3

Polynomials

3.01. Introduction

In earlier classes, we have studied various operation of algebraic expressions in which algebraic experssions are included in these classes, we have used the following algebraic indentities for factorisation.

$$(x+y)^2 = x^2 + 2xy + y^2,$$

$$(x-y)^2 = x^2 - 2xy + y^2 \text{ and}$$

$$x^2 - y^2 = (x+y)(x-y).$$

In this chapter we shall study a particular type of algebraic expressions, called *polynomials,* and the terminology related to it. We shall also study some algebraic identities to factorization of polynomials.

3.02. Polynomials

We know that a variable is denoted by the symbols x, y, z, etc. A variable can take any real value. When any constant and variables are expressed with four main operations, then the combination is called the algebraic expression. For examples, $3x$, $5x$, $-x$, $-\frac{3}{2}x$ etc., are algebraic expressions. General form of an algebraic expression is ax, where a is constant and x is variable. $3x$, x^2+3x, x^3+2x^2-4x+5 etc. are algebraic expressions. In all of these the exponents of variable x is a whole number. These type of expressions are called the polynomials in one variable. In all of above examples x is a variable. A polynomial is denoted as $p(x)$, $g(x)$, $q(y)$ ..... etc.

For example

$$p(x) = 3x^2 + 4x - 5$$

$$g(x) = x^3 + 1$$

$$q(y) = y^3 + 2y - 1$$
$$s(t) = 3 - t - 2t^2 + 5t^3$$

A polynomial can have a lot of but finite number of terms

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_2 x^2 + a_1 x + a_0$$

where $a_n, a_{n-1}, ... a_2, a_1, a_0$ are the constants and $a_n \neq 0$. In polynomial $x^2 + 3x$; x^2 and $3x$ are terms of the polynomial. In a polynomial every term has a coefficient.

In polynomial $5x^3 - 2x^2 + x + 3$

coefficient of $x^3 = 5$, coefficient of $x^2 = -2$

coefficient of $x = 1$, coefficient of $x^0 = 3$

Is 3 is a polynomial?

3, –7, 9 etc., are called **constant polynomials.**

0 is called zero polynomial.

Polynomials having only one term are called **monomials.**

For example, $3x,\ 5x^2,\ -3x^3,\ 2,\ t^2,\ y$ etc.

Polynomials having two terms are called **binomials.**

For example, $x + 2,\ x^2 - 2x,\ y^n + 2,\ t^{30} - t^3$ etc.

Similarly, polynomials having three terms are called trinomials.

For example, $p(x) = x^2 + x + 1$

$$g(x) = x - x^2 + \sqrt{3}$$
$$t(y) = y^3 + y + 3$$
$$s(t) = t^4 + t^2 - 2$$

The highest power of variable in a polynomial is called **the degree of the polynomial.**

In $p(x) = 4x^3 - 2x^2 + 8x - 21;$ the hightest power of the term $4x^3 = 3$.

In $q(y) = 3y^7 - 4y^6 + y + 9$ the hightest power of the term $3y^7 = 7$.

So, the degree of polynomials $p(x)$ and $q(y)$ are 3 and 7 respectively.

In the constant monomial $g(x) = 2$, the term 2 with highest power $= 2x^0$. So its degree = 0

Conclusion : The degree of a non-zero constant polynomial is zero.

Now analyse the following $p(x) = 5x + 4,\ g(y) = 12y,\ r(t) = 4 - 2t$ and

$s(u) = \sqrt{3} + 2u$.

Degree of all these polynomials is 1 (one).

Polynomials having degree 1 are called the linear polynomials.

Generally, a linear polynomial is expressed as :

$$p(x) = ax + b,\ a \neq 0$$

The maximum terms of a linear polynomial is two. It means a linear polynomial is either a binomial or a monomial.

Observe the following polynomial :

$p(x) = 2x^2 - 3x + 15,\quad g(x) = 5x^2 + 3$ and $g(y) = y^2 + 2y$

These polynomials have degree 2 are called the quadratic polynomials.

Generally, a quadratic polynomial is expressed as :

$$p(x) = ax^2 + bx + c,\ a \neq 0$$

A quadratic polynomial in one variable has at most three terms. It means a quadratic polynomial can be monomial, binomial or trinomial.

A polynomial of degree three is called a cubic polynomial. A cubic polynomial is expressed as $p(x) = ax^3 + bx^2 + cx + d$, $a \neq 0$ can have at most four terms. A polynomial in one varible x of degree n is an expression of the form :

$p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$ where $a_n \neq 0$, $a_n, a_{n-1}, \ldots a_1, a_0$ are constants.

In particular if $a_0 = a_1 = a_2 = a_3 = \ldots = a_n = 0$ (all the constants are zero), we get the zero polynomial, that is denoted by 0. The degree of the zero polynomial is not defined.

Consider an algebraic expression $x + \frac{1}{x}$.

$$x + \frac{1}{x} = x + x^{-1}$$

The exponent of 2nd term of expression is -1 which is not a whole number.

$$\sqrt{x} + 5 = x^{1/2} + 5$$

The exponent of $x^{1/2}$ is $\frac{1}{2}$, which is not a whole number.

$$\sqrt[3]{y} + y^3 = y^{1/3} + y^3$$

Here, exponent of $y^{1/3}$ is $\frac{1}{3}$, which is not a whole number.

All the above expressions are not polynomials because as none of them has exponent as a whole number.

We have studied polynomials having one variables. There are polynomials having more then one variable for example, x^2+y^2+xyz, $p^2+8q^3+r^4$, t^2+s^3. These polynomials have 3, 3 and 2 variables respectively. We shall study these polynomials leter on.

