Chapter – 2

Algebra

Resolve into partial fractions for the following:

Ex 2.1

Question 1.
$$\frac{3x+7}{x^2-3x+2}$$

Solution:
Here the denominator $x^2 - 3x + 2$ is not a linear factor.
So if possible we have to factorise it then only we can split up into partial fraction.
$x^2 - 3x + 2 = (x - 1)(x - 2)$

$$\frac{3x+7}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2} \text{ .... } (1)$$

Multiply both side by $(x - 1)(x - 2)$
$3x + 7 = A(x - 2) + B(x - 1)$ .... (2)
Put $x = 2$ in (2) we get
$3(2) + 7 = A(2 - 2) + B(2 - 1)$
$6 + 7 = 0 + B(1)$
$\therefore B = 13$
Put $x = 1$ in (2) we get
$3(1) + 7 = A(1 - 2) + B(1 - 1)$
$3 + 7 = A(-1) + 0$
$10 = A(-1)$
$\therefore A = -10$
Using $A = -10$ and $B = 13$ in (1) we get

$$\frac{3x+7}{(x-1)(x-2)}=\frac{-10}{x-1}+\frac{13}{x-2}$$
$$=\frac{13}{x-2}+\frac{-10}{x-1}$$
Thus $\frac{3x+7}{(x-1)(x-2)}=\frac{13}{x-2}-\frac{10}{x-1}$

Note: When the denominator is only two linear factors we can adopt the following method.

$$\frac{3x+7}{(x-1)(x-2)}=\frac{1}{(x-1)}\left[\frac{3x+7}{x-2}\right]_{x=1}+\frac{1}{x-2}\left[\frac{3x+7}{x-1}\right]_{x=2}$$
$$=\frac{1}{x-1}\left[\frac{3(1)+7}{1-2}\right]+\frac{1}{x-2}\left[\frac{3(2)+7}{2-1}\right]$$
$$=\frac{1}{x-1}\left[\frac{3+7}{-1}\right]+\frac{1}{x-2}\left[\frac{6+7}{1}\right]$$
$$=\frac{1}{x-1}[-10]+\frac{1}{x-2}[13]$$
$$=\frac{13}{x-2}-\frac{10}{x-1}$$

Question 2.

$$\frac{4x+1}{(x-2)(x+1)}$$

Solution:

Let $\frac{4x+1}{(x-2)(x+1)}=\frac{A}{x-2}+\frac{B}{x+1}$ ……… (1)

Multiply both sides by (x – 2) (x + 1) we get

4x + 1 = A(x + 1) + B(x – 2) ……. (2)

Put x = -1 in (2) we get

4(-1) + 1 = A(-1 + 1) + B(-1 – 2)

-4 + 1 = A(0) + B(-3)

-3 = B(-3)

B = $\frac{-3}{-3}$ = 1

Put x = 2 in (2) we get

4(2) + 1 = A(2 + 1) + B(2 – 2)

8 + 1 = A(3) + B(0)

9 = 3A

A = 3

Using A = 3, B = 1 in (1) we get

$$\frac{4x+1}{(x-2)(x+1)} = \frac{3}{x-2} + \frac{1}{x+1}$$

Question 3.

$$\frac{1}{(x-1)(x+2)^2}$$

Solution:

Here the denominator has repeated factors. So we write

$$\frac{1}{(x-1)(x+2)^2} = \frac{A}{x-1} + \frac{B}{(x+2)} + \frac{C}{(x+2)^2} \text{ ...... (1)}$$

Multiply both sides by (x – 1) $(x + 2)^2$ we get

1 = A$(x + 2)^2$ + B(x – 1) (x + 2) + C(x – 1) ...... (2)

Put x = 1 in (2) we get

1 = A$(1 + 2)^2$ + B(1 – 1) (1 + 2) + C(1 – 1)

1 = A(3^2) + 0 + 0

1 = 9A

A = $\frac{1}{9}$

Put x = -2 in (2) we get

$1 = A(-2 + 2)^2 + B(-2 - 1)(-2 + 2) + C(-2 - 1)$

$1 = 0 + 0 + C(-3)$

$C = \frac{-1}{3}$

From (2) we have

$1 = A(x + 2)^2 + B(x - 1)(x + 2) + C(x - 1)$

$0x^2 + 1 = A(x^2 + 4x + 4) + B(x^2 + x - 2) + C(x - 1)$

Equating coefficient of x^2 on both sides we get

$0 = A + B$

$0 = \frac{1}{9}$ $(\therefore A = \frac{1}{9})$

$B = -\frac{1}{9}$

Using $A = \frac{1}{9}$, $B = -\frac{1}{9}$, $C = -\frac{1}{3}$ in (1) we get,

$$\frac{1}{(x-1)(x+2)(x+2)^2} = \frac{\frac{1}{9}}{x-1} + \frac{\frac{-1}{9}}{x+2} + \frac{\frac{-1}{3}}{(x+2)^2}$$

$$= \frac{1}{9(x-1)} - \frac{1}{9(x+2)} - \frac{1}{3(x+2)^2}$$

Question 4.

$\frac{1}{x^2-1}$

Solution:

$$\text{Let } \frac{1}{x^2-1} = \frac{1}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1} \quad \text{...... (1)}$$

$$\frac{1}{(x+1)(x-1)} = \frac{A(x-1)+B(x+1)}{(x+1)(x-1)}$$

$1 = A(x - 1) + B(x + 1)$ .... (2)

Put $x = 1$ in (2) we get

$1 = 0 + B(1 + 1)$

$1 = B(2)$

$B = \frac{1}{2}$

Put x = -1 in (2) we get

1 = A(-1 – 1) + B(-1 + 1)

1 = -2A + 0

$A = \frac{-1}{2}$

Using $A = \frac{-1}{2}$, $B = \frac{1}{2}$ in (1) we get

$$\frac{1}{(x+1)(x-1)} = \frac{\frac{-1}{2}}{x+1} + \frac{\frac{1}{2}}{x-1}$$

$$\frac{1}{x^2-1} = \frac{1}{2(x-1)} - \frac{1}{2(x+1)}$$

Question 5.

$$\frac{x-2}{(x+2)(x-1)^2}$$

Solution:

$$\text{Let } \frac{x-2}{(x+2)(x-1)^2} = \frac{A}{x+2} + \frac{B}{x-1} + \frac{C}{(x-1)^2} \quad \text{........ (1)}$$

$$\frac{x-2}{(x+2)(x-1)^2} = \frac{A(x-1)^2 + B(x+2)(x-1) + C(x+2)}{(x+2)(x-1)^2}$$

$x - 2 = A(x - 1)^2 + B(x + 2)(x - 1) + C(x + 2)$ ......(2)

Put x = 1 in (2) we get

$1 - 2 = A(1 - 1)^2 + B(1 + 2)(1 - 1) + C(1 + 2)$

-1 = 0 + 0 + 3C

$C = -\frac{1}{3}$

Put x = -2 in (2) we get

$-2 - 2 = A(-2 - 1)^2 + B(-2 + 2)(-2 - 1) + C(-2 + 2)$

$-4 = A(-3)^2 + 0 + 0$

-4 = 9A

$A = \frac{-4}{9}$

From (2) we have,

$0x^2 + x - 2 = A(x - 1)^2 + B(x + 2)(x - 1) + C(x + 2)$

Equating coefficients of x^2 on both sides we get

$0 = A + B$

$0 = \frac{-4}{9} + B$ ($\because A = \frac{-4}{9}$)

$B = \frac{4}{9}$

Using $A = \frac{-4}{9}$, $B = \frac{4}{9}$, $C = -\frac{1}{3}$ in (1) we get

$$\frac{x-2}{(x+2)(x-1)^2} = \frac{\frac{-4}{9}}{x+2} + \frac{\frac{4}{9}}{x-1} + \frac{\frac{-1}{3}}{(x-1)^2} = \frac{4}{9(x-1)} - \frac{4}{9(x+2)} - \frac{1}{3(x-1)^2}$$

Question 6.

$$\frac{2x^2-5x-7}{(x-2)^3}$$

Solution:

$$\text{Let } \frac{2x^2-5x-7}{(x-2)^3} = \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{(x-2)^3} \quad \text{.........} \ \mathbf{(1)}$$

$$\frac{2x^2-5x-7}{(x-2)^3} = \frac{A(x-2)^2 + B(x-2) + C}{(x-2)^3}$$

$2x^2 - 5x - 7 = A(x - 2)^2 + B(x - 2) + C$

$2x^2 - 5x - 7 = A(x^2 - 4x + 4) + B(x - 2) + C$ …….. (2)

