CHAPTER 06

# Methods of Indeterminate Analysis : Basic Methods

## 6.1 Introduction

If a structure is determinate then equations of equilibrium are enough to find reactions, shear force, bending moment and axial thrust at supports and members. The equilibrium conditions does not involved the use of properties of material and cross-section. But if, the structure is indeterminate then apart from equilibrium conditions extra compatibility conditions are required which depends upon the properties of material and cross-section. If properties of material are known then exact methods can be used to analyse the structure but if properties of material are not known then approximate methods may be used. There are two basic methods for analysis of structure

(i) Force methods (ii) Displacement methods

**(i) Force Method/Compatibility Method/Flexibility Method**

In this method, the redundant forces are chosen as unknowns and additional equations are obtained by considering the geometrical conditions imposed on the deformations of the structure. The examples of force methods are:

(a) Method of consistent deformation
(b) Virtual work/Unit load method
(c) Strain energy method
(d) Method of minimum potential energy
(e) Three moment theorem
(f) Column analogy method
(g) Flexibility matrix method

**(ii) Displacement Method/Equilibrium Method/Stiffness Method**

In this method, displacement of joints are taken as unknowns. The equilibrium equations are expressed in terms of these displacements and internal loads. These equations are solved to give the actual joint displacements, from which redundant forces can be calculated. The examples of displacement methods are:

(a) Moment distribution method
(b) Slope deflection method
(c) Kani's method
(d) Stiffness matrix method

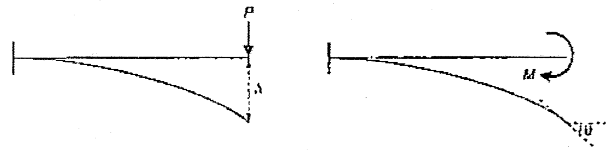
Comparison between Force and Displacement Method

| Force/Compatibility method | Displacement/Stiffness method |
|---|---|
| 1. The unknowns are taken as internal member forces or reactions. | 1. The unknowns are taken as joint displacement. |
| 2. To find unknowns compatibility conditions are written. Number of compatibility conditions needed = Number of redundant forces = Degree of static indeterminacy | 2. To find unknowns displacement, joint equilibrium conditions are written. Joint equilibrium conditions needed = degree of kinematic indeterminacy |
| 3. This method is suitable when $D_s < D_k$ | 3. This method is suitable when $D_k > D_s$ |
| 4. **Examples:** Strain energy method or minimum potential energy method, Castingliano's theorem, unit load method, virtual work method, column analogy method, three moment method, flexibility method. | 4. **Examples:** Moment distribution method, slope deflection method, kani's method, stiffness matrix method. |

## 6.2 Castigliano's Theorem

### 6.2.1 Castigliano's First Theorem

In any structure, the material of which is linear elastic, temperature is constant, support are unyielding, then "the first partial derivative of total strain energy with respect to any displacement component is equal to external force applied at that point in the direction of that displacement".

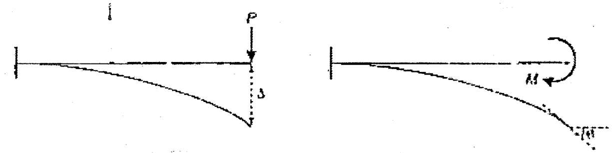



$$\frac{\partial U}{\partial \Delta} = P \qquad \frac{\partial U}{\partial \theta} = M$$

*Fig. 6.1*

### 6.2.2 Castigliano's Second Theorem

In any structure, the material of which is linear elastic, temperature is constant, supports are unyielding then "the first partial derivative of total strain energy with respect to any force is equal to deflection at that point in the direction of that force".




$$\frac{\partial U}{\partial P} = \Delta \qquad \frac{\partial U}{\partial M} = \theta$$

*Fig. 6.2*

## 6.3 Principle of Minimum Potential Energy

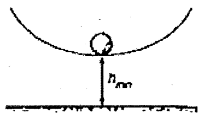



Fig.6.3

"Among all the geometrically compatible state of a structure, which satisfy deflection boundary conditions and force equilibrium requirement, will have final stable condition when its total potential energy is minimum".

OR

"If a structure is loaded and there are redundant reaction then true value of redundant reaction will be that for which total potential energy is minimum".

Consider a portal frame shown below.

Number of redundant in above frame

= Degree of static indeterminacy

$= D_S = 1$

∴ Redundant is one (say $R$)

If potential energy stored in portal frame is $U$ then the true value of redundant $R$ will be that for which $U$ is minimum.

If $U$ is minimum,

then, $\frac{\partial U}{\partial R} = 0$

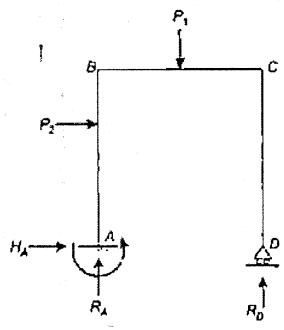



Fig.6.4

## 6.4 Betti's Law

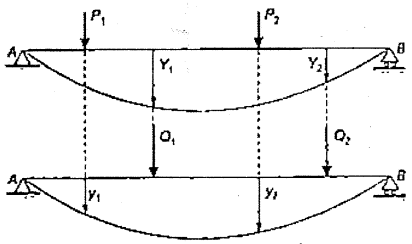



Fig.6.5

In any structure the material of which is linear elastic, and follow Hook's law, supports are unyielding and the temperature is constant, the external virtual work done by P-system of forces $P_1$, $P_2$, $P_3$ ...... during the distortion caused by Q-system of forces $Q_1$, $Q_2$, $Q_3$ ...... is equal to the external virtual work done by the Q-system of forces $Q_1$, $Q_2$, $Q_3$ ...... during the distortion caused by the P-system of forces $P_1$, $P_2$, $P_3$......

Virtual work done by $P$-system of forces is given by,

$$(W_e)_P = P_1y_1 + P_2y_2$$

and virtual work done by Q-system of forces is given by,

$$(W_e)_Q = Q_1Y_1 + Q_2Y_2$$

according to Betti's law,

$$P_1y_1 + P_2y_2 = Q_1Y_1 + Q_2Y_2$$

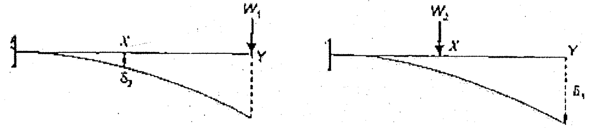
**Special Case-1:** If $\theta_1^*$ is rotational displacement in the direction of moment $M_1$ produced by some other moment $M_2$ and $\theta_2^*$ is rotational displacement in the direction of $M_2$ due to some other moment $M_1$; then external virtual work done by $M_1$ is equal to external virtual work done by $M_2$

$$M_1\theta_1^* = M_2\theta_2^*$$

**Special Case-2:** If $\theta^*$ is rotational displacement in the direction of any moment $M$ given by a linear force $P$ acting a some other point and if $\Delta^*$ is linear displacement in the direction of force $P$ due to any moment $M$ acting at some other point then external work done by $M$ is equals to external virtual work done by $P$.

$$M\theta^* = P\Delta^*$$

**Example 6.1** In the cantilever beam shown in the given figure, $\delta_2$ is the deflection under $X$ due to load $W_1$ at $Y$ and $\delta_1$ is the deflection under $Y$ due to load $W_2$ at $X$. The ratio of $\delta_1/\delta_2$ is




(a) $\frac{W_1}{W_2}$ (b) $\frac{W_2}{W_1 + W_2}$

(c) $\frac{W_2}{W_1}$ (d) $\frac{W_1}{W_1 + W_2}$

*Ans. (c)*

Virtual work done by $W_1$ during deflection caused by $W_2$ is given by,

$$(W_e)_{W_1} = W_1 \times \delta_1$$

and virtual work done by $W_2$ during deflection caused by $W_1$ is given by,

$$(W_e)_{W_2} = W_2 \times \delta_2$$

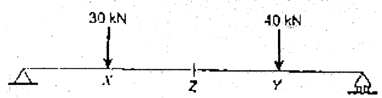
According to Betti's theorem

$$(W_e)_{W_1} = (W_e)_{W_2} \times \delta_2$$

$$W_1\delta_1 = W_2 \times \delta_2$$

$$\frac{\delta_1}{\delta_2} = \frac{W_2}{W_1}$$

**Example 6.2** The beam shown in figure carries load of 30 kN and 40 kN at points $X$ and $Y$ respectively and produces a deflection of 8 mm at $Z$.




To produce deflections of 8 mm and 5 mm at $X$ and $Y$ respectively, the load required at $Z$ would be

(a) 20 kN (b) 40 kN

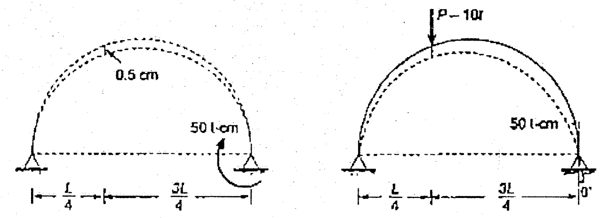
(c) 50 kN (d) 80 kN

*Ans. (c)*

According to Betti's theorem $W_z \times 8 = 30 \times 8 + 40 \times 5$

∴ $W_z = \frac{240 + 200}{8} = 55$ kN

**Example 6.3** For a semicircular 2 hinge arch shown in figure, a moment of 50 t-m is applied at $B$ which produces a displacement of 0.5 cm in vertical direction at $A$. If a concentrated load of 10t is applied at $A$ in vertical direction then rotation at $D$ will be




(a) 0.01 rad (b) 0.05 rad
(c) 0.1 rad (d) 0.5 rad

*Ans. (a)*

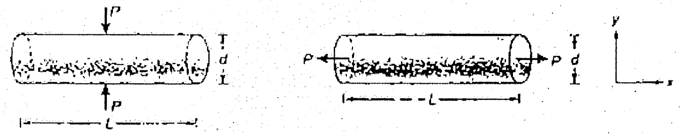
According to Betti's theorem,

$$P \cdot \delta^{*} = M\,\theta^{*}$$
$$-10 \times 0.5 \times 10^{-2} = -50 \times \theta^{*}$$
$$\theta^{*} = 0.01 \text{ radian}$$

**Example 6.4** Using Betti's theorem, determine the elongation of a bar 10 mm in diameter and 1 m long when it is subjected to a force of 10 kN at diametrically opposite points at midpoint of its length. $E = 2 \times 10^5$ MPa, $\mu = 0.3$.

*Solution:*

The loading on the bar may be shown as



Bettis' theorem says that the deflection along the axis of the bar due to force $P$ at centre is equal to the deflection at the centre due to force $P$ at the ends (axis).

The elongation along $x$-direction is given as

$$\Delta_x = \frac{PL}{AE}$$
$$\frac{\Delta_x}{L} = \frac{P}{AE}$$
$$\Rightarrow \quad \varepsilon_x = \frac{P}{AE}$$

$\therefore$ Strain in $y$-direction will be

$$\varepsilon_y = \mu\varepsilon_x$$
$$\Rightarrow \quad \varepsilon_y = \frac{\mu P}{AE}$$

But we know that,
$$\frac{\Delta_y}{D} = \varepsilon_y$$
$$\Rightarrow \quad \Delta_y = \frac{\mu P}{AE} \times D \qquad \left[\because A = \frac{\pi D^2}{4}\right]$$
$$\Rightarrow \quad \Delta_y = \frac{\mu P \times 4}{\pi D E}$$

Now if $\Delta$ is the deflection along the ends due to force P at the centre, then by Betti's theorem

$$\Delta = \Delta_y = \frac{4\mu P}{\pi D E}$$
$$= \frac{4 \times 0.3 \times 10 \times 10^3}{\pi \times 10 \times 2 \times 10^5} = 1.91 \times 10^{-3} \text{ mm}$$

## 6.5 Maxwell's Reciprocal Theorem

It is a special case of Betti's law in which $P = Q$. For example, in a beam or structure the deflection at any point $D$ due to load $W$ at any other point $C$ is the same as the deflection at $C$ due to the same load applied at $D$ i.e.

$$\delta_c = \Delta_d$$

Virtual work done by $W$ when applied at $C$
$$= W \times \delta_c$$
Virtual work done by $W$ when applied at $D$
$$= W \times \Delta_d$$

As per Betti's law,
$$W \times \delta_c = W \times \Delta_d$$
$$\delta_c = \Delta_d$$
(Known as Maxwell's reciprocal Law)

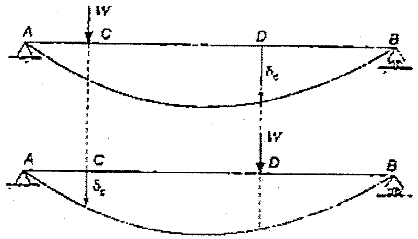
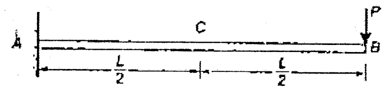


*Fig. 6.6*

This method can be applied for determinate and indeterminate beams both. Its special utility is seen in cantilever beams.

**Example 6.5** Find deflection at a distance $L/2$ from fixed end when load $P$ is applied at free end.




*Solution:*

According to Maxwell's reciprocal theorem, the deflection at $B$ due to load $P$ applied at $C$ will be same in above case i.e.

$$\delta_{BC} = \delta_{CB}$$

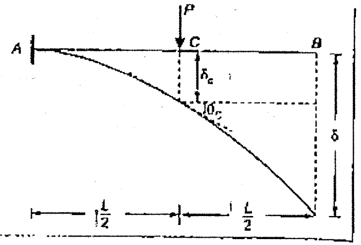
Here, $\delta_{BC}$ = deflection of $B$ when load at $C$

and $\delta_{CB}$ = deflection of $C$ when load at $B$

Applying load $P$ at $C$ and calculating deflection at $B$,

$$\delta_{C/B} = \delta_{B/C} = \delta_C + \theta_C \times \frac{L}{2}$$

$$= \frac{P\left(\frac{L}{2}\right)^3}{3EI} + \frac{P\left(\frac{L}{2}\right)^2}{2EI}\cdot\frac{L}{2} = \frac{5}{48}\frac{PL^3}{EI}$$




## 6.6 Principal of Virtual Work

**Case-1: For deformable bodies or elastic bodies:**

The total virtual work done is equal to zero

$$W_e + W_i = 0$$

where, $W_e$ = External virtual work done

$W_i$ = Internal virtual work done

The internal virtual work done ($W_i$) by internal resistance (such as stress) is negative.

Hence, $|W_e| = |W_i|$

**Case-2: For rigid bodies:**

In rigid bodies there is no internal deformation. Hence, internal virtual work done is always zero. Therefore external virtual work done is also zero.

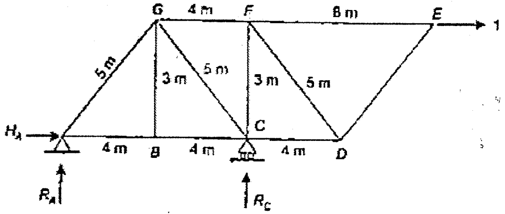
$$W_e = 0$$

**Example 6.6** For the truss shown in figure. Calculate the horizontal component of deflection at joint $E$ due to following movements of supports.

