x-3y+9)^2=3$.

30. $11x^2+16xy-y^2-70x-40y+82=0$.

Find the foci and the eccentricity of the conics

31. $x^2-3xy+4ax=2a^2$. **32.** $4xy-3x^2-2ay=0$.

33. $5x^2+6xy+5y^2+12x+4y+6=0$.

34. $x^2+4xy+y^2-2x+2y-6=0$.

35. Shew that the latus rectum of the parabola

$$(a^2+b^2)(x^2+y^2)=(bx+ay-ab)^2$$

is

$$2ab\div\sqrt{a^2+b^2}.$$

36. Prove that the lengths of the semi-axes of the conic

$$ax^2+2hxy+ay^2=d$$

are

$$\sqrt{\frac{d}{a+h}} \text{ and } \sqrt{\frac{d}{a-h}}$$

respectively, and that their equation is $x^2-y^2=0$.

37. Prove that the squares of the semi-axes of the conic

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

are

$$2\Delta\div\{(ab-h^2)(a+b\pm\sqrt{(a-b)^2+4h^2})\},$$

where Δ is the discriminant.

38. If λ be a variable parameter, prove that the locus of the vertices of the hyperbolas given by the equation $x^2-y^2+\lambda xy=a^2$ is the curve $(x^2+y^2)^2=a^2(x^2-y^2)$.

39. If the point $(at_1^2, 2at_1)$ on the parabola $y^2=4ax$ be called the point t_1, prove that the axis of the second parabola through the four points t_1, t_2, t_3, and t_4 makes with the axis of the first an angle

$$\cot^{-1}\left(\frac{t_1+t_2+t_3+t_4}{4}\right).$$

Prove also that if two parabolas meet in four points the distances of the centroid of the four points from the axes are proportional to the latera recta.

40. If the product of the semi-axes of the conic $x^2+2xy+17y^2=8$ be unity, shew that the axes of coordinates are inclined at an angle $\sin^{-1}\frac{1}{2}$.

41. Sketch the curve $6x^2-7xy-5y^2-4x+11y=2$, the axes being inclined at an angle of 30°.

42. Prove that the eccentricity of the conic given by the general equation satisfies the relation

$$\frac{e^4}{1-e^2}+4=\frac{(a+b-2h\cos\omega)^2}{(ab-h^2)\sin^2\omega},$$

where ω is the angle between the axes.

43. The axes being changed in any way, without any change of origin, prove that in the general equation of the second degree the quantities c, $\dfrac{f^2+g^2-2fg\cos\omega}{\sin^2\omega}$, $\dfrac{af^2+bg^2-2fgh}{\sin^2\omega}$, and $\dfrac{\Delta}{\sin^2\omega}$ are invariants, in addition to the quantities in Art. 137.

[On making the most general substitutions of Art. 132 it is clear that c is unaltered; proceed as in Art. 137, but introduce the condition that the resulting expressions are equal to the product of two linear quantities (Art. 116); the results will then follow.]

ANSWERS

7. $\left(\dfrac{-1503}{676}, \dfrac{-23}{169}\right)$. **9.** Two coincident straight lines.

10. $\tan\theta_1=-\frac{2}{3}$, $\tan\theta_2=\frac{3}{2}$, $r_1=\sqrt{3}$, and $r_2=4$.

11. $\theta_1=45°$, $\theta_2=135°$, $r_1=\sqrt{2}$, and $r_2=2$.

12. $\tan\theta_1=7+5\sqrt{2}$; $\tan\theta_2=7-5\sqrt{2}$,

$$r_1=\sqrt{\frac{-6}{5}(2\sqrt{2}-2)},\quad r_2=\sqrt{\frac{6}{5}(2\sqrt{2}+2)}.$$

28. 2. **29.** $\frac{1}{60}\sqrt{3}$. **30.** $\frac{4}{3}\sqrt{-3}$.

31. $\left(\mp\dfrac{2a}{3}\sqrt{\sqrt{10}+1}, \dfrac{4a}{3}\pm\dfrac{2a}{3}\sqrt{\sqrt{10}-1}\right)$; $\frac{1}{3}\sqrt{20+2\sqrt{10}}$.

32. $\left(\dfrac{a}{2}\pm\dfrac{a}{4}\sqrt{3}, \dfrac{3a}{4}\pm\dfrac{a}{2}\sqrt{3}\right)$; $\frac{1}{2}\sqrt{5}$.

33. $(-\frac{3}{2}\mp\frac{1}{4}\sqrt{6}, \frac{1}{2}\pm\frac{1}{4}\sqrt{6})$; $\frac{1}{2}\sqrt{3}$.

34. $(-1\pm\frac{2}{3}\sqrt{6}, 1\pm\frac{2}{3}\sqrt{6})$; 2.