Put $x = 2$ in (2) we get

$2(2^2) - 5(2) - 7 = A(0) + B(0) + C$

$8 - 10 - 7 = 0 + 0 + C$

$-9 = C$

$C = -9$

Equating coefficient of x2 on both sides of (2) we get

$2 = A$

$A = 2$

Equating coefficient of x on both sides of (2) we get

$-5 = A(-4) + B(1)$

$-5 = 2(-4) + B$ ($\because A = 2$)

-5 = -8 + B

B = 8 – 5 = 3

Using A = 2, B = 3, C = -9 in (1) we get

$$\frac{2x^2-5x-7}{(x-2)^3} = \frac{2}{x-2}+\frac{3}{(x-2)^2}+\frac{-9}{(x-2)^3}$$

$$= \frac{2}{x-2}+\frac{3}{(x-2)^2}-\frac{9}{(x-2)^3}$$

Question 7.

$$\frac{x^2-6x+2}{x^2(x+2)}$$

Solution:

Here the denominator has three factors. So given fraction can be expressed as a sum of three simple fractions.

$$\text{Let } \frac{x^2-6x+2}{x^2(x+2)} = \frac{A}{x}+\frac{B}{x^2}+\frac{C}{x+2} \text{ ...... (1)}$$

Multiply both sides by $x^2(x + 2)$ we get

$$\frac{(x^2-6x+2)}{x^2(x+2)} \times x^2(x+2) = \frac{A}{x}x^2(x+2)+\frac{B}{x^2}x^2(x+2)+\frac{C}{x+2}x^2(x+2)$$

$x^2 - 6x + 2 = Ax(x + 2) + B(x + 2) + C(x^2)$ .... (2)

Put x = 0 in (2) we get

0 – 0 + 2 = 0 + B(0 + 2) + 0

2 = B(2)

B = 1

Put x = -2 in (2) we get

$(-2)^2 - 6(-2) + 2 = 0 + 0 + C(-2)^2$

4 + 12 + 2 = C(4)

18 = 4C

C = 9/2

Comparing coefficient of x^2 on both sides of (2) we get, 1 = A + C

1 = A + $\frac{9}{2}$

A = 1 – $\frac{9}{2}$ = $\frac{2-9}{2} = \frac{-7}{2}$

Using A = $\frac{-7}{2}$, B = 1, C = $\frac{9}{2}$ in (1) we get,

$$\frac{(x^2-6x+2)}{x^2(x+2)} = \frac{-7}{2x} + \frac{1}{x^2} + \frac{9}{2(x+2)}$$

Question 8.

$$\frac{x^2-3}{(x+2)(x^2+1)}$$

Solution:
Here the quadratic factor $x^2 + 1$ is not factorisable.

Let $\frac{x^2-3}{(x+2)(x^2+1)} = \frac{A}{x+2} + \frac{(Bx+C)}{x^2+1}$ ….. (1)

Multiply both sides by $(x + 2)(x^2 + 1)$ we get,

$x^2 – 3 = A(x^2 + 1) + (Bx + C)(x + 2)$

Put x = -2 we get

$(-2)^2 – 3 = [A(-2)^2 + 1] + 0$

4 – 3 = A(4 + 1)

1 = 5A

A = $\frac{1}{5}$

Equating coefficient of x^2 on both sides of (2) we get

1 = A + B

1 = $\frac{1}{5}$ + B

B = 1 – $\frac{1}{5}$ = $\frac{4}{5}$

Equating coefficients of x on both sides of (2) we get

0 = 2B + C

0 = 2($\frac{4}{5}$) + C

C = $\frac{-8}{5}$

Using A, B, C's values in (1) we get

$$\frac{x^2-3}{(x+2)(x^2+1)} = \frac{1}{5(x+2)} + \frac{\frac{4}{5}x - \frac{8}{5}}{x^2+1}$$

$$= \frac{1}{5(x+2)} + \frac{\frac{4}{5}(x-2)}{x^2+1}$$

Question 9.

$$\frac{x+2}{(x-1)(x+3)^2}$$

Solution:

Here the denominator has three factors. So given fraction can be expressed as a sum of three simple fractions.

Let $\frac{x+2}{(x-1)(x+3)^2} = \frac{A}{x-1} + \frac{B}{x+3} + \frac{C}{(x+3)^2}$ .... (1)

Multiply both sides by (x – 1) $(x + 3)^2$ we get

$\frac{x+2}{(x-1)(x+3)^2}$ (x – 1) $(x + 3)^2$ = $\frac{A}{x-1}$ (x – 1) $(x + 3)^2$ +

$\frac{B}{x+3}$ (x – 1) $(x + 3)^2$ + $\frac{C}{(x+3)^2}$ (x – 1) $(x + 3)^2$

x + 2 = A$(x + 3)^2$ + B(x – 1) (x + 3) + C(x – 1) .... (2)

Put x = 1 in (2) we get

1 + 2 = A$(1 + 3)^2$ + 0 + 0

3 = A$(4)^2$

A = $\frac{3}{16}$

Put x = -3 in (2) we get

-3 + 2 = 0 + 0 + C(-3 – 1)

-1 = C(-4)

C = $\frac{1}{4}$

Comparing coefficient of x^2 on both sides of (2) we get,

0 = A + B

0 = $\frac{3}{16}$ + B

B = $-\frac{3}{16}$

Using A = $\frac{3}{16}$, B = $-\frac{3}{16}$, C = $\frac{1}{4}$ in (1) we get,

$$\frac{x+2}{(x-1)(x+3)^2} = \frac{3}{16(x-1)} - \frac{3}{16(x+3)} + \frac{1}{4(x+3)^2}$$

Question 10.

$$\frac{1}{(x^2+4)(x+1)}$$

Solution:

Here the quadratic factor $x^2 + 4$ is not factorisable.

Let $\frac{1}{(x+1)(x^2+4)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+4}$ .... (1)

Multiply both sides by (x + 1) (x^2 + 4) we get

1 = A(x^2 + 4) + (Bx + C) (x + 1) .... (2)

Put x = -1 in (2) we get

1 = A($(-1)^2$ + 4) + 0

1 = A(1 + 4)

A = $\frac{1}{5}$

Equating coefficient of x^2 on both sides of (2) we get,

0 = A + B

0 = $\frac{1}{5}$ + B

B = $\frac{-1}{5}$

Equating coefficient of x on both sides of (2) we get,

{$\because$ (Bx + C) (x + 1) = Bx^2 + Cx = Bx + C}

0 = B + C

0 = $\frac{-1}{5}$ + C

C = $\frac{1}{5}$

Using A = $\frac{1}{5}$, B = $\frac{-1}{5}$, C = $\frac{1}{5}$ we get,

$$\frac{1}{(x+1)(x^2+4)} = \frac{1}{5(x+1)} + \frac{-\frac{1}{5}x+\frac{1}{5}}{x^2+4}$$

$$= \frac{1}{5(x+1)} + \frac{1}{5}\frac{(-x+1)}{(x^2+4)}$$

$$= \frac{1}{5(x+1)} + \frac{1}{5}\frac{(1-x)}{(x^2+4)}$$

Ex 2.2

Question 1.

Find x if $\frac{1}{6!} + \frac{1}{7!} = \frac{x}{8!}$

Solution:

Given that $\frac{1}{6!} + \frac{1}{7!} = \frac{x}{8!}$

$\frac{1}{6!} + \frac{1}{7 \cdot 6!} = \frac{x}{8 \cdot 7 \cdot 6!}$

Cancelling all 6! we get

$\frac{1}{1} + \frac{1}{7} = \frac{x}{8 \times 7}$

$\frac{7+1}{7} = \frac{x}{8 \times 7}$

$\frac{8}{7} = \frac{x}{8 \times 7}$

x = $\frac{8}{7}$ × 7 × 8 = 64

Question 2.

Evaluate $\frac{n!}{r!(n-r)!}$ when n = 5 and r = 2.

Solution:

$$\frac{n!}{r!(n-r)!} = \frac{5!}{2!(5-2)!} = \frac{5!}{2!\times 3!} = \frac{5\times 4\times 3\times 2\times 1}{2\times 1\times 3\times 2\times 1} = 10$$

Question 3.

If (n + 2)! = 60[(n – 1)!], find n.

Solution:

Given that (n + 2)! = 60(n – 1)!
(n + 2) (n + 1) n (n – 1)! = 60(n – 1)!
Cancelling (n – 1)! we get,
(n + 2)(n + 1)n = 60
(n + 2)(n + 1)n = 5 × 4 × 3
Both sides we consecutive product of integers
∴ n = 3

Question 4.

How many five digits telephone numbers can be constructed using the digits 0 to 9 If each number starts with 67 with no digit appears more than once?

Solution:

Given that each number starts at 67, we need a five-digit number. So we have to fill only one's place, 10's place, and 100th place. From 0 to 9 there are 10 digits. In these digits, 6 and 7 should not be used as a repetition of digits is not allowed. Except for these two digits, we have 8 digits. Therefore one's place can be filled by any of the 8 digits in 8 different ways. Now there are 7 digits are left.

