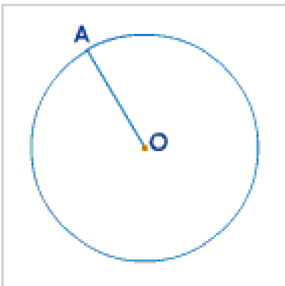
# Circles

## Introduction

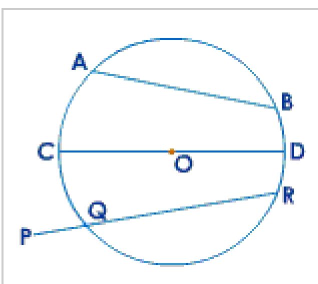
### Circle

A circle is defined as the locus of the points at a given distance from a certain fixed point.




### Chord

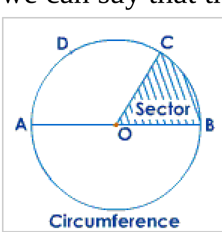
The straight line joining any 2 points on the circle is called a chord.
AB is a chord.
The longest chord is called the diameter if passes through the centre of the circle.




A diameter is twice the length of the radius. CD is a diameter.
A secant is a line cutting a circle into two parts. PQR is a secant.

### Circumference

The set of all the points on a circle constitute the circumference of the circle. In simple language we can say that the boundary curve of the circle (or perimeter) is its circumference.



**Arc**

Any part of the circumference is called an arc.

A diameter cuts a circle into 2 equal parts. An arc less than a semicircle is called a minor arc. An arc more than a semicircle is called a major arc.

$\overset{\frown}{ADC}$ is a minor arc and $\overset{\frown}{ABC}$ is a major arc.

**Sector**

A portion cut off by two radii is called a sector.

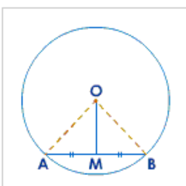
Segment: a portion of a circle cut off by a chord is called a segment.

**Concentric circles**

Circles having the same centre are called concentric circles.

**Theorem 1**

A straight line drawn from the centre of a circle to bisect a chord which is not a diameter, is at right angles to the chord.




**Data:**

AB is a chord of a circle with centre O.

M is the mid-point of AB. OM is joined

**To Prove:**

$\angle AMO = \angle BMO = 90^0$

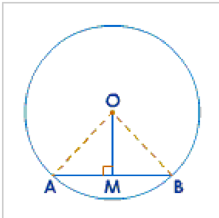
**Construction:**

Join AO and BO.

**Proof:**

| Statement | Reason |
|---|---|
| In $\Delta^s$ AOM and BOM | |
| 1. AO = BO | radii |
| 2. AM = BM | data |
| 3. OM = OM | common |
| 4. $\Delta AOM \cong \Delta BOM$ | (S.S.S.) |
| 5. $\therefore \angle AMO = \angle BMO$ | statement (4) |
| 6. But $\angle AMO + \angle BMO = 180^0$ | linear pair |
| 7. $\therefore \angle AMO = \angle BMO = 90^0$ | statements (5) and (6) |

**Theorem 2: (Converse of theorem 1)**
The perpendicular to a chord from the centre of a circle bisects the chord.




**Data:**
AB is a chord of a circle with centre O,
OM ⊥ AB.

**To Prove:**
AM = BM.

**Construction:**
Join AO and BO.

**Proof:**

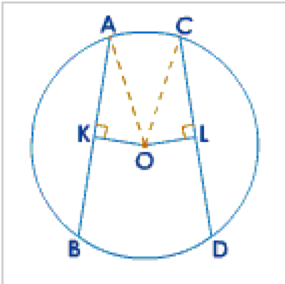
| Statement | Reason |
|---|---|
| In $\Delta^s$ AOM and BOM | |
| 1. ∠AMO = ∠BMO | each $90^0$ (data) |
| 2. AO = BO | radii |
| 3. OM = OM | Common |
| 4. ΔAOM ≅ ΔBOM | (R.H.S.) |
| 5. AM = BM | Statement (4) |

Converse of a theorem is the transposition of a statement consisting of 'data' and 'to prove'.
We elaborate it from the example of previous two theorems:

| Theorem | Converse of theorem |
|---|---|
| 1. Data: M is the mid-point of AB | To prove: M is the mid-point of AB. |
| 2. To prove: OM ⊥ AB | Data: OM ⊥ AB |

**Theorem 3**

Equal chords of a circle are equidistant from the centre.




