CHAPTER XXII.

UNDETERMINED COEFFICIENTS.

309. In Art. 230 of the *Elementary Algebra*, it was proved that if any rational integral function of x vanishes when $x=a$, it is divisible by $x-a$. [See also Art. 514. Cor.]

Let $p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots\ldots + p_n$

be a rational integral function of x of n dimensions, which vanishes when x is equal to each of the unequal quantities

$$a_1,\ a_2,\ a_3,\ \ldots\ldots\ a_n.$$

Denote the function by $f(x)$; then since $f(x)$ is divisible by $x-a_1$, we have

$$f(x) = (x-a_1)(p_0x^{n-1} + \ldots\ldots),$$

the quotient being of $n-1$ dimensions.

Similarly, since $f(x)$ is divisible by $x-a_2$, we have

$$p_0x^{n-1} + \ldots\ldots = (x-a_2)(p_0x^{n-2} + \ldots\ldots),$$

the quotient being of $n-2$ dimensions; and

$$p_0x^{n-2} + \ldots\ldots = (x-a_3)(p_0x^{n-3} + \ldots\ldots).$$

....

Proceeding in this way, we shall finally obtain after n divisions

$$f(x) = p_0(x-a_1)(x-a_2)(x-a_3)\ \ldots\ldots\ (x-a_n).$$

310. *If a rational integral function of* n *dimensions vanishes for more than* n *values of the variable, the coefficient of each power of the variable must be zero.*

Let the function be denoted by $f(x)$, where

$$f(x) = p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots\ldots + p_n;$$

and suppose that $f(x)$ vanishes when x is equal to each of the unequal values $a_1, a_2, a_3 \ldots\ldots a_n$; then

$$f(x) = p_0 (x - a_1)(x - a_2)(x - a_3) \ldots\ldots (x - a_n).$$

Let c be another value of x which makes $f(x)$ vanish; then since $f(c) = 0$, we have

$$p_0 (c - a_1)(c - a_2)(c - a_3) \ldots\ldots (c - a_n) = 0;$$

and therefore $p_0 = 0$, since, by hypothesis, none of the other factors is equal to zero. Hence $f(x)$ reduces to

$$p_1 x^{n-1} + p_2 x^{n-2} + p_3 x^{n-3} + \ldots\ldots + p_n.$$

By hypothesis this expression vanishes for more than n values of x, and therefore $p_1 = 0$.

In a similar manner we may shew that each of the coefficients $p_2, p_3, \ldots\ldots p_n$ must be equal to zero.

This result may also be enunciated as follows:

If a rational integral function of n *dimensions vanishes for more than* n *values of the variable, it must vanish for every value of the variable.*

Cor. If the *function* $f(x)$ vanishes for more than n values of x, the *equation* $f(x) = 0$ has more than n roots.

Hence also, *if an equation of* n *dimensions has more than* n *roots it is an identity.*

Example. Prove that

$$\frac{(x-b)(x-c)}{(a-b)(a-c)} + \frac{(x-c)(x-a)}{(b-c)(b-a)} + \frac{(x-a)(x-b)}{(c-a)(c-b)} = 1.$$

This equation is of *two* dimensions, and it is evidently satisfied by each of the *three* values a, b, c; hence it is an identity.

311. *If two rational integral functions of* n *dimensions are equal for more than* n *values of the variable, they are equal for every value of the variable.*

Suppose that the two functions

$$p_0 x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots\ldots + p_n,$$

$$q_0 x^n + q_1 x^{n-1} + q_2 x^{n-2} + \ldots\ldots + q_n,$$

are equal for more than n values of x; then the expression

$$(p_0 - q_0) x^n + (p_1 - q_1) x^{n-1} + (p_2 - q_2) x^{n-2} + \ldots\ldots + (p_n - q_n)$$

vanishes for more than n values of x; and therefore, by the preceding article,

$$p_0 - q_0 = 0,\ p_1 - q_1 = 0,\ p_2 - q_2 = 0, \ldots\ldots p_n - q_n = 0;$$

that is,

$$p_0 = q_0,\ p_1 = q_1,\ p_2 = q_2, \ldots\ldots p_n = q_n.$$

Hence the two expressions are *identical*, and therefore are equal for every value of the variable. Thus

if two rational integral functions are identically equal, we may equate the coefficients of the like powers of the variable.

This is the principle we assumed in the *Elementary Algebra*, Art. 227.