Exercise 3.1

1. Which of the following expressions are polynomials in one variable. Find the number of terms also :?

   (i) $3x^2-5x+13$ (ii) $y^2+2\sqrt{3}$ (iii) $y+\frac{3}{y}$

   (iv) 3 (v) $2\sqrt{x}+\sqrt{3}\,x$ (vi) $x^{12}+y^3+t^{20}$

2. Write the coefficient of x^2 in following expressions :

   (i) $12+3x+5x^2$ (ii) $7-11x+x^3$ (iii) $\sqrt{3}x-7$ (iv) $\frac{\pi}{2}x^2+x$

3. Write an example of binomial of degree 45.
4. Write an example of monomial of degree 120.
5. Write an example of trinomial of degree 8.
6. Can you give some examples other than questions number 3, 4, and 5 have been given? If yes, then give two more examples for each.
7. Write the degree of each of the following polynomials :

   (i) $12-3x+2x^3$ (ii) $5y-\sqrt{2}$ (iii) 9 (iv) $3+4t^2$

3.03. Zeros of a Polynomial

Consider a polynomial

$$p(x)=2x^3-3x^2+4x-2$$

If we replace *x* by 2 everywhere in $p(x)$, we get

$$p(2)=2\times(2)^3-3\times(2)^2+4\times2-2$$
$$=2\times8-3\times4+4\times2-2$$
$$=16-12+8-2=10$$

So, we can say that the value of *p(x)* at $x=2$ is 10.

Similarly $$p(0)=2\times(0)^3-3\times(0)^2+4\times0-2=-2$$

and $$p(-1)=2(-1)^3-3(-1)^2+4\times(-1)-2$$
$$=2\times-1-3\times1-4\times1-2=-11$$

We can say that the value of polynomial $p(x)$ can be obtained by replacing the $x=\alpha$ in $p(\alpha)$.

Example 1. Find the value of the polynomial $p(x)=8x^2-3x+7$ **, at** $x=-1$ and $x=2$ **.**

Solution : $p(x)=8x^2-3x+7$

The value of the polynomial $p(x)$ at $x = -1$ is :

$$p(-1)=8(-1)^2-3(-1)+7$$
$$=8+3+7=18$$

Again, the value of the polynomial $p(x)$ at $x = 2$ is :

$$p(2)=8(2)^2-3(2)+7$$
$$=32-6+7=33$$

Example 2. Find the value of the polynomial $p(x)=2x^3-13x^2+17x+12$, **at** $x=-\frac{1}{2}$.

Solution : $p(x)=2x^3-13x^2+17x+12$

Put $x=-\frac{1}{2}$

$$p\left(-\frac{1}{2}\right)=2\left(-\frac{1}{2}\right)^3-13\left(-\frac{1}{2}\right)^2+17\left(-\frac{1}{2}\right)+12$$
$$=2\times\frac{-1}{8}-13\times\frac{1}{4}+17\times\frac{-1}{2}+12$$
$$=-\frac{1}{4}-\frac{13}{4}-\frac{17}{2}+12=0$$

Example 3. Find the value of the polynomial $p(x)=x^3-6x^2+11x-6$ **at** $x=1$ **.**

Solution : $p(x)=x^3-6x^2+11x-6$

$$p(1)=(1)^3-6(1)^2+11(1)-6$$
$$=1-6+11-6=-12+12=0$$

In the above example since $p(1)=0$, so we say that 1 is a zero of polynomial $p(x)$.

In general we can say that a zero of a polynomial $p(x)$ is a number "α" (alpha) such that $p(\alpha)=0$.

What is the value of $p(1)=0$, for polynomial $p(x)=x-1$

$p(1)=1-1=0$

We observed that the zero of the polynomial $p(x)=x-1$ is obtained by equating it to 0 i.e., $x-1=0$, which gives $x=1$. We say $p(x)=0$ is a polynomial equation and 1 is the root of the polynomial equation $p(x)=0$. So we can say that 1 is the zero of the polynomial $x-1$, or a root of the polynomial equation $x-1=0$.

What is the zero of the constant polynomial 7?

It has no zero because replacing x by any number in $7x^0$ still gives us 7. In fact, a non-zero constant polynomial has no zero.

What about the zero of the zero polynomial ? By convention, every real number is a *zero of the zero polynomial.*

Examples 4. Check whether 3 and -3 are zeroes of the polynomial $p(x)$ = $x+3$.

Solution : $p(x)=x+3$

$\therefore \quad p(3)=3+3=6$

$p(-3)=-3+3=0$

Therefore, -3 is a zero of the polynomial $p(x)=x+3$ but 3 is not a zero.

Example 5. Find a zero of the polynomial p(x) = 3x+2.

Solution : Finding a zero of $p(x)$ is the same solving the equation $p(x)=0$

$\therefore \quad p(x)=3x+2=0 \quad \Rightarrow 0=3x+2$

$\Rightarrow \quad 3x=-2 \quad \Rightarrow x=\frac{-2}{3}$

Now, if $p(x)=ax+b,\ a\neq 0$, is a linear polynomial, then we can find a zero of $p(x)$ from above examples.

It means finding a zero of the polynomial *p(x)* is to solve the polynomial equation *p(x) = 0*.

Now, $p(x)=0 \Rightarrow ax+b=0,\quad a\neq 0$

$ax=-b$

$$x = -\frac{b}{a}$$

So, $x = -\frac{b}{a}$ is the only one zero of $p(x)$ i.e., a linear polynomial has one and only one zero.

Example 6. Verify that 3 and 0 are the zeroes of the polynomial $x^2 - 3x$.

Solution : Let $p(x) = x^2 - 3x$

Then $\quad p(3) = (3)^2 - 3(3) = 9 - 9 = 0$

and $\quad p(0) = (0)^2 - 3(0) = 0 - 0 = 0$

Hence, 3 and 0 are both zero of the polynomial $x^2 - 3x$.

We can get the follwing conclusion.