**Data:**

AB and CD are equal chords of a circle with centre O. OK $\perp$ AB and OL $\perp$ CD.

**To Prove:**

OK = OL

**Construction:**

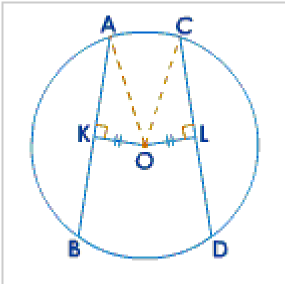
Join AO and CO.

**Proof:**

| Statement | Reason |
|---|---|
| 1. $AK = \frac{1}{2} AB$ | $\perp$ from the centre bisects the chord. |
| 2. $CL = \frac{1}{2} CD$ | $\perp$ from the centre bisects the chord. |
| 3. But AB = CD | data |
| 4. $\therefore$ AK = CL | statements (1), (2) and (3) |
| In $\Delta^s$ AOK and COL | |
| 5. $\angle AKO = \angle CLO$ | each $90^0$ (data) |
| 6. AO = CO | radii |
| 7. AK = CL | statement (4) |
| 8. $\therefore \Delta AOK \cong \Delta COL$ | (R.H.S.) |
| 9. $\therefore$ OK = OL. | statement (8) |

**Theorem 4 (Converse of 3)**

Chords which are equidistant from the centre of a circle are equal.




**Data:**

AB, CD are chords of a circle with centre O.

$OK \perp AB$, $OL \perp CD$ and OK = OL.

**To Prove:**

AB = CD.

**Construction:**

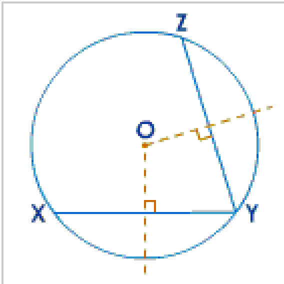
Join AO and CO.

**Proof:**

| Statement | Reason |
|---|---|
| In $\Delta^s AOK$ and $COL$ | |
| 1. $\angle AKO = \angle CLO$ | each $90^0$ (data) |
| 2. AO = CO | radii |
| 3. OK = OL | data |
| 4. $\Delta AOK \cong \Delta COL$ | (R.H.S.) |
| 5. $\therefore$ AK = CL | statement (4) |
| 6. But $AK = \frac{1}{2} AB$ | $\perp$ from centre bisects the chord. |
| 7. $CL = \frac{1}{2} CD$ | $\perp$ from centre bisects the chord |
| 8. $\therefore$ AB = CD | statements (5), (6) and (7) |

**Theorem 5**

There is one circle, and only one, which passes through three given points not in a straight line.




**Data:**

X, Y and Z are three points not in a straight line.

**To Prove:**

A unique circle passes through X, Y and Z.

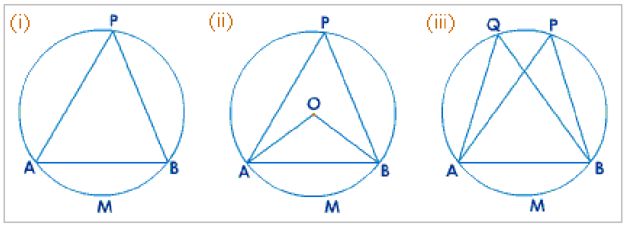
**Construction:**

Join XY and YZ. Draw perpendicular bisectors of XY and YZ to meet at O.

**Proof:**

| Statement | Reason |
|---|---|
| 1. OX = OY | O lies on the ⊥ bisector of XY. |
| 2. OY = OZ | O lies on the ⊥ bisector of YZ |
| 3. OX = OY = OZ | statements (1) and (2) |
| 4. O is the only point equidistant from X, Y and Z. | statement (3) |
| 5. With O as centre and radius OX, a circle can be drawn to pass through X, Y and Z. | statement (4) |
| 6. ∴ the circle with centre O is a unique circle passing through X, Y and Z. | statement (5) |

**Angle Properties (Angle, Cyclic Quadrilaterals and Arcs)**
In fig.(i), the straight line AB students ∠APB on the circumference.



