Algebra

Learning Objectives

- Ratio
- Proportion
- Percentage
- Algebraic Expressions
- Simple Equations

Ratio and Proportion

Ratio

We often have to compare quantities in our daily life. They may be heights, weights, salaries, marks etc. While comparing salaries of two persons i.e., salaries 6000 per month and 9000 per month, they may be written as the ratio 6000 : 9000 or 2 : 3. In general, the ratio of two quantities a and b in the same units is the fraction $\frac{a}{b}$ and we write it as a : b. In the ratio a: b, we call a as the first term or antecedent and b, the second term or consequent. For example, the ratio 5 : 9 represents $\frac{5}{9}$ with antecedent = 5, consequent = 9.

Note: The multiplication or division of each term of a ratio by the same non-zero number does not affect the ratio. For example, 4 : 5 = 8 : 10 = 12 : 15. Also, 4 : 6 = 2 : 3.

Proportion

The equality of two ratios is called proportion. If a : b = c : d, we write a : b :: c : d and we say that a, b, c, d are in proportion. Here a and d are called extremes, while b and c are called mean terms.
Product of means = Product of extremes.
Thus, $a : b :: c : d \Leftrightarrow (b \times c) = (a \times d)$.

Fourth Proportional: If a : b = c : d, then d is called the fourth proportional to a, b, c.
Third Proportional: a : b = c : d, then c is called the third proportion to a and b.
Mean Proportional: Mean proportional between a and b is ab.

Comparison of Ratios

We say that (a : b) > (c : d) $\Leftrightarrow$ $\frac{\mathbf{a}}{\mathbf{b}} > \frac{\mathbf{c}}{\mathbf{d}}$.

Compounded Ratio: The compounded ratio of the ratios: (a : b), (c : d), (e : f) is (ace : bdf).
Duplicate Ratios:

Duplicate ratio of (a : b) is $\left(a^2 : b^2\right)$.

Sub-duplicate ratio of (a : b) is $\left(\sqrt{a}, \sqrt{b}\right)$.

Triplicate ratio of (a : b) is $\left(a^3 : b^3\right)$.

Sub-triplicate ratio of (a : b) is $\left(a^{1/3} : b^{1/3}\right)$.

If $\frac{a}{b} = \frac{c}{d}$, *then* $\frac{a+b}{c-b} = \frac{c+d}{c-d}$ [componendo and dividend].

Example

If 4x + 3y : 6x + 5y = $\frac{11}{17}$, then find x : y.

(a) 0 : 1 (b) 2 : 1

(c) 1 : 0 (d) 5 : 0

(e) None of these

Ans. (b)

Explanation: $\frac{4x+3y}{6x+5y} = \frac{11}{17} \Rightarrow 17(4x+3y) = 11(6x+5y)$

$\Rightarrow 68x + 51y = 66x + 55y \Rightarrow 68x - 66x = 55y - 51y$

$\Rightarrow$ 2x = 4y $\Rightarrow \frac{x}{y} = \frac{4}{2} \Rightarrow$ x : y = 2 : 1.

If A, B, C, D are quantities of same kind and the ratio of A to B is 3 : 4, B to C is 5 : 7 and C to D is 8:

1. A to C is 15 : 28

2. B to D is 40 : 63

3. A to D is 10 : 21

Which one of the following options is correct?

(a) 1, 2 and 3 (b) 1 and 2

(c) 2 and 3 (d) 1 and 5

(e) None of These

Ans. (a)

Explanation: $\frac{A}{B} = \frac{3}{4}, \frac{B}{C} = \frac{5}{7} \ \& \ \frac{C}{D} = \frac{8}{9} \Rightarrow A = \frac{3}{4}B \ \& \ B = \frac{5}{7}C \Rightarrow A = \frac{3}{4} \times \frac{5}{7}C \ or \ \frac{A}{C} = \frac{15}{28}$

$$\Rightarrow C = \frac{8}{9}D \Rightarrow B = \frac{5}{7} \times \frac{8}{9}D \Rightarrow B = \frac{40}{63}D \ or \ \frac{B}{D} = \frac{40}{63}$$

$$\Rightarrow A = \frac{3}{4}B = \frac{3}{4} \times \frac{40}{63}D = \frac{10}{21}D \Rightarrow \frac{A}{D} = \frac{10}{21}.$$

The ratio between two quantities is 7 : 9. If the first quantity is 511, then find the other quantity.

(a) 655 (b) 555

(c) 657 (d) 656

(e) None of these

Ans. (c)

Let the other quantity be x then

$$7 : 9 = 511 : x \Rightarrow x = \frac{511 \times 9}{7} = 657$$

Find two numbers so that their mean proportional is 14 and their proportional is 112.

(a) 6 and 27 (b) 7 and 28

(c) 9 and 29 (d) 10 and 30

(e) None of these

Ans. (b)

Let the number be x and y, then according to question

$\sqrt{xy} = 14 \Rightarrow xy = 196$ ....(*i*)

$\Rightarrow \frac{x}{y} = \frac{y}{112} \Rightarrow y^2 = 112x$ ....(ii)

from (i) $y = \frac{196}{x} \Rightarrow \frac{(196)^2}{x^2} = 112x$

$\Rightarrow x^3 = 343 \Rightarrow x = 7$ *and* $y = \frac{196}{7} = 28$

Hence, the required numbers are 7 and 28.