1. A zero of a polynomial need not to be a 0.
2. 0 may be a zero of a polynomial.
3. Every linear polynomial has one and only one zero.
4. A polynomial may have more than one zero.

Exercise 3.2

1. Find the value of the polynomial $2x^3 - 13x^2 + 17x + 12$ at the following value of x :

   (i) $x = 2$ (ii) $x = -3$ (iii) $x = 0$ (iv) $x = -1$

2. Find the $P(2), P(1)$ and $P(0)$ for each of the following polynomials :

   (i) $p(x) = x^2 - x + 1$ (ii) $p(y) = (y+1)(y-1)$

   (iii) $p(x) = x^3$ (iv) $p(t) = 2 + t + t^2 - t^3$

3. Verify whether the followings are zeroes of the polynomial indicated against them :

   (i) $p(x) = x^2 - 1;\ x = 1,\ -1$ (ii) $p(x) = 2x + 1;\ x = -\frac{1}{2}$

   (iii) $p(x) = 4x + 5\ ;\ x = \frac{-5}{4}$ (iv) $p(x) = 3x^2;\ x = 0$

   (v) $p(x) = (x-3)(x+5);\ x = 3,\ -5$ (vi) $p(x) = ax + b;\ x = -\frac{b}{a}$

   (vii) $p(x) = 3x^2 - 1;\ x = -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$ (viii) $p(x) = 3x + 2;\ x = \frac{-2}{3}$

4. Find the zeros of the following polynomials :

(i) $p(x) = x - 4$ (ii) $p(x) = 4x$

(iii) $p(x) = bx,\ b \neq 0$ (iv) $p(x) = x + 3$

(v) $p(x) = 2x - 1$ (vi) $p(x) = 3x + 7$

(vii) $p(x) = cx + d,\ c \neq 0,\ c, d$ are real numbers.

Remainder Theorem :

We know that when we divide 25 by 7, we get quotient 3 and remainder 4. Mathematically we express it as

$$25 = (3 \times 7) + 4$$

Similarly if we divide 48 by 8, we get

$$48 = (6 \times 8) + 0$$

Here, remainder is (0) and we say that 8 is s factor of 48 or 48 is a multiple of 8. In the same way we can divide a polynomial by another polynomial. In first case if divisor is monomial. For example on divide polynomial $3x^3 + 2x^2 + x$ by monomial x.

$$(3x^3 + 2x^2 + x) \div x = \frac{3x^3}{x} + \frac{2x^2}{x} + \frac{x}{x} = 3x^2 + 2x + 1,$$

Here, you have noticed that x is common to each term of $(3x^3 + 2x^2 + x)$ $3x^3 + 2x^2 + x = x(3x^2 + 2x + 1)$. We say that x and $(3x^2 + 2x + 1)$ are factors of $3x^3 + 2x^2 + x$ and $3x^3 + 2x^2 + x$ is a multiple of x as well as a multiple of $3x^2 + 2x + 1$.

Now divide $5x^2 + x + 1$ by x.

$$(5x^2 + x + 1) \div x = (5x^2 \div x) + (x \div x) + (1 \div x)$$

Here, when 1 is divided by x, we don't get a polynomial term. So, in this case we stop here, and note that 1 is the remainder. Thus, we have

$$5x^2 + x + 1 = [(5x + 1) \times x] + 1$$

Here, we get $(5x + 1)$ as a quotient and 1 as a remainder. So $(5x - 1)$ is not a factor of $5x^2 + x + 1$. Since, the remainder is not zero.

Dividend = (Divisor × Quotient) + Remainder.

In general, if $p(x)$ and $g(x)$ are two such polynomials that the degree of $p(x)$ is greater then $g(x)$ and $g(x) \neq 0$, then we get two polynomial $q(x)$ and $r(x)$.

$$p(x) = g(x)\cdot q(x) + r(x) \text{ where } r(x) = 0$$

Or the degree of $r(x)$ is smaller than that of degree of $g(x)$;

When $p(x)$ is divided by $g(x)$, we get quotient $q(x)$ and remainder $r(x)$.

Example 7. Divide p(x) by g(x); where p(x) = 7x + 5x^2 + 3 and g(x) = x + 1.

Solution :

$$\begin{array}{r|l} & 5x+2 \\ \hline x+1 & 5x^2+7x+3 \\ & 5x^2+5x \\ & - \quad - \\ \hline & 2x+3 \\ & 2x+2 \\ & - \quad - \\ \hline & \quad 1 \end{array}$$

Note : We take the following steps for above division operation.

Step I : We write the dividend $7x+5x^2+3$ and divisor $x+1$ in the standard form arranging the terms in the descending order i.e. dividend as $5x^2+7x+3$ and divisor as $x+1$.

Step II : We divide the first term of the dividend by the first term of the divisor; i.e., we divide $5x^2$ by x get 5x. This gives us the first term of the quotient.

Step III : We multiply the divisor by the first term of quotient 5x and subtract this product $5x^2+5x$ form the dividend. This gives us the remainder as $2x+3$.

Step IV : We take this remainder $2x+3$ as the new dividend. We repeat the step II to get 2 is the second term of the quotient.

Step V : Similarly as step III. We multiply the divisor $x+1$ by the second terms of quotient 2 and subtract the product $2x+2$ from the dividend $2x+3$. This gives us 1 as remainder.

This process continues till the remainder is 0 or the degree of the new dividend is less than the degree of the divisor. At the last stage, dividend becomes the remainder and the sum of the quotients gives us the whole quotient.

In this example divisor is a linear polynomial. Let us see the relation between the remainder and certain values of dividend.

In $p(x) = 5x^2+7x+3$, substituting –1 in place of x, we get

$$p(-1) = 5(-1)^2 + 7(-1) + 3 = 5 - 7 + 3 = 1$$

Hence the remainder obtained on dividing $p(x) = 5x^2+7x+3$ by $(x+1)$ is the same as the value of the polynomial $p(x)$ at the zero of the polynomial $(x+1)$ i.e., –1.