SOLUTIONS/HINTS

1. Take $x-4y=0$ as a new axis of x, *i.e.* turn the axes through an angle θ, where $\tan\theta=\frac{1}{4}$, so that

$$\sin\theta = \frac{1}{\sqrt{17}} \text{ and } \cos\theta = \frac{4}{\sqrt{17}}.$$

For x we must substitute $\dfrac{4X - Y}{\sqrt{17}}$,

and for y " " $\dfrac{X + 4Y}{\sqrt{17}}$.

$$\therefore\ x - 4y = -\sqrt{17}\,.\,Y.$$

The equation then becomes

$$\left\{Y - \frac{6}{\sqrt{17}}\right\}^2 = \frac{3}{\sqrt{17}}\left(X + \frac{12}{\sqrt{17}}\right).$$

Hence etc., as in Art. 363.

2. Take $x - y = 0$ as a new axis of x, *i.e.* turn the axes through an angle of $45°$.

For x and y we must put $\dfrac{X - Y}{\sqrt{2}}$ and $\dfrac{X + Y}{\sqrt{2}}$,

$$\therefore\ x - y = -\sqrt{2}Y.$$

The equation then becomes $Y^2 = \dfrac{1}{\sqrt{2}}\left(X + \dfrac{1}{\sqrt{2}}\right)$.

Hence etc., as in Art. 363.

3. Take $5x - 12y = 0$ as a new axis of x, *i.e.* turn the axes through an angle θ, where $\tan\theta = \frac{5}{12}$, so that $\sin\theta = \frac{5}{13}$, and $\cos\theta = \frac{12}{13}$.

For x we must substitute $\dfrac{12X - 5Y}{13}$,

and for y " " $\dfrac{5X + 12Y}{13}$.

$$\therefore\ 5x - 12y = -13Y.$$

The equation then becomes $\left(Y - \dfrac{a}{13}\right)^2 = \dfrac{a}{13}\left(X + \dfrac{2a}{13}\right)$.

Hence etc., as in Art. 363.

4. The equation may be written

$$\left\{\frac{4x+3y+15}{5}\right\}^2 = \frac{3x-4y}{5}.$$

Since the lines $4x+3y+15=0$ and $3x-4y=0$ are at right angles, the equation represents a parabola having these lines for axis and tangent at the vertex respectively, and whose latus rectum is 1.

5. Take $4x+3y=0$ as a new axis of x, *i.e.* turn the axes through an angle θ, where $\tan\theta = -\frac{4}{3}$, so that $\sin\theta = \frac{4}{5}$, and $\cos\theta = -\frac{3}{5}$.

For x we must substitute $\dfrac{-3X-4Y}{5}$,

and for y ,, ,, $\dfrac{4X-3Y}{5}$.

$$\therefore\ 4x+3y=-5Y.$$

The equation then becomes $(Y+\frac{1}{5})^2 = \dfrac{X}{5}$.

Hence etc., as in Art. 363.

6. Take $3x+4y=0$ as a new axis of x, *i.e.* turn the axes through an angle θ, where $\tan\theta = -\frac{3}{4}$, so that $\sin\theta = \frac{3}{5}$, and $\cos\theta = -\frac{4}{5}$.

For x we must substitute $\dfrac{-4X-3Y}{5}$,

and for y ,, ,, $\dfrac{3X-4Y}{5}$.

$$\therefore\ 3x+4y=-5Y.$$

The equation then becomes $Y^2 + \dfrac{19Y}{125} = \dfrac{8X}{125} - \dfrac{7}{25}$.

Hence etc., as in Art. 363.

7. The equation may be written

$$(12x-5y+\lambda)^2 = x(24\lambda-619) + y(272-10\lambda) + \lambda^2 - 663,$$

and the lines $12x-5y+\lambda=0$,

and $\quad x(24\lambda - 619) + y(272 - 10\lambda) + \lambda^2 - 663 = 0,$

will be at right angles if

$$12(24\lambda - 619) - 5(272 - 10\lambda) = 0,$$

whence $\quad \lambda = 26,$

and the equation becomes

$$\left(\frac{12x - 5y + 26}{13}\right)^2 = \frac{1}{13}\left(\frac{5x + 12y + 13}{13}\right).$$

Hence etc., as in Art. 363.

If (h, k) be the coordinates of the focus the perpendicular from (h, k) on the tangent at the vertex $= \frac{1}{52}$.

$$\therefore \frac{5h + 12k + 13}{13} = \frac{1}{52} \ldots\ldots\ldots\ldots\ldots (i)$$

(It will be clear from a figure that the focus and the origin are on the same side of the line $5x + 12y + 13 = 0$.)

Also, since (h, k) lies on the axis,

$$\therefore 12h - 5k + 26 = 0. \quad \ldots\ldots\ldots\ldots\ldots (ii)$$

Solving (i) and (ii), we obtain

$$h = -\tfrac{1503}{676}, \ k = -\tfrac{23}{169}.$$

8. Take $4x - 3y = 0$ as a new axis of x, *i.e.* turn the axes through an angle θ, where $\tan\theta = \frac{4}{3}$, so that $\sin\theta = \frac{4}{5}$, and $\cos\theta = \frac{3}{5}$.