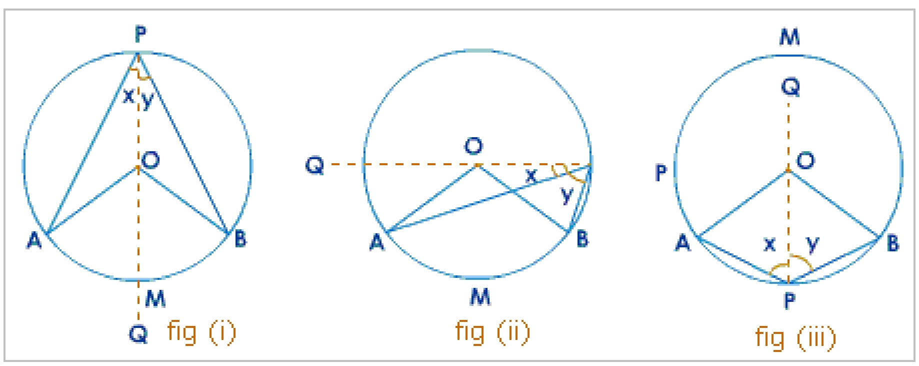

∠APB can be said to be subtended by arc AMB, on the remaining part of the circumference.

In fig.(ii), arc AMB subtends ∠APB on the circumference, and it subtends ∠AOB at the centre.

In fig. (iii), ∠APB and ∠AQB are in the same segment.

Let us study the theorems based on the angle properties of the circles.

**Theorem 6**
The angle which an arc of a circle subtends at the centre is double the angle which it subtends at any point on the remaining part of the circumference.




**Data:**
Arc AMB subtends □AOB at the centre O of the circle and □APB on the remaining part of the circumference.

**To Prove:**
∠AOB = 2 ∠APB

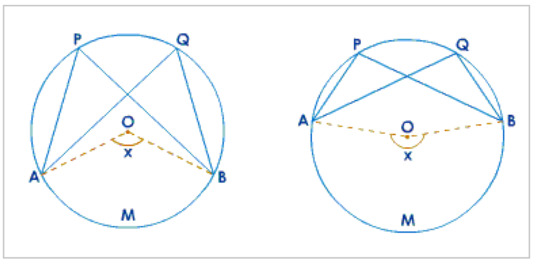
**Construction:**
Join PO and produce it to Q. Let □APQ = x and □BPQ = y.

**Proof:**

| Statement | Reason |
|---|---|
| 1. $\angle AOQ = \angle x + \angle A$ | ext. $\angle$ = sum of the int. opp. $\angle$s |
| 2. $\angle x = \angle A$ | $\because$ OA = OP (radii) |
| 3. $\therefore \angle AOQ = 2\angle x$ | statements (1) and (2) |
| 4. $\angle BOQ = 2\angle y$ | same way as statement (3) |
| For fig.(i) and fig.(iii) | |
| 5. $\angle AOQ + \angle BOQ = 2\angle x + 2\angle y$ | statements (3) and (4) |
| 6. $\Rightarrow \angle AOB = 2(\angle x + \angle y)$ | statement (5) |
| 7. For fig.(ii) | statements (3) and (4) |
| $\angle BOQ - \angle AOQ = 2\angle y - 2\angle x$ | |
| 8. $\angle AOB = 2(\angle y - \angle x)$ | statement (8) |
| 9. $\therefore \angle AOB = 2\angle APB$ | statement (9) |

**Theorem 7**

Angles in the same segment of a circle are equal.




**Data:**

$\angle APB$ and $\angle AQB$ are in the same segment of a circle with centre O.

**To Prove:**

$\angle APB = \angle AQB$

**Construction:**

Join AO and BO.

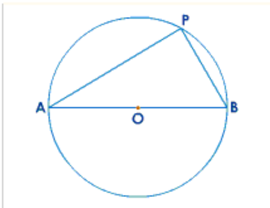
Let arc AMB subtend angle x at the centre O.

**Proof:**

| Statement | Reason |
|---|---|
| 1. $\angle x = 2\angle APB$ | $\angle$ at centre = 2 x $\angle$ on the circumference |
| 2. $\angle x = 2\angle AQB$ | $\angle$ at centre = 2 x $\angle$ on the circumference |
| 3. $\therefore \angle APB = \angle AQB$ | statements (1) and (2) |

**Theorem 8**

The angle in a semicircle is a right angle.