Let us consider some more examples.

Example 8. Divide the polynomial $2x^4 - 3x^3 + 3x + 1$ by $x + 1$.

Solution :

$$
\begin{array}{r|l}
 & 2x^3 - 5x^2 + 5x - 2 \\
\hline
 & 2x^4 - 3x^3 + 3x + 1 \\
x+1 & 2x^4 + 2x^3 \\
 & - \quad - \\
\hline
 & -5x^3 \qquad + 3x + 1 \\
 & -5x^3 - 5x^2 \\
 & + \quad + \\
\hline
 & 5x^2 + 3x + 1 \\
 & 5x^2 + 5x \\
 & - \quad - \\
\hline
 & -2x + 1 \\
 & -2x - 2 \\
 & + \quad + \\
\hline
 & 3
\end{array}
$$

Remainder $= 3$

Here the zero of divisor $x + 1$ is -1. So on putting $x = -1$ in $p(x)$.

$$p(-1) = 2(-1)^4 - 3(-1)^3 + 3(-1) + 1$$

$$= 2 + 3 - 3 + 1 = 3$$

= Remainder

Example 9. Find the remainder obtained on dividing $p(x) = x - 1$ by $x^3 - 1$.

Solution :

$$
\begin{array}{r|l}
 & x^2 + x + 1 \\
\hline
 & x^3 - 1 \\
x-1 & x^3 \qquad - x^2 \\
 & - \qquad + \\
\hline
 & x^2 \qquad - 1 \\
 & x^2 - x \\
 & - \quad + \\
\hline
 & x - 1 \\
 & x - 1 \\
 & - \quad + \\
\hline
 & 0
\end{array}
$$

Now the remainder is 0.

The root of divisor $x - 1$ is $x = 1$ and $p(x) = x^3 - 1$

$\therefore \quad p(1) = (1)^3 - 1 = 1 - 1 = 0$

So $p(1) = 0$ is equal to the remainder obtained by actual division.

In this way, it is a simple method to find the remainder obtained on dividing a polynomial by a linear polynomial. We shall now generalise this fact in the form of a theorem.

Remainder Theorem

Let $p(x)$ be any polynomial of degree greater than or equal to one and let a be any real number. If $p(x)$ is divided by the linear polynomial $x - a$, then the remainder is $p(a)$.

Proof : Let $p(x)$ be any polynomial with degree greater than or equal to 1. Suppose that when $p(x)$ is divided by $x - a$, the quotient is $q(x)$ and the remainder is $r(x)$, i.e.,

$$p(x) = (x-a)q(x) + r(x)$$

Since, the degree of $x - a$ is 1 and the degree of $r(x)$ is less then degree of $(x-a)$, the degree of $r(x) = 0$. It means that $r(x)$ is a constant, say r.

So for every value of x, $r(x) = r$

Therefore $\quad p(x) = (x-a)q(x) + r$

In particular, if $x = a$, this equation gives us

$$p(a) = (a-a)q(x) + r$$

$$= 0 \times q(x) + r = r$$

$$= r$$

Hence Proved.

Example 10. Find the remainder when $x^4 - 4x^2 + x^3 + 2x + 1$ is divided by $x - 1$.

Solution : Let $p(x) = x^4 - 4x^2 + x^3 + 2x + 1$

The zero of $x - 1$ is 1.

Thus, $\quad p(1) = (1)^4 - 4(1)^2 + (1)^3 + 2(1) + 1$

$$= 1 - 4 + 1 + 2 + 1 = 5 - 4 = 1$$

Thus, remainder = 1

Example 11. Verify whether the polynomial $p(x) = 4x^3 - 12x^2 + 13x - 4$ is a multiple of $g(x) = 2x - 1$.

Solution : As we know, $p(x)$ will be a multiple of $g(x)$ only when $g(x)$ divides $p(x)$ completely *i.e.*, remainder is zero.

So, $g(x) = 2x - 1 = 0$

$$x = \frac{1}{2}$$

$$\therefore \quad p\left(\frac{1}{2}\right) = 4\left(\frac{1}{2}\right)^3 - 12\left(\frac{1}{2}\right)^2 + 13\left(\frac{1}{2}\right) - 4$$

$$= \frac{1}{2} - 3 + \frac{13}{2} - 4 = 0$$

Hence, $g(x)$ is a factor of $p(x)$, means $p(x)$ is multiple of $g(x)$.

Exercise 3.3

1. Find the remainder, if the polynmomial $x^4 + x^3 - 3x^2 + 3x + 1$ is divided by following linear expression :

   (i) $x - 1$ (ii) $x - \frac{1}{2}$ (iii) $x + \pi$ (iv) $3 + 2x$ (v) x

2. Find the remainder, when $2x^3 + 2ax^2 - 5x + a$ is divided by $x + a$.

3. Check whether $x + 1$ is factor of $x^3 + 3x^2 + 3x + 1$ or not.

4. We get the same remainder if polynomials $x^3 + x^2 - 4x + a$ and $2x^3 + ax^2 + 3x - 3$ are divided by $x - 2$. Find the value of a.

Factorisation of Polynomials :

It is found by observing the example 11. Since, the remainder $p\left(\frac{1}{2}\right) = 0$,

therefore, $g(x) = (2x - 1)$, is factor of $p(x)$. So for a polynomial $p(x)$

$$p(x) = (2x - 1)q(x)$$

This is the particular case of the theorem that is given below.

Factor Theroem : If $p(x)$ is a polynomial of degree $n \geq 1$ and a is any real number such that $p(a) = 0$ then $(x - a)$ is a factor of $p(x)$, i.e. if $(x - a)$ is a factor of $p(x)$ then $p(a) = 0$.