For x we must substitute $\dfrac{3X - 4Y}{5}$,

and for y " " $\dfrac{4X + 3Y}{5}$.

$$\therefore 4x - 3y = -5Y,$$

and the equation then becomes $(Y + \frac{13}{25})^2 = -\frac{88}{25}(X - \frac{13}{25})$.

Hence etc., as in Art. 363.

9. The equation is $(2x - y - 3)^2 = 0$.

10. As in Art. 365, $\tan 2\theta = \dfrac{2h}{a - b} = \dfrac{-12}{12 - 7} = -\dfrac{12}{5}$.

$$\therefore \frac{2\tan\theta}{1 - \tan^2\theta} = -\frac{12}{5}, \text{ or } 6\tan^2\theta - 5\tan\theta - 6 = 0.$$

$$\therefore\ \tan\theta_1 = -\tfrac{2}{3},\ \text{and}\ \tan\theta_2 = \tfrac{3}{2}.$$

Also $$r^2 = \frac{48(1+\tan^2\theta)}{12-12\tan\theta+7\tan^2\theta} = 3 \text{ or } 16.$$

11. As in Art. 365, $\tan 2\theta = \dfrac{2h}{a-b} = \dfrac{2}{3-3} = \infty$,

$$\therefore\ \theta_1 = 45^\circ,\ \text{and}\ \theta_2 = 135^\circ.$$

Also $$r^2 = \frac{8(1+\tan^2\theta)}{3+2\tan\theta+3\tan^2\theta} = 2 \text{ or } 4.$$

12. As in Art. 365, $\tan 2\theta = \dfrac{2h}{a-b} = \dfrac{-1}{7}$.

$$\therefore\ \frac{2\tan\theta}{1-\tan^2\theta} = \frac{-1}{7};\ \therefore\ \tan^2\theta - 14\tan\theta - 1 = 0;$$

$$\therefore\ \tan\theta = 7 \pm 5\sqrt{2}\ \text{and}\ r^2 = \frac{6(1+\tan^2\theta)}{1-\tan\theta-6\tan^2\theta} = \text{etc.}$$

13. Since $\cos^2 2a - 1$ is negative, the curve is an ellipse. The direction of the axes is given by

$$\tan 2\theta = \frac{2h}{a-b} = \frac{-2\cos 2a}{1-1} = \infty.$$

$$\therefore\ \theta_1 = 45^\circ,\ \theta_2 = 135^\circ,$$

and $$r^2 = \frac{2a^2(1+\tan^2\theta)}{1-2\cos 2a\,.\tan\theta+\tan^2\theta} = a^2\operatorname{cosec}^2 a \text{ or } a^2\sec^2 a.$$

14. Since $\operatorname{cosec}^2 2a - 1$ is positive, the curve is a hyperbola. The direction of the axes is given by

$$\tan 2\theta = \frac{2h}{a-b} = \frac{-2\operatorname{cosec} 2a}{1-1} = \infty.$$

$$\therefore\ \theta_1 = 45^\circ,\ \theta_2 = 135^\circ,$$

and $$r^2 = \frac{a^2(1+\tan^2\theta)}{1-2\operatorname{cosec} 2a\tan\theta+\tan^2\theta} = \frac{a^2}{1\mp\operatorname{cosec} 2a}.$$

15. The equation is equivalent to $(x-a)(y-a) = a^2$, which is a rectangular hyperbola, whose asymptotes are parallel to the axes, and whose centre is the point (a, a).

16. The curve is a hyperbola, the direction of whose axes is given by $\tan 2\theta = 1$.

$$\therefore\ \theta_1 = 22\tfrac{1}{2}^\circ \text{ and } \theta_2 = 112\tfrac{1}{2}^\circ.$$

$$\therefore\ \tan\theta_1 = \sqrt{2} - 1,\ \tan\theta_2 = -\sqrt{2} - 1 \text{ and}$$

$$r^2 = \frac{a^2(1+\tan^2\theta)}{\tan\theta - \tan^2\theta} = \text{etc.}$$

17. Since $1^2 - 2$ is negative the curve is an ellipse.

The centre is given by

$$2\bar{x} - \bar{y} + 1 = 0, \text{ and } -\bar{x} + \bar{y} - 1 = 0,$$

whence $\bar{x} = 0,\ \bar{y} = 1$ and $\therefore\ c' = -1$.

Therefore the equation referred to parallel axes through the centre is $y^2 - 2xy + 2x^2 = 1$.

