Chapter 8

Binomial Theorem

Exercise 8.2

Q. 1 Find the coefficient of

x^5 in $(x + 3)^8$

Answer:

It is known that $(r + 1)^n$ term, (T_{r+1}), in the binomial expression of $(a + b)^n$ is given by

$T r +1 = {}^nC_r a^{n-r} b^r$

Assuming that x5 occurs in the $(r + 1)^n$ term of the expression $(x + 3)^8$, we obtain

$T_{r+1} = {}^8C_r(x)^{8-r} (3)^r$

Comparing the indices of x in x^5 in T_{r+1}

We, obtain r = 3

Thus, the coefficient of x5 is ${}^8C_3 (3)^3 = \frac{8!}{3!5!} \times 3^3 = \frac{8.7.6.5!}{3.2.5!}.3^3 = 1512$

Q. 2 Find the coefficient of

a^5b^7 in $(a - 2b)^{12}$

Answer:

It is known that $(r + 1)^n$ term (T_{r+1}), in the binomial expression of $(a + b)^n$ is given by

$T_{r+1} = {}^nC_r a^{n-r} b^r$

Assuming that a^5b^7 occurs in the $(r + 1)^{12}$ term of the expression $(a - 2b)^{12}$, we obtain

$T_{r+1} = {}^{12}C_r(a)^{12-r}\ (-2b)^{r} = {}^{12}C_r\ (a)^{12-r}\ (b)^r$

Comparing the indices of a and b in a^5b^7 in T_{r+1}

We, obtain r = 7

Thus, the coefficient of a^5b^7 is

${}^{12}C_r\ (-2)^7 = \frac{12!}{7!5!}.2^7 = \frac{12.11.10.9.8.7!}{5.4.3.2.7!}.(-2)^7$ = - (792)(128) = - 101376

Q. 3 Write the general term in the expansion of$(x^2 - y^6)^6$

Answer:

It is known that the general term T_{r+1} {which is the $(r + 1)^{n}$ term} in the binomial expression of $(a + b)^{n}$ is given by $T_{r+1} = {}^{n}C_r\ a^{n-r}\ b^{r}$.

Thus, the general term in the expansion of $(x^2 - y^6)$ is

$T^{r+1} = {}^{6}C_r\ (x^2)^{6-r}\ (-y)^{r} = (-1)^{r}\ {}^{6}C_r . x^{12-2r}.\ y^{r}$

Q. 4 Write the general term in the expansion of$(x^2 - y\ x)^{12}$, $x \neq 0$.

Answer:

It is known that the general term T_{r+1} {which is the $(r + 1)^{n}$ term} in the binomial expansion of $(a + b)^{n}$ is given by $T_{r+1} = {}^{n}C_r\ a^{n-r}\ b^{r}$

Thus, the general term in the expansion of $(x^2 - y\ x)^{12}$ is

$T_{r+1} = {}^{12}C_r\ (x^2)^{12-r}\ (-y\ x)^{r} = (-1)^{r}\ {}^{12}C_r.\ x^{24-2r}.\ y^{r} = (-1)^{r}\ {}^{12}C_r.\ x^{24-r}.\ y^{r}$

Q. 5 Find the 4th term in the expansion of $(x - 2y)^{12}$.

Answer:

It is known $(r +1)^{n}$ term, T_{r+1} in the binomial expansion of (a + b) n is given by $T_{r+1} = {}^{n}C_{r}\ a^{n-r}\ b^{r}$

Thus, the 4^{th} term is the expansion of $(x^2 - 2y)^{12}$ is

$T_4 = T_{3+1} = {}^{12}C_3\ (x)^{12-3}\ (-2y)^3 = (-1)^3.\frac{12!}{3!9!}.x^9.\ (2)^3.\ y^{3} = \frac{12.11.10}{3.2}.(2)^3\ x^9\ y^3$
$= - 1760x^9\ y^3$

Q. 6 Find the 13th term in the expansion of $\left\{9x - \frac{1}{3\sqrt{x}}\right\}^{18}$

Answer:

It is known (r + 1) ^{n} term, T_{r+1} in the binomial expansion of (a + b) n is given by $T_{r+1} = {}^{n}C_{r}\, a^{n-r}\, b^{r}$

Thus, the 13th term in the expansion of $\left\{9X - \frac{1}{3\sqrt{X}}\right\}^{18}$ is

$T_{13} = T_{12-1} = {}^{18}C_{12}\,(9x)^{18-12}\left\{-\frac{1}{3\sqrt{x}}\right\}^{12}$

$= (-1)^{12}\frac{18!}{12!6!}(9)^{6}(x)^{6}\left(\frac{1}{3}\right)^{12} x\left(\frac{1}{\sqrt{x}}\right)^{12}$

$= \frac{18.17.16.15.14.13.12!}{12!6.5.4.3.2.}\,.\,x^{6}\,\frac{1}{x^{6}}\,.3^{12}\,\frac{1}{3^{12}}$

= 18564

Q. 7 Find the middle terms in the expansions of $\left(3 - \frac{x^{3}}{6}\right)^{7}$

Answer:

It is known that in the expansion of $(a + b)^{n}$ in n is odd, then there are two middle terms

Namely $\left(\frac{n+1}{2}\right)^{n}$ term and $\left(\frac{n+1}{2} + 1\right)^{n}$ term.

Therefore, the middle terms in the expansion $\left(3 - \frac{x^{3}}{6}\right)^{7}$ are $\left(\frac{7+1}{2}\right)^{n}$ = 4th and $\left(\frac{7+1}{2} + 1\right)^{n}$ = 5th term.

$T_{4} = T_{3+1} = {}^{7}C_{3}(3)^{7-3} - \frac{x^{3}}{6} = (-1)^{3}\,\frac{7!}{3!4!}.\,3^{4}.\,\frac{x^{9}}{6^{3}}$

$= \frac{7.6.5.4!}{3.2.4!}.\,3^{4}.\,\frac{1}{2^{3}.3^{3}}.\,x^{9} = -\,\frac{105}{8}\,x^{4}$

$T_5 = T_{4+1} = {}^7C_4(3)^{7-4}\left(-\frac{x^3}{6}\right)^4 = (-1)^4 \frac{7!}{4!3!}. 3^3. \frac{x^{12}}{6^4}$

$= \frac{7.6.5.4!}{4!3.2}. \frac{3^3}{2^4.3^4}.x^{12} = \frac{35}{48}x^{12}$

Thus, the middle terms in the expansion of $\left(3 - \frac{x^3}{6}\right)^7$ are - $\frac{105}{8}$ x^9 and $\frac{35}{48}$ x^{12}

Q. 8Find the middle terms in the expansions of $\left(\frac{x}{3} + 9y\right)^{10}$

Answer:

It is known that in the expansion of (a + b)n, $in\ n$ is even the middle term is $\left(\frac{n}{2} + 1\right)^n$ term.

Therefore, the middle term in the expansion of $\left\{\frac{x}{3} + 9y\right\}^{10}$ is $\left(\frac{10}{2} + 1\right)^n$ = 6^{th}

$T_4 = T_{5+1} = {}^{10}C_5\left(\frac{x}{3}\right)^{10-5}(9y)^5 = \frac{10!}{5!5!}.\frac{x^5}{3^6}. 9^5. Y^5$

$= \frac{10.9.8.7.6.5!}{5.4.3.2.5!}.\frac{1}{3^6}.3^{10}.x^5y^5$ [$9^5 = (3^2)^5 = 3^{10}$]

$= 252 \times 3^6.x^5. y^5 = 6123x^5y_5$

Thus, the middle term in the expansion of $\left(\frac{x}{3} + 9y\right)^{10}$ is $6123x^5y^5$

Q. 9 In the expansion of $(1 + a)^{m+n}$, prove that coefficients of a^m and a^n are equal.

Answer:

It is known that $(r + 1)^n$ term, (T_{r+1}) in the binomial expansion of $(a + b)^n$ is given by

$T_{r+1} = {}^nC_r a^{n-r} b^r$

Assuming that a n occurs in the (r +1) ^{n} term of the expansion (1 + a) m + n, we obtain

$T_{r+1} = {}^{m+n}C_r (1)^{m+n-r} (a)^r = {}^{m+n}C_r a^r$

Comparing the indices of a in a^n in T_{r+1}

We, obtain r = m

Therefore, the coefficient of an is

$^{M+n}C_r = \frac{(m+n)!}{m!(m+n-m)!} = \frac{(m!+n!)}{m!n!} \ldots (1)$

Assuming that an occurs in the (k + 1) ^{n} term of the expansion (1 + a) m + n, we obtain

$T_{k+1} = {}^{m+n}C_k (1)^{m+n-k} (a)^k = {}^{m+n}C_k (a)^k$

Comparing the indices of a in a^n and T_{k+1}

We, obtain

K = n

Therefore, the coefficient of an is

$^{M+n}C_n = \frac{(m+n)!}{n!(m+n-n)!} = \frac{(m+n)!}{n!m!} \ldots (2)$

Thus, from (1) and (2), it can be observed that the coefficient of a^n in the expansion of $(1 + a)^{m+n}$ is equal.

Q. 10 The coefficients of the (r – 1) th, r th and (r + 1) th terms in the expansion of $(x + 1)^n$ are in the ratio 1: 3: 5. Find n and r.

Answer:

It is known that (k + 1) ^{n} term (T_{k+1}) in the binomial expansion of (a + b) n is given by $T_{k+1} = {}^nC_k a^{n-k} b_k$

Therefore, (r – 1) ^{n} term in the expansion of $(x + 1)^1$ is

$T_{r-1} = {}^nC_{r-2} (x)^{n-(r-2)} (1)^{(r-2)} = {}^nC_{r-2} x^{n-r-2}$

(r + 1) term in the expansion of $(x + 1)^n$ is

$T_{r-1} = {}^nC_r (x)^{n-r} (1)^r = {}^nC_1 x^{n-r}$

r^{th} term in the expansion of $(x + 1)^n$ is

$T_r = {}^nC_{r-1} (x)^{n-(r-1)} = {}^nC_{r-1} x^{n-r+1}$

Therefore, the coefficient of the $(r - 1)\ th$, r^{th} and $(r + 1)^{th}$ term in the expansion of $(x + 1)^n$

${}^nc_r - 2$, ${}^nc_r - 1$, and n c r are respectively. Since these - coefficient are in the ratio 1: 3: 5, we obtain

$= \frac{n_{c_{r-2}}}{n_{c_{r-1}}} = \frac{1}{3}$ and $\frac{n_{c_{r-1}}}{n_{c_r}} = \frac{3}{5}$

$\frac{n_{c_{r-2}}}{n_{c_{r-1}}} = \frac{n!}{(r-1)!(n-r+1)} \times \frac{r!(n-r)!}{n!} = \frac{(r-1)!(r-2)!9n-r+1)}{(r-2)!(n-r+1)!(n-r+2)!}$

$= \frac{r}{n-r+2}$

$\therefore \frac{r-1}{n-r+2} = \frac{1}{3}$

$= 3r - 3 = n - r + 2$

$= n - 4r + 5 = 0 \ldots. (1)$

$\frac{n_{c_{r-1}}}{n_{c_r}} = \frac{n!}{(r-1)!(n-r+1)} \times \frac{r!(n-r)!}{n!} = \frac{r(r-1)(n-r)!}{(r-1)!(n-r+1)(n-r)!}$

$= \frac{r}{n-r+1}$

$\therefore \frac{r}{n-r+1} = \frac{3}{5}$

$= 5r = 3n - 3r + 3$

$= 3n - 8r + 3 = 0 \ldots (2)$

Multiplying (1) by 3 and subtracting it from (2), we obtain

$4r - 12 = 0$

$= r = 3$

Putting the value of r in (1), we obtain n

- 12 + 5 = 0
- n = 7

thus, n = 7 and r = 3

Q. 11 Prove that the coefficient of x^n in the expansion of (1 + x)2n is twice the coefficient of x^n in the expansion of (1 + x)2n – 1.