**Data:**

AB is a diameter of a circle with centre O.P is any point on the circle

**To Prove:**

$\angle APB = 90^0$

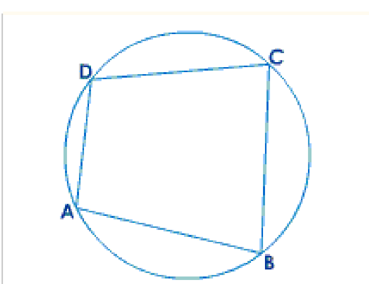
**Proof:**

| Statement | Reason |
|---|---|
| 1. $\angle APB = \frac{1}{2}\angle AOB$ | $\angle$ at the centre = 2 x $\angle$ on the Oce. |
| 2. $\angle AOB = 180^o$ | AOB is a straight line |
| 3. $\therefore \angle APB = \frac{1}{2} \times 180^o$ | Statements (1) and (2) |
| 4. $\therefore \angle APB = 90^o$ | Statement (3) |

**Cyclic Quadrilaterals**

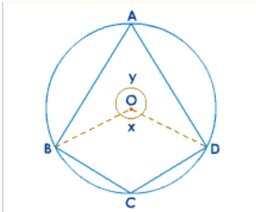
If the vertices of a quadrilateral lie on a circle, the quadrilateral is called a cyclic quadrilateral. The vertices are called concyclic points.

In the given figure, ABCD is a cyclic quadrilateral. The vertices A,B,C and D are concyclic points.



**Theorem 9**

The opposite angles of a quadrilateral inscribed in a circle (cyclic) are supplementary.




**Data:**

ABCD is a cyclic quadrilateral; O is the centre of the circle.

**To Prove:**

(i) $\angle A + \angle C = 180^o$

(ii) $\angle B + \angle D = 180^o$

**Construction:**

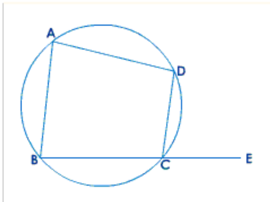
Join BO and DO.

Let $\angle BOD = x$ and reflex $\angle BOD = y$

**Proof:**

| Statement | Reason |
|---|---|
| 1. $\angle A = \frac{1}{2}\angle x$ | $\angle$ at the centre = 2($\angle$ on the circumference) |
| 2. $\angle C = \frac{1}{2}\angle y$ | $\angle$ at the centre = 2($\angle$ on the circumference) |
| 3. $\angle A + \angle C = \frac{1}{2}\angle x + \frac{1}{2}\angle y$ | Statements (1) and (2) |
| 4. $\angle A + \angle C = \frac{1}{2}(\angle x + \angle y)$ | Statement (3) |
| 5. But $\angle x + \angle y = 360^o$ | $\angle$s at a point |
| 6. $\therefore \angle A + \angle C = \frac{1}{2} \times 360^o$ | statements (4) and (5) |
| 7. $\therefore \angle A + \angle C = 180^o$ | statement (6) |
| 8. Also $\angle ABC + \angle ADC = 180^o$ | same way as statement (7) |

**Corollary:**
The exterior angle of a cyclic quadrilateral is equal to the interior opposite angle.




**Data:**
ABCD is a cyclic quadrilateral. BC is produced to E.

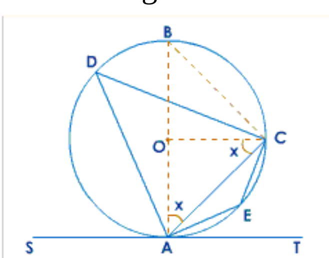
**To Prove:**
∠DCE = ∠A

**Proof:**

| Statement | Reason |
|---|---|
| 1. ∠A + ∠BCD = $180^{0}$ | Opp. ∠s of a cyclic quad. |
| 2. ∠BCD + ∠DCE = $180^{0}$ | linear pair |
| 3. ∴ ∠BCD + ∠DCE = ∠A + ∠BCD | Statements (1) and (2) |
| 4. ∴ ∠DCE = ∠A . | Statement (2) |

**Alternate Segment Property**
**Theorem 10:**

The angle between a tangent and a chord through the point of contact is equal to the angle in the alternate segment.