The direction of the axes is given by

$$\tan 2\theta = \frac{-2}{2-1} = -2.$$

$$\therefore\ \tan^2\theta - \tan\theta - 1 = 0,$$

$$\therefore\ \tan\theta_1 = \frac{1+\sqrt{5}}{2}, \text{ and } \tan\theta_2 = \frac{1-\sqrt{5}}{2}.$$

$$\therefore\ r^2 = \frac{1+\tan^2\theta}{2 - 2\tan\theta + \tan^2\theta} = \frac{6 \pm 2\sqrt{5}}{4}, \text{ etc.}$$

18. Since $(\frac{1}{2})^2 - 1$ is negative, the curve is an ellipse. The centre is given by

$$2\bar{x} + \bar{y} + 1 = 0, \text{ and } \bar{x} + 2\bar{y} + 1 = 0.$$

$$\therefore\ \bar{x} = -\tfrac{1}{3}, \text{ and } \bar{y} = -\tfrac{1}{3} \text{ and } c' = -\tfrac{1}{6} - \tfrac{1}{6} - 1 = -\tfrac{4}{3}.$$

Therefore the equation referred to parallel axes through the centre is $x^2 + xy + y^2 = \frac{4}{3}$.

Hence $\tan 2\theta = \dfrac{2h}{a-b} = \infty$; $\therefore\ \theta_1 = 45^\circ$, and $\theta_2 = 135^\circ$.

$$r^2 = \frac{\frac{4}{3}(1+\tan^2\theta)}{1+\tan\theta+\tan^2\theta};\ \therefore\ r_1 = \frac{2\sqrt{2}}{3},\ r_2 = \frac{2\sqrt{6}}{3}.$$

19. Since $a+b=0$, the curve is a rectangular hyperbola. The centre is given by

$$4\bar{x}+3\bar{y}-7=0, \text{ and } 3\bar{x}-4\bar{y}+1=0.$$

$$\therefore \bar{x}=1, \text{ and } \bar{y}=1 \text{ and } c'=-\tfrac{7}{2}+\tfrac{1}{2}-2=-5.$$

Therefore the equation referred to parallel axes through the centre is $2x^2+3xy-2y^2=5$.

$$\therefore \tan 2\theta=\frac{2h}{a-b}=\tfrac{3}{4}, \text{ whence } \tan\theta=\tfrac{1}{3} \text{ or } -3.$$

$$r^2=\frac{5(1+\tan^2\theta)}{2+3\tan\theta-2\tan^2\theta}=2 \text{ or } -2; \therefore r=\sqrt{2} \text{ or } \sqrt{-2}.$$

20. Since $18^2-40.25$ is negative, the curve is an ellipse. The centre is given by

$$20\bar{x}+9\bar{y}-49=0, \text{ and } 18\bar{x}+25\bar{y}-61=0.$$

$$\therefore \bar{x}=2, \text{ and } \bar{y}=1 \text{ and } c'=-98.2-61.1+205=-52.$$

Therefore the equation referred to parallel axes through the centre is $40x^2+36xy+25y^2=52$.

$$\therefore \tan 2\theta=\frac{2h}{a-b}=\frac{36}{15}=\frac{12}{5}.$$

$$\therefore 6\tan^2\theta+5\tan\theta-6=0.$$

$$\therefore \tan\theta_1=\tfrac{2}{3}, \tan\theta_2=-\tfrac{3}{2},$$

and $$r^2=\frac{52(1+\tan^2\theta)}{40+36\tan\theta+25\tan^2\theta}=1 \text{ or } 4.$$

21. Since $16^2-9.9$ is positive, the curve is a hyperbola. The centre is given by

$$9\bar{x}-16\bar{y}+30=0, \text{ and } -16\bar{x}+9\bar{y}+5=0.$$

$$\therefore \bar{x}=2, \text{ and } \bar{y}=3 \text{ and } c'=2.30+3.5-64\tfrac{1}{2}=10\tfrac{1}{2}.$$

Therefore the equation referred to parallel axes through the centre is $9x^2-32xy+9y^2=-10\frac{1}{2}$.

$$\tan 2\theta=\frac{2h}{a-b}=\infty; \therefore \theta_1=45^\circ, \theta_2=135^\circ,$$

and $$r^2=\frac{-10\frac{1}{2}(1+\tan^2\theta)}{9-32\tan\theta+9\tan^2\theta}=\frac{3}{2} \text{ or } -\frac{21}{50}.$$

22. Since $1^2 - 8$ is negative, the curve is an ellipse. The centre is given by

$$2\bar{x} - \bar{y} - 2a = 0, \text{ and } -\bar{x} + 4\bar{y} - 6a = 0.$$

$$\therefore \bar{x} = 2a, \ \bar{y} = 2a \text{ and } c' = -a \,.\, 2a - 3a \,.\, 2a + 7a^2 = -a^2.$$

Therefore the equation referred to parallel axes through the centre is $x^2 - xy + 2y^2 = a^2$.

$$\therefore \tan 2\theta = \frac{2h}{a-b} = \frac{-1}{1-2} = 1 ; \ \therefore \theta_1 = 22\tfrac{1}{2}^\circ, \ \theta_2 = 112\tfrac{1}{2}^\circ,$$

and $$r^2 = \frac{a^2(1+\tan^2\theta)}{1 - \tan\theta + 2\tan^2\theta}, \text{ etc.}$$