Percentage

- A way of comparing quantities is called percentage. Percentages are numerators of fractions with denominator 100. Percent means per hundred. For example 38% marks means 38 marks out of hundred.
- Fractions can be converted to percentages and vice-versa.

  For example, $\frac{1}{4} = \frac{1}{4} x\ 100\% = 25\%$ whereas, $75\% = \frac{75}{100} = \frac{3}{4}$.
- Decimals too can be converted to percentages and vice-versa. For example, $0.25 = 0.25 \times 100\% = 25\%$.
- The increase or decrease in a certain quantity can also be expressed as percentage.
- The profit or loss incurred in a certain transaction can be expressed in terms of percentages.

Example

Due to reduction of $6\frac{1}{4}\%$ in the price of sugar, a man is able to buy 1 kg more sugar for Rs. 120. The reduced rate of sugar is:

(a) Rs. 8 per kg
(b) Rs. 6.5 per kg
(c) Rs. 7.5 per kg
(d) Rs. 9 per kg
(e) None of these

Ans. (c)

Explanation: Suppose original rate of sugar be Rs. x per kg.

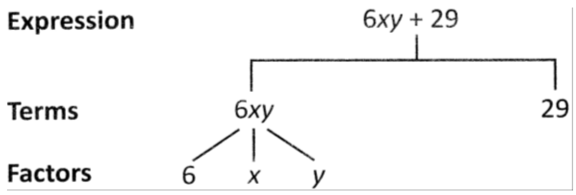
Reduced rate $= \left[\left(100 - \frac{25}{4}\right) \times \frac{1}{100} \times x\right] = \frac{15x}{16}$

According to question, $\frac{120}{\frac{15x}{16}} - \frac{120}{x} = 1 = 8 \Rightarrow \frac{128}{x} - \frac{120}{x} = 1 \Rightarrow x = 8$

Reduced rate $= \frac{15}{16} \times 8$ = Rs. 7.5 per kg.

Algebraic Expressions

- Algebraic expressions are formed from variables and constants. The various parts of algebraic expressions are shown below in the diagram.



- Expressions are made up of terms which is a product of factors. For example, the term 5xy in the expression shown above is a product of factors x, y and 5. Factors containing variables are said to be algebraic factors. The numerical factor in the term is called coefficient.
- Any expression with one or more terms is called a polynomial. Specifically a one-term expression is called a monomial (e.g., 2x); a two-term expression is called a binomial (e.g. 2x - 3) and a three-term expression is called a trinomial (e.g., $5x^2+6x+1$).
- The terms which have the same algebraic factors are called like terms. Terms which have different algebraic factors are unlike terms. Only like terms can be added or subtracted but the unlike terms are left as they are.

Some important algebraic identities are

1. $(a+b)^2 = a^2+2ab+b^2$
2. $(a-b)^2 = a^2-2ab+b^2$
3. $(a+b+c)^2 = a^2+b^2+c^2+2(ab+bc+ca)$
4. $(a+b)^3 + a^3+b^3+3ab(a+b)$
5. $(a-b)^3 = a^3-b^3-3ab(a-b)$
6. $a^2-b^2 = (a+b)(a-b)$
7. $a^3-b^3 = (a+b)(a^2+ab+b^2)$
8. $a^3+b^3 = (a+b)(a^2-ab+b^2)$

These identities can be used for expanding an algebraic expansion as well as to factorise the given algebraic expression.

Example
Find the value of
$\mathbf{4xy\ (x - y) - 6x^2(y - y^2) - 3y^2(2x^2 - x)\ 4 - 2xy\ (x - y)}$ **for** $\mathbf{x = 5}$ **and** $\mathbf{y = 13}$.

(a) -195 (b) 2535
(c) -2535 (d) 215
(e) None of these

Ans. (c)

$4xy(x-y)-6x^2(y-y^2)-3y^2(2x^2-x)+2xy(x-y)$

$4x^2y-4xy^2-6x^2y+6x^2y^2-6y^2x^2+3xy^2+2x^2y-2xy^2$

After simplification, we get

$-3xy^2 = -3\ 5\ 13\ 13 = -2535$

The expanded form of $\mathbf{(2x+3y-5z)^2}$ **is:**

(a) $4x^2+9y^2+25z^2+12xy-30yz-20zx$
(b) $5x^2-6y^3+15z^3+12xy-36y^6+21xy^2$

(c) $8a^2+a^3-c^2+7x^2+2c^2+9c^2y$

(d) $1z^2-z^4-8^C-75^2+2c^2-98c^2$

(e) None of these

Ans. (a)

Explanation:

$(2x+3y-5z)^2=(2x)^2+(3y)^2+(-5z)^2+2(2x)(3y)+2(3y)(-5z)+2(5z)(2x)=4x^2+9y^2+25z^2+$
$12xy-30yz-20zx$.