**Data:**
A straight line SAT touches a given circle with centre O at A. AC is a chord through the point of contact A. ∠ADC is an angle in the alternate segment to ∠CAT and ∠AEC is an angle in the alternate segment to ∠CAS.
**To Prove:**
(i) ∠CAT = ∠ADC
(ii) ∠CAS = ∠AEC

**Construction:**

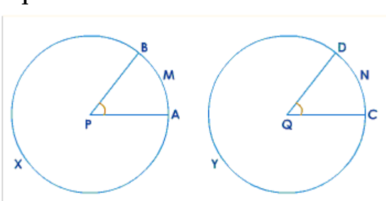
Draw AOB as diameter and join BC and OC.

**Proof:**

| Statement | Reason |
|---|---|
| 1. $\angle OAC = \angle OCA = x$ | $\because$ OA = OC and supposition |
| 2. $\angle CAT + \angle x = 90^{o}$ | $\because$ tangent-radius property |
| 3. $\angle AOC + \angle x + \angle x = 180^{o}$ | sum of the angles of a $\Delta$ |
| 4. $\angle AOC = 180^{o} - 2\angle x$ | statement (3) |
| 5. Also $\angle AOC = 2\angle ADC$ | $\angle$ at the centre = 2$\angle$ on the Oce. |
| 6. $\angle CAT = 90^{o} - x$ | Statement (2) |
| 7. $2\angle CAT = 180^{o} - 2x$ | Statement (6) |
| 8. $\therefore 2\angle CAT = 2\angle ADC$ | Statement (4), (5) and (7) |
| 9. $\angle CAT = \angle ADC$ | Statement (8) |
| 10. $\angle CAS + \angle CAT = 180^{o}$ | Linear pair |
| 11. $\angle ADC + \angle AEC = 180^{o}$ | Opp. angles of a cyclic quad |
| 12. $\angle CAS + \angle CAT = \angle ADC + \angle AEC$ | Statements (10) and (11) |
| 13. $\therefore \angle CAS = \angle AEC$ | Statements (9) and (12) |

**Theorem 11**

In equal circles (or in the same circle), if two arcs subtend equal angles at the centres, they are equal.




**Data:**

AXB and CYD are equal circles with centres P and Q; arcs AMB, CND subtend equal angles APB, CQD.

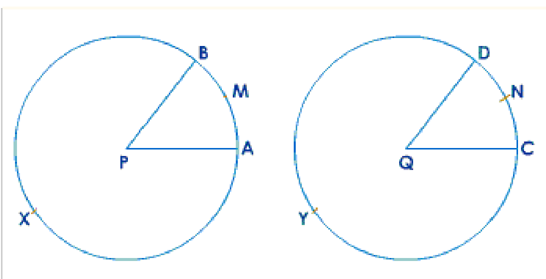
**To Prove:**

arc AMB = arc CND.

**Proof:**

| Statement | Reason |
|---|---|
| 1. Apply ⊙ CYD to ⊙ AXB so that centre Q falls on centre P and QC along PA and D on the same side as B. | ∵ ⊙s are equal (data) |
| ∴ Oce. CYD overlaps Oce. AXB. | |
| 2. ∴ C falls on A. | ∵ PA = QC (data) |
| 3. ∠APB = ∠CQD | data |
| 4. ∴ QD falls along PB | statements (1) and (3) |
| 5. ∴ D falls on B | ∵ QD = PB (data) |
| 6. ∴ arc CND coincides with arc AMB. | statements (2) and (5) |
| 7. arc AMB = arc CND | statement (6) |

**Theorem 12 (Converse of 11)**




In equal circles (or in the same circle) if two arcs are equal, they subtend equal angles at the centres.

**Data:**

In equal circles AXB and CYD, equal arcs AMB and CND subtend ∠APB and ∠CQD at the centres P and Q respectively.

**To Prove:**

∠APB = ∠CQD.