Example 12. Examine whether $x - 3$ is a factor of polynomials

$x^3 - 3x^2 + 4x - 12$ and $3x - 9$.

Solution : Given, $p(x) = x^3 - 3x^2 + 4x - 12,\ q(x) = 3x - 9$

According to factor theorem if $(x-3)$ is a factor of $p(x)$ and $q(x)$, then :

$$p(3) = q(3) = 0$$

The zero of $(x-3)$ is 3.

$$p(3) = (3)^3 - 3(3)^2 + 4(3) - 12$$

$$= 27 - 27 + 12 - 12 = 0$$

Thus, $x-3$ is a factor of $p(x)$

Similarly, ` $q(3) = 3 \times 3 - 9 = 0$

Thus, $x-3$ is a factor of $q(x)$ also.

Example 13. Find the value of a if x - 5 is a factor of the polynomial $x^3 - 3x^2 + ax - 10$.

Solution : As $x-5$ is a factor of $p(x) = x^3 - 3x^2 + ax - 10$

$\therefore$ $p(5) = 0$

Now, $p(5) = (5)^3 - 3(5)^2 + a(5) - 10 = 0$

$\Rightarrow$ $125 - 75 + 5a - 10 = 0$

$\Rightarrow$ $40 + 5a = 0$

Thus, $a = -\frac{40}{5} = -8$

The factor theroem is used to factorise some polynomials of degree 2 and 3. We are already familiar with the factorisation of quadratic polynomials like $ax^2 + bx + c$ where $a \neq 0$ and a, b, c are constants by splitting the middle term.

Let $ax^2 + bx + c = (px + q)(rx + s)$

$= prx^2 + (ps + qr)x + qs$

Comparing the coeffficients of both side, we get

$a = pr$

$b = ps + qr$

$c = qs$

Where b is the sum of two numbers ps and qr, whose product is

$$(ps)(qr) = (pr)(qs) = a \cdot c$$

Therefore, we can say that to factorise $ax^2 + bx + c$, we have to write b as the sum of those two numbers whose product is ac.

Example 14. Factorise $6x^2+17x+5$ by splitting the middle term and using the factor theorem.

Solution : 1. By spliting the middle term :

We have to split middle term 17 into such numbers whose sum is 17 and product is 6 × 5= 30

Factors of 30 are, $1\times 30=30$

$2\times 15=30$

$3\times 10=30$

$5\times 6=30$

The sum of the pair of 2 and 15 is 17. So

$$6x^2+17x+5=6x^2+(2+15)x+5$$

$$=6x^2+2x+15x+5$$

$$=2x(3x+1)+5(3x+1)$$

$$=(3x+1)(2x+5)$$

2. By factorisation theorem

$$6x^2+17x+5=6\left(x^2+\frac{17}{6}x+\frac{5}{6}\right)=6\cdot p(x)$$

Let the zeroes of $p(x)$ are a and b, then

$$6x^2+17x+5=6(x-a)(x-b)$$

So, $$ab=\frac{5}{6}$$

Now, possible values of a and b $=\pm\frac{1}{2},\ \pm\frac{1}{3},\ \pm\frac{5}{2},\pm\frac{5}{3},\pm 1$

Putting the values respectively, we have

$$p\left(\frac{1}{2}\right)=\frac{1}{4}+\frac{17}{12}+\frac{5}{6}\neq 0$$

$$p\left(-\frac{1}{3}\right)=\frac{1}{9}+\frac{17}{6}\times\frac{-1}{3}+\frac{5}{6}$$

$$=\frac{1}{9}-\frac{17}{18}+\frac{5}{6}=0$$

So $\left(x+\frac{1}{3}\right)$ is a factor of $p(x)$

Similarly, by trial, we can find that $\left(x+\frac{5}{2}\right)$ is a factor of $p(x)$

Therefore, $$6x^2+17x+5=6\left(x+\frac{1}{3}\right)\left(x+\frac{5}{2}\right)$$

$$=6\left(\frac{3x+1}{3}\right)\left(\frac{2x+5}{2}\right)$$

$$=(3x+1)(2x+5)$$

Example 15. Factorise $x^2-7x+12$ with the help of factor theorem.

Solution : Let $p(x)=x^2-7x+12$

Now, if $p(x)=(x-a)(x-b)$, then

Here, constant term $ab=12$

So, to look for the factors of $p(x)$ we find the factors of 12.

Factors of 12 = 1, 2, 3, 4, 6

$$p(3)=(3)^2-7(3)+12=0$$

So, $(x-3)$ is factor of $p(x)$

Similarly, $$p(4)=(4)^2-7(4)+12=0$$

So, $(x-4)$ is factor of $p(x)$

Thus, $x^2-7x+12=(x-3)(x-4)$

Example 16. Using the factor theorem, factorise the polynomial $x^4+x^3-7x^2-x+6$.

Solution : Let $p(x)=x^4+x^3-7x^2-x+6$

The factors of constant term 6 = $\pm 1, \pm 2, \pm 3$ and ± 6

$$p(1)=(1)^4+(1)^3-7(1)^2-1+6=8-8=0$$

So, $(x-1)$ is factor of $p(x)$.

Similarly, $$p(-1)=(-1)^4+(-1)^3-7(-1)^2-(-1)+6=8-8=0$$

So, $(x+1)$ is also a factor of $p(x)$

$$p(2)=(2)^4+(2)^3-7(2)^2-(2)+6=30-30=0$$

Therefore, $(x-2)$ is another factor of $p(x)$

$$p(-2)=(-2)^4+(-2)^3-7(-2)^2-(-2)+6=24-36\neq 0$$

So, $(x+2)$ is not a factor of $p(x)$

$$p(-3)=(-3)^4+(-3)^3-7(-3)^2-(-3)+6=90-90=0$$

So, $(x+3)$ is a factor of $p(x)$ ।

Since, $p(x)$ is a polynomial of degree 4, therefore it can not have the factors more than 4.