Therefore 10's place can be filled by any of the 7 digits in 7 different ways. Similarly, 100th place can be filled in 6 different ways. By multiplication principle, the number of telephone numbers constructed is 8 × 7 × 6 = 336.

Question 5.

How many numbers lesser than 1000 can be formed using the digits 5, 6, 7, 8, and 9 if no digit is repeated?

Solution:
The required numbers are lesser than 1000.
They are one digit, two-digit or three-digit numbers.
There are five numbers to be used without repetition.
One digit number: One-digit numbers are 5.
Two-digit number: 10th place can be filled by anyone of the digits by 5 ways and 1's place can be 4 filled by any of the remaining four digits in 4 ways.
∴ Two-digit number are $5 \times 4 = 20$.
Three-digit number: 100th place can be filled by any of the 5 digits, 10th place can be filled by 4 digits and one's place can be filled by 3 digits.
∴ Three digit numbers are $= 5 \times 4 \times 3 = 60$
∴ Total numbers $= 5 + 20 + 60 = 85$.

Ex 2.3

Question 1.
If ${}^nP_4 = 12({}^nP_2)$, find n.

Solution:
Given that ${}^nP_4 = 12({}^nP_2)$
$n(n - 1)(n - 2)(n - 3) = 12n(n - 1)$
Cancelling $n(n - 1)$ on both sides we get
$(n - 2)(n - 3) = 4 \times 3$
We have product of consecutive number on both sides with decreasing order.
$n - 2 = 4$
∴ $n = 6$

Question 2.
In how many ways 5 boys and 3 girls can be seated in a row so that no two girls are together?

Solution:
5 boys can be seated among themselves in ${}^5P_5 = 5!$ Ways. After this arrangement, we have to arrange the three girls in such a way that in between two girls there atleast one boy. So the possible places girls can be placed with the × symbol given below.
$\times B \times B \times B \times B \times B \times$
∴ There are 6 places to seated by 3 girls which can be done 6P3 ways.
∴ Total number of ways $= 5! \times {}^6P_3$

$= 120 \times (6 \times 5 \times 4)$
$= 120 \times 120$
$= 14400$

Question 3.
How many 6-digit telephone numbers can be constructed with the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 if each number starts with 35 and no digit appears more than once?

Solution:
Given that each number starts with 35. We need a 6 digit number. So we have to fill only one's place, 10's place, 100th place, and 1000th places. We have to use 10 digits.

In these digits, 3 and 5 should not be used as a repetition of digits is not allowed. Except for these two digits, we have to use 8 digits. One's place can be filled by any of the 8 digits in different ways, 10's place can be filled by the remaining 7 digits in 7 different ways.

100th place can be filled by the remaining 6 different ways and 1000th place can be filled by the remaining 5 digits in 5 different ways.

$\therefore$ Number of 6 digit telephone numbers $= 8 \times 7 \times 6 \times 5 = 1680$

Question 4.
Find the number of arrangements that can be made out of the letters of the word "ASSASSINATION".

Solution:
The number of letters of the word "ASSASSINATION" is 13.
The letter A occurs 3 times
The letter S occurs 4 times
The letter I occur 2 times
The letter N occurs 2 times
The letter T occurs 1 time
The letter O occurs 1 time

$\therefore$ Number of arrangements $= \frac{13!}{3!4!2!2!1!1!} = \frac{13!}{3!4!2!2!}$

Question 5.
(a) In how many ways can 8 identical beads be strung on a necklace?
(b) In how many ways can 8 boys form a ring?

Solution:
(a) Number of ways 8 identical beads can be stringed by $\frac{(8-1)!}{2}=\frac{7!}{2}$

(b) Number of ways 8 boys form a ring = (8 – 1)! = 7!

Question 6.
Find the rank of the word 'CHAT' in the dictionary.

Solution:
The letters of the word CHAT in alphabetical order are A, C, H, T. To arrive the word CHAT, first, we have to go through the word that begins with A. If A is fixed as the first letter remaining three letters C, H, T can be arranged among themselves in 3! ways. Next, we select C as the first letter and start arranging the remaining letters in alphabetical order. Now C and A is fixed remaining two letters can be arranged in 2! ways. Next, we move on H with C, A, and H is fixed the letter T can be arranged in 1! ways.
∴ Rank of the word CHAT = 3! + 2! + 1! = 6 + 2 + 1 = 9

Note: The rank of a given word is basically finding out the position of the word when possible words have been formed using all the letters of the given word exactly once and arranged in alphabetical order as in the case of dictionary. The possible arrangement of the word CHAT are (i) ACHT, (ii) ACTH, (iii) AHCT, (iv) AHTC, (v) ATCH, (vi) ATHC, (vii) CAHT, (viii) CATH, (ix) CHAT. So the rank of the word occurs in the ninth position.
∴ The rank of the word CHAT is 9.

Ex 2.4

Question 1.
If ${}^nP_r$ = 1680 and ${}^nC_r$ = 70, find n and r.

Solution:

Given that $^nP_r = 1680$, $^nC_r = 70$

We know that $^nC_r = \frac{nP_r}{r!}$

$70 = \frac{1680}{r!}$

$r! = \frac{1680}{70} = 24$

$r! = 4 \times 3 \times 2 \times 1 = 4!$

$\therefore r = 4$

Question 2.
Verify that $^8C_4 + {}^8C_3 = {}^9C_4$.

Solution:

LHS $= {}^8C_4 + {}^8C_3$

$= \frac{8 \cdot 7 \cdot 6 \cdot 5}{4 \cdot 3 \cdot 2 \cdot 1} + \frac{8 \cdot 7 \cdot 6}{3 \cdot 2 \cdot 1}$

$= 7 \times 2 \times 5 + 8 \times 7$

$= 70 + 56$

$= 126$

RHS $= {}^9C_4$

$= \frac{9 \times 8 \times 7 \times 6}{4 \times 3 \times 2 \times 1}$

$= 9 \times 7 \times 2$

$= 126$

$\therefore$ LHS = RHS

Hence verified.

Question 3.
How many chords can be drawn through 21 points on a circle?

Solution:
To draw a chord we need two points on a circle.

$\therefore$ Number chords through 21 points on a circle $= {}^{21}C_2 = \frac{21 \times 20}{2 \times 1} = 210.$

Question 4.
How many triangles can be formed by joining the vertices of a hexagon?

Solution:
A hexagon has six vertices. By joining any three vertices of a hexagon we get a triangle.
∴ Number of triangles formed by joining the vertices of a hexagon =
$^6C_3 = \frac{6\times5\times4}{3\times2\times1} = 20.$

Question 5.
Out of 7 consonants and 4 vowels, how many words of 3 consonants and 2 vowels can be formed?

Solution:
In this problem first, we have to select consonants and vowels.
Then we arrange a five-letter word using 3 consonants and 2 vowels.
Therefore here both combination and permutation involved.
The number of ways of selecting 3 consonants from 7 is $^7C_3$.
The number of ways of selecting 2 vowels from 4 is $^4C_3$.
The number of ways selecting 3 consonants from 7 and 2 vowels from 4 is $^7C_3 \times {}^4C_2$.
Now with every selection number of ways of arranging 5 letter word
$= 5! \times {}^7C_3 \times {}^4C_2$
$= 120 \times \frac{7\times6\times5}{3\times2\times1} \times \frac{4\times3}{2\times1}$
$= 25200$

Question 6.
If four dice are rolled, find the number of possible outcomes in which atleast one die shows 2.

Solution:
When a die is rolled number of possible outcomes is selecting an event from 6 events = $^6C_1$

When four dice are rolled number of possible outcomes = $^6C_1 \times {}^6C_1 \times {}^6C_1 \times {}^6C_1$
When a die is rolled number of possible outcomes in which '2' does not appear is selecting an event from 5 events = $^5C_1$
When four dice are rolled number of possible outcomes in which 2 does not

appear = $^5C_1 \times {}^5C_1 \times {}^5C_1 \times {}^5C_1$
Therefore the number of possible outcomes in which atleast one die shows 2
= $^6C_1 \times {}^6C_1 \times {}^6C_1 \times {}^6C_1 - {}^5C_1 \times {}^5C_1 \times {}^5C_1 \times {}^5C_1$
= $6 \times 6 \times 6 \times 6 - 5 \times 5 \times 5 \times 5$
= 1296 – 625
= 671

Note: when two dice are rolled number of possible outcomes is 36 and the number of possible outcomes in which 2 doesn't appear = 25. When two dice are rolled the number of possible outcomes in which atleast one die shows 2 = 36 – 25 = 11. Use the sample space, S = {(1, 1), (1, 2),… (6, 6)}.