**(a)** Support $A$ moves horizontally by 0.0050 m from right to left and 0.0075 m vertically down.

**(b)** Support $C$ moves vertically down by 0.00250 m.

**Assume there is no change in length of any member because all members are rigid and joint displacement is due to movement of support only.**




***Solution:***

Let us apply unit load in horizontal direction at $E$. Let $R_A$, $H_A$, $R_C$ are reactions developed at $A$ and $C$ as shown in figure due to unit load at $E$.

$$\Sigma F_x = 0$$

$$\Rightarrow \quad H_A + 1 = 0$$

$$\therefore \quad H_A = -1 \quad (\rightarrow) \text{ or } 1\ (\leftarrow)$$

Also, $\Sigma F_y = 0$

$$\Rightarrow \quad R_A + R_C = 0 \quad \ldots(i)$$

Also, $\Sigma M_A = 0$

$$\Rightarrow \quad R_C \times 8 - 1 \times 3 = 0$$

$$\therefore \quad R_C = \frac{3}{8}(\uparrow)$$

From equation ($i$), $R_A = -\frac{3}{8}(\downarrow)$

Since displacement of support $A$ and $C$ is due to some other agency. Hence total external virtual work done should be zero (body is rigid).

$$W_e = \Sigma P\Delta^* = 0$$

$$\Rightarrow R_A \times \delta^*_{VA} + H_A \times \delta^*_{HA} + R_C \times \delta^*_{VC} + 1 \times \delta^*_{HE} = 0$$

$$\Rightarrow -\frac{3}{8}\times(-0.0075) + (-1)\times(0.0050) + \frac{3}{8}\times(-0.0025) + \delta^*_{HE} = 0$$

$$\therefore \quad \delta^*_{HE} = -0.006875 \text{ m} \quad \text{(From right to left)}$$

## 6.7 Principle of Superposition

Assumptions:

($i$) Material is isotropic homogenous and linearly elastic in which Hooke's law is valid.

($ii$) Temperature is constant

($iii$) Supports are unyielding

"In a beam, truss or frame which may be determinate or indeterminate. The resultant value of any stress function due to multiple loading is equal to the sum of effects of individual loading. The stress function may be SF, BM, reaction, slope, deflection, stress or strain".

## 6.8 Consistent Deformation of Method

This is the basic method to analyse Redundant structures taking force or Reaction as unknown. To find unknown force or reactions suitable compatibility conditions are written.

Consider a propped cantilever subjected to UDL over entire span as shown below:

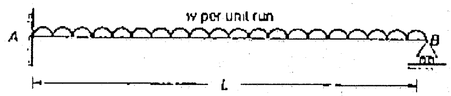



*Fig. 6.7*

For above propped cantilever, $D_S = r_e - 2$

$$= (2 + 1) - 2 = 1$$

Hence above beam is statically indeterminate to first degree. So SF and BM at any section cannot be obtained by using equations of static equilibrium alone.

If we consider Reaction $R_B$ as redundant and remove restraint offered by $R_B$, then remaining cantilever is statically determinate which is called primary structure.

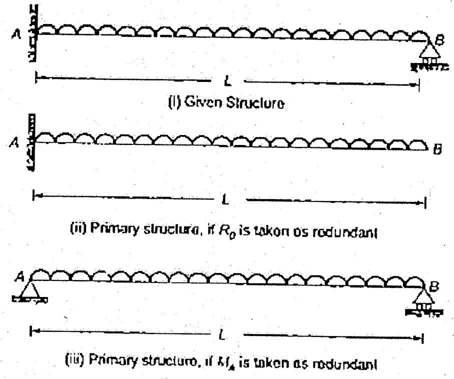



*Fig. 6.8*

Consider figure (ii) as primary structure.

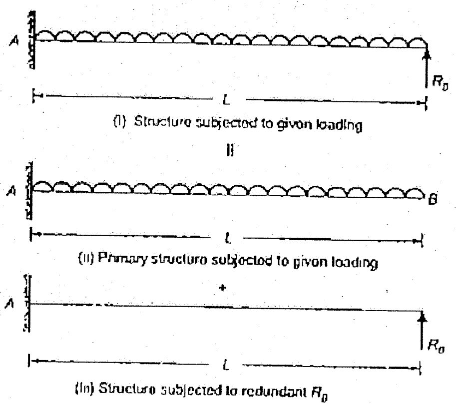



*Fig. 6.9*

Since at B, there is a unyielded support. Hence Net vertical deflection at $B$ will be zero.

$$\Delta_B = 0 \qquad \text{... (compatibility condition)}$$

$$\Rightarrow \quad \frac{wL^4}{8EI} - \frac{R_B L^3}{3EI} = 0$$

$$\Rightarrow \quad R_B = \frac{3}{8}wL$$

Now $R_B$ is known, the BM and SF diagrams can be drawn for the structure by superimposing the BM and SF diagrams of primary structures.

Shear Force Diagram:

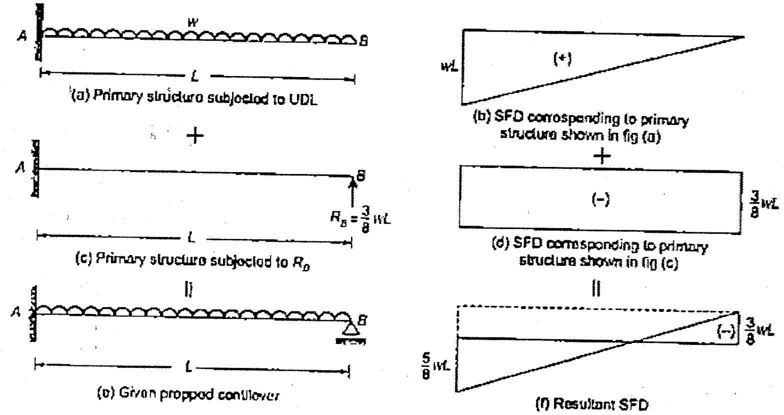



*Fig. 6.10*

Bending Moment Diagram:

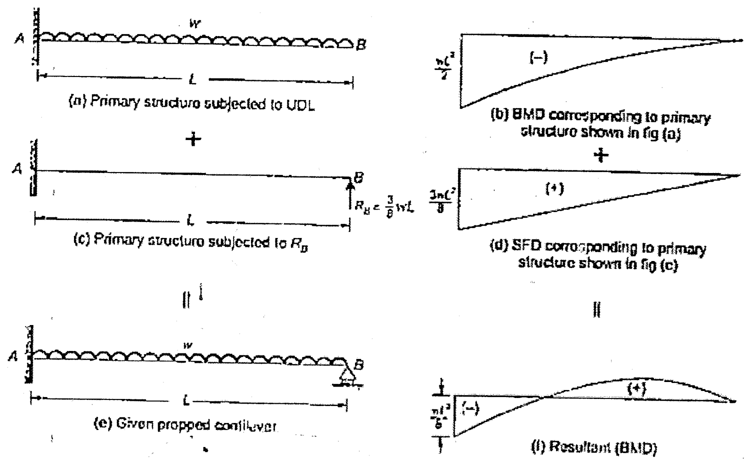



*Fig. 6.11*

If we consider $M_A$ as redundant in the analysis of propped cantilever as shown below:

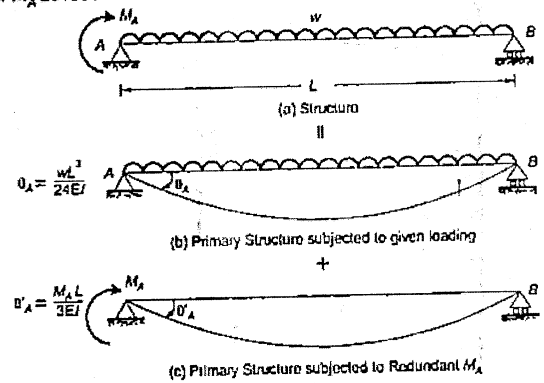



*Fig.6.12*

Since end $A$ in actual structure is fixed, hence at end $A$ net rotation (slope) will be zero.

$$\theta_A = 0 \quad \text{...(compatibility condition)}$$

$\therefore$

$$\theta_A + \theta_A' = 0$$

$$\frac{wL^3}{24EI} + \frac{M_A L}{3EI} = 0$$

$\therefore$

$$M_A = -\frac{wL^2}{8}$$

Here Negative sign indicate $M_A$ is just opposite as we assumed.

### Shear Force Diagram:

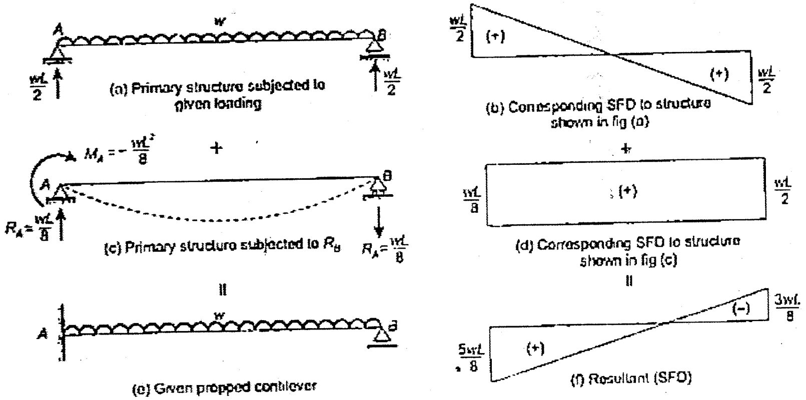



*Fig.6.13*

### Bending Force Diagram:

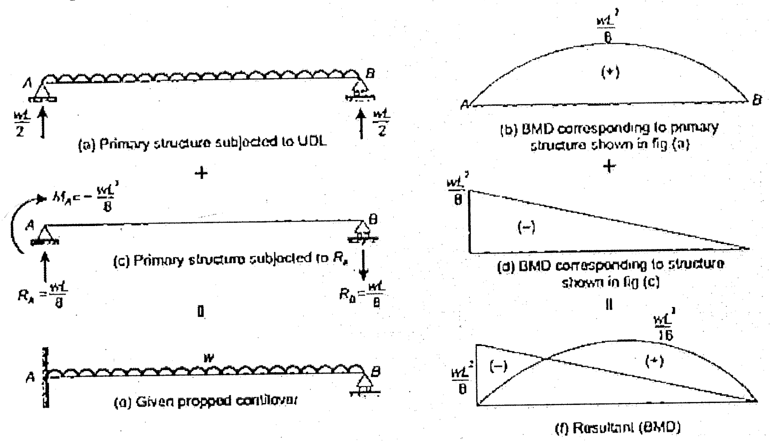
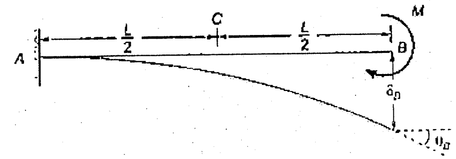
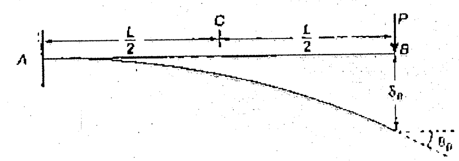
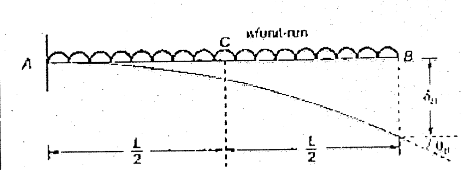
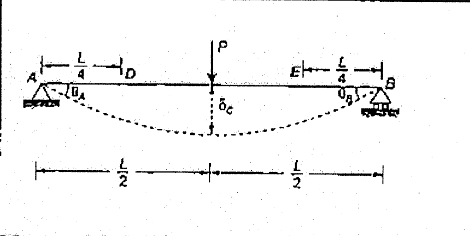
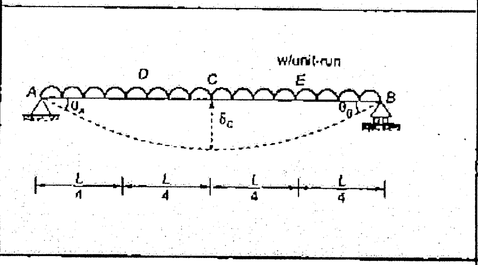
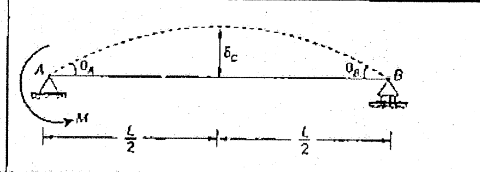
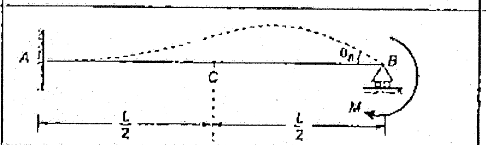


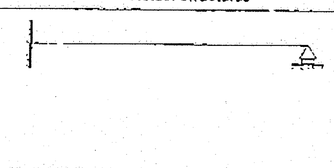
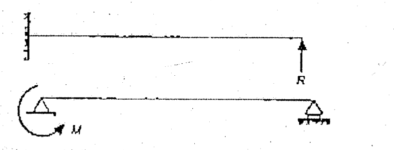
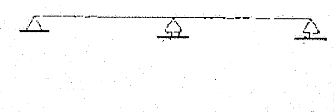
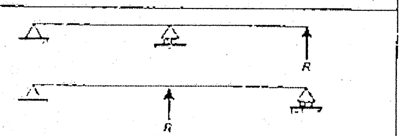
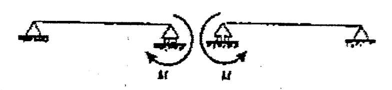
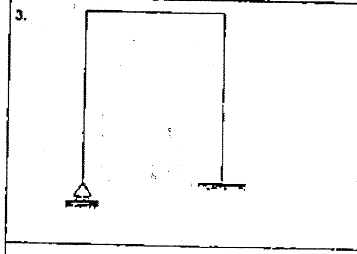
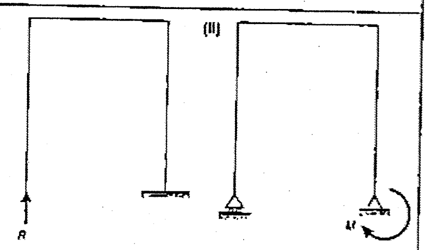
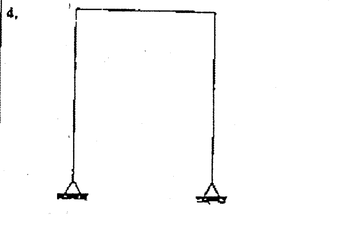
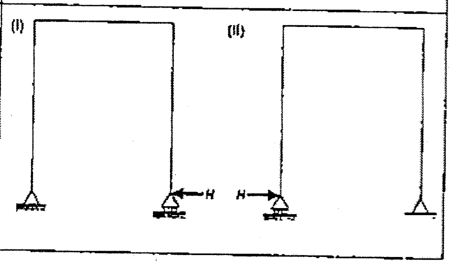
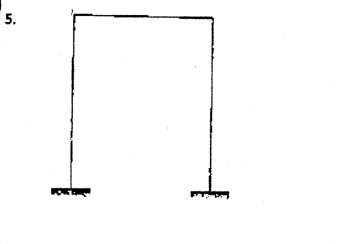
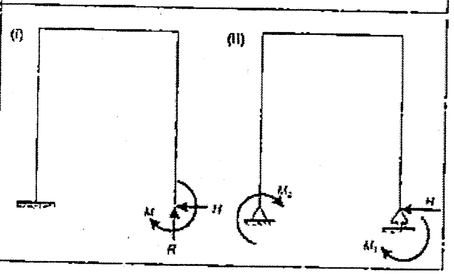