$\therefore \qquad p(x)=k(x-1)(x+1)(x-2)(x+3)$

$\Rightarrow \quad x^4+x^3-7x^2-x+6=k(x-1)(x+1)(x-2)(x+3)$ . . . (1)

put $x=0$, we get

$0+0+0-0+6=k(-1)(1)(-2)(3)$

$\Rightarrow \quad 6=6k$

$\Rightarrow \quad k=1$

Replacing 1 for k in the equation (i) we get

$x^4+x^3-7x^2-x+6=(x-1)(x+1)(x-2)(x+3)$

Exercise 3.4

1. Determine which of the following polynomials has $(x-1)$ as a factor :

(i) $x^4-2x^3-3x^2+2x+2$ (ii) $x^4+x^3+x^2+x+1$

(iii) $x^4+3x^3-3x^2+x-2$ (iv) $x^3-x^2-(2+\sqrt{3})x+\sqrt{3}$

2. Using the factor theorem, find if $g(x)$ is a factor of $p(x)$?

(i) $p(x)=3x^3-x^2-3x+1;\ g(x)=x+1$

(ii) $p(x)=2x^4-7x^3-13x^2+63x-45;\ g(x)=x-1$

(iii) $p(x)=3x^3+3x^2+3x+1;\ g(x)=x+2$

(iv) $p(x)=2x^3+x^2-2x-1;\ g(x)=2x+1$

3. Find the value of k, when $(x-5)$ is a factor of the polynomial $x^3-3x^2+kx-10$.

4. Find the value of k, when $(x-1)$ is a factor of the polynomial $2x^2+kx+\sqrt{2}$.

5. Find the value of a and b, if $(x+1)$ and $(x-1)$ are the factors of the polynomial $x^4+ax^3-3x^2+2x+b$.

6. Factorise :

(i) $3x^2+7x+2$ (ii) $4x^2-x-3$

(iii) $12x^2-7x+1$ (iv) $6x^2+5x-6$

7. Find the zeroes of the polynomials :

(i) $x^3+6x^2+11x+6$ (ii) x^3+2x^2-x-2

(iii) $x^4-2x^3-7x^2+8x+12$ (iv) x^3-2x^2-x+2

(v) x^3-3x^2-9x-5 (vi) $x^3-23x^2+142x-120$

Algebraic Identities :

In our earlier classes, we studied that an algebraic identity is an algebraic equation that is true for all values of the variables occuring in it. We have already studied the identities given below in our previous classes.

Identitiy I : $(x+y)^2=x^2+2xy+y^2$

Identitiy II : $(x-y)^2=x^2-2xy+y^2$

Identitiy III : $x^2-y^2=(x+y)(x-y)$

Identitiy IV : $(x+a)(x+b)=x^2+(a+b)x+ab$

All of above identities involved product of binomials. Let us extend the Identity I to a trimonial $x+y+z$. We shall compute $(x+y+z)^2$.

Let $x+y=t$, then

$$\therefore \quad (x+y+z)^2=(t+z)^2$$

$$=t^2+2tz+z^2 \quad \text{(Using Identity I)}$$

$$=(x+y)^2+2(x+y)z+z^2$$

Substituting the value of t, we have

$$=x^2+2xy+y^2+2xz+2yz+z^2$$

$$=x^2+y^2+z^2+2xy+2yz+2zx$$

So, we get the following identity :

Identity V : $(x+y+z)^2 = x^2+y^2+z^2+2xy+2yz+2zx$

Example 17. Expand : $(2x+4y+3z)^2$

Solution : On compairing with identity V.

$x = 2x, y = 4y, z = 3z$

On using the Identity V.

$$(2x+4y+3z)^2 = (2x)^2+(4y)^2+(3z)^2+2(2x)(4y)+2(4y)(3z)+2(3z)(2x)$$

$$= 4x^2+16y^2+9z^2+16xy+24yz+12zx$$

Example 18. Expand : $(2a-3b-4c)^2$

Solution : Using Identity V

$$(2a-3b-4c)^2 = [2a+(-3b)+(-4c)]^2$$

$$= (2a)^2+(-3b)^2+(-4c)^2+2(2a)(-3b)+2(-3b)(-4c)+2(-4c)(2a)$$

$$= 4a^2+9b^2+16c^2-12ab+24bc-16ac$$

Example 19. Factorise : $4x^2+y^2+z^2-4xy-2yz+4xz$

Solution : $4x^2+y^2+z^2-4xy-2yz+4xz$

$$= (2x)^2+(-y)^2+(z)^2+2(2x)(-y)+2(-y)(z)+2(z)(2x)$$

$$= (2x-y+z)^2 \quad \text{(Using Identity V)}$$

$$= (2x-y+z)(2x-y+z)$$

Now, let us extend identity I to compute $(x+y)^3$.

Here, $(x+y)^3 = (x+y)(x+y)^2$

$$= (x+y)(x^2+2xy+y^2)$$

$$= x(x^2+2xy+y^2)+y(x^2+2xy+y^2)$$

$$= x^3+2x^2y+xy^2+x^2y+2xy^2+y^3$$

$$= x^3+3x^2y+3xy^2+y^3$$

$$= x^3+y^3+3xy(x+y)$$

So, we get the following identites :

Identity VI : $(x+y)^3 = x^3+y^3+3xy(x+y)$

Replacing y by $-y$ in the above identity, we get

Identity VII : $(x-y)^3 = x^3 - y^3 - 3xy(x-y) = x^3 - y^3 - 3x^2y + 3xy^2$

Example 20. Using the identities expand the following expressions :