COR. This proposition still holds if one of the functions is of lower dimensions than the other. For instance, if

$$p_0x^n + p_1x^{n-1} + p_2x^{n-2} + p_3x^{n-3} + \ldots\ldots + p_n$$
$$= q_2x^{n-2} + q_3x^{n-3} + \ldots\ldots + q_n,$$

we have only to suppose that in the above investigation $q_0 = 0$, $q_1 = 0$, and then we obtain

$$p_0 = 0,\ p_1 = 0,\ p_2 = q_2,\ p_3 = q_3, \ldots\ldots p_n = q_n.$$

312. The theorem of the preceding article is usually referred to as the *Principle of Undetermined Coefficients.* The application of this principle is illustrated in the following examples.

Example 1. Find the sum of the series

$$1.2 + 2.3 + 3.4 + \ldots\ldots\ldots\ldots + n(n+1).$$

Assume that

$$1.2 + 2.3 + 3.4 + \ldots + n(n+1) = A + Bn + Cn^2 + Dn^3 + En^4 + \ldots,$$

where A, B, C, D, E,... are quantities independent of n, whose values have to be determined.

Change n into $n+1$; then

$$1.2 + 2.3 + \ldots + n(n+1) + (n+1)(n+2)$$
$$= A + B(n+1) + C(n+1)^2 + D(n+1)^3 + E(n+1)^4 + \ldots.$$

By subtraction,

$$(n+1)(n+2) = B + C(2n+1) + D(3n^2+3n+1) + E(4n^3+6n^2+4n+1) + \ldots.$$

This equation being true for all integral values of n, the coefficients of the respective powers of n on each side must be equal; thus E and all succeeding coefficients must be equal to zero, and

$$3D = 1; \quad 3D + 2C = 3; \quad D + C + B = 2;$$

whence $$D = \frac{1}{3}, \quad C = 1, \quad B = \frac{2}{3}.$$

Hence the sum $$=A+\frac{2n}{3}+n^2+\frac{1}{3}n^3.$$

To find A, put $n=1$; the series then reduces to its first term, and

$$2=A+2, \text{ or } A=0.$$

Hence $$1.2+2.3+3.4+\ldots+n(n+1)=\frac{1}{3}n(n+1)(n+2).$$

NOTE. It will be seen from this example that when the n^{th} term is a rational integral function of n, it is sufficient to assume for the sum a function of n which is of one dimension higher than the n^{th} term of the series.

Example 2. Find the conditions that x^3+px^2+qx+r may be divisible by x^2+ax+b.

Assume $$x^3+px^2+qx+r=(x+k)(x^2+ax+b).$$

Equating the coefficients of the like powers of x, we have

$$k+a=p, \quad ak+b=q, \quad kb=r.$$

From the last equation $k=\frac{r}{b}$; hence by substitution we obtain

$$\frac{r}{b}+a=p, \text{ and } \frac{ar}{b}+b=q;$$

that is, $$r=b(p-a), \text{ and } ar=b(q-b);$$

which are the conditions required.

EXAMPLES. XXII. a.

Find by the method of Undetermined Coefficients the sum of

1. $1^2+3^2+5^2+7^2+\ldots$ to n terms.

2. $1.2.3+2.3.4+3.4.5+\ldots$ to n terms.

3. $1.2^2+2.3^2+3.4^2+4.5^2+\ldots$ to n terms.

4. $1^3+3^3+5^3+7^3+\ldots$ to n terms.

5. $1^4+2^4+3^4+4^4+\ldots$ to n terms.

6. Find the condition that $x^3-3px+2q$ may be divisible by a factor of the form $x^2+2ax+a^2$.

7. Find the conditions that ax^3+bx^2+cx+d may be a perfect cube.

8. Find the conditions that $a^2x^4+bx^3+cx^2+dx+f^2$ may be a perfect square.

9. Prove that $ax^2+2bxy+cy^2+2dx+2ey+f$ is a perfect square, if $b^2=ac$, $d^2=af$, $e^2=cf$.

10. If ax^3+bx^2+cx+d is divisible by x^2+h^2, prove that $ad=bc$.

11. If $x^5-5qx+4r$ is divisible by $(x-c)^2$, shew that $q^5=r^4$.

12. Prove the identities:

(1) $$\frac{a^2(x-b)(x-c)}{(a-b)(a-c)}+\frac{b^2(x-c)(x-a)}{(b-c)(b-a)}+\frac{c^2(x-a)(x-b)}{(c-a)(c-b)}=x^2.$$

(2) $$\frac{(x-b)(x-c)(x-d)}{(a-b)(a-c)(a-d)}+\frac{(x-c)(x-d)(x-a)}{(b-c)(b-d)(b-a)}$$
$$+\frac{(x-d)(x-a)(x-b)}{(c-d)(c-a)(c-b)}+\frac{(x-a)(x-b)(x-c)}{(d-a)(d-b)(d-c)}=1.$$