*Fig.6.14*

### Some Important Beam Displacement:

| S.No. | Beam | Displacement |
|---|---|---|
| 1. |  | $\theta_B = \frac{ML}{EI}$ <br> $\delta_B = \frac{ML^2}{2EI}$ <br> $\delta_C = \frac{3ML^2}{8EI}$ |
| 2. |  | $\theta_B = \frac{PL^2}{2EI}$ <br> $\delta_B = \frac{PL^3}{3EI}$ <br> $\delta_C = \frac{5}{48}\frac{PL^3}{EI}$ |
| 3. |  | $\theta_B = \frac{wL^3}{6EI}$ <br> $\delta_B = \frac{wL^4}{8EI}$ <br> $\delta_C = \frac{7wL^4}{192EI}$ |

| | | |
|---|---|---|
| 4. |  | $\theta_A = \frac{PL^2}{16EI}$<br>$\theta_B = -\frac{PL^2}{16EI}$<br>$\delta_C = \frac{PL^3}{48EI}$<br>$\delta_D = \delta_E = \frac{11PL^3}{768EI}$ |
| 5. |  | $\theta_A = \frac{wL^3}{24EI}$<br>$\theta_B = -\frac{wL^3}{24EI}$<br>$\delta_C = \frac{5}{384}\frac{wL^4}{EI}$<br>$\delta_D = \delta_E = \frac{19wL^4}{2048EI}$ |
| 6. |  | $\theta_A = -\frac{ML}{3EI}$<br>$\theta_B = \frac{ML}{6EI}$<br>$\delta_C = -\frac{ML^2}{16EI}$ |
| 7. |  | $\theta_B = \frac{ML}{4EI}$<br>$\delta_C = -\frac{ML^2}{32EI}$ |

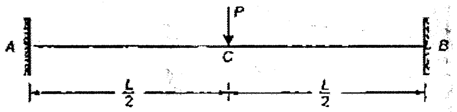
Some Possible Primary Structures:

| | Actual Structures | Possible Primary Structure |
|---|---|---|
| 1. |  | (I)  |
| 2. |  | (I)  |
| | | (III)  |
| 3. |  | (I)  |
| 4. |  | (I)  |
| 5. |  | (I)  |

### 6.8.1 Procedure of Analysis by Consistent Deformation Method

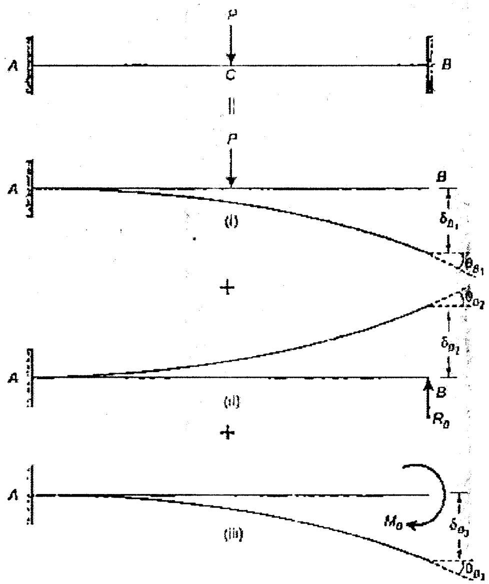
1. Find degree of indeterminacy.
2. Choose statically determinate and stable primary structure.
3. Identify the redundants and compute displacement of primary structure subjected to external loading and redundants separately at those points where conditions of compatibility may exists.
4. By using compatibility conditions get the values of those assumed redundants.
5. Plot SF and BM diagram by superimposing effect of redundants and given loading on the primary structure.

**Example 6.7** Analyse the Fixed beam shown in figure below:




***Solution:***

Let us assumed $M_B$ and $R_B$ as redundant. The corresponding primary structure is shown below.




Since end B is fixed in actual structure.

Hence,

(i) $\theta_B = 0$ ... First compatibility condition

(ii) $\delta_B = 0$ ... Second compatibility condition

Slope at end B: $$\theta_{B_1} = \frac{P(L/2)^2}{2EI} = +\frac{PL^2}{8EI} \ (\circlearrowright)$$

and $$\theta_{B_2} = -\frac{R_B L^2}{2EI} \ (\circlearrowleft)$$

and $$\theta_{B_3} = +\frac{M_B L}{EI} \ (\circlearrowright)$$

From compatibility condition (i),

$$\theta_B = 0$$

$$\Rightarrow \quad \theta_{B_1} + \theta_{B_2} + \theta_{B_3} = 0$$

$$\Rightarrow \quad \frac{PL^2}{8EI} - \frac{R_B L^2}{2EI} + \frac{M_B L}{EI} = 0$$

$$\Rightarrow \quad PL - 4R_B L + 8M_B = 0 \qquad \text{...(i)}$$

Deflection at end B:

$$\delta_{B_1} = \delta_C + \theta_C \times \frac{L}{2}$$

$$= \frac{P(L/2)^3}{3EI} + \frac{PL^2}{8EI} \times \frac{L}{2} = \frac{PL^3}{24EI} + \frac{PL^3}{16EI}$$

$$= \frac{5}{48}\frac{PL^3}{EI} (\downarrow)$$

and $$\delta_{B_2} = \frac{-R_B L^3}{3EI} (\uparrow)$$

and $$\delta_{B_3} = \frac{+M_B L^2}{2EI} (\downarrow)$$

From compatibility condition (ii),

$$\delta_B = 0$$

$$\Rightarrow \quad \delta_{B_1} + \delta_{B_2} + \delta_{B_3} = 0$$

$$\Rightarrow \quad \frac{5}{48}\frac{PL^3}{EI} - \frac{R_B L^3}{3EI} + \frac{M_B L^3}{2EI} = 0$$

$$\Rightarrow \quad 5\,PL - 16\,R_B L + 24\,M_B = 0 \qquad \text{...(ii)}$$

From (i), we have

$$4R_B L = PL + 8M_B \qquad \text{...(iii)}$$

Substituting value of $4R_B$ in (ii), we get

$$\Rightarrow \quad 5PL - 4(PL + 8M_B) + 24\,M_B = 0$$

$$M_B = \frac{PL}{8}$$

From (iii), we get

$$4R_B L = PL + 8\left(\frac{PL}{8}\right)$$

$$\therefore \quad R_B = \frac{P}{2}$$

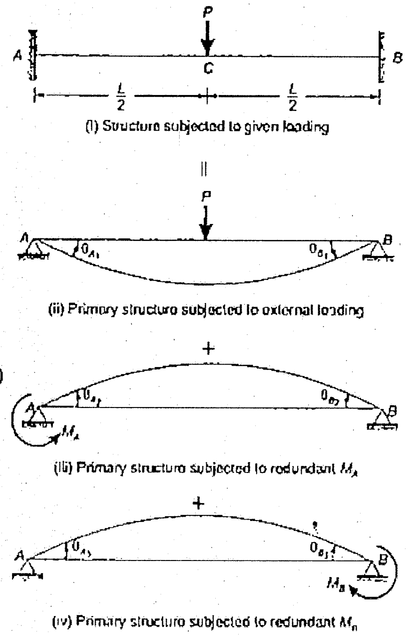
**Alternative Approach:** Let us assume $M_A$ and $M_B$ as redundant. The corresponding primary structure is shown below.

In actual structure Both, ends are fixed. Hence

(i) $\theta_A = 0$ ... compatibility condition (i)

(ii) $\theta_B = 0$ ... compatibility condition (ii)

Slope at end A:




$$\theta_{A_1} = +\frac{PL^2}{16EI}$$

$$\theta_{A_2} = -\frac{M_A L}{3EI}$$

and $$\theta_{A_3} = -\frac{M_B L}{6EI}$$

From compatibility condition (i),

$$\theta_A = 0$$

$$\Rightarrow \quad \theta_{A_1} + \theta_{A_2} + \theta_{A_3} = 0$$

$$\Rightarrow \quad \frac{PL^2}{16EI} - \frac{M_A L}{3EI} - \frac{M_B L}{6EI} = 0$$

$$\Rightarrow \quad 16M_A + 8M_B = 3PL \quad ...(i)$$

Slope at end B:

$$\theta_{B_1} = -\frac{PL^2}{16EI}$$

$$\theta_{B_2} = +\frac{M_A L}{3EI}$$

and $$\theta_{B_3} = +\frac{M_B A}{3EI}$$

From compatibility condition (ii),

$$\theta_B = 0$$

$$\Rightarrow \quad \theta_{B_1} + \theta_{B_2} + \theta_{B_3} = 0$$

$$\Rightarrow \quad -\frac{PL^2}{16EI} + \frac{M_A L}{6EI} + \frac{M_B L}{3EI} = 0$$

$$\Rightarrow \quad 8M_A = (3PL - 16M_B)$$

Substituting value of $8M_A$ into (i), we get

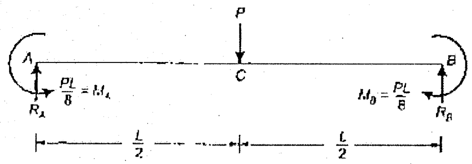
$$\Rightarrow \quad 2(3PL - 16M_B) + 8M_B = 3PL$$

$$\Rightarrow \quad M_B = \frac{PL}{8}$$

From (i),

$$\Rightarrow \quad 16M_A + 8\left(\frac{PL}{8}\right) = 3PL$$

$$\Rightarrow \quad M_A = \frac{PL}{8}$$




$$\Sigma F_y = 0; \quad R_A + R_B = P \quad ...(A)$$

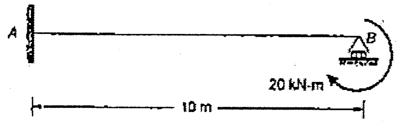
$$\Sigma M_B = 0; R_A \times L - \frac{PL}{8} - \frac{P \times L}{2} + \frac{PL}{8} = 0$$

$$R_A = \frac{P}{2}(\uparrow)$$

From (A), we get

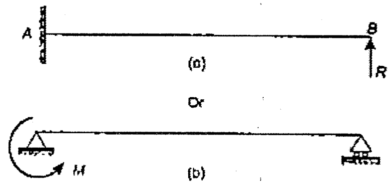
$$\therefore \quad R_B = \frac{P}{2}(\uparrow)$$

**Example 6.8** Analyse the propped cantilevers shown in figure.

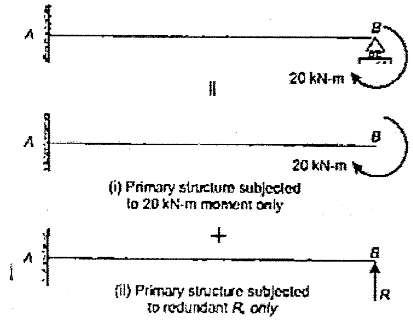



***Solution:***

(i) Possible redundant reactions are




Considering figure (a) as primary structure




Since at end B, there is a unyielded support. Hence Net deflection at end B will be zero.

$\Delta_B = 0$ Compatibility condition... (i)

Downward deflection at B due to 20 kN-m moment at B,

$$\Delta_{BM} = \frac{ML^2}{2EI} = \frac{20 \times 10^2}{2EI} = \frac{1000}{EI}(\downarrow)$$

Upward deflection at $B$ due to redundant Reaction $R$ at $B$,

$$\Delta_{BR} = \frac{RL^3}{3EI} = \frac{R(10)^3}{3EI} = \frac{1000R}{3EI}$$

From compatibility condition (i),

$$\Delta_{BR} - \Delta_{BM} = 0$$

$\Rightarrow$ $$\frac{1000R}{3EI} - \frac{1000}{EI} = 0$$

$\Rightarrow$ $$R = 3 \text{ kN} (\uparrow)$$

$$\Sigma F_y = 0$$

$\Rightarrow$ $$R_A + R_B = 0$$

$\Rightarrow$ $$R_A + 3 = 0$$

$\therefore$ $$R_A = -3 \text{ kN} (\uparrow) \text{ or } 3 \text{ kN} (\downarrow)$$

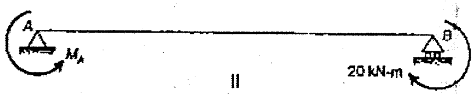
Now, $$\Sigma M_A = 0$$

$\Rightarrow$ $$M_A + 20 - R_B \times 10 = 0$$

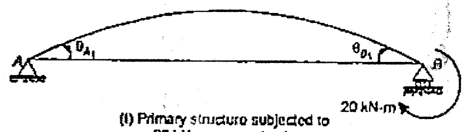
$$M_A = 10 \text{ kN-m (clockwise direction)}$$

**Alternative solution:**

Considering figure (b) as primary structure

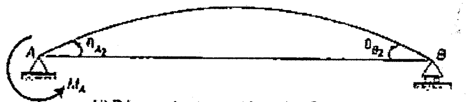



||




(i) Primary structure subjected to 20 kN-m moment only

+




(ii) Primary structure subjected to Redundant $M_A$ only

Since support $A$ is fixed. Hence Net rotation at $A$ will be

$\therefore$ $$\theta_A = 0 \qquad \text{Compatibility condition ...(i)}$$

Rotation at $A$ due to 20 kN-m moment at $B$.

$$\theta_{A_1} = \frac{ML}{6EI} = \frac{20 \times 10}{6EI} = \frac{110}{3EI} (\circlearrowright)$$

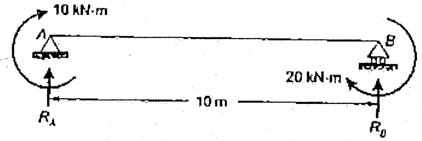
$$\theta_{A_2} = \frac{ML}{3EI} = \frac{M_A \times 10}{3EI} = \frac{10M_A}{3EI} (\circlearrowright)$$

From compatibility condition (i),

$$\theta_{A_1} + \theta_{A_2} = 0$$

$\Rightarrow$ $$\frac{100}{3EI} + \frac{10M_A}{3EI} = 0$$

$\Rightarrow$ $$M_A = -10 \text{ kN-m} (\circlearrowright) \text{ or } 10 \text{ kN-m} (\circlearrowleft)$$




$$\Sigma F_y = 0$$

$\Rightarrow$ $$R_A + R_B = 0 \qquad \text{...(A)}$$

$$\Sigma M_A = 0$$

$\Rightarrow$ $$10 + 20 - R_B \times 10 = 0$$

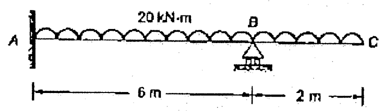
$\Rightarrow$ $$R_B = 3 \text{ kN} (\uparrow)$$

From (A),

$$R_A = -R_B = -3 \text{ kN} (\uparrow)$$

or $$R_A = 3 \text{ kN} (\downarrow)$$

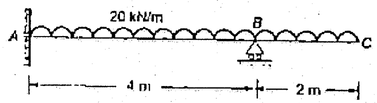
**Example 6.9** Determine the reaction components in the beam shown in figure.



