

6. Application of Definite Integral

- Area of the region bounded by the curve $y = f(x)$, x -axis, and the lines $x = a$ and $x = b$ ($b > a$) is given by
 $A = \int_a^b y \, dx$ or $A = \int_a^b f(x) \, dx$
- The area of the region bounded by the curve $x = g(y)$, y -axis, and the lines $y = c$ and $y = d$ is given by
 $A = \int_c^d x \, dy$ or $A = \int_c^d g(y) \, dy$
- If a line $y = mx + p$ intersects a curve $y = f(x)$ at $x = a$ and $x = b$, ($b > a$), then the area (A) of region bounded by the curve $y = f(x)$ and the line $y = mx + p$ is

$$A = \int_a^b (y_1 - y_2) \, dx, \text{ where } y_1 = mx + p \text{ and } y_2 = f(x)$$

$$A = \int_a^b [(mx + p) - f(x)] \, dx$$

- If a line $y = mx + p$ intersects a curve $x = g(y)$ at $y = c$ and $y = d$, ($d > c$), then the area (A) of region bounded by the curve $x = g(y)$ and the line $y = mx + p$ is

$$A = \int_c^d (x_1 - x_2) \, dy, \text{ where } x_1 = \frac{y-p}{m} \text{ and } x_2 = g(y)$$

$$A = \int_c^d \left[\left(\frac{y-p}{m} \right) - g(y) \right] \, dy$$

Example 1: Find the area of the region in the first and third quadrant enclosed by the x -axis and the line

$$y = \sqrt{3x}, \text{ and the ellipse } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Solution: The given equations are

$$y = \sqrt{3x} \quad \dots (1)$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \dots (2)$$

Substituting $y = \sqrt{3x}$ in equation (2), we obtain

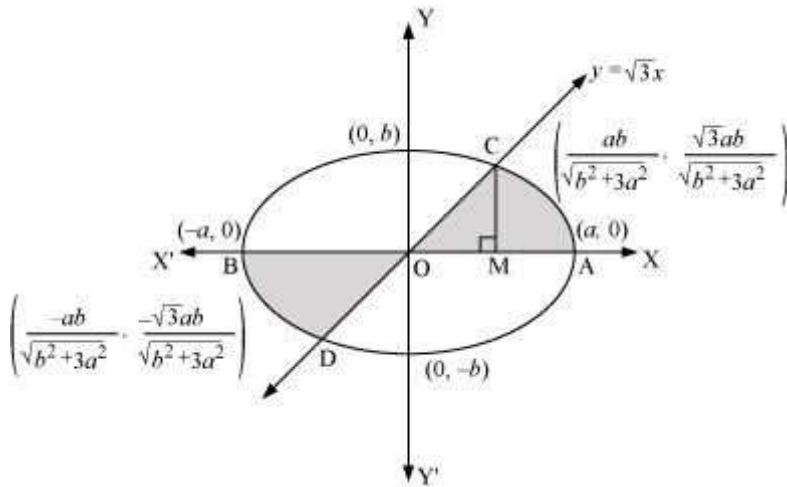
$$\frac{x^2}{a^2} + \frac{3x^2}{b^2} = 1$$

$$\Rightarrow x^2(b^2 + 3a^2) = a^2b^2$$

$$\Rightarrow x = \pm \frac{ab}{\sqrt{b^2 + 3a^2}}$$

$$\therefore y = \pm \frac{\sqrt{3ab}}{\sqrt{b^2 + 3a^2}}$$

Hence, the line meets the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at C $\left(\frac{ab}{\sqrt{b^2+3a^2}}, \frac{\sqrt{3}ab}{\sqrt{b^2+3a^2}} \right)$ and D $\left(\frac{-ab}{\sqrt{b^2+3a^2}}, \frac{-\sqrt{3}ab}{\sqrt{b^2+3a^2}} \right)$ in the first and third quadrant respectively.



In the figure, $CM \perp XX'$

$$\text{Now, area OCMO} = \int_0^{\frac{ab}{\sqrt{b^2+3a^2}}} \sqrt{3}x \, dx = \frac{\sqrt{3}}{2} \left[x^2 \right]_0^{\frac{ab}{\sqrt{b^2+3a^2}}} = \frac{\sqrt{3}a^2b^2}{2(b^2+3a^2)}$$