Question 7.
There are 18 guests at a dinner party. They have to sit 9 guests on either side of a long table, three particular persons decide to sit on one side and two others on the other side. In how many ways can the guests to be seated?

Solution:
Let A and B be two sides of the table 9 guests sit on either side of the table in $9! \times 9!$ ways.

Out of 18 guests, three particular persons decide to sit namely inside A and two on the other side B. remaining guest = 18 – 3 – 2 = 13.
From 13 guests we can select 6 more guests for side A and 7 for the side.
Selecting 6 guests from 13 can be done in $^{13}C_6$ ways.

Therefore total number of ways the guest to be seated = $^{13}C_6 \times 9! \times 9!$
$$= \frac{13!}{6!(13-6)!} \times 9! \times 9!$$
$$= \frac{13!}{6! \times 7!} \times 9! \times 9!$$

Question 8.
If a polygon has 44 diagonals, find the number of its sides.

Solution:
A polygon of n sides has n vertices. By joining any two vertices of a polygon, we obtain either a side or a diagonal of the polygon.

A number of line segments obtained by joining the vertices of a n sided polygon taken two at a time = Number of ways of selecting 2 out of n.

$= {}^{n}C_2$
$= \frac{n(n-1)}{2}$

Out of these lines, n lines are the sides of the polygon, Sides can't be diagonals.

∴ Number of diagonals of the polygon = $\frac{n(n-1)}{2} - n = \frac{n(n-3)}{2}$

Given that a polygon has 44 diagonals.
Let n be the number of sides of the polygon.
$\frac{n(n-3)}{2} = 44$

$\Rightarrow n(n - 3) = 88$
$\Rightarrow n^2 - 3n - 88 = 0$
$\Rightarrow (n + 8)(n - 11)$
$\Rightarrow n = -8$ (or) $n = 11$
n cannot be negative.
∴ n = 11 is number of sides of polygon is 11.

Question 9.
In how many ways can a cricket team of 11 players be chosen out of a batch of 15 players?
(i) There is no restriction on the selection.
(ii) A particular player is always chosen.
(iii) A particular player is never chosen.

Solution:
(i) Number of ways choosing 11 players from 15 is ${}^{15}C_{11} = {}^{15}C_4$

$= \frac{15\times14\times13\times12}{4\times3\times2\times1}$
$= 15 \times 7 \times 13$
$= 1365.$

(ii) If a particular is always chosen there will be only 14 players left put, in which 10 are to selected in ${}^{14}C_{10}$ ways.

$^{14}C_{10} = {}^{14}C_4$

$= \frac{14\times13\times12\times11}{4\times3\times2\times1}$

$= \frac{14\times13\times11}{2}$

$= 91 \times 11$

$= 1001$ ways

(iii) If a particular player is never chosen we have to select 11 players out of remaining 14 players in $^{14}C_{11}$ ways.

i.e., $^{14}C_3$ ways $= \frac{14\times13\times12}{3\times2\times1} = 364$ ways.

Question 10.
A committee of 5 is to be formed out of 6 gents and 4 ladies. In how many ways this can be done when
(i) atleast two ladies are included.
(ii) atmost two ladies are included.

Solution:
(i) A committee of 5 is to be formed.

Possibilities	Ladies (4)	Gents (6)	Combinations
1	2	3	$4C_2 \times 6C_3 = \frac{4\times3}{2\times1}\times\frac{6\times5\times4}{3\times2\times1} = 120$
	1	4	$4C_1 \times 6C_4 = 4 \times 6C_2 = 4 \times \frac{6\times5}{2\times1} = 60$
	0	5	$4C_0 \times 6C_5 = 1 \times 6C_1 = 1 \times 6 = 6$
			Total number of ways = 186

(ii) Almost two ladies are included means maximum of two ladies are included.

Possibilities	Ladies (4)	Gents (6)	Combinations
1	2	3	$4C_2 \times 6C_3 = \frac{4\times3}{2\times1}\times\frac{6\times5\times4}{3\times2\times1} = 120$
2	3	2	$4C_3 \times 6C_2 = \frac{4\times3\times2}{3\times2\times1}\times\frac{6\times5}{2\times1} = 60$
3	4	1	$4C_4 \times 6C_1 = 1 \times 6 = 6$
			Total number of ways = 186

Ex 2.5

Question 1.

$1^3 + 2^3 + 3^3 + \ldots\ldots + n^3 = \frac{n^2(n+1)^2}{4}$ for all x ∈ N.

Solution:

Let P(n) be the statement $1^3 + 2^3 + \ldots\ldots + n^3 = \frac{n^2(n+1)^2}{4}$ for all n ∈ N.

i.e., $p(n) = 1^3 + 2^3 + \ldots\ldots + n^3 = \frac{n^2(n+1)^2}{4}$, for all n ∈ N

Put n = 1

LHS = $1^3 = 1$

RHS = $\frac{1^2(1+1)^2}{4}$

$= \frac{1\times2^2}{4}$

$= \frac{4}{4}$

= 1

∴ P(1) is true.

Assume that P(n) is true n = k

P(k): $1^3 + 2^3 + \ldots\ldots + k^3 = \frac{k^2(k+1)^2}{4}$

To prove P(k + 1) is true.

i.e., to prove $1^3 + 2^3 + \ldots\ldots + k^3 + (k + 1)^3 = \frac{(k+1)^2((k+1)+1)^2}{4} = \frac{(k+1)^2(k+2)^2}{4}$

Consider $1^3 + 2^3 + \ldots\ldots + k^3 + (k + 1)^3 = \frac{k^2(k+1)^2}{4} + (k + 1)^3$

$= (k + 1)^2 [\frac{k^2}{4} + (k + 1)]$

$= (k + 1)^2 \left[\frac{k^2+4(k+1)}{4}\right]$

$= \frac{(k+1)^2(k+2)^2}{4}$

$\Rightarrow$ P(k + 1) is true, whenever P(k) is true.

Hence, by the principle of mathematical induction P(n) is true for all n $\in$ N.

Question 2.

$1.2 + 2.3 + 3.4 + \ldots\ldots + n(n + 1) = \frac{n(n+1)(n+2)}{3}$, for all n $\in$ N.

Solution:

Let P(n) denote the statement

$1.2 + 2.3 + 3.4 + \ldots\ldots + n(n + 1) = \frac{n(n+1)(n+2)}{3}$

Put n = 1

LHS = 1(1 + 1) = 2

RHS = $\frac{1(1+1)(1+2)}{3} = \frac{1(2)(3)}{3} = 2$

$\therefore$ P(1) is true.

Now assume that the statement be true for n = k

(i.e.,) assume P(k) be true

(i.e.,) assume $1.2 + 2.3 + 3.4 + \ldots\ldots + k(k + 1) = \frac{k(k+1)(k+2)}{3}$ be true

To prove: P(k + 1) is true

(i.e.,) to prove: $1.2 + 2.3 + 3.4 + \ldots\ldots + k(k + 1) + (k + 1)(k + 2) = \frac{(k+1)(k+2)(k+3)}{3}$

Consider $1.2 + 2.3 + 3.4 + \ldots\ldots + k(k + 1) + (k + 1)(k + 2)$

$= [1.2 + 23 + \ldots\ldots + k(k + 1)] + (k + 1)(k + 2)$

$= \frac{k(k+1)(k+2)+3(k+1)(k+2)}{3}$

$= \frac{(k+1)(k+2)(k+3)}{3}$

∴ P(k + 1) is true.

Thus if P(k) is true, P(k + 1) is true.

By the principle of Mathematical 'induction, P(n) is true for all n ∈ N.

$1.2 + 2.3 + 3.4 + \ldots\ldots + n(n + 1) = \frac{n(n+1)(n+2)}{3}$

Question 3.
4 + 8 + 12 + ……. + 4n = 2n(n + 1), for all n ∈ N.

Solution:
Let P(n) denote the statement 4 + 8 + …….. + 4n = 2n(n + 1)
i.e., P(n) : 4 + 8 + 12 + … + 4n = 2n(n + 1)
Put n = 1,
P(1): LHS = 4
RHS = 2 (1)(1 + 1) = 4
P(1) is true.

Assume that P(n) is true for n = k
P(k): 4 + 8 + 12 + ……. + 4k = 2k(k + 1)
To prove P(k + 1)
i.e., to prove 4 + 8 + 12 + ……. + 4k + 4(k + 1) = 2(k + 1) (k + 1 + 1)
4 + 8 + 12 + …… + 4k + (4k + 4) = 2(k + 1) (k + 2)
Consider, 4 + 8 + 12 + …….. + 4k + (4k + 4) = 2k(k + 1) + (4k + 4)
= 2k(k + 1) + 4(k + 1)
= $2k^2$ + 2k + 4k + 4
= $2k^2$ + 6k + 4
= 2(k + 1)(k + 2)
P(k + 1) is also true.
∴ By Mathematical Induction, P(n) for all value n ∈ N.