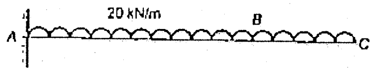

***Solution:***

$$D_S = r_e - 2 \quad \text{(for vertical loading)}$$
$$= 3 - 2 = 1$$

Considering reaction $R_B$ as redundant

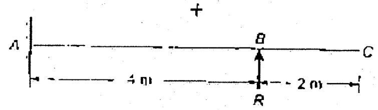



||




(i) Primary structure subjected to UDL only

+

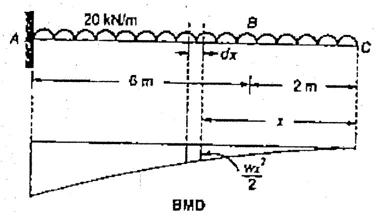



(ii) Primary structure subjected to Redundant R only

Since at end $B$, there is a unyielding support. Hence net deflection at $B$ will be zero.

$$\Delta_B = 0 \qquad \text{Compatibility condition ...(i)}$$

Downward deflection at B due to given UDL:




$$\Delta_{BL} = \frac{\partial U}{\partial W} = \int_2^8 \frac{wx^2}{2EI}(x-2)\,dx$$

$$= \frac{w}{2EI}\int_2^8 \left(x^3 - 2x^2\right)dx$$

$$= \frac{20}{2EI}\left[\frac{x^4}{4} - \frac{2\cdot x^3}{3}\right]_2^8$$

$$= \frac{10}{EI} \times 684 = \frac{6840}{EI}(\downarrow)$$

Upward deflection at $B$ due to redundant $R$:

$$\Delta_{BR} = \frac{RL_{AB}^3}{3EI} = \frac{R \times 6^3}{3EI} = \frac{72R}{EI}(\uparrow)$$

From compatibility condition (i),

$$\Delta_{BR} - \Delta_{BL} = 0$$

$$\Rightarrow \quad \frac{72R}{EI} - \frac{6840}{EI} = 0$$

$$\Rightarrow \quad R = 95 \text{ kN} (\uparrow)$$

$$\Sigma F_y = 0$$

$$\Rightarrow \quad R_A + R_B = 20 \times 8$$

$$\Rightarrow \quad R_A = 160 - 95$$

$$\Rightarrow \quad R_A = 65 \text{ kN} (\uparrow)$$

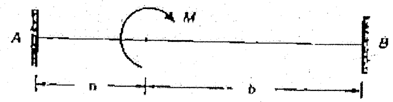
$$\Sigma M_A = 0$$

$$\Rightarrow \quad M_A + 20 \times 8 \times 4 - R_B \times 6 = 0$$

$$\Rightarrow \quad M_A + 640 - 95 \times 6 = 0$$

$$\Rightarrow \quad M_A = -70 \text{ kN-m or } 70 \text{ kN-m } (\circlearrowleft)$$

**Example 6.10** Determine the fixed end moment developed in the fixed beam shown in figure below:



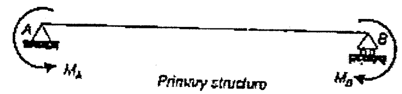

***Solution:***

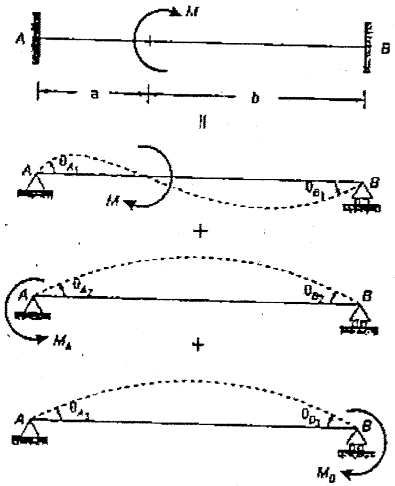
$$D_S = r_e - 2 \qquad \text{(vertical loading)}$$

Here $\quad r_e = 4$

$\therefore \quad D_S = 4 - 2 = 2$

Thus the above beam is indeterminate to 2nd degree. Assuming moment reaction at $A$ and $B$ as redundant, resulting primary structure will be simply supported beam as shown below.



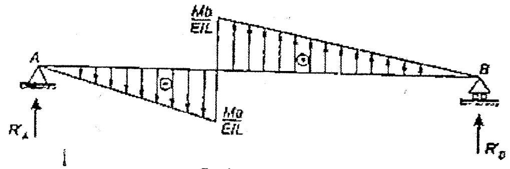




Since both ends $A$ and $B$ are fixed. Hence Net rotations at $A$ and $B$ will be zero.

$$\theta_A = 0 \qquad \text{...(i) (Compatibility condition)}$$

$$\theta_B = 0 \qquad \text{...(ii) (Compatibility condition)}$$

Rotation $\theta_{A_1}$ in figure due to eccentric moment on simply supported beam can be found out by conjugate beam method.




*Conjugate beam*

$$\theta_A = R'_A = \frac{\Sigma M_B}{L}$$

$$= \frac{1}{L}\left[-\frac{Ma}{2EIL}a\left(b + \frac{a}{3}\right) + \frac{Mb}{EIL} \times \frac{b}{2} \times \frac{2b}{3}\right]$$

$$= -\frac{Ma}{6EIL^2}\left[2b^3 - a^2(3b + a)\right] \circlearrowleft$$

$$= \frac{Ma}{L^3}\left[2b^2 - 2a^2 + a^2 + ab\right]$$

Similarly,

$$\theta_{B_1} = R'_B = \frac{\Sigma M_A}{L}$$

$$= \frac{1}{L}\left[-\frac{1}{2}\frac{Ma}{EIL}a\cdot\frac{2a}{3} - \frac{1}{2}b\frac{Mb}{EIL}\left(a + \frac{b}{3}\right)\right]$$

$$= -\frac{M}{6EIL^2}\left[2a^3 - b^2(3a + b)\right]$$

$$= \frac{M}{6EIL^2}\left[b^2(3a + b) - 2a^3\right] \ (\circlearrowleft)$$

Rotation $\theta_{A_2}$ and $\theta_{B_2}$ for figure is given by

$$\theta_{A_2} = \frac{M_A L}{3EI} \ (\circlearrowright)$$

and

$$\theta_{B_2} = \frac{M_A L}{6EI} \ (\circlearrowleft)$$

Rotation $\theta_{A_3}$ and $\theta_{B_3}$ for figure is given by

$$\theta_{A_3} = \frac{M_B L}{6EI}$$

and

$$\theta_{B_3} = \frac{M_B L}{3EI}$$

From compatibility condition (i), we get

$$\theta_{A_1} + \theta_{A_2} + \theta_{A_3} = 0$$

$$\Rightarrow \quad -\frac{M}{6EIL^2}\left[2b^3 - a^2(3b + a)\right] + \frac{M_A L}{3EI} + \frac{M_B L}{6EI} = 0$$

$$\Rightarrow \quad \frac{M}{6EIL^2}\left[2b^3 - a^2(3b + a)\right] = \frac{L}{6EI}\left[2M_A + M_B\right]$$

$$\Rightarrow \quad 2M_A + M_B = \frac{M}{L^3}\left[2b^3 - a^2(2b + a)\right] \quad \text{...(A)}$$

From compatibility condition (ii), we get

$$\theta_{B_1} + \theta_{B_2} + \theta_{B_3} = 0$$

$$\Rightarrow \quad -\frac{M}{6EIL^2}\left[b^2(3a + b) - 2a^3\right] + \frac{M_A L}{6EI} + \frac{M_B L}{3EI} = 0$$

$$\Rightarrow \quad M_A + 2M_B = \frac{M}{L^3}\left[b^2(3a + b) - 2a^3\right] \quad \text{...(B)}$$

Subtracting (B) from twice (A), we get

$$3M_A = \frac{2M}{L^3}\left[2b^2 - a^2(3b + a)\right] - \frac{M}{L^3}\left[b^2(3a + b) - 2a^3\right]$$

$$\Rightarrow \quad M_A = \frac{M}{L^3}\left[b^3 - 2a^2 b - ab^2\right]$$

$$= \frac{Mb}{L^3}\left[b^2 - 2a^2 - ab\right]$$

$$= \frac{Mb}{L^3}\left[b^2 - a^2 - a^2 - ab\right]$$

$$= \frac{Mb}{L^3}\left[(b^2 - a^2) - a(a + b)\right]$$

$$= \frac{Mb}{L^3}\left[(b + a)(b - a) - a(a + b)\right]$$

$$= \frac{Mb}{L^3}(a + b)(b - 2a) \qquad [\because L = a + b]$$

$$= \frac{Mb(L - 3a)}{L^2} \ \circlearrowright \ \text{or} \ \frac{Mb(3a - L)}{L^3} \ \circlearrowleft$$

From (B),

$$2M_B = \frac{M}{L^3}\left[b^2(3a + b) - 2a^3\right] - M_A$$

$$\Rightarrow \quad M_B = \frac{M}{L^3}\left[2b^2 a - a^3 + a^2 b\right]$$

$$= \frac{Ma}{L^3}\left[2b^2 - a^2 + ab\right]$$

$$= \frac{Ma}{L^3}\left[2b^2 - 2a^2 + a^2 + ab\right]$$

$$= \frac{Ma}{L^3}\left[2(b^2 - a^2) + a(a + b)\right]$$

$$= \frac{Ma}{L^3}\left[2(a + b)(b - a) + a(a + b)\right]$$

$$= \frac{Ma}{L^3}\left[(a + b)\{2(b - a) + a\}\right]$$

$$= \frac{Ma}{L^3}(a + b)(2b - a)$$

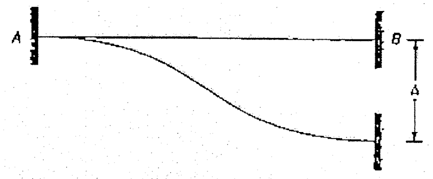
$$= \frac{Ma}{L^2}\cdot L\left[2b - (L - b)\right] \qquad [\because L = a + b]$$

$$= \frac{Ma}{L^2}(3b - L) \ (\circlearrowleft)$$

Thus, $\bar{M}_{AB} = M_A = \frac{Mb}{L^2}(3a - L)$ (↺)

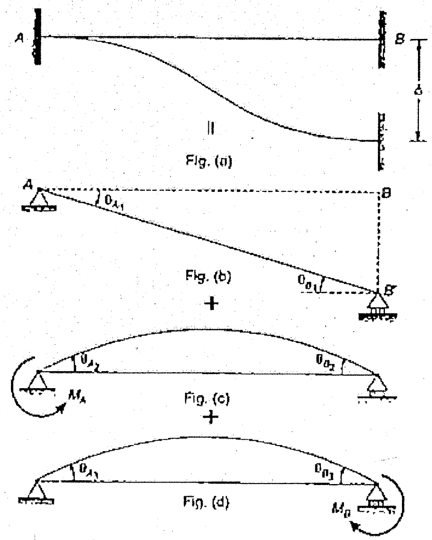
and $\bar{M}_{BA} = M_B = \frac{Ma}{L^2}(3b - L)$ (↻)

**Example 6.11** Determine the fixed end moments developed in a fixed beam of span $L$ and flexural rigidity $EI$ when the right hand side support settles down by $\Delta$.




***Solution:***

Let the moment reaction at $A$ and $B$ as redundant, Resulting primary structure will be simply supported beam as shown below:




Since both ends $A$ and $B$ are fixed. Hence Net rotations at $A$ and $B$ will be zero.

$\theta_A = 0$ ...(i) (Compatibility condition)

$\theta_B = 0$ ...(ii) (Compatibility condition)

$\theta_{A_1}$ and $\theta_{B_1}$ for fig (b) is given by

$$\theta_{A_1} = \frac{\Delta}{L} \text{ (↺)}$$

and $$\theta_{B_1} = \frac{\Delta}{L} \text{ (↺)}$$

$\theta_{A_2}$ and $\theta_{B_2}$ for fig (c) is given by

$$\theta_{A_2} = \frac{M_A L}{3EI} \text{ (↻)}$$

and $$\theta_{B_2} = \frac{M_A L}{6EI} \text{ (↺)}$$

$\theta_{A_3}$ and $\theta_{B_3}$ for fig (d) is given by

$$\theta_{A_3} = \frac{M_B L}{6EI} \text{ (↻)}$$

and $$\theta_{B_3} = \frac{M_B L}{3EI} \text{ ↺}$$

From compatibility condition (i), we get

$$\theta_{A_1} + \theta_{A_2} + \theta_{A_3} = 0$$

$$\Rightarrow \quad -\frac{\Delta}{L} + \frac{M_A L}{3EI} + \frac{M_B L}{6EI} = 0$$

$$\Rightarrow \quad 2M_A + M_B = \frac{6EI\Delta}{L^2} \qquad \text{...(A)}$$

From compatibility condition (ii), we get

$$\theta_{B_1} + \theta_{B_2} + \theta_{B_3} = 0$$

$$\Rightarrow \quad -\frac{\Delta}{L} + \frac{M_A L}{6EI} - \frac{M_B L}{3EI} = 0$$

$$\Rightarrow \quad M_A + 2M_B = -\frac{6EI\Delta}{L^2} \qquad \text{...(B)}$$

Substracting ($B$) from twice $A$, we get

$$3M_A = \frac{12EI\Delta}{L^2} + \frac{LEI\Delta}{L^2}$$

$$\therefore \quad M_A = \frac{6EI\Delta}{L^2} \text{ ↺}$$

From ($A$) & ($B$), we get

$$M_B = \frac{6EI\Delta}{L^2} - 2M_A = \frac{6EI\Delta}{L^2} - \frac{12EI\Delta}{L^2}$$

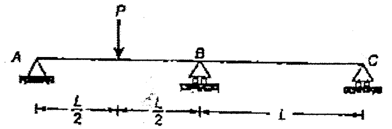
$$= -\frac{6EI\Delta}{L^2} \text{ ↺}$$

$$= \frac{6EI\Delta}{L^2} \text{ ↻}$$

Thus $\quad \bar{M}_{AB} = M_A = \dfrac{6EI\Delta}{L^2}$ ↻

and $\quad \bar{M}_{BA} = M_B = \dfrac{6EI\Delta}{L^2}$ ↻

**Example 6.12** Analyse the continuous beam shown in figure




*EI* is constant

***Solution:***

$$D_S = r_e - 2 \qquad \text{(vertical loading)}$$

Here, $\quad r_e = 1 + 1 + 1 = 3$

$\therefore \quad D_S = 3 - 2 = 1$

Above beam is indeterminate to first degree. Considering Reaction $R_B$ as redundant.