23. Since $a + b = 0$, the curve is a rectangular hyperbola. The centre is given by

$$10\bar{x} - 24\bar{y} + 19 = 0, \text{ and } -24\bar{x} - 10\bar{y} + 22 = 0.$$

$$\therefore \bar{x} = \tfrac{1}{2}, \ \bar{y} = 1, \text{ and } c' = \tfrac{1}{2} \,.\, 19 + 22 - 5\tfrac{1}{2} = 26.$$

Therefore the equation referred to parallel axes through the centre is $5x^2 - 24xy - 5y^2 = -13$.

$$\therefore \tan 2\theta = \frac{2h}{a-b} = -\frac{12}{5},$$

whence $$\tan\theta_1 = \tfrac{3}{2}, \text{ and } \tan\theta_2 = -\tfrac{2}{3}.$$

$$\therefore r^2 = \frac{-13(1+\tan^2\theta)}{5 - 24\tan\theta - 5\tan^2\theta} = 1 \text{ or } -1.$$

24. Since $(\tfrac{27}{2})^2 - 4 \,.\, 35$ is positive, the curve is a hyperbola. The centre is given by

$$8\bar{x} + 27\bar{y} - 14 = 0, \text{ and } 27\bar{x} + 70\bar{y} - 31 = 0.$$

Whence $$\bar{x} = -\tfrac{11}{13}, \ \bar{y} = \tfrac{10}{13},$$

and $$c' = 7 \,.\, \tfrac{11}{13} - \tfrac{31}{2} \,.\, \tfrac{10}{13} - 6 = -12.$$

Therefore the equation referred to parallel axes through the centre is $4x^2 + 27xy + 35y^2 = 12$.

$$\therefore \tan 2\theta = \frac{2h}{a-b} = \frac{27}{4-35} = \frac{27}{-31}.$$

$$\therefore 27\tan^2\theta - 62\tan\theta - 27 = 0.$$

$$\therefore \tan\theta = \frac{31 \pm 13\sqrt{10}}{27}, \text{ etc.}$$

25. The equation may be written

$$\left(\frac{3x-4y+a}{5}\right)\left(\frac{4x+3y+a}{5}\right)=\frac{a^2}{25},$$

or referred to $3x-4y+a=0$, and $4x+3y+a=0$, as axes (which are at right angles)

$$X\,.\,Y=\frac{a^2}{25}.$$

The equation therefore represents a rectangular hyperbola, having these lines for asymptotes and semi-axis

$$=\frac{\sqrt{2}\,.\,a}{5}.$$

26. The equation may be written

$$\frac{1}{2}\cdot\left(\frac{2x-3y+4}{\sqrt{13}}\right)^2+\frac{1}{3}\cdot\left(\frac{3x+2y-5}{\sqrt{13}}\right)^2=1.$$

Hence etc., as in Art. 369.

The curve is an ellipse whose centre is $(\frac{7}{13},\ \frac{22}{13})$, and semi-axes $\sqrt{3}$ and $\sqrt{2}$.

27. The equation may be written

$$\frac{1}{3}\left(\frac{3x-4y+5}{5}\right)^2-\frac{1}{2}\left(\frac{4x+3y-10}{5}\right)^2=1.$$

Hence etc., as in Art. 369.

The curve is a hyperbola whose centre is (1, 2), and semi-axes $\sqrt{3}$ and $\sqrt{2}$.

28. See Art. 366. $\alpha\beta=\dfrac{1}{\sqrt{ab-h^2}}=\dfrac{1}{\sqrt{\frac{1}{2}\cdot\frac{5}{2}-1^2}}=2.$

29. The equation may be written

$$\frac{100}{3}\cdot\left(\frac{3x+4y-7}{5}\right)^2+25\,.\left(\frac{4x-3y+9}{5}\right)^2=1.$$

This is an ellipse whose axes are the straight lines

$$3x+4y-7=0 \text{ and } 4x-3y+9=0,$$

and the lengths of whose corresponding semi-axes are $\sqrt{\frac{3}{100}}$ and $\sqrt{\frac{1}{25}}$ (cf. Art. 369, Ex. 1).

$$\therefore \alpha\beta = \sqrt{\frac{3}{100}\cdot\frac{1}{25}} = \frac{1}{50}\cdot\sqrt{3}.$$

30. The centre is given by

$$11\bar{x} + 8\bar{y} - 35 = 0, \text{ and } 8\bar{x} - \bar{y} - 20 = 0.$$

Whence $\bar{x} = \frac{13}{5},\ \bar{y} = \frac{4}{5}$,

and $c' = -35 \cdot \frac{13}{5} - 20 \cdot \frac{4}{5} + 82 = -25.$

Therefore the equation referred to parallel axes through the centre is $11x^2 + 16xy - y^2 = 25$.

$$\alpha\beta = \frac{1}{\sqrt{ab - h^2}} = \frac{1}{\sqrt{-\frac{11}{25^2} - \frac{8^2}{25^2}}} = \frac{25}{\sqrt{-75}} = -\frac{5}{3}\sqrt{-3}.$$