Question 4.
$1 + 4 + 7 + \ldots\ldots + (3n - 2) = \frac{n(3n-1)}{2}$ for all n ∈ N.

Solution:

Let P(n) : 1 + 4 + 7 + ... + (3n – 2) = $\frac{n(3n-1)}{2}$

Put n = 1,

LHS = 1

RHS = $\frac{1(3-1)}{2}$ = 1

∴ P(1) is true.

Assume P(k) is true for n = k

P(k): 1 + 4 + 7 + ... + (3k – 2) = $\frac{k(3k-1)}{2}$

To prove P(k + 1) is true, i.e., to prove

1 + 4 + 7 + ... + (3k – 2) + (3(k + 1) – 2) = $\frac{(k+1)(3(k+1)-1)}{2}$

1 + 4 + 7 + ... + (3k – 2) + (3k + 3 – 2) = $\frac{(k+1)(3k+2)}{2}$

1 + 4 + 7 + ... + (3k + 1) = $\frac{(k+1)(3k+2)}{2}$

1 + 4 + 7 + ... + (3k – 2) + (3k + 1) = $\frac{k(3k-1)}{2}$ + (3k + 1)

= $\frac{k(3k-1)+2(3k+1)}{2}$

= $\frac{3k^2-k+6k+2}{2}$

= $\frac{3k^2+5k+2}{2}$

= $\frac{(k+1)(3k+2)}{2}$

∴ P(k + 1) is true whenever P(k) is true.

∴ By the Principle of Mathematical Induction, P(n) is true for all n ∈ N.

Question 5.
$3^{2n} - 1$ is divisible by 8, for all n ∈ N.

Solution:
Let P(n) denote the statement $3^{2n} - 1$ is divisible by 8 for all n ∈ N
Put n = 1
P(1) is the statement $3^{2(1)} - 1 = 3^2 - 1 = 9 - 1 = 8$, which is divisible by 8
∴ P(1) is true.
Assume that P(k) is true for n = k.

i.e., $3^{2k} - 1$ is divisible by 8 to be true.
Let $3^{2k} - 1 = 8m$
To prove $P(k + 1)$ is true.
i.e., to prove $3^{2(k+1)} - 1$ is divisible by 8
Consider $3^{2(k+1)} - 1 = 3^{2k+2} - 1$
$= 3^{2k} . 3^2 - 1$
$= 3^{2k} (9) - 1$
$= 3^{2k} (8 + 1) - 1$
$= 3^{2k} \times 8 + 3^{2k} \times 1 - 1$
$= 3^{2k} (8) + 3^{2k} - 1$
$= 3^{2k} (8) + 8m$ ($\because 3^{2k} - 1 = 8m$)
$= 8(3^{2k} + m)$, which is divisible by 8.
$\therefore$ $P(k + 1)$ is true wherever $P(k)$ is true.
$\therefore$ By principle of Mathematical Induction, $P(n)$ is true for all $n \in N$.

Question 6.
$a^n - b^n$ is divisible by $a - b$, for all $n \in N$.

Solution:
Let $P(n)$ denote the statement $a^n - b^n$ is divisible by $a - b$.
Put $n = 1$. Then $P(1)$ is the statement: $a^1 - b^1 = a - b$ is divisible by $a - b$
$\therefore$ $P(1)$ is true. Now assume that the statement be true for $n = k$
(i.e.,) assume $P(k)$ be true, (i.e.,) $a^k - b^k$ is divisible by $(a - b)$ be true.

$\Rightarrow \frac{a^k - b^k}{a - b} = m$ (say) where $m \in N$
$\Rightarrow a^k - b^k = m(a - b)$
$\Rightarrow a^k = b^k + m(a - b)$ .... (1)
Now to prove $P(k + 1)$ is true, (i.e.,) to prove: $a^{k+1} - b^{k+1}$ is divisible by $a - b$
Consider $a^{k+1} - b^{k+1} = a^k . a - b^k . b$
$= [b^k + m(a - b)] a - b^k . b$ [$\because a^k = bm + k(a - b)$]
$= b^k . a + am(a - b) - b^k . b$
$= b^k . a - b^k . b + am(a - b)$
$= b^k(a - b) + am(a - b)$

$= (a - b)(b^k + am)$ is divisible by $(a - b)$

$\therefore$ P(k + 1) is true.

By the principle of Mathematical induction. P(n) is true for all $n \in N$.

$\therefore a^n - b^n$ is divisible by $a - b$ for $n \in N$.

Question 7.
$5^{2n} - 1$ is divisible by 24, for all $n \in N$.

Solution:
Let P(n) be the proposition that $5^{2n} - 1$ is divisible by 24.
For n = 1, P(1) is: $5^2 - 1 = 25 - 1 = 24$, 24 is divisible by 24.
Assume that P(k) is true.
i.e., $5^{2k} - 1$ is divisible by 24
Let $5^{2k} - 1 = 24m$

To prove P(k + 1) is true.
i.e., to prove $5^{2(k+1)} - 1$ is divisible by 24.
P(k): $5^{2k} - 1$ is divisible by 24.
$P(k + 1) = 5^{2(k+1)} - 1$
$= 5^{2k} . 5^2 - 1$
$= 5^{2k} (25) - 1$
$= 5^{2k} (24 + 1) - 1$
$= 24 . 5^{2k} + 5^{2k} - 1$
$= 24 . 5^{2k} + 24m$
$= 24 [5^{2k} + 24]$
which is divisible by 24 $\Rightarrow$ P(k + 1) is also true.
Hence by mathematical induction, P(n) is true for all values $n \in N$.

Question 8.
n(n + 1) (n + 2) is divisible by 6, for all $n \in N$.

Solution:
P(n): n(n + 1) (n + 2) is divisible by 6.
P(1): 1 (2) (3) = 6 is divisible by 6
$\therefore$ P(1) is true.
Let us assume that P(k) is true for n = k
That is, k (k + 1) (k + 2) = 6m for some m
To prove P(k + 1) is true i.e. to prove (k + 1) (k + 2)(k + 3) is divisible by 6.

$P(k + 1) = (k + 1) (k + 2) (k + 3)$
$= (k + 1)(k + 2)k + 3(k + 1)(k + 2)$
$= 6m + 3(k + 1)(k + 2)$
In the second term either $k + 1$ or $k + 2$ will be even, whatever be the value of k.
Hence second term is also divisible by 6.
$\therefore$ P (k + 1) is also true whenever P(k) is true.
By Mathematical Induction P (n) is true for all values of n.

Question 9.
$2^n > n$, for all $n \in N$.

Solution:
Let P(n) denote the statement $2^n > n$ for all $n \in N$
i.e., P(n): $2^n > n$ for $n \geq 1$
Put n = 1, P(1): $2^1 > 1$ which is true.
Assume that P(k) is true for n = k
i.e., $2^k > k$ for $k \geq 1$
To prove P(k + 1) is true.
i.e., to prove $2^{k+1} > k + 1$ for $k \geq 1$
Since $2^k > k$
Multiply both sides by 2
$2 . 2^k > 2k$
$2^{k+1} > k + k$
i.e., $2^{k+1} > k + 1$ ($\because k \geq 1$)
$\therefore$ P(k + 1) is true whenever P(k) is true.
$\therefore$ By principal of mathematical induction P(n) is true for all $n \in N$.