13. Find the condition that

$$ax^2+2hxy+by^2+2gx+2fy+c$$

may be the product of two factors of the form

$$px+qy+r,\quad p'x+q'y+r'.$$

14. If $\xi=lx+my+nz$, $\eta=nx+ly+mz$, $\zeta=mx+ny+lz$, and if the same equations are true for all values of x, y, z when ξ, η, ζ are interchanged with x, y, z respectively, shew that

$$l^2+2mn=1,\quad m^2+2ln=0,\quad n^2+2lm=0.$$

15. Shew that the sum of the products $n-r$ together of the n quantities a, a^2, a^3, ... a^n is

$$\frac{(a^{r+1}-1)(a^{r+2}-1)\ldots(a^n-1)}{(a-1)(a^2-1)\ldots(a^{n-r}-1)}\,a^{\frac{1}{2}(n-r)(n-r+1)}.$$

313. *If the infinite series* $a_0+a_1x+a_2x^2+a_3x^3+\ldots\ldots$ *is equal to zero for every finite value of* x *for which the series is convergent, then each coefficient must be equal to zero identically.*

Let the series be denoted by S, and let S_1 stand for the expression $a_1+a_2x+a_3x^2+\ldots\ldots$; then $S=a_0+xS_1$, and therefore, by hypothesis, $a_0+xS_1=0$ for all finite values of x. But since S is convergent, S_1 cannot exceed some finite limit; therefore by taking x small enough xS_1 may be made as small as we please. In this case the limit of S is a_0; but S is *always* zero, therefore a_0 must be equal to zero identically.

Removing the term a_0, we have $xS_1=0$ for all finite values of x; that is, $a_1+a_2x+a_3x^2+\ldots\ldots$ vanishes for all finite values of x.

Similarly, we may prove in succession that each of the coefficients a_1, a_2, a_3,...... is equal to zero identically.

314. *If two infinite series are equal to one another for every finite value of the variable for which both series are convergent, the coefficients of like powers of the variable in the two series are equal.*

Suppose that the two series are denoted by

$$a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots\ldots$$

and
$$A_0 + A_1x + A_2x^2 + A_3x^3 + \ldots\ldots;$$

then the expression

$$a_0 - A_0 + (a_1 - A_1)\,x + (a_2 - A_2)\,x^2 + (a_3 - A_3)\,x^3 + \ldots\ldots$$

vanishes for all values of x within the assigned limits; therefore by the last article

$$a_0 - A_0 = 0,\ a_1 - A_1 = 0,\ a_2 - A_2 = 0,\ a_3 - A_3 = 0, \ldots\ldots$$

that is,
$$a_0 = A_0,\ a_1 = A_1,\ a_2 = A_2,\ a_3 = A_3, \ldots\ldots;$$

which proves the proposition.

Example 1. Expand $\dfrac{2+x^2}{1+x-x^2}$ in a series of ascending powers of x as far as the term involving x^5.

Let
$$\frac{2+x^2}{1+x-x^2} = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots,$$

where a_0, a_1, a_2, a_3,... are constants whose values are to be determined; then

$$2 + x^2 = (1 + x - x^2)\,(a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots).$$

In this equation we may equate the coefficients of like powers of x on each side. On the right-hand side the coefficient of x^n is $a_n + a_{n-1} - a_{n-2}$, and therefore, since x^2 is the highest power of x on the left, for all values of $n > 2$ we have

$$a_n + a_{n-1} - a_{n-2} = 0;$$

this will suffice to find the successive coefficients after the first three have been obtained. To determine these we have the equations

$$a_0 = 2,\ a_1 + a_0 = 0,\ a_2 + a_1 - a_0 = 1;$$

whence
$$a_0 = 2,\ a_1 = -2,\ a_2 = 5.$$

Also
$$a_3 + a_2 - a_1 = 0, \text{ whence } a_3 = -7;$$

$$a_4 + a_3 - a_2 = 0, \text{ whence } a_4 = 12;$$

and
$$a_5 + a_4 - a_3 = 0, \text{ whence } a_5 = -19;$$

thus
$$\frac{2+x^2}{1+x-x^2} = 2 - 2x + 5x^2 - 7x^3 + 12x^4 - 19x^5 + \ldots$$

Example 2. Prove that if n and r are positive integers

$$n^r - n(n-1)^r + \frac{n(n-1)}{\underline{|2}}(n-2)^r - \frac{n(n-1)(n-2)}{\underline{|3}}(n-3)^r + \ldots$$

is equal to 0 if r be less than n, and to $\underline{|n}$ if $r = n$.