31. The equations for the centre are

$$2\bar{x} - 3\bar{y} + 4a = 0, \text{ and } \bar{x} = 0.$$

$$\therefore \bar{y} = \frac{4a}{3},\ \bar{x} = 0, \text{ and } c' = -2a^2.$$

Therefore the equation referred to parallel axes through the centre is $x^2 - 3xy = 2a^2$.

$$\therefore \tan 2\theta = -3;\ \therefore \tan\theta_1 = \frac{1 - \sqrt{10}}{3}, \text{ and } \tan\theta_2 = \frac{1 + \sqrt{10}}{3}.$$

$$\therefore \sin\theta_1 = -\sqrt{\frac{\sqrt{10} - 1}{2\sqrt{10}}}, \text{ and } \cos\theta_1 = -\sqrt{\frac{\sqrt{10} + 1}{2\sqrt{10}}}.$$

Also $$r^2 = \frac{2(1 + \tan^2\theta)}{1 - 3\tan\theta} = \frac{2\left(1 + \frac{11 \pm 2\sqrt{10}}{9}\right)}{\mp\sqrt{10}}$$

$$= \frac{2}{9}\cdot\frac{20 \pm 2\sqrt{10}}{\mp\sqrt{10}} = \frac{2}{9}(\mp 2\sqrt{10} - 2).$$

$$\therefore r_1^2 - r_2^2 = \frac{8}{9}\sqrt{10},\ \therefore \sqrt{r_1^2 - r_2^2} = \frac{2\sqrt{2}}{3}\sqrt{\sqrt{10}}, \text{ etc.}$$

See Art. 367.

The equation for the eccentricity is [Art. 366, equation 5]

$$\frac{e^2}{2-e^2} = -\sqrt{1^2+3^2} = -\sqrt{10},$$

so that $$e = \tfrac{1}{3}\sqrt{20+2\sqrt{10}}.$$

32. The equations for the centre are

$$3\bar{x} - 2\bar{y} = 0, \quad -2\bar{x} + a = 0.$$

$$\therefore\ \bar{x} = \frac{a}{2},\quad \bar{y} = \frac{3a}{4}, \text{ and } c' = -\frac{3a^2}{4}.$$

Therefore the equation referred to parallel axes through the centre is

$$4xy - 3x^2 = \frac{3a^2}{4},\ \ i.e.\ \ 3x^2 - 4xy = -\frac{3a^2}{4}.$$

$$\therefore\ \tan 2\theta = \frac{2h}{a-b} = -\frac{4}{3}.$$

$$2\tan^2\theta - 3\tan\theta - 2 = 0.$$

$$\therefore\ \tan\theta_1 = 2,\quad \tan\theta_2 = -\tfrac{1}{2}.$$

$$r^2 = \frac{-\frac{3}{4}a^2(1+\tan^2\theta)}{3-4\tan\theta} = \tfrac{3}{4}a^2 \text{ or } -\tfrac{3}{16}a^2.$$

$$\therefore\ r_1^2 - r_2^2 = \tfrac{15}{16}a^2, \text{ and } \cos\theta_1 = \frac{1}{\sqrt{5}},\ \sin\theta_1 = \frac{2}{\sqrt{5}}.$$

Therefore the coordinates of the foci are

$$\left(\frac{a}{2} \pm \frac{\sqrt{3}a}{4},\ \frac{3a}{4} \pm \frac{\sqrt{3}a}{2}\right). \quad \text{[Art. 367.]}$$

Also $$e^2 = \frac{\alpha^2+\beta^2}{\alpha^2} \quad \text{[Art. 366]}$$

$$= \frac{\frac{3}{4}+\frac{3}{16}}{\frac{3}{4}} = \frac{5}{4};\ \therefore\ e = \tfrac{1}{2}\sqrt{5}.$$

33. The equations for the centre are

$$5\bar{x} + 3\bar{y} + 6 = 0, \text{ and } 3\bar{x} + 5\bar{y} + 2 = 0.$$

$$\therefore \bar{x} = -\tfrac{3}{2}, \ \bar{y} = \tfrac{1}{2},$$

and $$c' = -\tfrac{3}{2} \cdot 6 + \tfrac{1}{2} \cdot 2 + 6 = -2.$$

Therefore the equation referred to parallel axes through the centre is $5x^2 + 6xy + 5y^2 = 2$,

$$\tan 2\theta = \frac{2h}{a-b} = \frac{6}{5-5} = \infty; \ \therefore \ \theta_1 = 45°, \ \theta_2 = 135°,$$

and $$r^2 = \frac{2(1+\tan^2\theta)}{5 + 6\tan\theta + 5\tan^2\theta} = \frac{1}{4} \text{ or } 1.$$

$$\therefore \sqrt{r_1^2 - r_2^2} = \frac{\sqrt{3}}{2}, \ \cos\theta_2 = -\frac{1}{\sqrt{2}}, \ \sin\theta_2 = \frac{1}{\sqrt{2}}.$$