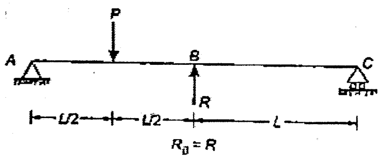
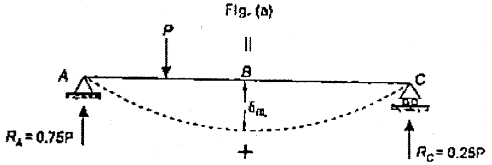



Fig. (a)

||




\+

Fig. (b)

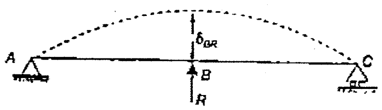



Fig. (c)

Since there is a unyielding support at *B*. Hence net deflection at *B* will be zero.

$$\Delta B = 0 \qquad \text{...(i) Compatibility condition}$$

Downward deflection at *B* due to given loading:

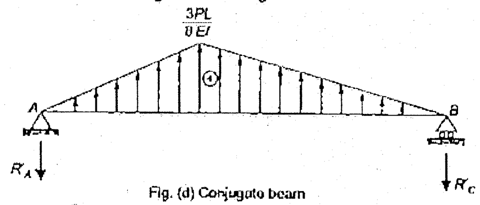



Fig. (d) Conjugate beam

Reactions in conjugate beam:

$$R'_A + R'_C = \frac{1}{2} \times 2L \times \frac{3PL}{8EI} = \frac{3PL}{8EI} \qquad \text{...(1)}$$

$$\Sigma M_C = 0$$

$$\Rightarrow \quad 2LR'_A = \left(\frac{1}{2} \times \frac{1}{2} \times \frac{3PL}{8EI}\right) \times \left(\frac{3L}{2} + \frac{L}{6}\right) + \left(\frac{1}{2} \times \frac{3PL}{8EI} \times \frac{3L}{2} \times \frac{2}{3} \times \frac{3L}{2}\right)$$

$$\Rightarrow \quad R'_A = \frac{7PL^2}{16EI}$$

and $\quad R'_C = \dfrac{5PL^2}{32EI}$

$\Delta_{BL}$ = BM at *B* in conjugate beam

$$\Rightarrow \quad \Delta_{BL} = \frac{11}{96} \frac{PL^3}{EI}$$

Upward deflection at *B* due to redundant $R_B$:

$$\Delta_{BR} = \frac{R(2L)^3}{48EI} = \frac{RL^3}{6EI}$$

From compatibility condition (i), we get

$$\Delta_{BR} - \Delta_{BL} = 0$$

$$\Rightarrow \quad \frac{RL^3}{6EI} - \frac{11}{96}\frac{PL^3}{EI} = 0$$

$$\Rightarrow \quad R_B = \frac{11}{16}P(\uparrow)$$

$$\Sigma F_y = 0$$

$$\Rightarrow \quad R_A + R_B + R_C = P$$

$$\Rightarrow \quad R_A + R_C = \frac{5}{16}P \qquad \text{...(A)}$$

$$\Sigma M_A = 0$$

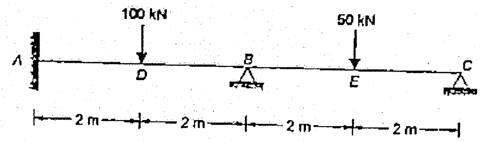
$$\Rightarrow \quad R_C \times 2L - P \times \frac{3L}{2} + \frac{11}{16}PL = 0$$

$$\Rightarrow \quad R_C = \frac{13}{32}P(\uparrow)$$

From equation (A),

$$R_A = \frac{5}{16}P - R_C = -\frac{3P}{32} \text{ or } \frac{3P}{32}(\downarrow)$$

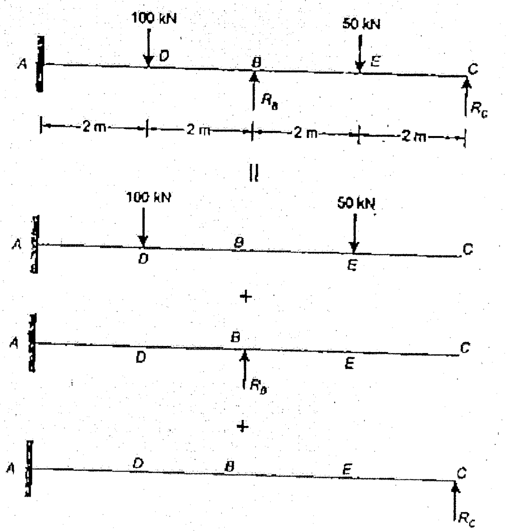
**Example 6.13** Analyse the beam shown in figure by consistent deformation method.




*Solution:*

$$D_S = r_e - 2 \quad \text{...(For vertical loading)}$$
$$= 4 - 2$$
$$= 2$$

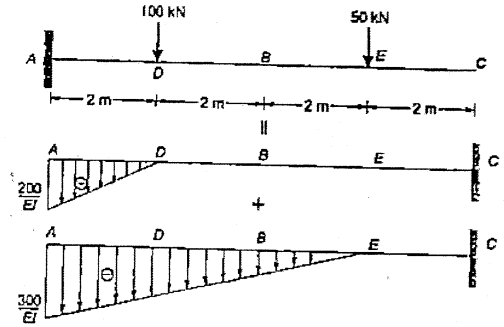
Thus the above beam is indeterminate to 2nd degree. Assuming $R_B$ and $R_C$ as redundant.




Since there are unyielding supports at $B$ and $C$. Hence net vertical deflection at $A$ and $C$ will be zero.

$$\Delta_B = 0 \quad \text{...(i) (compatibility condition)}$$
$$\Delta_C = 0 \quad \text{...(ii) (compatibility condition)}$$

Downward deflection at $B$ and $C$ due to given loading be found by conjugate beam method.




*Conjugate beam by part*

$\Delta_{BL}$ = BM at $B$ in conjugate beam

$$= -\left[\frac{1}{2}\times 2\times\frac{200}{EI}\right]\times\left(2+2\times\frac{2}{3}\right)-\left[\frac{1}{2}\times 6\times\frac{300}{EI}\right]\times\left(\frac{2}{3}\times 6\right)$$

$$= \frac{4266.67}{EI}(\downarrow)$$

and $\Delta_{CL}$ = BM at $C$ in conjugate beam

$$= \frac{200}{EI}\times\left(6+\frac{2}{3}\times 2\right)+\frac{900}{EI}\times\left(2+\frac{2}{3}\times 6\right) = \frac{6866.67}{EI}(\downarrow)$$

Vertically upward deflection at $B$ and $C$ due to redundant can be given as

Deflection at $B$ due to $R_B$, $\quad \Delta_{B,RB} = \frac{R_B(4)^3}{3EI} = \frac{64}{3}\frac{R_B}{EI}(\uparrow)$

and deflection at $C$ due to $R_B$, $\quad \Delta_{C,RB} = \Delta_{B,RB} + \theta_B \times 4$

$$= \frac{64R_B}{3EI}+\frac{R_B\times(4)^2}{2EI}\times 4 = \frac{320R_B}{6EI}(\uparrow)$$

Deflection at $B$ due to $R_C$, $\quad \Delta_{B,RC} = \Delta_C$ when $R_C$ is at $B$

$$= \frac{R_C(4)^3}{3EI}+\frac{R_C(4)^2}{2EI}\times 4 = \frac{320R_C}{6EI}(\uparrow)$$

and deflection at $C$ due to $R_C$, $\quad \Delta_{C,RC} = \frac{R_C(8)^3}{3EI} = \frac{512R_C}{3EI}(\uparrow)$

From compatibility condition (i)

$$\Delta_{BL} - \Delta_{B,RB} - \Delta_{B,RC} = 0$$

$$\Rightarrow \quad \frac{4266.67}{EI} - \frac{64}{3}\frac{R_B}{EI} - \frac{320R_C}{6EI} = 0$$

$$\Rightarrow \quad 128\,R_B + 320\,R_C = 25600 \quad \text{...(A)}$$

From compatibility condition (ii),

$$\Delta_{C} - \Delta_{CRB} - \Delta_{CRC} = 0$$

$$\Rightarrow \quad \frac{6866.67}{EI} - \frac{320R_B}{6EI} - \frac{512R_C}{3EI} = 0$$

$$\Rightarrow \quad 320\,R_B + 1024\,R_C = 41200 \qquad \text{...(B)}$$

Solving (*A*) and (*B*), we get

$$R_B = 454.46 \text{ kN } (\uparrow)$$

$$R_C = -101.78 \text{ kN or } 101.78 \text{ kN } (\downarrow)$$

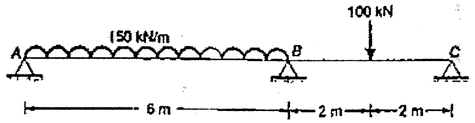
$$\Sigma F_y = 0$$

$$\Rightarrow \quad R_A + R_B + R_C = 100 + 50$$

$$\Rightarrow \quad R_A = 150 - 454.46 + 101.78$$

$$\Rightarrow \quad R_A = -202.22 \text{ kN or } 202.22 \text{ kN } (\downarrow)$$

**Example 6.14** Analyse the continuous beam shown in figure using consistent deformation method.




*EI* is constant

***Solution:***

$$D_S = r_e - 2 \qquad \text{...(For vertical loading)}$$

$$\therefore \quad = 3 - 2 = 1$$

Assuming internal moment at *B* as redundant

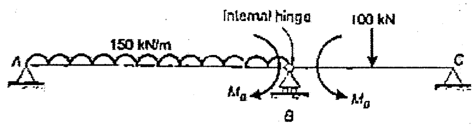



Fig. (i) Primary Structure

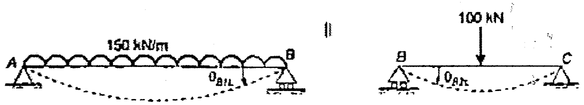



Fig. (ii) Primary structure subjected to external loading only

+

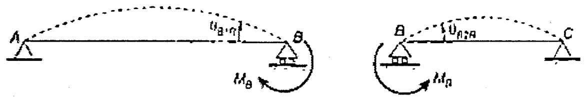



Fig. (iii) Primary structure subjected to redundant only

At intermediate support *B*,

$$\theta_{B1L} + \theta_{B2L} + \theta_{B1R} + \theta_{B2R} = 0 \qquad \text{...(i) (Compatibility)}$$

Where,

$$\theta_{B1L} = \frac{wl^3}{24EI} = \frac{150 \times 6^3}{24EI} = \frac{1350}{EI}$$

and

$$\theta_{B2L} = \frac{Pl^2}{16EI} = \frac{100 \times 4^2}{16EI} = \frac{100}{EI}$$

and

$$\theta_{B1R} = \frac{M_B l}{3EI} = \frac{M_B \times 6}{3EI} = \frac{2M_B}{EI}$$

and

$$\theta_{B2R} = \frac{M_B l}{3EI} = \frac{M_B \times 4}{3EI}$$

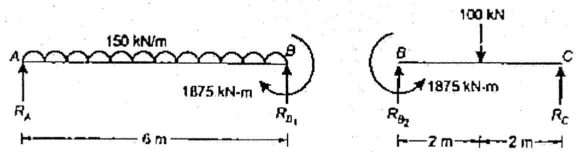
From compatibility condition (i)

$$\Rightarrow \quad -\frac{1350}{EI} + \frac{100}{EI} + \frac{2M_B}{EI} - \frac{4}{3}\frac{M_B}{EI} = 0$$

$$\Rightarrow \quad -1350 \times 3 + 300 + 6\,M_B - 4\,M_B = 0$$

$$\Rightarrow \quad M_B = 1875 \text{ kN-m}$$

Support reactions:




Portion AB:

$$\Sigma F_y = 0$$

$$\Rightarrow \quad R_A + R_{B_1} = 150 \times 6 = 900 \qquad \text{...(i)}$$

$$\Sigma M_B = 0$$

$$\Rightarrow \quad R_A \times 6 + 1875 - 150 \times 6 \times 3 = 0$$

$$\Rightarrow \quad R_A = 137.5 \text{ kN } (\uparrow)$$

From eq. (i),

$$R_{B_1} = 900 - 137.5 = 762.5 \text{ kN } (\uparrow)$$

Portion BC:

$$\Sigma F_y = 0$$

$$\Rightarrow \quad R_{B_2} + R_C = 100 \qquad \text{...(ii)}$$

$$\Sigma M_B = 0$$

$$\Rightarrow \quad -1875 + 100 \times 2 - R_C \times 4 = 0$$

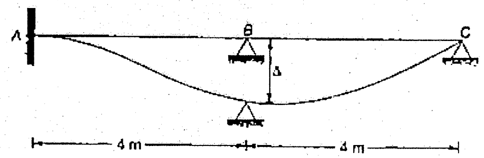
$$\Rightarrow \quad R_C = -418.75 \text{ kN or } 418.75 \text{ kN } (\downarrow)$$

and

$$R_{B_2} = 100 - R_C = 100 + 418.75 = 518.75 \text{ kN } (\uparrow)$$

$$\therefore \quad R_B = R_{B_1} + R_{B_2} = 762.5 + 518.75 = 1281.25 \text{ kN } (\uparrow)$$

**Example 6.15** Using consistent deformation method, determine all induced reactions due to a vertical settlement of 3 mm of the intermediate support $B$ as shown below. $E = 2 \times 10^5$ N/mm² and $I = 160 \times 10^{-6}$ m⁴.

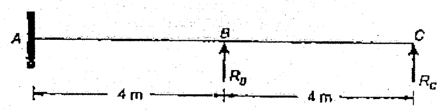



*Solution:*

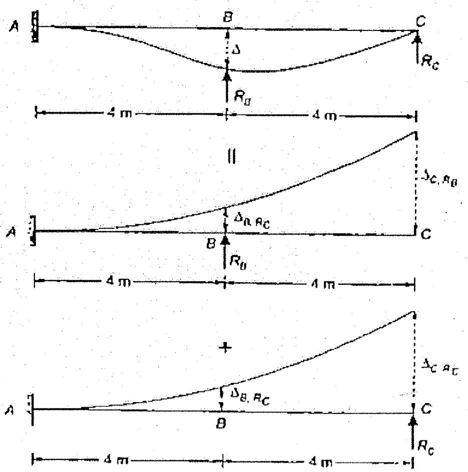
$$D_S = r_e - 2 \qquad \text{(For vertical loading)}$$

$\Rightarrow$ $= 4 - 2 = 2$

Assuming reaction $R_B$ and $R_C$ redundant.




*Primary structure*




Compatibility conditions,

$$\Delta_B = 3.0 \text{ mm} \qquad \text{...(i)}$$

$$\Delta_C = 0 \qquad \text{...(ii)}$$

Vertical deflection at $B$ due to $R_B$,

$$\Delta_{B,R_B} = \frac{R_B(4)^3}{3EI} = \frac{64R_B}{3EI}(\uparrow)$$

Vertical deflection at $C$ due to $R_B$,

$$\Delta_{C,R_B} = \frac{R_B(4)^3}{3EI} + \frac{R_B(4)^2}{2EI} \times 4 = \frac{64}{3}\frac{R_B}{EI} + \frac{64R_B}{2EI} = \frac{320R_B}{6EI}(\uparrow)$$

Vertical deflection at $C$ due to $R_C$,

$$\Delta_{C,R_C} = \frac{R_C(8)^3}{3EI} = \frac{512R_C}{3EI}(\uparrow)$$

Vertical deflection at $B$ due $R_C$,

$$\Delta_{B,R_C} = \Delta_C \text{ when } R_C \text{ at } B \qquad \text{(Maxwell's reciprocal theorem)}$$

$$= \frac{R_C(4)^3}{3EI} + \frac{R_C(4)^2}{2EI} \times 4 = \frac{320R_C}{6EI}(\uparrow)$$

From compatibility condition (i), we get

$$\frac{64R_B}{3EI} + \frac{320R_C}{6EI} - 3 \times 10^{-3} = 0$$

$\Rightarrow$ $128\,R_B + 320\,R_C = 18 \times 10^{-3} \times 200 \times 10^6 \times 160 \times 10^{-6}$

$\Rightarrow$ $128\,R_B + 320\,R_C = 576 \qquad \text{...(A)}$

From compatibility condition (ii), we have

$$\frac{320R_B}{6EI} + \frac{512R_C}{3EI} = 0$$

$\Rightarrow$ $320\,R_B + 1024\,R_C = 0$

$\Rightarrow$ $R_B = -\left(\frac{1024}{320}\right)R_C = -3.2R_C$

Substituting value of $R_B$ into (A).

$\Rightarrow$ $128 \times (-3.2\,R_C) + 320\,R_C = 576$

$R_C = -6.42$ kN or 6.42 kN (↓)

$\Rightarrow$ $R_B = -3.2 \times (-6.42) = 20.57$ kN (↑)

$\Sigma F_y = 0$

$\Rightarrow$ $R_A + R_B + R_C = 0$

$\Rightarrow$ $R_A + 20.57 - 6.49 = 0$

$R_A = -14.08$ or 14.08 kN(↓)

$\Sigma M_A = 0$

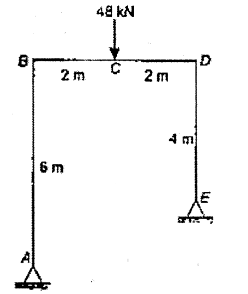
$\Rightarrow$ $M_A - R_B \times 4 - R_C \times 8 = 0$

$\Rightarrow$ $M_A = R_B \times 4 + R_C \times 8$

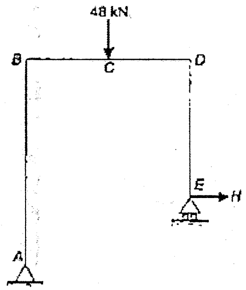
$= 20.57 \times 4 - 6.42 \times 8$

$= 30.92$ kN-m

**Example 6.16** Analyse the portal frame shown in figure by using consistent deformation method.




***Solution:***




$$D_S = r_e - 3$$

Here $r_e = 2 + 2 = 4$

$\therefore$ $D_S = 4 - 3 = 1$

Assuming horizontal reaction at $D$ as redundant. After removing redundant rest of structure is determinate.