**Proof:**

| Statement | Reason |
|---|---|
| 1. Apply ⊙CYD to AXB so that centre Q falls on centre P and QC along PA, and D on the same side as B. ∴ ⊙ce. CYD overlaps ⊙ce. AXB | ⊙s are equal (data) |
| 2. ∴ C falls on A | PA = QC (data) |
| 3. arc AMB = arc CND | Data |
| 4. ∴ D falls on B. | Statements (1), (2) and (3) |
| 5. ∴ QD coincides with PB and QC coincides with PA | Statements (1), (2) and (4) |
| 6. ∠APB = ∠CQD . | Statement (5) |

In case of the same circle:

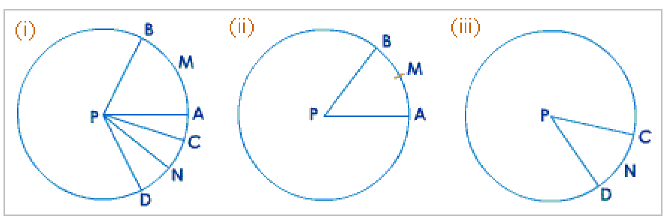
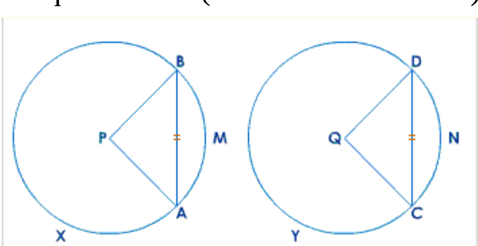



Fig.(ii) and fig.(iii) may be considered to be two equal circles obtained from fig.(i) and then the above proofs may be applied.

**Theorem 13**

In equal circles (or in the same circle), if two chords are equal, they cut off equal arcs.




**Data:**

In equal circles AXB and CYD, with centres P and Q, chord AB = chord CD.

**To Prove:**

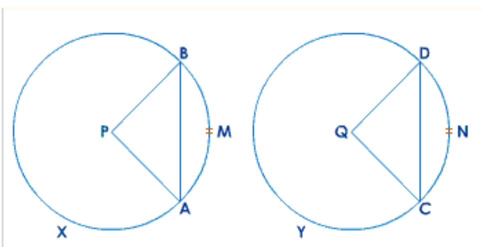
arc AMB = arc CND; arc AXB = arc CYD

**Proof:**

| Statement | Reason |
|---|---|
| In $\Delta^s$ ABP and CDQ | |
| 1. AP = CQ | radii of equal. ⊙s. |
| 2. BP = DQ | radii of equal ⊙s |
| 3. AB = CD | data |
| 4. $\Delta ABP \equiv \Delta CDQ$ | (S.S.S.) |
| 5. $\therefore \angle APB = \angle CQD$ | statement (4) |
| 6. arc AMB = arc CND | statement (5) |
| 7. ⊙AXB - arc AMB = ⊙CYD - arc CND | equal arcs [statement (6)] |
| 8. ∴ arc AXB = arc CYD. | statement (7) |

**Theorem 14 (Converse of 13)**

In equal circles (or in the same circle) if two arcs are equal, the chords of the arcs are equal.




**Data:**

Equal circles AXB, CYD with centres P and Q have arc AMB = arc CND.

**To Prove:**

chord AB = chord CD

**Construction:**

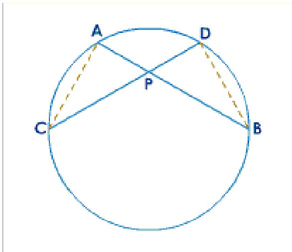
Join AP, BP, CQ and DQ.

**Proof:**

| Statement | Reason |
|---|---|
| In $\Delta^s$ ABP and CDQ | |
| 1. AP = CQ | radii of equal ⊙s |
| 2. BP = DQ | radii of equal ⊙s |
| 3. $\angle APB$ = CQD | ∵ arc AMB = arc CND |
| 4. $\therefore \Delta ABP \cong \Delta CDQ$ | (S.A.S.) |
| 5. ∴ AB = CD | statement (4) |

**Theorem 15**

If two chords of a circle intersect internally, then the product of the length of the segments are equal.




**Data:**

AB and CD are chords of a circle intersecting internally at P.

**To Prove:**

AP x BP = CP x DP.

**Construction:**

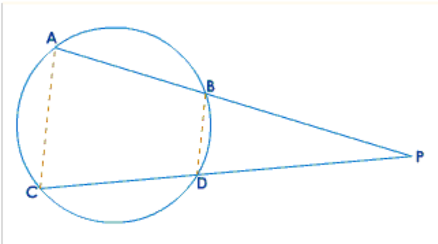
Join AC and BD.