Ex 2.6

Question 1.
Expand the following by using binomial theorem:
(i) $(2a – 3b)^4$
(ii) $\left(x+\frac{1}{y}\right)^7$
(iii) $\left(x+\frac{1}{x^2}\right)^6$

Solution:

(i) $(x+a)^n = nC_0\, x^n a^0 + nC_1\, x^{n-1} a^1 + nC_2\, x^{n-2} a^2 + \ldots + nC_r\, x^{n-r} a^r + \ldots + nC_n\, a^n$

$$(2a-3b)^4 = 4C_0(2a)^4 - 4C_1(2a)^3(3b)^1 + 4C_2(2a)^2(3b)^2 - 4C_3(2a)^1(3b)^3 + 4C_4(3b)^4$$

$$= 1 \times 2^4 a^4 - (4)\, 2^3 a^3 (3b) + \frac{4\times 3}{2\times 1} 2^2 a^2 3^2 b^2 - \frac{4\times 3\times 2}{3\times 2\times 1}(2a)\, 3^3 b^3 + 1(3^4 b^4)$$

$$= 16a^4 - 96a^3 b + 216a^2b^2 - 216ab^3 + 81b^4$$

(ii) $(x+a)^n = nC_0\, x^n a^0 + nC_1\, x^{n-1} a^1 + nC_2\, x^{n-2} a^2 + \ldots + nC_r\, x^{n-r} a^r + \ldots + nC_n\, a^n$

$$\left(x+\frac{1}{y}\right)^7 = 7C_0 x^7 - 7C_1 x^6\left(\frac{1}{y}\right)^1 + 7C_2 x^5\left(\frac{1}{y}\right)^2 + 7C_3 x^4\left(\frac{1}{y}\right)^3 + 7C_4 x^3\left(\frac{1}{y}\right)^4 + 7C_5 x^2\left(\frac{1}{y}\right)^5$$

$$+ 7C_6 x\left(\frac{1}{y}\right)^6 + 7C_7\left(\frac{1}{y}\right)^7$$

$$= x^7 + 7x^6\left(\frac{1}{y}\right) + \frac{7\times 6}{2\times 1}x^5\left(\frac{1}{y^2}\right) + \frac{7\times 6\times 5}{3\times 2\times 1}x^4\left(\frac{1}{y^3}\right) + \frac{7\times 6\times 5\times 4}{4\times 3\times 2\times 1}x^3\left(\frac{1}{y^4}\right)$$

$$+ \frac{7\times 6\times 5\times 4\times 3}{5\times 4\times 3\times 2\times 1}x^2\left(\frac{1}{y^5}\right) + 7x\left(\frac{1}{y^6}\right) + \left(\frac{1}{y^7}\right)$$

$$= x^7 + \frac{7x^6}{y} + \frac{21x^5}{y^2} + \frac{35x^4}{y^3} + \frac{35x^3}{y^4} + \frac{21x^2}{y^5} + \frac{7x}{y^6} + \frac{1}{y^7}$$

(iii) $(x+a)^n = nC_0\, x^n a^0 + nC_1\, x^{n-1} a^1 + nC_2\, x^{n-2} a^2 + \ldots + nC_r\, x^{n-r} a^r + \ldots + nC_n\, a^n$

$$\left(x+\frac{1}{x^2}\right)^6 = 6C_0 x^6 + 6C_1 x^5\left(\frac{1}{x^2}\right)^1 + 6C_2 x^4\left(\frac{1}{x^2}\right)^2 + 6C_3 x^3\left(\frac{1}{x^2}\right)^3 + 6C_4 x^2\left(\frac{1}{x^2}\right)^4 + 6C_5 x\left(\frac{1}{x^2}\right)^5$$

$$+ 6C_6\left(\frac{1}{x^2}\right)^6$$

$$= x^6 + 6x^3 + \frac{6\times 5}{2\times 1}x^4\frac{1}{x^4} + \frac{6\times 5\times 4}{3\times 2\times 1}x^3\left(\frac{1}{x^6}\right) + \frac{6\times 5}{2\times 1}x^2\left(\frac{1}{x^8}\right) + 6x\left(\frac{1}{x^{10}}\right) + 1\left(\frac{1}{x^{12}}\right)$$

$$= x^6 + 6x^3 + 15 + 20\frac{1}{x^3} + 15\frac{1}{x^6} + 6\left(\frac{1}{x^9}\right) + \frac{1}{x^{12}}$$

$$= x^6 + 6x^3 + 15 + \frac{20}{x^3} + \frac{15}{x^6} + \frac{6}{x^9} + \frac{1}{x^{12}}$$

Question 2.

Evaluate the following using binomial theorem:

(i) $(101)^4$

(ii) $(999)^5$

Solution:
(i) $(x + a)^n = nC_0\, x^n\, a^0 + nC_1\, x^{n-1}\, a^1 + nC_2\, x^{n-2}\, a^2 + \dots\dots\dots + nC_r\, x^{n-r}\, a^r + \dots\dots + nC_n\, a^n$
$(101)^4 = (100 + 1)^4 = 4C_0\,(100)^4 + 4C_1\,(100)^3\,(1)^1 + 4C_2\,(100)^2\,(1)^2 + 4C_3\,(100)^1\,(1)^3 + 4C_4\,(1)^4$
$= 1 \times (100000000) + 4 \times (1000000) + 6 \times (10000) + 4 \times 100 + 1 \times 1$
$= 100000000 + 4000000 + 60000 + 400 + 1$
$= 10{,}40{,}60{,}401$

(ii) $(x + a)^n = nC_0\, x^n\, a^0 + nC_1\, x^{n-1}\, a^1 + nC_2\, x^{n-2}\, a^2 + \dots\dots\dots + nC_r\, x^{n-r}\, a^r + \dots\dots + nC_n\, a^n$
$(999)^5 = (1000 - 1)^5 = 5C_0\,(1000)^5 - 5C_1\,(1000)^4\,(1)^1 + 5C_2\,(1000)^3\,(1)^2 - 5C_3\,(1000)^2\,(1)^3 + 5C_4\,(1000)^5\,(1)^4 - 5C_5\,(1)^5$
$= 1(1000)^5 - 5(1000)^4 - 10(1000)^3 - 10(1000)^2 + 5(1000) - 1$
$= 1000000000000000 - 5000000000000 + 10000000000 - 10000000 + 5000 - 1$
$= 995009990004999$

Question 3.
Find the 5th term in the expansion of $(x - 2y)^{13}$.

Solution:
General term is $t_{r+1} = nC_r\, x^{n-r}\, a^r$
$(x - 2y)^{13} = (x + (-2y))^{13}$
Here x is x, a is (-2y) and n = 13
5th term $= t_5 = t_{4+1} = 13C_4\, x^{13-4}\, (-2y)^4$

$= 13C_4\, x^9\, 2^4\, y^4$
$= \frac{13\times12\times11\times10}{4\times3\times2\times1} \times 2 \times 2 \times 2 \times 2 \times x^9y^4$
$= 13 \times 11 \times 10 \times 8x^9y^4$
$= 13 \times 880x^9y^4$
$= 11440x^9y^4$

Question 4.

Find the middle terms in the expansion of

(i) $\left(x+\frac{1}{x}\right)^{11}$

(ii) $\left(3x+\frac{x^2}{2}\right)^{8}$

(iii) $\left(2x^2-\frac{3}{x^3}\right)^{10}$

Solution:

(i) General term is $t_{r+1} = nC_r\, x^{n-r}\, a^r$

Here x is x, a is $\frac{1}{x}$ and n = 11, which is odd.

So the middle terms are $t_{\frac{n+1}{2}} = t_{\frac{11+1}{2}}, t_{\frac{n+3}{2}} = t_{\frac{11+3}{2}}$

i.e. the middle terms are t_6, t_7

$$\text{Now } t_6 = t_{5+1} = 11C_5\, x^{11-5}\left(\frac{1}{x}\right)^5 = 11C_5\, x^6\left(\frac{1}{x^5}\right) = 11C_5\, x$$

$$t_7 = t_{6+1} = 11C_6\, x^{11-6}\left(\frac{1}{x}\right)^6$$

$$= 11C_6\, x^5\left(\frac{1}{x^6}\right) = 11C_6\left(\frac{1}{x}\right) = 11C_{11-6}\left(\frac{1}{x}\right) = 11C_5\left(\frac{1}{x}\right)$$

(ii) Here x is 3x, a is $\frac{x^2}{2}$, n = 8, which is even.

∴ The only one middle term = $t_{\frac{n+1}{2}} = t_{\frac{8+1}{2}}$ = t_5

General term $t_{r+1} = nC_r\, x^{n-r}\, a^r$

$$t_5 = t_{4+1} = 8C_4\,(3x)^{8-4}\left(\frac{x^2}{2}\right)^4 = 8C_4\,(3x)^4\frac{(x^2)^4}{2^4}$$

$$= 8C_4\, 3^4 x^4\, \frac{x^8}{2^4} = 8C_4\, \frac{3^4}{2^4}\, x^{12} = 8C_4\, \frac{81}{16}\, x^{12}$$

(iii) $\left(2x^2 - \frac{3}{x^3}\right)^{10} = \left(2x^2 + \frac{-3}{x^3}\right)^{10}$ compare with the $(x + a)^n$

Here x is $2x^2$, a is $\frac{-3}{x^3}$, n = 10, which is even.