Answer:

It is known that $(r + 1)^{th}$ term, (T_{r+1}), in the binomial expansion of (a + b) n is given by

$T_{r+1} = {}^{n}c_{r}\, a^{n-r}\, b^{r}$

Assuming that x n occurs in the $(r + 1)^{th}$ term of the expansion of $(1 + x)^{2n}$, we obtain

$T_{r+1} = {}^{2n}c_{r}\,(1)^{2n-r}\,(x)^{r} = {}^{2n}c_{r}\,(x)^{r}$

Comparing the indices of x in x^n and in T_{r+2}, we obtain r = n

Therefore, the coefficient of x^n in the expansion of $(1+x)^{2n}$ is

${}^{2n}c_{n} = \frac{(2n)!}{n!(2n-n)!} = \frac{(2n)!}{n!n!} = \frac{(2n)!}{n!^2}$ … (1)

Assuming that x n occurs in the $(k + 1)^{th}$ term of the expansion of $(1 + x)^{2n-2}$, we obtain

$T_{k+1} = {}^{2n}c_{k}\,(1)^{2n-r-k}\,(x)^{k} = {}^{2n}c_{k}\,(x)^{k}$

Comparing the indices of x in x^n and in T_{k+1}, we obtain k = n

Therefore, the coefficient of x n in the expansion of $(1 + x)^{2n-1}$ is

2n – 1 c n = $\frac{(2n-1)!}{n!(2n-1-n)!} = \frac{(2n-1)!}{n!(n-1)!}$

$= \frac{2n(2n-1)!}{2n.n!(n-1)!} = \frac{(2n)!}{2n!n!} = \frac{1}{2}\left[\frac{(2n)!}{(n!)^2}\right]$ … (2)From (1) and (2), it is observed that

$\frac{1}{2}$ $({}^{2n}c_r) = {}^{2n-1}c_n$

$= {}^{2n}c_n = 2\,({}^{2n-1}c_n)$

Therefore, the coefficient of x^n expansion of $(1 + x)^{2n}$ is twice the coefficient of x^n in the expansion of $(1 + x)^{2n-1}$

Hence, proved.

Q. 12 Find a positive value of m for which the coefficient of x^2 in the expansion $(1 + x)^m$ is 6.

Answer:

It is known that $(r + 1)^{th}$ term, (T_{r+1}) in the binomial expansion of $(a + b)^n$ is given by

$T_{r+1} = {}^nc_r\, a^{n-r}\, b^r$

Assuming that x^2 occurs in the $(r + 1)^{th}$ term of the expansion of $(1 + x)^n$, we obtain

$T_{r+1} = {}^nC_r\,(1)^{n-r}\,(x)^r = {}^nc_r\,(x)^r$

Comparing, the coefficient of x in x^2 and in T_{r+1}, we obtain r = 2

Therefore, the coefficient of x^2 is ${}^nc_2$

It is given that the coefficient of x2 in the expansion (1 + x) n is 6.

$= {}^nC_2 = 6$

$= \frac{m!}{2!(m-2)!} = 6$

$= \frac{m(m+1)(m-2)!}{2\times(m-2)!} = 6$

$= m\,(m-1) = 12$

$= m^2 - m - 12 = 0$

$= m^2 - 4m + 3m - 12 = 0$

$= m (m – 4) + 3 (m – 4) = 0$

$= (m – 4) (m + 3) = 0$

$= (m – 4) = 0$ or $(m + 3) = 0$

$= m = 4$ or $m = - 3$

Thus, the positive value of m, for which the coefficient of x^2 in the expansion $(1 + x)_n$ is 6, is 4.