Since there is an unyielding and non movable support at $E$ hence net horizontal movement of support $E$ will be zero.

$\Delta_{EH} = 0$ ...(i) (compatibility condition)

Horizontal displacement of support E due to given loading.

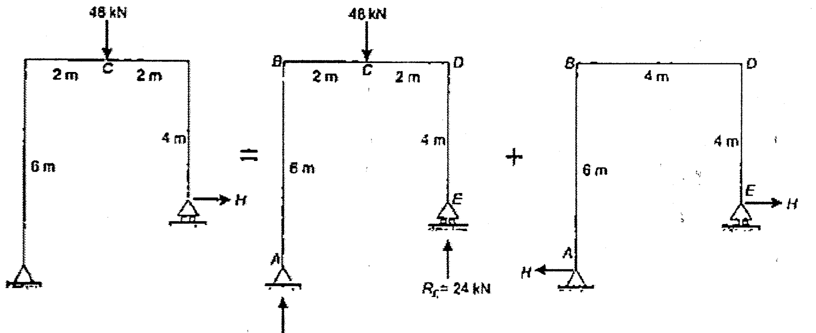



Fig (a) Fig. (b) Fig. (c)

Using unit load method.

| Portion | AB | BC | CD | DE |
|---|---|---|---|---|
| Origin | A | B | D | E |
| Limit | 0-6 | 0-2 | 0-2 | 0-4 |
| M | 0 | $24x$ | $24x$ | 0 |
| $m_1$ | $x$ | $-\frac{x}{2}+6$ | $\frac{x}{2}+4$ | $x$ |

Where,

$M$ = BM at any section due to given loading, figure (*b*)

$m_1$ = BM at any section when horizontal unit load is applied at $E$, figure (*d*)

$$\Delta_{EL} = \int \frac{M m_1 dx}{EI}$$

$$EI\Delta_{EL} = 0 + \int_0^2 24x\left(6 - \frac{x}{2}\right)dx + \int_0^2 24x\left(4 + \frac{x}{2}\right)dx + 0$$

$$= \int_0^2 \left[144x - 12x^2 + 96x + 12x^2\right]dx$$

$$= \int_0^2 240x\,dx$$

$$= 240\left[\frac{x^2}{2}\right] = \frac{240 \times 2^2}{2}$$

$\therefore$ $$\Delta_{EL} = \frac{480}{EI}(\rightarrow)$$

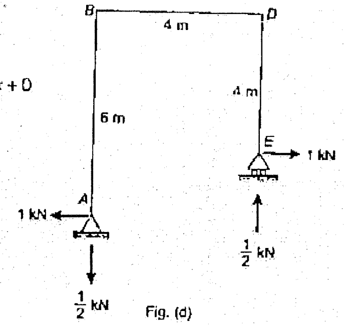



Fig. (d)

Horizontal displacement of support $E$ due to horizontal reaction $H$.

$\Delta_{ER}$ = Horizontal displacement at $E$ due to unit horizontal unit load $\times H$

$$\Delta_{ER} = \int \frac{m_1^2 dx}{EI} \times H$$

$$EI\Delta_{ER} = \left[\int_0^6 x^2 dx + \int_0^2 \left(6 - \frac{x}{2}\right)^2 dx + \int_0^2 \left(4 + \frac{x}{2}\right)^2 dx + \int_0^4 x^2 dx\right] \times H$$

$$\frac{EI\Delta_{ER}}{H} = 72 + 60.67 + 40.67 + 21.33$$

$\therefore$ $$\Delta_{ER} = \frac{194.67H}{EI}(\rightarrow)$$

From compatibility condition (i)

$$\Delta_{EH} = 0$$

$\Rightarrow$ $$\Delta_{ER} + \Delta_{EL} = 0$$

$\Rightarrow$ $$\frac{194.67H}{EI} + \frac{480}{EI} = 0$$

$\Rightarrow$ $$H = -2.46 \text{ kN } (\rightarrow) \text{ or } 2.46 \text{ kN } (\leftarrow)$$

## 6.9 Three Moment Equation

The three moment equation express the relationship between the moment at the three successive supports. The support moments can be determined by the application of three moment equations. This method is most suitable for the analysis of continuous beam.

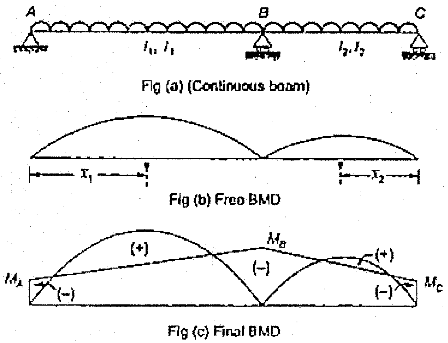



*Fig. 6.15*

According to three moment equation, support moment $M_A$, $M_B$ and $M_C$ at the support A, B and C are given by the relation.

$$M_A\left(\frac{l_1}{I_1}\right)+2M_B\left(\frac{l_1}{I_1}+\frac{l_2}{I_2}\right)+M_C\left(\frac{l_2}{I_2}\right) = \frac{6a_1\bar{x}_1}{l_1 I_1}+\frac{6a_2\bar{x}_2}{l_2 I_2}$$

Where, $a_1$ = area of free BMD for span AB

$a_2$ = area of free BMD for span BC

$\bar{x}_1$ = centroidal distance of free BMD on AB from A

$\bar{x}_2$ = centroidal distance of free BMD on BC from C

Special Case-1: When span carries UDL over entire span.

$a_1$ = Area of free BMD on AB

$$= \frac{2}{3}\times l_1\times\frac{w_1 l_1^2}{8} = \frac{w_1 l_1^3}{12}$$

$\bar{x}_1$ = centroidal distance of free BMD on AB from A

$$= \frac{l_1}{2}$$

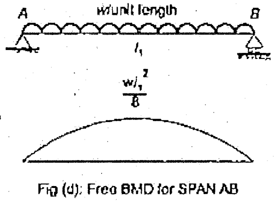



*Fig. 6.16*

$$\therefore \quad \frac{6a_1\bar{x}_1}{l_1} = \frac{6\times\frac{w_1 l_1^3}{12}\times\frac{l_1}{2}}{l_1} = \frac{w_1 l_1^3}{4}$$

Hence, the three moment equation can be written as when span AB and AC carries UDL over entire span.

$$M_A l_1 + 2M_B(l_1+l_2)+M_C l_2 = \frac{w_1 l_1^3}{4}+\frac{w_2 l_2^3}{4}$$

Special case 2: When EI is not constant

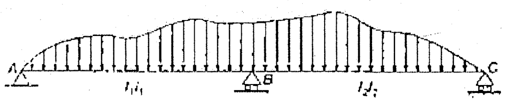
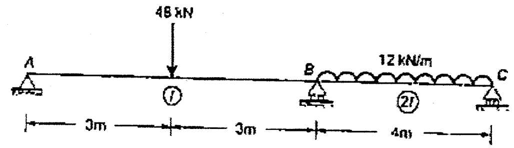



*Fig. 6.17*

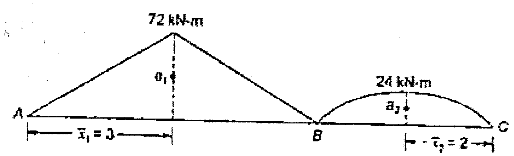
then three moment equation will be,

$$M_A\frac{l_1}{I_1}+2M_B\left(\frac{l_1}{I_1}+\frac{l_2}{I_2}\right)+M_C\frac{l_2}{I_2} = \frac{6a_1\bar{x}_1}{l_1 I_1}+\frac{6a_2\bar{x}_2}{l_2 I_2}$$

**Example 6.17** Analyse the continuous beam shown in figure.




*Solution:*




Maximum ordinate of free BMD on $AB = \frac{48\times 6}{4} = 72$ kN-m

Maximum ordinate of free BMD on $BC = \frac{12\times 4^2}{8} = 24$ kN-m

Area of free BMD on AB, $a_1 = \frac{1}{2}\times 6\times 72 = 216$ unit

Area of free BMD on BC, $a_2 = \frac{1}{2}\times 4\times 24 = 48$ unit

Centroidal distance of free BMD on AB from A,

$\bar{x}_1$ = 3 m

Centroidal distance of free BMD on BC from B,

$\bar{x}_2$ = 2 m

Applying three moment equation for span AB and BC

$$M_A\frac{l_1}{I_1}+2M_B\left(\frac{l_1}{I_1}+\frac{l_2}{I_2}\right)+M_C\frac{l_2}{I_2} = \frac{6a_1\bar{x}_1}{l_1 I_1}+\frac{6a_2\bar{x}_2}{l_2 I_2}$$

Since A and C are simple supports,

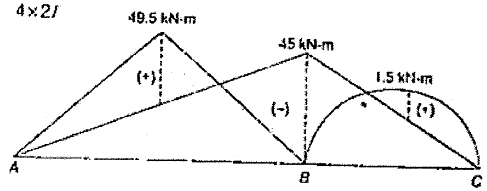
Hence, $M_A = M_C = 0$

$$0+2M_B\left[\frac{6}{I}+\frac{4}{2I}\right]+0 = \frac{6\times 216\times 3}{6\times I}+\frac{6\times 48\times 2}{4\times 2I}$$

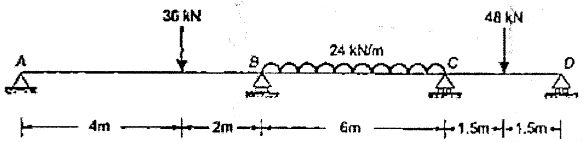
$$16\frac{M_B}{I} = \frac{648}{I}+\frac{72}{I}$$

$16 M_B = 720$

$M_B = 45$ kN-m



**Example 6.18** Analyse the continuous beam *ABCD* shown in figure.




*EI* is constant

*Solution:*

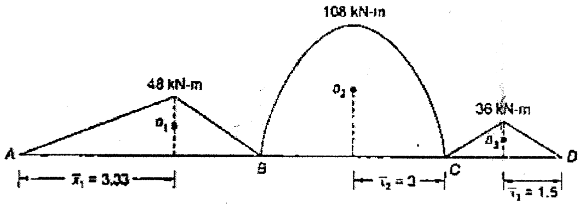



Fig. Free BMD

Area of free BMD on *AB*, $a_1 = \frac{1}{2}\times 6\times 48 = 144$ unit

Area of free BMD on *BC*, $a_2 = \frac{2}{3}\times 6\times 108 = 432$ unit

Area of free BMD on *CD*, $a_3 = \frac{1}{2}\times 3\times 36 = 54$ unit

Centroidal distance of free BMD on *AB* from *A*,

$$\bar{x}_1 = 6-\left(\frac{6+2}{3}\right) = 3.33 \text{ m}$$

Centroidal distance of free BMD on *BC* from *C*,

$$\bar{x}_2 = 3 \text{ m}$$

Centroidal distance of free BMD on *CD* from *D*,

$$\bar{x}_3 = 1.5 \text{ m}$$

Applying three moment equation for span *AB* and *BC*,

$$M_A l_1 + 2M_B(l_1+l_2) + M_C l_2 = \frac{6a_1\bar{x}_1}{l_1} + \frac{6a_2\bar{x}_2}{l_2}$$

Since end *A* is simply supported.

Hence, $M_A = 0$

$$0 + 2M_B(6+6) + M_C\times 6 = \frac{6\times 144\times 3.33}{6} + \frac{6\times 432\times 3}{6}$$

$$24M_B + 6M_C = 1775.52 \qquad ...(i)$$

Applying three moment equation for span *BC* and *CD*.

$$M_B l_2 + 2M_C(l_2+l_3) + M_D\times l_3 = \frac{6a_2\bar{x}_2}{l_2} + \frac{6a_3\bar{x}_3}{l_3}$$

Since end *D* is simply supported.

Hence, $M_D = 0$

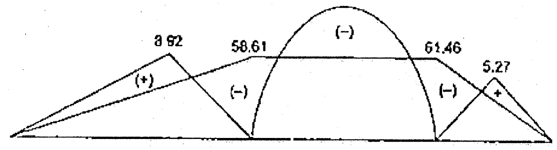
$$6M_B + 2M_C(6+3) + 0 = \frac{6\times 432\times 3}{6} + \frac{6\times 54\times 1.5}{3}$$

$$6M_B + 18M_C = 1458 \qquad ...(i)$$

On solving (i) and (ii), we get

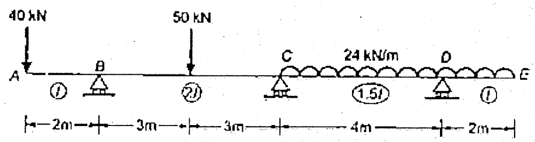
$$M_B = 58.61 \text{ kN-m}$$

$$M_C = 61.46 \text{ kN-m}$$




BMD

**Example 6.19** Analyse the continuous beam shown in figure.




*Solution:*

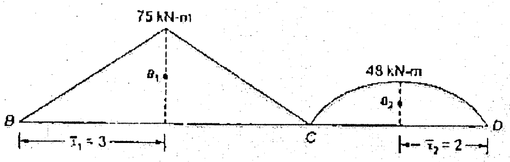



Fig. (a) Free BMD

Area of free BMD on *BC*, $a_1 = \frac{1}{2}\times 6\times 75 = 225$ unit

Area of free BMD on *CD*, $a_2 = \frac{2}{3}\times 4\times 48 = 128$ unit

Centroidal distance of free BMD on *BC* from *B*,

$$\bar{x}_1 = 3 \text{ m}$$

Centroidal distance of free BMD on *CD* from *D*,

$$\bar{x}_2 = 2\text{ m}$$

$$M_B\,(\text{from } A) = -40\times2 = -80\text{ kN-m}$$

$$M_D\,(\text{from } E) = -24\times2\times1 = -48\text{ kN-m}$$

Applying three moment equation for span *BC* and *CD*,

$$M_B\left[\frac{l_1}{I_1}\right]2M_C\left[\frac{l_1+l_2}{I_1+I_2}\right]+M_D\left[\frac{l_2}{I_2}\right] = \frac{6a_1\bar{x}_1}{l_1I_1}+\frac{6a_2\bar{x}_2}{l_2I_2}$$

$$80\left(\frac{6}{2I}\right)+2M_C\left[\frac{6}{2I}+\frac{4}{1.5I}\right]+\left[\frac{4}{1.5I}\right] = \frac{6\times225\times3}{6\times2I}+\frac{6\times128\times2}{4\times1.5I}$$

$$240+\frac{34}{3}M_C+128 = 337.5+256$$

$$\frac{34}{3}M_C = 225.5$$

$$M_C = 19.89\text{ kN-m}$$

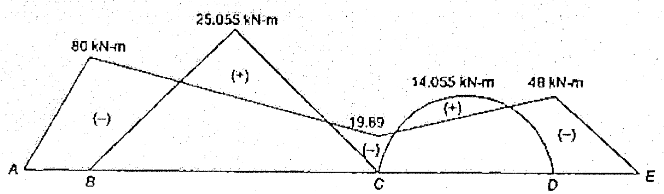


Fig (b) BMD

### 6.9.1 Application of Three Moment Equation to Continuous Beams with Fixed Ends

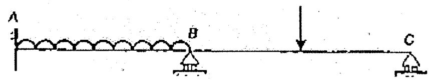


*Fig. 6.18*

Above beam can be analyzed by three moment equation if an imaginary zero span *A'A* of length zero with flexural rigidity ($EI = \infty$) is added at fixed end *A*.