So the only middle term is $\frac{t_{n+1}}{2} = \frac{t_{10}}{2} + 1 = t_6$

General term $t_{r+1} = nC_r\, x^{n-r}\, a^r$

$t_6 = t_{5+1} = t_{r+1}$

$$= 10C_5\,(2x^2)^{10-5}\left(\frac{-3}{x^3}\right)^5$$

$$= 10C_5\,(2x^2)^5\,\frac{(-3)^5}{(x^3)^5} = 10C_5\,2^5x^{10}\,\frac{(-3)^5}{x^{15}}$$

$$= 10C_5\,2^5 \times (-3)^5 \times x^{10-15} = 10C_5\,2^5 \times (-3)^5 \times x^{-5}$$

$$= 10C_5\,2^5 \times (-1)^5 \times 3^5 \times \frac{1}{x^5} = -\,10C_5 \times (2 \times 3)^5 \times \frac{1}{x^5} = -\,10C_5\,(6^5)\,\frac{1}{x^5}$$

Question 5.
Find the term in dependent of x in the expansion of
(i) $\left(x^2 - \frac{2}{3x}\right)^9$
(ii) $\left(x - \frac{2}{x^2}\right)^{15}$
(iii) $\left(2x^2 + \frac{1}{x}\right)^{12}$

Solution:
(i) Let the independent form of x occurs in the general term, $t_{r+1} = nC_r\, x^{n-r}\, a^r$
Here x is x^2, a is $-2/3x$ and $n = 9$

$$\therefore\ t_{r+1} = 9C_r\,(x^2)^{9-r}\left(\frac{-2}{3x}\right)^r = 9C_r\,x^{2(9-r)}\,\frac{(-2)^r}{3^r x^r}$$

$$= 9C_r\,x^{18-2r} \cdot x^{-r}\,\frac{(-2)^r}{3^r}$$

$$= 9C_r\,x^{18-2r-r}\,\frac{(-2)^r}{3^r} = 9C_r\,x^{18-3r}\,\frac{(-2)^r}{3^r}$$

Independent term occurs only when x power is zero.

$18 - 3r = 0$

$\Rightarrow 18 = 3r$

$\Rightarrow r = 6$

Put r = 6 in (1) we get the independent term as $9C_6\ x^0\ \frac{(-2)^6}{3^6} = 9C_3\left(\frac{2}{3}\right)^6$

[$\because 9C_6 = 9C_{9-6} = 9C_3$]

(ii) $\left(x - \frac{2}{x^2}\right)^{15} = \left(x + \frac{-2}{x^2}\right)^{15}$ compare with the $(x + a)^n$

Here x is x, a is $\frac{-2}{x^2}$, n = 15.

Let the independent term of x occurs in the general term

$$t_{r+1} = nC_r\, x^{n-r}\, a^r$$

$$t_{r+1} = 15C_r\, x^{15-r}\left(\frac{-2}{x^2}\right)^r = 15C_r\, x^{15-r}\frac{(-2)^r}{(x^2)^r}$$

$$= 15C_r\, x^{15-r}\frac{(-2)^r}{x^{2r}}$$

$$= 15C_r\, x^{15-r}\cdot x^{-2r}(-2)^r = 15C_r\, x^{15-r-2r}\cdot(-2)^r$$

$$= 15C_r\, x^{15-3r}\cdot(-2)^r$$

Independent term occurs only when x power is zero.

$15 - 3r = 0$

$15 = 3r$

$r = 5$

Using r = 5 in (1) we get the independent term

$= 15C_5\ x^0\ (-2)^5$ [$\because (-2)^5 = (-1)^5\ 2^5 = -2^5$]

$= -32(15C_5)$

(iii) $\left(2x^2 + \frac{1}{x}\right)^{12}$ Compare with the $(x + a)^n$.

Here x is $2x^2$, a is $\frac{1}{x}$, n = 12.

Let the independent term of x occurs in the general term.

$$t_{r+1} = nC_r\, x^{n-r}\, a^r$$

$$t_{r+1} = 12C_r\,(2x^2)^{12-r}\left(\frac{1}{x}\right)^r = 12C_r\, 2^{12-r}\, x^{2(12-r)}\, x^{-r}$$

$$= 12C_r\, 2^{12-r}\, x^{24-2r}\, x^{-r}$$

$$= 12C_r\, 2^{12-r}\, x^{24-3r}$$

Independent term occurs only when x power is zero

24 – 3r = 0

24 = 3r

r = 8

Put r = 8 in (1) we get the independent term as

$= 12C_8\, 2^{12-8}\, x^0$

$= 12C_4 \times 2^4 \times 1$

= 7920

Question 6.

Prove that the term independent of x in the expansion of $\left(x + \frac{1}{x}\right)^{2n}$ is $\frac{1 \cdot 3 \cdot 5 \cdots (2n-1) 2^n}{n!}$

Solution:

There are (2n + 1) terms in expansion.

∴ t_{n+1} is the middle term.

$$t_{n+1} = 2_nC_n\,(x)^{2n-n}\left(\frac{1}{x}\right)^n$$

$$t_{n+1} = 2_nC_n\, x^n \frac{1}{x^n} = 2nC_n = \frac{\lfloor 2n}{\lfloor n \lfloor n}$$

$$= \frac{(2n)(2n-1)(2n-2)\ldots 4 \cdot 3 \cdot 2 \cdot 1}{n(n-1)\ldots 2 \cdot 1 \lfloor n}$$

$$= \frac{(2n-1)(2n-3)\ldots3\cdot1n(n-1)(n-2)\ldots3\cdot2\cdot1\,2^n}{n(n-1)(n-2)\ldots3\cdot2\cdot1\lfloor n}$$

$$= \frac{(2n-1)\ldots3\cdot1}{\lfloor n}2^n = \frac{1\cdot3\cdot5\ldots(2n-1)}{\lfloor n}2^n$$

Question 7.
Show that the middle term in the expansion of is $(1 + x)^{2n}$ is
$\frac{1\cdot3\cdot5\ldots(2n-1)2^n x^n}{n!}$

Solution:
There are $2n + 1$ terms in expansion of $(1 + x)^{2n}$.
∴ The middle term is t_{n+1}.

$$t_{n+1} = 2nC_n\,(1)^{2n-n}\,x^n = 2nC_n\,x^n = \frac{\lfloor 2n}{\lfloor n\lfloor n}x^n$$

$$= \frac{(2n)(2n-1)(2n-2)(2n-3)\ldots5\cdot4\cdot3\cdot2\cdot1}{\lfloor n\lfloor n}x^n$$

$$= \frac{((2n-1)(2n-3)\ldots3\cdot1)2n(2n-2)\ldots4\cdot2}{\lfloor n\lfloor n}$$

$$= \frac{(2n-1)(2n-3)\ldots1\,2^n n(n-1)1\ldots3\cdot2\cdot1}{\lfloor n\lfloor n}$$

$$= \frac{(2n-1)(2n-3)\ldots3\cdot1}{\lfloor n}\,2^n x^n$$

$$= \frac{1\cdot3\cdot5\cdot7\ldots(2n-1)2^n x^n}{\lfloor n}$$

Ex 2.7

Question 1.
If $nC_3 = nC_2$ then the value of nC_4 is:
(a) 2
(b) 3
(c) 4

(d) 5

Answer:
(d) 5
Hint:
Given that $nC_3 = nC_2$
We know that if $nC_x = nC_y$ then $x + y = n$ or $x = y$
Here $3 + 2 = n$
$\therefore n = 5$

Question 2.
The value of n, when $np_2 = 20$ is:
(a) 3
(b) 6
(c) 5
(d) 4

Answer:
(c) 5
Hint:
$nP_2 = 20$
$n(n - 1) = 20$
$n(n - 1) = 5 \times 4$
$\therefore n = 5$

Question 3.
The number of ways selecting 4 players out of 5 is:
(a) 4!
(b) 20
(c) 25
(d) 5

Answer:
(d) 5
Hint:
$5C_4 = 5C_1 = 5$

Question 4.
If $nP_r = 720(nC_r)$, then r is equal to:

(a) 4
(b) 5
(c) 6
(d) 7

Answer:
(c) 6
Hint:
Given $nP_r = 720(nC_r)$
$$\frac{n!}{(n-r)!} = 720\frac{n!}{r!(n-r)!}$$
$$1 = \frac{720}{r!}$$
$$r! = 720$$
$$r! = 6 \times 5 \times 4 \times 3 \times 2 \times 1$$
$$r! = 6!$$
$$r = 6$$

Question 5.
The possible outcomes when a coin is tossed five times:
(a) 25
(b) 52
(c) 10
(d) 5.2

Answer:
(a) 25
Hint:
Number of possible outcomes When a coin is tossed is 2
∴ When five coins are tossed (same as a coin is tossed five times)
Possible outcomes = $2 \times 2 \times 2 \times 2 \times 2 = 2^5$

Question 6.
The number of diagonals in a polygon of n sides is equal to:
(a) nC_2
(b) $nC_2 - 2$
(c) $nC_2 - n$
(d) $nC_2 - 1$

Answer:
(c) $nC_2 - n$

Question 7.
The greatest positive integer which divide $n(n + 1) (n + 2) (n + 3)$ for all $n \in N$ is:
(a) 2
(b) 6
(c) 20
(d) 24

Answer:
(d) 24
Hint:
Put $n = 1$ in $n(n + 1) (n + 2) (n + 3)$
$= 1 \times 2 \times 3 \times 4$
$= 24$

Question 8.
If n is a positive integer, then the number of terms in the expansion of $(x + a)^n$ is:
(a) n
(b) $n + 1$
(c) $n - 1$
(d) 2n

Answer:
(b) $n + 1$

Question 9.
For all $n > 0$, $nC_1 + nC_2 + nC_3 + \ldots\ldots + nC_n$ is equal to:
(a) 2n
(b) $2^n - 1$
(c) n^2
(d) $n^2 - 1$

Answer:
(b) $2^n - 1$
Hint:

Sum of binomial coefficients 2n
i.e., $nC_0 + nC_1 + nC_2 + nC_3 + ... + nC_n = 2^n$
$nC_1 + nC_2 + nC_3 + ... + nC_n = 2^n - nC_0 = 2^n - 1$

Question 10.
The term containing x^3 in the expansion of $(x - 2y)^7$ is:
(a) 3rd
(b) 4th
(c) 5th
(d) 6th

Answer:
(c) 5th

Hint:
First-term contains x^7.
The second term contains x^6.
The fifth term contains x^3.