Hence the coordinates of the foci are

$$(-\tfrac{3}{2} \mp \tfrac{1}{4}\sqrt{6}, \ \tfrac{1}{2} \pm \tfrac{1}{4}\sqrt{6}).$$

$$e^2 = \frac{\alpha^2 - \beta^2}{\alpha^2} = \frac{1 - \frac{1}{4}}{1} = \frac{3}{4}; \ \therefore \ e = \tfrac{1}{2}\sqrt{3}.$$

34. The equations for the centre are

$$\bar{x} + 2\bar{y} - 1 = 0, \text{ and } 2\bar{x} + \bar{y} + 1 = 0.$$

$$\therefore \bar{x} = -1, \ \bar{y} = 1, \text{ and } c' = 1 + 1 - 6 = -4.$$

Therefore the equation referred to parallel axes through the centre is $x^2 + 4xy + y^2 = 4$.

$$\tan 2\theta = \frac{2h}{a-b} = \frac{4}{1-1} = \infty; \ \therefore \ \theta_1 = 45°, \ \theta_2 = 135°.$$

$$\therefore r^2 = \frac{4(1+\tan^2\theta)}{1 + 4\tan\theta + \tan^2\theta} = \frac{4}{3} \text{ or } -4.$$

$$\therefore \sqrt{r_1^2 - r_2^2} = \tfrac{4}{3}\sqrt{3}, \ \sin\theta_1 = \frac{1}{\sqrt{2}}, \ \cos\theta_1 = \frac{1}{\sqrt{2}}.$$

Therefore the coordinates of the foci are

$$(-1 \pm \tfrac{2}{3}\sqrt{6},\ 1 \pm \tfrac{2}{3}\sqrt{6}).$$

Also $$e^2 = \frac{\alpha^2+\beta^2}{\alpha^2} = \frac{4+\frac{4}{3}}{\frac{4}{3}} = 4\ ;\ \therefore\ e = 2.$$

35. The latus rectum = twice the perpendicular from the focus (0, 0) upon the directrix $(bx + ay - ab = 0)$

$$= \frac{2ab}{\sqrt{a^2+b^2}}.$$

36. Here $\tan 2\theta = \dfrac{2h}{a-a} = \infty\ ;\ \therefore\ \theta_1 = 45°,\ \theta_2 = 135°.$

Hence the equations of the axes are

$$x - y = 0 \text{ and } x + y = 0.$$

Also $$r^2 = \frac{d(1+\tan^2\theta)}{a + 2h\tan\theta + a\tan^2\theta} = \frac{d}{a \pm h}.$$

37. The equation referred to parallel axes through the centre is [Art. 352]

$$ax^2 + 2hxy + by^2 = \frac{-\Delta}{ab-h^2} = C.$$

Hence as in Art. 364,

$$\frac{1}{\alpha^2} + \frac{1}{\beta^2} = \frac{a+b}{C}, \text{ and } \frac{1}{\alpha^2\beta^2} = \frac{ab-h^2}{C^2}.$$

$$\therefore\ \alpha^2+\beta^2 = \frac{(a+b)C}{ab-h^2} \text{ and } \alpha^2\beta^2 = \frac{C^2}{ab-h^2}.$$

$$\therefore\ \alpha^2-\beta^2 = \pm\sqrt{\frac{C^2(a+b)^2 - 4C^2(ab-h^2)}{(ab-h^2)^2}}$$

$$= \pm\frac{C}{ab-h^2}\sqrt{(a-b)^2+4h^2}.$$

Therefore the values of α^2 and β^2 are

$$\frac{1}{2}\frac{C}{ab-h^2}\left[a+b \pm \sqrt{(a-b)^2+4h^2}\right],$$

i.e. $$-\frac{1}{2}\frac{\Delta}{(ab-h^2)^2}\,\frac{4(ab-h^2)}{a+b\mp\sqrt{(a-b)^2+4h^2}},$$

i.e. $$-2\Delta \div \left[(ab-h^2)\{a+b\mp\sqrt{(a-b)^2+4h^2}\}\right].$$

38. If θ be the angle made with the axis by the axis of the curve,

$$\tan 2\theta = \frac{\lambda}{2}.$$

Hence the polar equation of the required locus is

$$r^2 = \frac{a^2(1+\tan^2\theta)}{1-\tan^2\theta+\lambda\tan\theta} = \frac{a^2}{\frac{1-\tan^2\theta}{1+\tan^2\theta}+\frac{\lambda}{2}\cdot\frac{2\tan\theta}{1+\tan^2\theta}}$$

$$= \frac{a^2}{\cos 2\theta + \tan 2\theta \,.\, \sin 2\theta} = a^2\cos 2\theta,$$

or, in Cartesians, $(x^2+y^2)^2 = a^2(x^2-y^2)$.