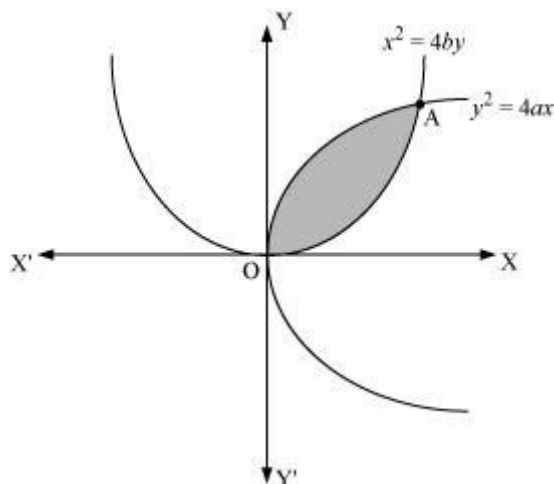
Area ACMA

$$\begin{aligned} &= \int_a^{\frac{ab}{\sqrt{b^2+3a^2}}} \frac{b}{a} \sqrt{a^2 - x^2} \, dx = \frac{b}{a} \left[\frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} \right]_{\frac{ab}{\sqrt{b^2+3a^2}}}^a \\ &= \frac{b}{a} \left[\frac{a}{2} \times 0 + \frac{a^2}{2} \times \sin^{-1} 1 - \left(\frac{ab}{2\sqrt{b^2+3a^2}} \times \frac{\sqrt{3}a^2}{\sqrt{b^2+3a^2}} + \frac{a^2}{2} \sin^{-1} \frac{b}{\sqrt{b^2+3a^2}} \right) \right] \\ &= \frac{\pi}{4}ab - \frac{\sqrt{3}a^2b^2}{2(b^2+3a^2)} - \frac{ab}{2} \sin^{-1} \frac{b}{\sqrt{b^2+3a^2}} \end{aligned}$$

- a. The area of the region enclosed between two curves $y = f(x)$ and $y = g(x)$ and the lines $x = a$ and $x = b$ is given by,

$$A = \left\{ \begin{aligned} &\int_a^b [f(x) - g(x)] \, dx, \text{ where } f(x) \geq g(x) \text{ in } [a, b] \\ &\int_a^c [f(x) - g(x)] \, dx + \int_c^b [g(x) - f(x)] \, dx \\ &\text{where } a < c < b \text{ and } f(x) \geq g(x) \text{ in } [a, c] \text{ and } f(x) \leq g(x) \text{ in } [c, b] \end{aligned} \right\}$$

Example 2: Show that region bounded by two parabolas (shown in the figure) $y^2 = 4ax$ and $x^2 = 4by$ is $\frac{16}{3}ab$.



Solution:

The point of intersection of the parabolas $y^2 = 4ax$ and $x^2 = 4by$ are $O(0, 0)$ and $A(4\sqrt[3]{ab^2}, 4\sqrt[3]{a^2b})$.

Here, $y^2 = 4ax \Rightarrow y = 2\sqrt{a}\sqrt{x} = f(x)$ and $x^2 = 4by \Rightarrow y = \frac{x^2}{4b} = g(x)$

It can be observed that $f(x) \geq g(x)$ in $[0, 4\sqrt[3]{ab^2}]$.

Therefore, required area of the shaded region

$$= \int_0^{4\sqrt[3]{ab^2}} [f(x) - g(x)] dx$$

$$= \int_0^{4(ab^2)^{\frac{1}{3}}} \left(2\sqrt{a}\sqrt{x} - \frac{x^2}{4b} \right) dx$$

$$= \left[\frac{2\sqrt{a}x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{1}{4b} \times \frac{x^3}{3} \right]_0^{4(ab^2)^{\frac{1}{3}}}$$

$$= \frac{4}{3}\sqrt{a} \left[8(ab^2)^{\frac{1}{2}} \right] - \frac{1}{12b} \cdot [64ab^2]$$

$$= \frac{32}{3}ab - \frac{16}{3}ab$$

$$= \frac{16}{3}ab$$

If we rotate a curve about the x -axis through 360° , then the curve maps out the surface of a solid as it rotates. Such solids are called **solids of revolution**. The fixed line about which the area is rotated is called the axis of solid of revolution.

- If a semicircle is revolved about its diameter, then the surface of a sphere is obtained.
- If a right angled triangle is revolved about a line making a right angle, then the surface of a cone is obtained.
- If a rectangle is revolved about one of its sides, then the surface of a right circular cylinder is obtained.

Volume of Solid Revolution

If the area bounded by the curve $y = f(x)$ and lines $x = a$ and $x = b$ is rotated about the x -axis, then the volume of the solid of revolution is given by

$$V = \int_a^b \pi y^2 dx = \int_{x=a}^{x=b} \pi f(x)^2 dx$$

If the area bounded by the curve $x = g(y)$ and lines $y = a$ and $y = b$ is rotated about the y -axis, then the volume of the solid of revolution is given by

$$V = \int_a^b \pi x^2 dy = \int_{y=a}^{y=b} \pi g(y)^2 dy$$