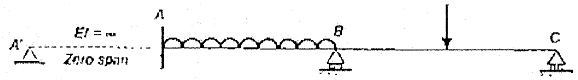
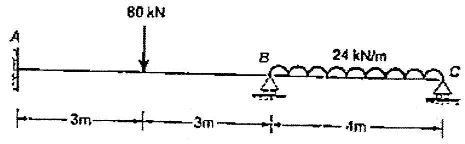


*Fig. 6.19*

Now $M_A$ and $M_B$ can be found by applying three moment equation for span *A'A*, *AB* and span *AB*, *BC* separately.

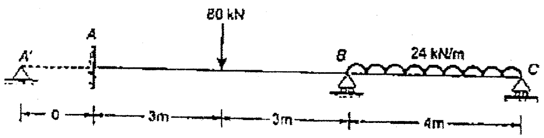
**Example 6.20** **Analyse the continuous beam shown in figure by using three moment equation.**



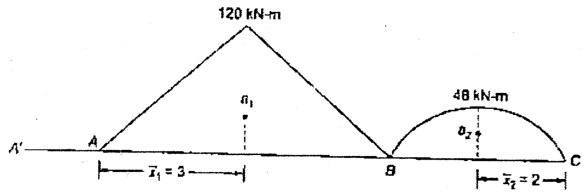
*EI* is constant

*Solution:*

Let us introduce an imaginary zero span *A'A*



*EI* is constant



Free BMD

Area of free BMD on *AB*,
$$a_1 = \frac{1}{2}\times6\times120 = 360\text{ unit}$$

Area of free BM on *BC*,
$$a_2 = \frac{2}{3}\times4\times48 = 128\text{ unit}$$

Centroidal distance of free BMD on *AB* from *A*,

$$\bar{x}_1 = 3\text{ m}$$

Centroidal distance of free BMD on *BC* from *B*,

$$\bar{x}_2 = 2\text{ m}$$

Applying three moment equation for span *A'A* and *AB*,

$$M_{A'}\times0+2M_A(0+6)+M_B\times6 = \frac{6a_1\bar{x}_1}{l_1}$$

$$\Rightarrow\quad 0+12M_A+6M_B = \frac{6\times360\times3}{6}$$

$$\Rightarrow\quad 12M_A+6M_B = 1080 \qquad \text{...(i)}$$

Applying three moment equation for span $AB$ and $BC$

$$\Rightarrow \quad M_A \times 6 + 2M_B(6+4) + M_C \times 4 = \frac{6a_1\bar{x}_1}{l_1} + \frac{6a_2\bar{x}_2}{l_2}$$

$$\Rightarrow \quad 6M_A + 20M_B + 4M_C = \frac{6\times360\times3}{6} + \frac{6\times128\times2}{4}$$

$$\Rightarrow \quad 6M_A + 20M_B + 4M_C = 1464$$

Since support $C$ is simply supported.

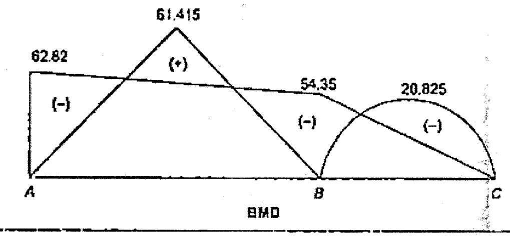
$$\therefore \quad M_C = 0$$

$$\Rightarrow \quad 6M_A + 20M_B = 1464 \quad \text{...(ii)}$$

On solving (i) and (ii), we get

$$M_A = 62.82 \text{ kN-m}$$

$$M_B = 54.35 \text{ kN-m}$$




### 6.9.2 Application of Three Moment Equation to Continuous Beam with Support at Different Level

Let the support $B$ settler by $\delta_1$ wrt support $A$ and $\delta_2$ w.r.t. support $C$.

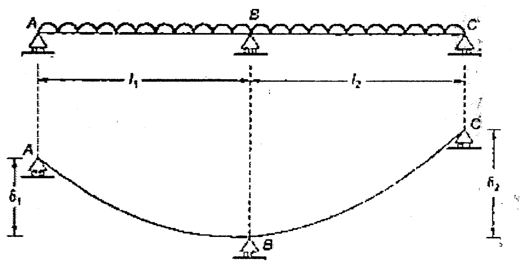
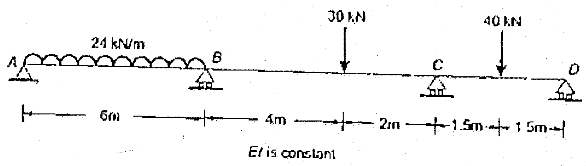



*Fig. 6.20*

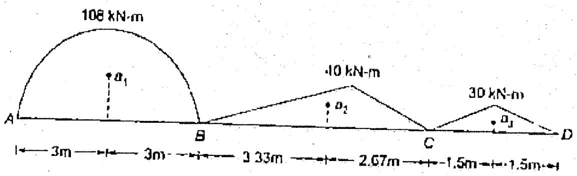
Then three moment equation can be written as

$$M_A l_1 + 2M_B(l_1 + l_2) + M_C l_2 = \frac{6a_1\bar{x}_1}{l_1} + \frac{6a_2\bar{x}_2}{l_2} - 6EI\left(\frac{\delta_1}{l_1} + \frac{\delta_2}{l_2}\right)$$

**Example 6.21** Analyse the continuous beam $ABCD$ shown in figure if support $C$ sinks by 5 mm. Take $EI = 7.5 \times 10^{10}$ kN-mm²




*Solution:*




$$\text{Max. ordinate of free BMD on } AB = \frac{24\times6^2}{8} = 108 \text{ kN-m}$$

Area of free BMD on $AB$,

$$a_1 = \frac{2}{3}\times6\times108 = 432 \text{ unit}$$

Centroidal distance of free BMD on $AB$ from $A$,

$$\bar{x}_1 = 3 \text{ m}$$

$$\text{Max. ordinate of free BMD on } BC, = \frac{30\times4\times2}{6} = 40 \text{ kN-m}$$

Area of free BMD on $BC$,

$$a_2 = \frac{1}{2}\times6\times4 = 120 \text{ unit}$$

Centroidal distance of free BMD on $BC$ from $C$.

$$\bar{x}_2 = \frac{6+2}{3} = 2.667 \text{ m}$$

Max. ordinate of free BMD on $CD$,

$$= \frac{40\times3}{4} = 30 \text{ kN-m}$$

Area of free BMD on $CD$,

$$a_3 = \frac{1}{2}\times3\times30 = 45 \text{ unit}$$

Centroidal distance of free BMD on $CD$ from $D$.

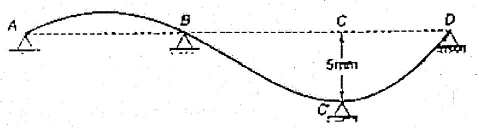
$$\bar{x}_3 = 1.5 \text{ m}$$

Applying three moment equation for span $AB$ and $BC$

$$M_A l_1 + 2M_B(l_1 + l_2) + M_C l_2 = \frac{6a_1\bar{x}_1}{l_1} + \frac{6a_2\bar{x}_2}{l_2} - 6EI\left(\frac{\delta_1}{l_1} + \frac{\delta_2}{l_2}\right)$$

Since end $A$ is simply supported.

Hence, $M_A = 0$




$\delta_1$ = settlement of support $B$ wrt support $A$

= 0

$\delta_2$ = settlement of support $B$ wrt support $C$

= -5 mm or 0.005 m

$$\Rightarrow \quad 0 \times 6 + 2M_B(6+6) + M_C \times 6 = \frac{6\times432\times3}{6} + \frac{6\times120\times2.667}{6} - 6EI\left(0 - \frac{0.005}{6}\right)$$

$$\Rightarrow \quad 24M_B + 6M_C = 1296 + 320.04 + \frac{6\times7.5\times10^{10}\times10^{-6}\times0.005}{6}$$

$$\Rightarrow \quad 24M_B + 6M_C = 1616.04 + 375$$

$$\Rightarrow \quad 24M_B + 6M_C = 1991.04 \qquad \text{...(i)}$$

Applying three moment equation for span $BC$ and $CD$,

$$M_B l_2 + 2M_C(l_2 + l_3) + M_D l_3 = \frac{6a_2\bar{x}_2}{l_2} + \frac{6a_3\bar{x}_3}{l_3} - 6EI\left(\frac{\Delta_1}{l_2} + \frac{\Delta_2}{l_3}\right)$$

Since end $D$ is simply supported.

Hence, $M_D = 0$

and $\Delta_1$ = settlement of support $C$ wrt support $B$

= +5 mm or 0.005 m

$\Delta_2$ = settlement of support $C$ wrt support $D$

= +5 mm or 0.005 m

$$\Rightarrow \quad 6M_B + 2M_C(6+3) + 0 \times 3 = \frac{6\times120\times3.33}{6} + \frac{6\times45\times1.5}{3} - 6EI\left(\frac{0.005}{6} + \frac{0.005}{3}\right)$$

$$6M_B + 18M_C = 399.6 + 135 - \frac{6\times7.5\times10^{10}\times10^{-6}\times0.005}{6}$$

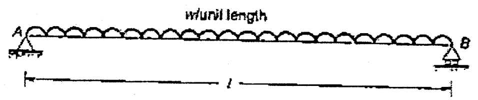
$$6M_B + 18M_C = 534.6 - 375$$

$$6M_B + 18M_C = 159.6 \qquad \text{...(ii)}$$

On solving eq (i) and (ii), we get

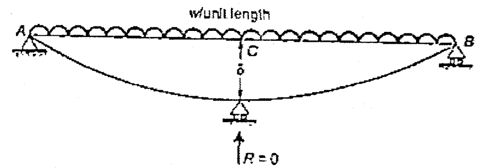
$$M_B = 88.08 \text{ kN-m}$$
$$M_C = -20.49 \text{ kN-m}$$

**Example 6.22** **Employing the equation of three moment to obtain the central deflection in a simply supported beam of span $l$ loaded with a UDL of $w$ per unit length. $EI$ is constant.**




*Solution:*

Let us introduce an imaginary support providing zero reaction at the centre of the beam $AB$. Assuming imaginary support sinks by $\delta$ at centre




Applying three moment equation for span $AC$ and $CB$,

$$M_A l_1 + 2M_C(l_1 + l_2) + M_B l_2 = \frac{w_1 l_1^3}{4} + \frac{w_2 l_2^3}{4} - 6EI\left(\frac{\delta_1}{l_1} + \frac{\delta_2}{l_2}\right)$$

Since end $A$ and $B$ are simply supported.

Hence, $M_A = M_B = 0$

and $M_C = -\frac{wl^2}{8}$ (Hogging)

$\delta_1$ = settlement of support $C$ wrt $A$

= $+\delta$

$\delta_2$ = settlement of support $C$ wrt $B$

= $+\delta$

Applying three moment equation,

$$\Rightarrow \quad 0 \times \frac{l}{2} + 2M_C\left(\frac{l}{2} + \frac{l}{2}\right) + 0 = \frac{wl^3}{4\times8} + \frac{wl^3}{4\times8} - 6EI\left(\frac{2\delta}{l} + \frac{2\delta}{l}\right)$$

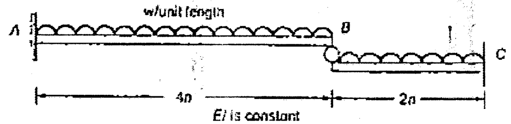
$$\Rightarrow \quad -2\times\frac{wl^2}{8}\times l = \frac{wl^3}{32} + \frac{wl^3}{32} - \frac{24EI\delta}{l}$$

$$\Rightarrow \quad -\frac{wl^3}{4} = \frac{wl^3}{16} - \frac{24EI\delta}{l}$$

$$\Rightarrow \quad \delta = \frac{5}{384}\frac{wl^4}{EI}$$

## Illustrative Examples

**Example 6.23** The free end of a cantilever $AB$ '$4a$' long is supported from below on the free end $C$ of the cantilever $CD$ '$2a$' long through a roller as shown in figure. The two beams have uniform and equal flexural rigidities and carry a UDL of intensity w/unit length. Draw the BMD for $AB$ and $CD$.




***Solution:***

$$D_S = r_a - 2 - r_r$$

Here, $r_a = 4$ and $r_r = 1$

$\therefore$ $$D_S = 4 - 2 - 1 = 1$$

Let the reaction at the roller as redundant (say $R$)

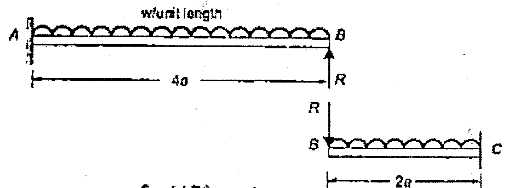



Fig (a) Primary structure

At $B$, the deflection will be equal for both portion $AB$ and $BC$

$\therefore$ $$(\Delta_B)_{AB} = (\Delta_B)_{BC}$$

$\Rightarrow$ $$\frac{w(4a)^4}{8EI} - \frac{R(4a)^3}{3EI} = \frac{w(2a)^4}{8EI} + \frac{R(2a)^3}{3EI}$$

$\Rightarrow$ $$\frac{w}{8EI}\left[256a^4 - 16a^4\right] = \frac{R}{3EI}\left(8a^3 + 64a^3\right)$$

$\Rightarrow$ $$\frac{w}{8} \times 240a^4 = \frac{R}{3} \times 72a^3$$

$\Rightarrow$ $$R = \frac{5}{4}wa$$

**Bending moment diagram:**

**Portion AB:**

$$M_x(x \text{ from B}) = Rx - \frac{wx^2}{2} \qquad [0 \le x \le 4a]$$

$$M_x = \frac{5}{4}wax - \frac{wx^2}{2} \qquad \text{(Parabolic)}$$

at $x = 0$, $M_B = 0$

at $x = 4a$, $M_A = -3wa^2$

For BM to be zero,

$$M_x = 0$$

$\Rightarrow$ $$\frac{5}{4}wax - \frac{wx^2}{2} = 0$$

$\Rightarrow$ $$x\left[\frac{5}{4}wa - \frac{wx}{2}\right] = 0$$

$\therefore$ $$x = 0$$

and $$\frac{5}{4}wa - \frac{wx}{2} = 0$$

$\Rightarrow$ $$x = 2.5a$$

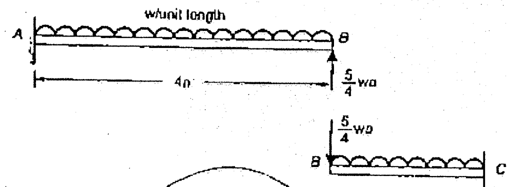
Portion CD:

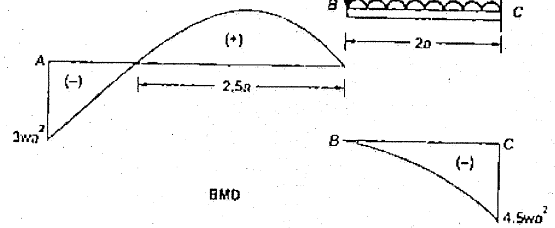
$$M_x(x \text{ from } C) = -Rx - \frac{wx^2}{2} \qquad [0 \le x \le 2a]$$

$$M_x = -\frac{5}{4}wax - \frac{wx^2}{2} \qquad \text{(Parabolic)}$$

at $x = 0$, $M_B = 0$

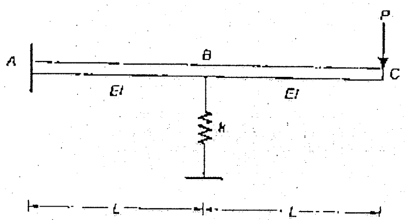
at $x = 2a$, $M_C = 4.5\,wa^2$



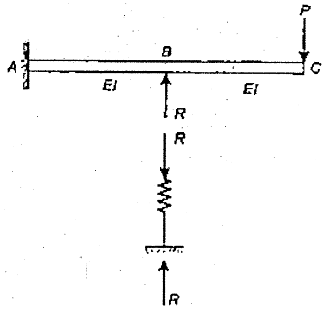

BMD

**Example 6.24** Using the force (flexibility/compatibility) method. Analyse the structure in the figure below:



*Solution:*

Let the stiffness of spring be $k$. Also let reaction at $B$ be redundant and say $R$.