Question 11.
The middle term in the expansion of $\left(x+\frac{1}{x}\right)^{10}$ is:

(a) $10C_4\left(\frac{1}{x}\right)$
(b) $10C_5$
(c) $10C_6$
(d) $10C_7\,x^2$

Answer:
(b) $10C_5$
Hint:
x is x, a = 1/x, n = 10 which is even.

So the middle term is

$$t_{\frac{n}{2}+1} = t_{\frac{10}{2}+1} = t_6$$

$$t_6 = t_{5+1} = 10C_5\,(x)^5 \times \left(\frac{1}{x}\right)^{10-5}$$

$$= 10C_5\,x^5 \times \frac{1}{x^5} = 10C_5$$

Question 12.

The constant term in the expansion of $\left(x + \frac{2}{x}\right)^6$ is:

(a) 156
(b) 165
(c) 162
(d) 160

Answer:
(d) 160

Hint:
Here x is x, a is 2/x (Note that each term x will vanish)
∴ Constant term occurs only in middle term
n = 6
∴ middle term = $t_{\frac{6}{2}+1} = t_{3+1}$

$$t_{3+1} = 6C_3\,(x)^3 \left(\frac{2}{x}\right)^3$$

$$= \frac{6\cdot5\cdot4}{3\cdot2\cdot1}\,x^3 \times \frac{2^3}{x^3} = 20 \times 8 = 160$$

Question 13.
The last term in the expansion of $(3 + \sqrt{2})^8$ is:
(a) 81
(b) 16
(c) 8

(d) 2

Answer:
(b) 16
Hint:

$$(\sqrt{2})^8 = \left(2^{\frac{1}{2}}\right)^8 = 2^4 = 16$$

Question 14.
If $\frac{kx}{(x+4)(2x-1)} = \frac{4}{x+4} + \frac{1}{2x-1}$ then k is equal to:

(a) 9
(b) 11
(c) 5
(d) 7

Answer:
(a) 9
Hint:

$$\frac{kx}{(x+4)(x-1)} = \frac{4}{x+4} + \frac{1}{2x-1}$$

$$kx = 8x - 4 + x + 4$$

$$kx = 9x$$

$$k = 9$$

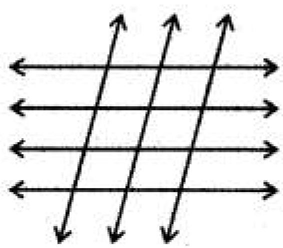
Question 15.
The number of 3 letter words that can be formed from the letters of the word 'NUMBER' when the repetition is allowed are:
(a) 206
(b) 133
(c) 216
(d) 300

Answer:
(c) 216
Hint:

Number of letters in NUMBER is 5
From 5 letters we can form 3 letter ways = 6 × 6 × 6 = 216.

Question 16.
The number of parallelograms that can be formed from a set of four parallel lines intersecting another set of three parallel lines is:
(a) 18
(b) 12
(c) 9
(d) 6

Answer:
(a) 18
Hint:



To form a parallelogram we need 2 parallel lines from 4 and 2 intersecting lines from 3.
Number of parallelograms = $4C_2 \times 3C_2$

$= \frac{4\times3}{2\times1} \times 3$
$= 18$

Question 17.
There are 10 true or false questions in an examination. Then these questions can be answered in
(a) 240 ways
(b) 120 ways
(c) 1024 ways
(d) 100 ways

Answer:
(c) 1024 ways
Hint:
For each question, there are two ways of answering it.

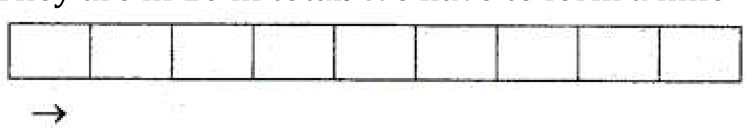
for 10 questions the numbers of ways to answer = 2 × 2 × 2 × 2 × 2 × 2 × 2 × 2 × 2 × 2
$= 2^{10}$
= 1024 ways

Question 18.
The value of $({}^5C_0 + {}^5C_1) + ({}^5C_1 + {}^5C_2) + ({}^5C_2 + {}^5C_3) + ({}^5C_3 + {}^5C_4) + ({}^5C_4 + {}^5C_5)$ is:
(a) $2^6 - 2$
(b) $2^5 - 1$
(c) 2^8
(d) 2^7

Answer:
(a) $2^6 - 2$
Hint:
$({}^5C_0 + {}^5C_1 + {}^5C_2 + {}^5C_3 + {}^5C_4 + {}^5C_5) + ({}^5C_1 + {}^5C_2 + {}^5C_3 + {}^5C_4)$
$= 2^5 + ({}^5C_0 + {}^5C_1 + {}^5C_2 + {}^5C_3 + {}^5C_4 + {}^5C_5) - ({}^5C_0 + {}^5C_5)$
$= 2^5 + 2^5 - (1 + 1)$ ($\because$ Adding and subtracting of ${}^5C_0$ and ${}^5C_5$)
$= 2(2^5) - 2$ ($\because {}^5C_0 = {}^5C_5 = 1$)
$= 2^6 - 2$

Question 19.
The total number of 9 digit number which has all different digit is:
(a) 10!
(b) 9!
(c) 9 × 9!
(d) 10 × 10!

Answer:
(c) 9 × 9!
Hint:
Here we can use the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9.
They are in 10 in total. We have to form a nine-digit number.



The first place from the left can be filled up by anyone of the digits other than zero in 9 ways. The second place can be filled up by anyone of the remaining

(10 – 1) digits (including zero) in 9 ways, the third place in 8 ways, fourth place in 7 ways, fifth place in 6 ways, sixth place in 5 ways, seventh place in 4 ways, eighth place in 3 ways and ninth place in 2 ways.
∴ The number of ways of making 9 digit numbers = $9 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 = 9 \times 9!$

Question 20.
The number of ways to arrange the letters of the word "CHEESE":
(a) 120
(b) 240
(c) 720
(d) 6

Answer:
(a) 120
Hint: Here there are 6 letters.
The letter C occurs one time
The letter H occurs one time
The letter E occurs three times
The letter S occurs one time
Number of arrangements = $\frac{6!}{1!1!3!1!} = \frac{6!}{3!} = \frac{6 \times 5 \times 4 \times 3!}{3!}$ = 120

Question 21.
Thirteen guests have participated in a dinner. The number of handshakes that happened in the dinner is:
(a) 715
(b) 78
(c) 286
(d) 13
(b) 78
Hint:
To handshakes, we need two guests.
Number of selecting 2 guests from 13 is $13C_2 = \frac{13 \times 12}{2 \times 1}$ = 78

Question 22.
The number of words with or without meaning that can be formed using letters of the word "EQUATION", with no repetition of letters is:
(a) 7!

(b) 3!
(c) 8!
(d) 5!

Answer:
(c) 8!
Hint:
There are 8 letters.
From 8 letters number of words is formed = $8P_8$ = 8!

Question 23.
Sum of binomial coefficient in a particular expansion is 256, then number of terms in the expansion is:
(a) 8
(b) 7
(c) 6
(d) 9

Answer:
(a) 8
Hint:
Sum of binomial coefficient = 256
i.e., $2^n = 256$
$2^n = 2^8$
n = 8

Question 24.
The number of permutation of n different things taken r at a time, when the repetition is allowed is:
(a) r^n
(b) n^r
(c) $\frac{n!}{(n-r)!}$
(d) $\frac{n!}{(n+r)!}$

Answer:
(b) n^r

Question 25.
The sum of the binomial coefficients is:
(a) 2^n
(b) n^2
(c) $2n$
(d) $n + 17$

Answer:
(a) 2^n