39. Let the axis of the second parabola be parallel to $y - mx = 0$, so that its equation is of the form

$$(y-mx)^2 + 2gx + 2fy + c = 0.$$

If this meets the first parabola in the point $(at^2, 2at)$, then, on substitution,

$$t^4m^2a^2 - 4ma^2t^3 + \ldots = 0.$$

$$\therefore\ t_1 + t_2 + t_3 + t_4 = \frac{4ma^2}{m^2a^2} = \frac{4}{m} = 4\cot\theta,$$

if θ be the inclination of the axes of the parabolas.

If y_1 and y_2 be the distances of the centroid from the axes,

$$y_1 = \frac{2a(t_1+t_2+t_3+t_4)}{4} = 2a\cot\theta.$$

Similarly, if $4b$ be the latus rectum of the second parabola, $y_2 = 2b\cot\theta$.

$$\therefore\ \frac{y_1}{y_2} = \frac{a}{b}.$$

40. Since $\dfrac{ab-h^2}{\sin^2\omega} = 1,$ [Art. 364]

$$\therefore\ \frac{17-1}{8^2} = \sin^2\omega. \quad \therefore\ \sin^2\omega = \frac{1}{4}; \quad \therefore\ \omega = \sin^{-1}\frac{1}{2}.$$

41. The equation breaks up into

$$(3x-5y+1)(2x+y-2) = 0,$$

and therefore gives two straight lines which are easily drawn.

42. See Art. 364.

$$\alpha^2 + \beta^2 = \frac{a + b - 2h\cos\omega}{ab - h^2}, \quad \ldots\ldots\ldots\ldots\ldots\text{(i)}$$

and
$$\alpha^2\beta^2 = \frac{\sin^2\omega}{ab - h^2}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}$$

Also $\dfrac{e^2}{2-e^2} = \dfrac{\alpha^2 - \beta^2}{\alpha^2 + \beta^2}$; $\therefore \dfrac{e^4}{(2-e^2)^2} = \dfrac{(\alpha^2+\beta^2)^2 - 4\alpha^2\beta^2}{(\alpha^2+\beta^2)^2}$;

$$\therefore \frac{4\alpha^2\beta^2}{(\alpha^2+\beta^2)^2} = 1 - \frac{e^4}{(2-e^2)^2} = \frac{4 + e^4 - 4e^2 - e^4}{(2-e^2)^2};$$

$$\therefore \frac{(\alpha^2+\beta^2)^2}{\alpha^2\beta^2} = \frac{4 - 4e^2 + e^4}{1 - e^2}.$$

Substituting, from (i) and (ii) we have

$$\frac{(a + b - 2h\cos\omega)^2}{(ab - h^2)\sin^2\omega} = 4 + \frac{e^4}{1 - e^2}.$$

43. As in Art. 137, we have

$$x^2(a+\lambda) + 2xy(h + \lambda\cos\omega) + y^2(b+\lambda) + 2gx + 2fy + c$$
$$= x'^2(a'+\lambda) + \text{etc.} \ldots$$

The left-hand side will be the product of linear factors if

$$c(a+\lambda)(b+\lambda) + 2fg(h + \lambda\cos\omega) - c(h+\lambda\cos\omega)^2$$
$$-(a+\lambda)f^2 - (b+\lambda)g^2 = 0,$$

or

$$\lambda^2 c\sin^2\omega + \lambda\{c(a+b-2h\cos\omega) + 2fg\cos\omega - f^2 - g^2\} + \Delta = 0.$$

Hence, as in Art. 137,

$$\frac{c(a+b-2h\cos\omega) + 2fg\cos\omega - f^2 - g^2}{c\sin^2\omega}$$
$$= \frac{c'(a'+b'-2h'\cos\omega') + 2f'g'\cos\omega' - f'^2 - g'^2}{c'\sin^2\omega'} \quad \ldots(1)$$

and
$$\frac{\Delta}{c\sin^2\omega} = \frac{\Delta'}{c'\sin^2\omega'}. \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

But, clearly, since the origin is unaltered, such substitutions as those of Art. 132 do not alter the constant terms, and thus $c = c'$.

Therefore from (2) $\frac{\Delta}{\sin^2 \omega} = \frac{\Delta'}{\sin^2 \omega'}$. ....(3)

Also, from Art. 137,

$$\frac{a+b-2h\cos\omega}{\sin^2\omega} = \frac{a'+b'-2h'\cos\omega'}{\sin^2\omega'},$$

and therefore (1) gives, on subtraction,

$$\frac{f^2+g^2-2fg\cos\omega}{\sin^2\omega} = \frac{f'^2+g'^2-2f'g'\cos\omega'}{\sin^2\omega'}. \quad \ldots(4)$$

Also Art. 137 gives $\frac{ab-h^2}{\sin^2\omega} = \frac{a'b'-h'^2}{\sin^2\omega'}$.

Multiplying the two sides of this by c and c' and subtracting from (3), we get

$$\frac{af^2+bg^2-2fgh}{\sin^2\omega} = \frac{a'f'^2+b'g'^2-2f'g'h'}{\sin^2\omega'}.$$