

Three Dimensional Geometry

- **Direction cosines (d.c.'s) of a line:**

- d.c.'s of a line are the cosines of angles made by the line with the positive direction of the coordinate axes.

- If l , m , and n are the d.c.'s of a line, then $l^2 + m^2 + n^2 = 1$

- d.c.'s of a line joining two points P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) are $\frac{x_2 - x_1}{PQ}, \frac{y_2 - y_1}{PQ}, \frac{z_2 - z_1}{PQ}$, where

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

- **Direction ratios (d.r.'s) of a line:**

- d.r.'s of a line are the numbers which are proportional to the d.c.'s of the line.

- d.r.'s of a line joining two points P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) are given by $x_1 - x_2, y_1 - y_2, z_1 - z_2$ or $x_2 - x_1, y_2 - y_1, z_2 - z_1$.

- If a , b , and c are the d.r.'s of a line and l , m , and n are its d.c.'s, then $\frac{l}{a} = \frac{m}{b} = \frac{n}{c}$

$$l = \pm \frac{a}{\sqrt{a^2 + b^2 + c^2}}, m = \pm \frac{b}{\sqrt{a^2 + b^2 + c^2}}, \text{ and } n = \pm \frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

- **Equation of a line through a given point and parallel to a given vector:**

- **Vector form:** Equation of a line that passes through the given point whose position vector is $\vec{a}$ and which is parallel to a given vector $\vec{b}$ is $\vec{r} = \vec{a} + \lambda \vec{b}$, where λ is a constant.

- **Cartesian form:**

- Equation of a line that passes through a point (x_1, y_1, z_1) having d.r.'s as a, b, c is given by

$$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$$

- Equation of a line that passes through a point (x_1, y_1, z_1) having d.c.'s as l, m, n is given by

$$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$$

- **Equation of a line passing through two given points:**

- **Vector form:** Equation of a line passing through two points whose position vectors are $\vec{a}$ and $\vec{b}$ is given by $\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$, where $\lambda \in \mathbf{R}$

- **Cartesian form:** Equation of a line that passes through two given points (x_1, y_1, z_1) and (x_2, y_2, z_2) is given by, $\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$

- **Co-planarity of two lines**

- **Vector form:** Two lines $\vec{r} = \vec{a}_1 + \lambda \vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu \vec{b}_2$ are co-planar, if $(\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_1 \times \vec{b}_2) = 0$

- **Cartesian form:** Two lines $\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ and $\frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$ and $\frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$ are co-planar, if $\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$

- **Angle between two Non-skew lines:**

- **Cartesian form:**

- If l_1, m_1, n_1 , and l_2, m_2, n_2 are the d.c.'s of two lines and θ is the acute angle between them, then $\cos \theta = |l_1 l_2 + m_1 m_2 + n_1 n_2|$

- If a_1, b_1, c_1 and a_2, b_2, c_2 are the d.r.'s of two lines and θ is the acute angle between them, then

$$\cos \theta = \left| \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2} \cdot \sqrt{a_2^2 + b_2^2 + c_2^2}} \right|$$

- **Vector form:** If θ is the acute angle between the lines $\vec{r} = \vec{a}_1 + \lambda \vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu \vec{b}_2$, then

$$\cos \theta = \left| \frac{\vec{b}_1 \cdot \vec{b}_2}{|\vec{b}_1| |\vec{b}_2|} \right|$$

- Two lines with d.r.'s a_1, b_1, c_1 and a_2, b_2, c_2 are

- perpendicular, if $a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$

- parallel, if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

- Two lines in space are said to be skew lines, if they are neither parallel nor intersecting. They lie in different planes.
- Angle between two skew lines is the angle between two intersecting lines drawn from any point (preferably from the origin) parallel to each of the skew lines.
- **Shortest Distance between two skew lines:** The shortest distance is the line segment perpendicular to both the lines.

- **Vector form:** Distance between two skew lines $\vec{r} = \vec{a}_1 + \lambda \vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu \vec{b}_2$ is given by,

$$d = \left| \frac{(\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1)}{|\vec{b}_1 \times \vec{b}_2|} \right|$$

- **Cartesian form:** The shortest distance between two lines $\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ and $\frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$ is given by,

$$d = \frac{\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2-b_2c_1)^2 + (c_1a_2-c_2a_1)^2 + (a_1b_2-a_2b_1)^2}}$$

- The shortest distance between two parallel lines $\vec{r} = \vec{a}_1 + \lambda \vec{b}$ and $\vec{r} = \vec{a}_2 + \mu \vec{b}$ is given by,

$$d = \left| \frac{\vec{b} \times (\vec{a}_2 - \vec{a}_1)}{|\vec{b}|} \right|$$

- **Equation of a plane in normal form:**

- **Vector form:** Equation of a plane which is at a distance of d from the origin, and the unit vector $\hat{n}$ normal to the plane through the origin is $\vec{r} \cdot \hat{n} = d$, where $\vec{r}$ is the position vector of a point in the plane