(i) $(4a+3b)^3$ (ii) $(3x-5y)^3$

Solution : (i) Comparing the expression with $(x+y)^3$, we find that

$$x = 4a \text{ and } y = 3b$$

$$(4a+3b)^3 = (4a)^3 + (3b)^3 + 3(4a)(3b)(4a+3b)$$

$$= 64a^3 + 27b^3 + 144a^2b + 108ab^2$$

(ii) Comparing the expression with $(x-y)^3$, we find that

$$x = 3x,\ \ y = 5y$$

$$\therefore \quad (3x-5y)^3 = (3x)^3 - (5y)^3 - 3(3x)(5y)(3x-5y)$$

$$= 27x^3 - 125y^3 - 135x^2y + 225xy^2$$

Example 21. Using the suitable identity evaluate each of the following :

(i) $(102)^3$ (ii) $(998)^3$

Solution : (i) $(102)^3 = (100+2)^3$

$= (100)^3 + (2)^3 + 3(100)(2)(100+2)$ (Using the Identity VI)

$= 1000000 + 8 + 60000 + 1200$

$= 1061208$

(ii) $(998)^3 = (1000-2)^3$

$= (1000)^3 - (2)^3 - 3(1000)(2)(1000-2)$

$= 1000000000 - 8 - 6000000 + 12000$

$= 994011992$

Example 22. Factorise : $8x^3 + 27y^3 + 36x^2y + 54xy^2$

Solution : $8x^3 + 27y^3 + 36x^2y + 54xy^2$

$= (2x)^3 + (3y)^3 + 3(2x)^2(3y) + 3(2x)(3y)^2$

$= (2x+3y)^3$ (Using Identity VI)

$= (2x+3y)(2x+3y)(2x+3y)$

Let us find out an important identity :

On expanding $(x+y+z)(x^2+y^2+z^2-xy-yz-zx)$, we get

$$= x\ (x^2 + y^2 + z^2 - xy - yz - zx) + y(x^2 + y^2 + z^2 - xy - yz - zx) +$$
$$z(x^2 + y^2 + z^2 - xy - yz - zx)$$
$$= x^3 + xy^2 + xz^2 - x^2y - xyz - x^2z + x^2y + y^3 + yz^2 - xy^2 - y^2z - xyz$$
$$+x^2z + y^2z + z^3 - xyz - yz^2 - xz^2$$
$$= x^3 + y^3 + z^3 - 3xyz \text{ (By simplification)}$$

So we obtain the following identity.

Identity VIII: $x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$

Example 23. Factorise : $27x^3 + y^3 + z^3 - 9xyz$

Solution : $27x^3 + y^3 + z^3 - 9xyz = (3x)^3 + (y)^3 + (z)^3 - 3(3x)(y)(z)$

$$= (3x + y + z)\left[(3x)^2 + y^2 + z^2 - 3x \cdot y - y \cdot z - z \cdot 3x\right]$$

(Using the identity (VIII))

$$= (3x + y + z)(9x^2 + y^2 + z^2 - 3xy - yz - 3zx)$$

Exercise 3.5

1. Use the suitable identities to find the product of :

   (i) $(x+3)(x+7)$ (ii) $(x-5)(x+8)$ (iii) $(2x+7)(3x-5)$

   (iv) $(5-3x)(3+2x)$ (v) $\left(x^2+\frac{3}{5}\right)\left(x^2-\frac{3}{5}\right)$ (vi) $(x+2)(x-5)$

2. Using the algebric identities, find the product of following :

   (i) 104×109 (ii) 94×97 (iii) 103×97

3. Using the suitable identities, factorise the following.

   (i) $x^2 + 6xy + 9y^2$ (ii) $x^2 - 4x + 4$ (iii) $\frac{x^2}{100} - y^2$

4. Expand the following with the help of suitable identities :

   (i) $(2a-3b-c)^2$ (ii) $(2+x-2y)^2$ (iii) $(a+2b+4c)$

   (iv) $(m+2n-5p)^2$ (v) $(3a-7b-c^2)^2$ (vi) $\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right)^2$

5. Factorise :

   (i) $9x^2 + 4y^2 + 16z^2 - 12xy - 16yz + 24xz$

(ii) $x^2+2y^2+8z^2+2\sqrt{2}xy-8yz-4\sqrt{2}xz$

6. Expand the following cubes :

(i) $(3a-2b)^3$ (ii) $(1+2x)^3$ (iii) $\left(\frac{2}{5}x+3\right)^3$ (iv) $\left(x-\frac{2}{3}y\right)^3$

7. Evaluate the following using suitable identities :

(i) $(98)^3$ (ii) $(103)^3$ (iii) $(999)^3$

8. Factorise :

(i) $x^3+8y^3+6x^2y+12xy^2$ (ii) $27a^3-8b^3-54a^2b+36ab^2$

(iii) $27-125x^3-135x+225x^2$ (iv) $125x^3-64y^3-300x^2y+240xy^2$

9. Factorise :

(i) $64a^3+27b^3$ (ii) $125x^3-8y^3$

10. Verify that :

(i) $x^3+y^3+z^3-3xyz=\frac{1}{2}(x+y+z)\left[(x-y)^2+(y-z)^2+(z-x)^2\right]$

(ii) $27a^3+b^3+c^3=(3a+b+c)\left[9a^2+b^2+c^2-3ab-bc-3ac\right]$

11. If $x+y+z=0$, then verify that $x^3+y^3+z^3=3xyz$

12. Using the suitable identitites compute :

(i) $(30)^3+(20)^3+(-50)^3$ (ii) $(-15)^3+(28)^3+(-13)^3$

[**Hint :**Use identity if $x+y+z=0$, then $x^3+y^3+z^3=3xyz$]

Important Points

1. A polynomial $p(x)$ in one variable x is an algebraic expression in x of the form

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$$

Where $a_0, a_1, a_2, \ldots$ are constant and $a_n \neq 0$

$a_0, a_1, a_2 \ldots a_n$ are respectively the coefficients of $x^0, x, x^2, \ldots$ and n is called the degree of the polynomial.