**Proof:**

| Statement | Reason |
|---|---|
| In $\Delta^s$ APC and DPB | |
| 1. $\angle A = \angle D$ | $\angle$s in the same segment |
| 2. $\angle C = \angle B$ | $\angle$s in the same segment |
| 3. $\therefore \Delta APC \sim \Delta DPB$ | AA similarity |
| 4. $\therefore \frac{AP}{DP} = \frac{CP}{BP}$ | Statement (3) |
| 5. $\therefore$ AP x BP = CP x DP | Statement (4) |

**Theorem 16**

If two chords of a circle intersect externally, then the product of the lengths of the segments are equal.




**Data:**

AB and CD are chords of a circle intersecting externally at P.

**To Prove:**

AP x BP = CP x DP.

**Construction:**

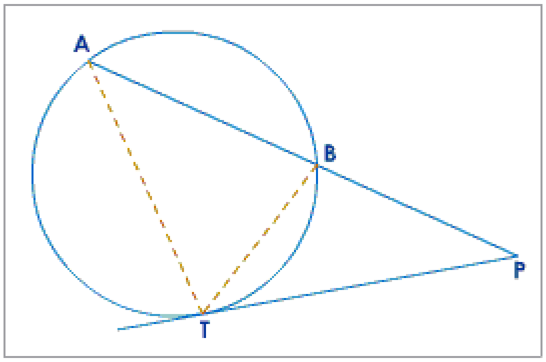
Join AC and BD.

**Proof:**

| Statement | Reason |
|---|---|
| In $\Delta^s$ ACP and DBP | |
| 1. $\angle A = \angle BDP$ | ext. $\angle$ of a cyclic quad. = int. opp. $\angle$ |
| 2. $\angle C = \angle DBP$ | ext. $\angle$ of a cyclic quad. = int. opp. $\angle$ |
| 3. $\therefore \Delta ACP \sim \Delta DBP$ | AA similarity |
| 4. $\therefore \dfrac{AP}{DP} = \dfrac{CP}{BP}$ | Statement (3) |
| 5. $\therefore AP \times BP = CP \times DP$ | Statement (4) |

**Theorem 17**

If a chord and a tangent intersect externally, then the product of the lengths of the segments of the chord is equal to the square on the length of the tangent from the point of contact to the point of intersection.




**Data:**

A chord AB and a tangent TP at a point T on the circle intersect at P.

**To Prove:**

AP x BP = PT²

**Construction:**

Join AT and BT.

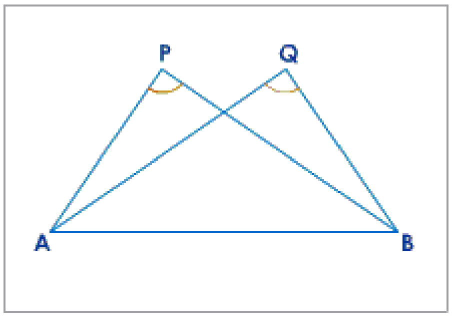
**Proof:**

| Statement | Reason |
|---|---|
| In $\Delta^s$APT and TPB | Angle in the alternate segment |
| 1. $\angle A = \angle BTP$ | |
| 2. $\angle P = \angle P$ | Common |
| 3. $\therefore \Delta APT \sim \Delta TPB$ | AA similarity |
| 4. $\frac{AP}{PT} = \frac{PT}{BP}$ | Statement (3) |
| 5. $AP \times BP = PT^2$ | Statement (4) |

**Test for Concyclic Points**

(a) Converse of the statement, 'Angles in the same segment of a circle are equal', is one test for concyclic points. We state:

If two equal angles are on the same side of a line and are subtended by it, then the four points are concyclic. In the figure, if $\angle P = \angle Q$ and the points P, Q are on the same side of AB, then the points A, B, Q and P are concyclic.




(b) Converse of 'opposite angles of a cyclic quadrilateral are supplementary' is one more test for concyclic points.

We state:

If the opposite angles of a quadrilateral are supplementary, then its vertices are concyclic. In the figure, if $\angle A + \angle C = 180^o$, then A,B,C and D are concyclic points.