- **Cartesian form:** Equation of a plane which is at a distance d from the origin and the d.c.'s of the normal to the plane as l, m, n is $lx + my + nz = d$

- **Equation of a plane perpendicular to a given vector and passing through a given point:**

- **Vector form:** Equation of a plane through a point whose position vector is $\vec{a}$ and perpendicular to the vector $\vec{N}$ is $(\vec{r} - \vec{a}) \cdot \vec{N} = 0$, where $\vec{r}$ is the position vector of a point in the plane
- **Cartesian form:** Equation of plane passing through the point (x_1, y_1, z_1) and perpendicular to a given line whose d.r.'s are A, B, C is $A(x - x_1) + B(y - y_1) + C(z - z_1) = 0$

- **Equation of a plane passing through three non-collinear points:**

- **Cartesian form:** Equation of a plane passing through three non-collinear points $(x_1, y_1, z_1), (x_2, y_2, z_2)$, and (x_3, y_3, z_3) is
$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$
- **Vector form:** Equation of a plane that contains three non-collinear points having position vectors $\vec{a}, \vec{b}$, and $\vec{c}$ is $(\vec{r} - \vec{a}) \cdot [(\vec{b} - \vec{a}) \times (\vec{c} - \vec{a})] = 0$, where $\vec{r}$ is the position vector of a point in the plane

- **Planes passing through the intersection of two planes:**

- **Vector form:** Equation of the plane passing through intersection of two planes $\vec{r} \cdot \vec{n}_1 = d_1$ and $\vec{r} \cdot \vec{n}_2 = d_2$ is given by, $\vec{r} \cdot (\vec{n}_1 + \lambda \vec{n}_2) = d_1 + \lambda d_2$, where λ is a non-zero constant

- **Cartesian form:** Equation of a plane passing through the intersection of two planes $A_1x + B_1y + C_1z + D_1 = 0$ and $A_2x + B_2y + C_2z + D_2 = 0$, is given by, $(A_1x + B_1y + C_1z + D_1) + \lambda (A_2x + B_2y + C_2z + D_2) = 0$, where λ is a non-zero constant

- **Angle between two planes:** The angle between two planes is defined as the angle between their normals.

- **Vector form:** If θ is the angle between the two planes $\vec{r} \cdot \vec{n}_1 = d_1$ and $\vec{r} \cdot \vec{n}_2 = d_2$, then

$$\cos \theta = \left| \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1| |\vec{n}_2|} \right|$$

Note that if two planes are perpendicular to each other, then $\vec{n}_1 \cdot \vec{n}_2 = 0$; and if they are parallel to each other, then $\vec{n}_1$ is parallel to $\vec{n}_2$.

- **Cartesian form:** If θ is the angle between the two planes

$$A_1x + B_1y + C_1z + D_1 = 0 \text{ and } A_2x + B_2y + C_2z + D_2 = 0, \text{ then } \cos \theta = \left| \frac{A_1A_2 + B_1B_2 + C_1C_2}{\sqrt{A_1^2 + B_1^2 + C_1^2} \sqrt{A_2^2 + B_2^2 + C_2^2}} \right|$$

Note that if two planes are perpendicular to each other, then $A_1A_2 + B_1B_2 + C_1C_2 = 0$; and if they are parallel to each other, then $\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2}$

- **Distance of a point from a plane:**

- **Vector form:** The distance of a point, whose position vector is $\vec{a}$, from the plane $\vec{r} \cdot \hat{n} = d$ is $|d - \vec{a} \cdot \hat{n}|$.

Note:

- If the equation of the plane is in the form of $\vec{r} \cdot \vec{N} = d$, where $\vec{N}$ is the normal to the plane, then the perpendicular distance is $\frac{|\vec{a} \cdot \vec{N} - d|}{|\vec{N}|}$.

- Length of the perpendicular from origin to the plane $\vec{r} \cdot \vec{N} = d$ is $\frac{|d|}{|\vec{N}|}$.

- **Cartesian form:** The distance from a point (x_1, y_1, z_1) to the plane $Ax + By + Cz + D = 0$ is

$$\left| \frac{Ax_1 + By_1 + Cz_1 + D}{\sqrt{A^2 + B^2 + C^2}} \right|.$$

- **Angle between a line and a plane:** The angle Φ between a line $\vec{r} = \vec{a} + \lambda \vec{b}$ and the plane $\vec{r} \cdot \vec{n} = d$ is the complement of the angle between the line and the normal to the plane and is given by $\Phi = \sin^{-1} \left| \frac{\vec{b} \cdot \vec{n}}{|\vec{b}| |\vec{n}|} \right|$