We have $$(e^x - 1)^n = \left(x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \frac{x^4}{\underline{|4}} + \ldots\ldots\right)^n$$

$$= x^n + \text{terms containing higher powers of } x \ldots (1).$$

Again, by the Binomial Theorem,

$$(e^x - 1)^n = e^{nx} - ne^{(n-1)x} + \frac{n(n-1)}{1 \cdot 2}e^{(n-2)x} - \ldots\ldots, \ldots\ldots\ldots (2).$$

By expanding each of the terms e^{nx}, $e^{(n-1)x}$,... we find that the coefficient of x^r in (2) is

$$\frac{n^r}{\underline{|r}} - n \cdot \frac{(n-1)^r}{\underline{|r}} + \frac{n(n-1)}{\underline{|2}} \cdot \frac{(n-2)^r}{\underline{|r}} - \frac{n(n-1)(n-2)}{\underline{|3}} \cdot \frac{(n-3)^r}{\underline{|r}} + \ldots$$

and by equating the coefficients of x^r in (1) and (2) the result follows.

Example 3. If $$y = ax + bx^2 + cx^3 + \ldots\ldots,$$
express x in ascending powers of y as far as the term involving y^3.

Assume $$x = py + qy^2 + ry^3 + \ldots\ldots,$$
and substitute in the given series; thus

$$y = a(py + qy^2 + ry^3 + \ldots) + b(py + qy^2 + \ldots)^2 + c(py + qy^2 + \ldots)^3 + \ldots\ldots$$

Equating coefficients of like powers of y, we have

$$ap = 1; \text{ whence } p = \frac{1}{a}.$$

$$aq + bp^2 = 0; \text{ whence } q = -\frac{b}{a^3}.$$

$$ar + 2bpq + cp^3 = 0; \text{ whence } r = \frac{2b^2}{a^5} - \frac{c}{a^4}.$$

Thus $$x = \frac{y}{a} - \frac{by^2}{a^3} + \frac{(2b^2 - ac)y^3}{a^5} + \ldots$$

This is an example of *Reversion of Series*.

Cor. If the series for y be given in the form

$$y = k + ax + bx^2 + cx^3 + \ldots$$

put $$y - k = z;$$

then $$z = ax + bx^2 + cx^3 + \ldots;$$

from which x may be expanded in ascending powers of z, that is of $y - k$.

EXAMPLES. XXII. b.

Expand the following expressions in ascending powers of x as far as x^3.

1. $\dfrac{1+2x}{1-x-x^2}$. 2. $\dfrac{1-8x}{1-x-6x^2}$. 3. $\dfrac{1+x}{2+x+x^2}$.

4. $\dfrac{3+x}{2-x-x^2}$. 5. $\dfrac{1}{1+ax-ax^2-x^3}$.

6. Find a and b so that the n^{th} term in the expansion of $\dfrac{a+bx}{(1-x)^2}$ may be $(3n-2)x^{n-1}$.

7. Find a, b, c so that the coefficient of x^n in the expansion of $\dfrac{a+bx+cx^2}{(1-x)^3}$ may be n^2+1.

8. If $y^2+2y=x(y+1)$, shew that one value of y is

$$\tfrac{1}{2}x+\tfrac{1}{8}x^2-\tfrac{1}{128}x^4+\ldots\ldots$$

9. If $cx^3+ax-y=0$, shew that one value of x is

$$\frac{y}{a}-\frac{cy^3}{a^4}+\frac{3c^2y^5}{a^7}-\frac{12c^3y^7}{a^{10}}\ldots\ldots$$

Hence shew that $x=\cdot 00999999$ is an approximate solution of the equation $x^3+100x-1=0$. To how many places of decimals is the result correct?

10. In the expansion of $(1+x)(1+ax)(1+a^2x)(1+a^3x)\ldots\ldots$, the number of factors being infinite, and $a<1$, shew that the coefficient of x^r is

$$\frac{1}{(1-a)(1-a^2)(1-a^3)\ldots\ldots(1-a^r)}\,a^{\frac{1}{2}r(r-1)}.$$

11. When $a<1$, find the coefficient of x^n in the expansion of

$$\frac{1}{(1-ax)(1-a^2x)(1-a^3x)\ldots\ldots\text{to inf.}}.$$

12. If n is a positive integer, shew that

(1) $n^{n+1}-n(n-1)^{n+1}+\dfrac{n(n-1)}{\lfloor 2}(n-2)^{n+1}-\ldots\ldots=\tfrac{1}{2}n\,\lfloor n+1$;

(2) $n^n-(n+1)(n-1)^n+\dfrac{(n+1)n}{\lfloor 2}(n-2)^n-\ldots\ldots=1$;

the series in each case being extended to n terms; and

(3) $1^n-n2^n+\dfrac{n(n-1)}{1\,.\,2}3^n-\ldots\ldots=(-1)^n\,\lfloor n$;

(4) $(n+p)^n-n(n+p-1)^n+\dfrac{n(n-1)}{\lfloor 2}(n+p-2)^n-\ldots\ldots=\lfloor n$;

the series in the last two cases being extended to $n+1$ terms.