*Primary Structure*

At $B$,

$$\Delta_{B\ cantilever} = \Delta_{B,\ spring} \quad \text{....(i) (compatibility condition)}$$

where, $\Delta_{B,\ cantilever} = \Delta_B$ due to $R - \Delta_B$ due to $P$ at $C$

$$= \frac{RL^3}{3EI} - \Delta_C \text{ when } P \text{ acting at } B$$

$$= \frac{RL^3}{3EI} - \left[\frac{PL^3}{3EI} + \frac{PL^2}{2EI} \times L\right] = \frac{RL^3}{3EI} - \frac{5PL^3}{6EI}$$

and $\quad \Delta_{B,\ spring} = \dfrac{R}{k}$

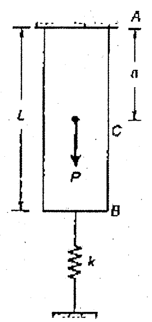
From (i), we have

$$\Rightarrow \quad \frac{1}{EI}\left[\frac{RL^3}{3} - \frac{5PL^3}{6}\right] = \frac{R}{k}$$

$$\Rightarrow \quad \frac{RL^3}{3EI} - \frac{5PL^3}{6EI} = \frac{R}{k}$$

$$R = \frac{5PL^3}{\left[2L^3 - \dfrac{6EI}{k}\right]}$$

**Example 6.25** A floating column $AB$ of length '$L$', is modelled with top end fixed and bottom end yielding under column load. The yielding support can be visualised as an axial spring of stiffness '$k$'. If an axial force '$P$' is applied at a distance '$a$' from the top, determine reactions at the ends $A$ and $B$ and displacement at the end $B$.



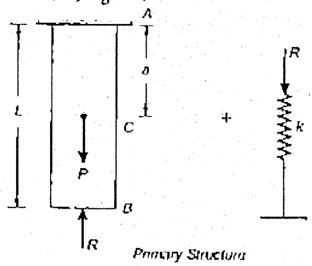

*Solution:*

$$D_S = r_e - E$$
$$= 2 - 1 = 1$$

$$\left[\because E = \Sigma F_y = 0, \text{ only}\right]$$

Let reactions at $A$ and $B$ be $R_A$ and $R_B$ respectively.

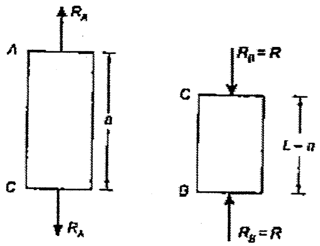
Assuming $R_B$ as redundant (say $R_B = R$)



*Primary Structure*

Net elongation in bar = Net compression in spring ...(i) (compatibility)

Net elongation in bar:




$$\Sigma F_y = 0$$
$$\Rightarrow \quad R_A + R_B = P$$
$$\Rightarrow \quad R_A + R = P$$
$$\therefore \quad R_A = (P - R)$$
$$\Delta_{Bar} = \Delta_{AC} + \Delta_{CB}$$
$$= \frac{R_A a}{AE} - \frac{R_B(L-a)}{AE}$$
$$= \frac{(P-R)a}{AE} - \frac{R(L-a)}{AE}$$
$$= \frac{Pa}{AE} - \frac{Ra}{AE} - \frac{RL}{AE} + \frac{Ra}{AE} = \frac{Pa}{AE} - \frac{RL}{AE}$$

Net compression in spring:

$$\Delta_S = \frac{R_B}{k} = \frac{R}{k}$$

From (i), we have

$$\frac{Pa}{AE} - \frac{RL}{AE} = \frac{R}{k}$$

$$\Rightarrow \quad \frac{Pa}{AE} = \frac{RL}{AE} + \frac{R}{k}$$

$$\Rightarrow \quad R = \frac{Pa/AE}{\left[\dfrac{L}{AE} + \dfrac{1}{k}\right]} = \frac{Pa}{\left[L + \dfrac{AE}{k}\right]}$$

and

$$R_A = P - R$$

$$= P - \frac{Pa}{\left(L - \dfrac{AE}{k}\right)} = P\left(1 - \frac{a}{\left[L - \dfrac{AE}{k}\right]}\right)$$

Displacement of End $B$,

$$\Delta_B = \Delta_S$$

$$= \frac{R}{k} = \frac{Pa}{k\left(L + \dfrac{AE}{k}\right)} = \frac{Pa}{\left(kL + \dfrac{AE}{k}\right)}$$

## Summary

- There are two basic methods for structure analysis
  (*i*) Force method (*ii*) Displacement method
- In force method unknowns are taken as internal forces of member or reaction. It is suitable when $D_S < D_K$.
- In displacement method, unknowns are taken joint displacements. It is suitable when $D_K < D_S$

- If a structure is loaded and there are redundant reaction, then true value of redundant reaction will be that for which total potential energy is minimum.
- In any structure, the deflection at any point $D$ due to load $W$ at any point $C$ is same as the deflection at $C$ due to the same load applied at $D$.

$$\Delta_{DC} = \Delta_{CD}$$

this condition is also known as Maxwell's reciprocal theorem.
- For rigid bodies external virtual work done is zero.

$$W_e = 0$$

## Objective Brain Teasers

**Q.1** Match List-I (Method of analysis) with List-II (Unknowns being evaluated) and select the correct answer using the codes given below the lists:

List-I
- A. Flexibility Method
- B. Stiffness Method
- C. Kani's Method
- D. Moment Distribution Method

List-II
1. Degrees of freedom
2. Redundant forces
3. Rotations by incremental iteration and unknown sways of plane frames
4. Displacement, rotations and sways of plane frames

Codes:

| | A | B | C | D |
|---|---|---|---|---|
| (a) | 2 | 1 | 4 | 3 |
| (b) | 3 | 4 | 1 | 2 |
| (c) | 2 | 4 | 1 | 3 |
| (d) | 3 | 1 | 4 | 2 |

**Q.2** Match List-I with List-II and select the correct answer using the codes given below the lists:

List-I
- A. Strain energy method
- B. Slope deflection
- C. Moment distribution
- D. Kani's method

List-II
1. Successive approximation
2. Flexibility method
3. Iteration process
4. Stiffness method

Codes:

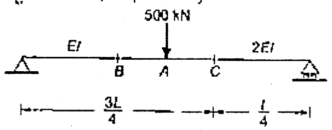
| | A | B | C | D |
|---|---|---|---|---|
| (a) | 1 | 4 | 2 | 3 |
| (b) | 2 | 3 | 1 | 4 |
| (c) | 1 | 3 | 2 | 4 |
| (d) | 2 | 4 | 1 | 3 |

**Q.3** Consider the following statements regarding the analysis of indeterminate structures:
1. The force method consists in applying displacement compatibility conditions at the nodes.
2. The stiffness method consists in formulating equilibrium equations at the nodes.

Which of these statements is/are correct?

(a) Only 1 (b) Only 2
(c) Both 1 and 2 (d) Neither 1 nor 2

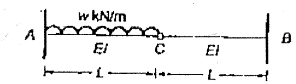
**Q.4** A load 500 kN applied at point $A$, as shown in the figure below, produces a vertical deflection at $B$ and $C$ of the beam as $\Delta_B = 10$ mm and $\Delta_C = 15$ mm, respectively.




What is the deflection at $A$ when loads of 100 kN and 300 kN are applied at $B$ and $C$, respectively?

(a) 6 mm (b) 8 mm
(c) 11 mm (d) 12.5 mm

**Q.5** What is the reaction on the hinge $C$ for a beam as shown in the figure




(a) $\frac{3}{8}wL$ (b) $\frac{1}{2}wL$
(c) $\frac{1}{4}wL$ (d) $\frac{3}{16}wL$

**Q.6** The theorem of three moment equation for a continuous beam expresses the condition of
(a) equilibrium at an intermediate support
(b) slope compatibility at an intermediate support
(c) zero support settlement
(d) None of the above

**Directions:** The following items consists of two statements; one labelled as 'Assertion (A)' and the other as 'Reason (R)'. You are to examine these two statements carefully and select the answers to these items using the codes given below:

Codes:
(a) both A and R are true and R is the correct explanation of A
(b) both A and R are true but R is not a correct explanation of A
(c) A is true but R is false
(d) A is false but R is true

**Q.7** **Assertion (A):** Force method of analysis is not convenient for computer programming
**Reason (R):** Band width of flexibility matrix is much larger compared to that of stiffness matrix

**Q.8** **Assertion (A):** The total virtual work done by a system of force acting on a rigid body in equilibrium during a virtual displacement is zero.
**Reason (R):** If a system of forces acting on a deformable body is in equilibrium, as the body is subjected to a small deformation, the external virtual work done by the force will also be zero.

**Q.9** **Assertion (A):** In a cantilever, the ILD for deflection at the free end is same as elastic curve of the beam due to unit load placed at the free end.
**Reason (R):** By Maxwell's Reciprocal theorem, the deflection at the free end, due to various positions of unit load on the span equals deflection at those places of unit moving load due to static unit load at the free end.

**Q.10** **Assertion (A):** In the analysis of pin-jointed plane frames, the force method is generally preferable to the displacement method.
**Reason (R):** The degrees of freedom for pin-jointed plane frames are generally much larger than the degrees of static indeterminacy and thus force method requires less formulation and computation than the displacement method.

**Answers**

| | | | | |
|---|---|---|---|---|
| 1. (a) | 2. (d) | 3. (c) | 4. (c) | 5. (b) |
| 6. (d) | 7. (c) | 8. (c) | 9. (a) | 10. (a) |

## Hints and Explanations:

**1. (a)**

Force or flexibility method uses redundant forces while stiffness or displacement method of analysis uses degrees of freedom.

**3. (c)**

(i) The force method considers redundant forces as unknowns. The displacement compatibility at nodes gives simultaneous equations for these unknowns.

(ii) Similarly the stiffness method considers degree of freedom (independent displacement components at joints) to be unknowns. The force equilibrium equations at the nodes gives the simultaneous equations for these unknowns

4. (c)

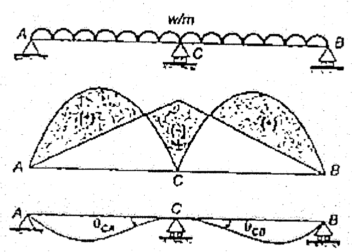
Using Maxwell-Betti's theorem

$$500\Delta_A = 100\Delta_B + 300\Delta_C$$

$$\therefore \quad \Delta_A = \frac{\Delta_B}{5} + \frac{3}{5}\Delta_C = \frac{10}{5} + \frac{15\times 3}{5}$$

$$= 11 \text{ mm}$$

5. (b)



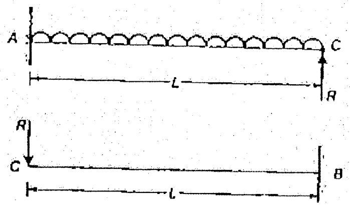

at intermediate support,

$$\theta_{CA} + \theta_{CB} = 0$$

Thus the three moment theorem for continuous beam express condition of slope compatibility at an intermediate support.

Hence option (b) is correct.

6. (d)




By compatibility,

$$(\Delta_C)_{AC} = (\Delta_C)_{CB}$$

$$\frac{wL^4}{8EI} - \frac{RL^3}{3EI} = \frac{RL^3}{3EI}$$

$$\frac{2RL^3}{3EI} = \frac{wL^4}{8EI}$$

$$R = \frac{3}{16}wL$$

Hence option (d) is correct.

8. (c)

Bernoulli's principle of virtual work states that for a rigid body in equilibrium by a system of forces and/or couples, the total virtual work done by this system of forces and/or couples during a virtual displacement is zero.

For a deformable body under equilibrium, the external virtual work is equal to internal virtual work.

9. (a)

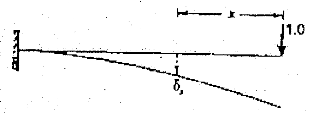



Figure-I

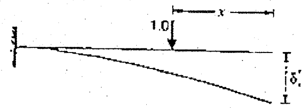



Figure-II

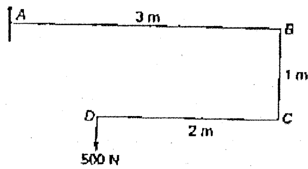
From Maxwell's reciprocal theorem

$$\delta_x = \delta'_x$$

Therefore the deflected shape in figure-I will be the ILD for deflection.

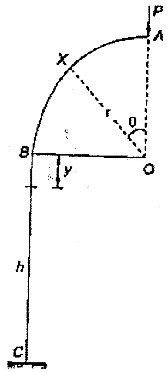
## Conventional Practice Questions

**Q.1** A mild steel bar 100 mm diameter is bent as shown in figure. It is fixed horizontally at $A$ and a load of 500 N hangs at $D$. Draw the bending moment diagram for the parts $AB$, $BC$ and $CD$ indicating the maximum values. Find the deflection at $D$. Take $E = 2\times 10^5$ N/mm$^2$.



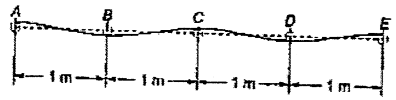

Ans. $\delta_v = 4.92$ mm

**Q.2** Find the vertical and horizontal deflection at A for the lamp post loaded as shown in figure. Assume uniform flexural rigidity.




Ans. $\frac{Pr^2}{4EI}(\pi r + 4h)$, $\frac{Pr(r+h)^2}{2EI}$

**Q.3** A 40 mm wide 3 mm deep strip of steel is placed through a row of five rigid pegs as shown in figure. The pegs which are 15 mm in diameter are provided at a spacing of 1 m. Find the thrust on each peg, neglecting the weight of the strip. Take E = 2 × 10$^5$ N/mm$^2$.

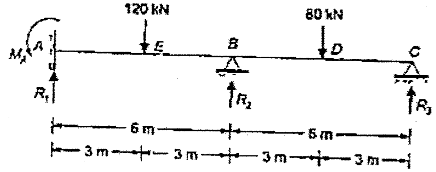



Ans. $R_a = 1.157$ N (↑), $R_b = -3.702$ N (↓)

$R_d = R_b = -3.702$ N (↓), $R_e = R_a = 1.157$ N (↓)

$R_c = 2(3.702) - 2(1.157) = 5.090$N (↑)

**Q.4** Find out the value of $R_2$ and $R_3$ by using method of Least Work.




Ans. $R_3 = 25$ kN, $R_2 = 115$ kN

■■■■