$a_n x^n, a_{n-1} x^{n-1}, \ldots, a_0$ where $a_n \neq 0$ is called a term of the polynomial $p(x)$

2. A polynomial of one term is called a monomial.
3. A polynomial of two terms is called a binomial.
4. A polynomial of three terms is called a trinomial.
5. A polynomial of degree one is called a linear polynomial.
6. A polynomial of degree two is called a quadratic polynomial.
7. A polynomial of degree three is called a cubic plynomial.
8. A real number 'a' is a zero of the polynomial $p(x)$, if $p(a) = 0$.
9. Every linear polynomial in one variable has a unique zero, a non-zero constant polynomial has no zero, and every real number is a zero of the zero polynomial.
10. Remainder Theorem : If $p(x)$ is any polynomial of degree greater than or equal to 1 and a is a real number. If $p(x)$ is divided by the linear polynomial $(x-a)$, then the remainder is $p(a)$
11. Factor theorem : $(x-a)$ is a factor of the polynomial $p(x)$, if $p(a) = 0$. Also if $x-a$ is a factor of $p(x)$, then $p(a) = 0$
12. $(x+y+z)^2 = x^2 + y^2 + z^2 + 2xy + 2yz + 2xz$
13. $(x+y)^3 = x^3 + y^3 + 3xy(x+y)$
14. $(x-y)^3 = x^3 - y^3 - 3xy(x-y)$
15. $x^3 + y^3 + z^3 - 3xyz = (x+y+z)(x^2 + y^2 + z^2 - xy - yz - zx)$

Answer

Exercise 3.1

1. (i) Polynomial, one variable.

(ii) Polynomial, one variable

(iii) and (v) are not polynomial, because each exponent is not a whole number.

(iv) Constant polynomial

(vi) Polynomial, three variables.

2. (i) 1 (ii) 0 (iii) 0 (iv) $\frac{\pi}{2}$

3. $5x^{45}+7$ (other polynomials may be possible)

4. $3x^{120}$ (other polynomials may be possible)

5. $2x^8+3x^4+5x$ (other polynomials are also possible)

6. Possible, write yourself.

7. (i) 3 (ii) 1 (iii) 0 (iv) 2

Exercise 3.2

1. (i) 10 (ii) –210 (iii) 12 (iv) –20

2. (i) 3, 1, 1 (ii) 3, 0, –1 (iii) 8, 1, 0 (iv) 0, 3, 2

4. (i) 4 (ii) 0 (iii) 0 (iv) –3

(v) $\frac{1}{2}$ (vi) $-\frac{7}{3}$ (vii) $-\frac{d}{c}$

Exercise 3.3

1. (i) 1 (ii) $\frac{31}{16}$ (iii) $\pi^4-\pi^3-3\pi^2-3\pi+1$

(iv) $-\frac{137}{16}$ (v) 1

2. 6a

3. Yes, because remainder is zero.

4. $a=-5$

Exercise 3.4

1. (i) and (iii) has a factor $(x-1)$;

(ii) and (iv) do not have a factor $(x-1)$

2. (i) Yes (ii) Yes (iii) No (iv) Yes

3. (i) $k = -8$

4. $k = -\left(2+\sqrt{2}\right)$

5. $a = -2, b = 2$

6. (i) $(x+2)(3x+1)$; (ii) $(x-1)(4x+3)$;

(iii) $(3x-1)(4x-1)$; (iv) $(2x+3)(3x-2)$

7. (i) $-1, -2, -3$; (ii) $-2, -1, 1$; (iii) $-2, -1, 2, 3$;

(iv) $-1, 1, 2$; (v) $-1, 5$; (vi) $1, 10, 12$

Exercise 3.5

1. (i) $x^2+10x+21$ (ii) $x^2+3x-40$ (iii) $6x^2+11x-35$

(iv) $15+x-6x^2$ (v) $x^4-\frac{9}{25}$ (vi) $x^2-3x-10$

2. (i) 11336 (ii) 9118 (iii) 9991

3. (i) $(x+3y)(x+3y)$ (ii) $(x-2)(x-2)$ (iii) $\left(\frac{x}{10}+y\right)\left(\frac{x}{10}-y\right)$

4. (i) $4a^2\times 9b^2+c^2-12ab+6bc-4ac$

(ii) $4+x^2+4y^2+4x-4xy-8y$

(iii) $a^2+4b^2+16c^2+4ab+16bc+8ac$

(iv) $m^2+4n^2+25p^2+4mn-20np-10pm$

(v) $9a^2+49b^2+c^2-42ab+14bc-6ac$

(vi) $\frac{x^2}{y^2}+\frac{y^2}{z^2}+\frac{z^2}{x^2}+\frac{2x}{z}+\frac{2y}{x}+\frac{2z}{y}$

5. (i) $(3x-2y+4z)^2$ (ii) $\left(x+\sqrt{2}y-2\sqrt{2}z\right)^2$

6. (i) $27a^3 - 8b^3 - 54a^2b + 36ab^2$ (ii) $1 + 8x^3 + 6x + 12x^2$

(iii) $\frac{8}{125}x^3 + 27 + \frac{36}{25}x^2 + \frac{54}{5}x$ (iv) $x^3 - \frac{8}{27}y^3 - 2x^2y + \frac{4}{3}xy^2$

7. (i) 941192 ; (ii) 1092727; (iii) 997002999

8. (i) $(x + 2y)^3$; (ii) $(3a - 2b)^3$; (iii) $(3 - 5x)^3$; (iv) $(5x - 4y)^3$

9. (i) $(4a + 3b)(16a^2 - 12ab + 9b^2)$; (ii) $(5x - 2y)(25x^2 + 10xy + 4y^2)$

12. (i) –90000; (ii) 16380