Factorize: $(5a+4b)^2-(3a-2b)^2$

(a) $4(a+3b)(4a+b)$ (b) $2(a+3b)(4a+b)$

(c) $3(a+3b)(4a+b)$ (d) $(a+3b)(4a+b)$

(e) None of these

Ans. (a)

Explanation: $(5a+4b)^2-(3a-2b)^2$

$=[(5a+4b)-(3a-2b)][(5a+4b)+(3a-2b)]$

$=(2a+6b)(8a+2b)$

$=2(a+3b)2(4a+b)$

$=4(a+3b)(4a+b)$.

Simple Equations

- An equation is a condition on a variable such that two expressions in the variable should have equal value.
- An equation remains the same if the L.H.S. and the R.H.S. are interchanged.
- The value of the variable in the equation for which L.H.S. remains equal to the R.H.S. is called the solution of the equation.
- To move the particular variables or constants from one side to another side is called transposition. When you transpose a number from one side of the equation to the other side, you change its sign. For example, transposing +3 from the L.H.S. to the R.H.S. in equation 7x - 3 = 4 gives 7x = 4 + 3.
- Carefully read the information given in the question and transform it accurately in the equation form for getting the correct answer

Example

Which one of the following options is the solution of the equation $\frac{4x-3}{2x+3}=\frac{5}{7}$?

(a) 2 (b) 4

(c) 10 (d) 9

(e) None of these

Ans. (a)

Explanation: $\frac{4x-3}{2x+3}=\frac{5}{7}$

Multiplying both sides of the above equation by (2x + 3)

$$\frac{4x-3}{2x+3}\times(2x+3)=\frac{5}{7}\times(2x+3)$$

$$\Rightarrow 4x-3=\frac{5}{7}(2x+3) \ \Rightarrow 4x-3=\frac{10x+15}{7} \ \Rightarrow 7(4x-3)=10x+15$$

$$\Rightarrow 28x-21=10x+15 \ \Rightarrow 28x-10x=15+21 \ \Rightarrow 18x=36 \Rightarrow x=\frac{36}{18}=2.$$

The denominator of a rational number is greater than its numerator by 6. If the numerator is increased by 5 and the denominator is decreased by 3 then the number obtained is $\frac{5}{4}$, find the rational number.

(a) $\frac{5}{11}$ (b) $\frac{11}{5}$

(c) $\frac{12}{3}$ (d) $\frac{9}{8}$

(e) None of these

Explanation: Let the numerator of the rational number be x.
Then the denominator of the rational number will be x + 6.
It is given that the numerator and denominator of the number are increased and decreased by 5 and 3 respectively then the number obtained is $\frac{5}{4}$.

$\therefore$ Numerator of the new rational number $= x+5$.

Denominator of the new rational number $=(x+6)-3=x+3$.

New rational number $= \frac{x+5}{x+3}$

But the new rational number is given as $\frac{5}{4}$.

$\frac{x+5}{x+3}=\frac{5}{4} \Rightarrow 4(x+5)=5(x+3)$ (By cross-multiplication)

$\Rightarrow 4x+20=5x+15.$

$4x-5x=15-20$ [transposing 5x to LH.S. and 20 to R.H.S.

$\Rightarrow -x=-5 \text{ or } x=5$

$\therefore$ Numerator of the rational number = 5.

Denominator of the rational number $=5+6=11$.

$\therefore$ The required rational number $=\frac{5}{11}$.

Exponents

The exponent of a number denotes how many times to use the number in multiplication. For example, 10^3. Here '3' means to use 10 thrice in multiplication.
So, 10,000 = 10^4 (can be read as 10 raise to power 4).
We can also say that 10,000 is the 4^{th} power of 10.
Exponents follow some laws which are as under.
For any non-zero integers a and b and whole number m and n.

(a) $a^m \times a^n = a^{m+n}$ (b) $a^m \div a^n = a^{m-n}$,$m > n$

(c) $(a^m)^n = a^{mn}$ (d) $a^m \times b^m = (ab)^m$

(e) $a^m \div b^m = \left(\frac{a}{b}\right)^m$ (f) $a^0 = 1$

(g) $(-1)^{\text{even number}} = 1; (-1)^{\text{odd number}} = -1$

Example

Evaluate:

$$\frac{(27)^{\frac{-2}{3}} \times (81)^{\frac{5}{4}}}{\left(\frac{1}{3}\right)^{-3}}$$

(a) 1 (b) 2

(c) 3 (d) 4

(e) None of these

Ans. (a)

Explanation:

$$\frac{(27)^{\frac{-2}{3}} \times (81)^{\frac{5}{4}}}{\left(\frac{1}{3}\right)^{-3}} = \frac{3^{\cancel{3}\times\left(\frac{2}{\cancel{3}}\right)} \times 3^{\cancel{4}\times\frac{5}{\cancel{4}}}}{3^3} = \frac{3^{-2} \times 3^5}{3^3} = \frac{3^{-2+5}}{3^3} = \frac{3^3}{3^3} = 1$$

Find the value of x so that $2^{2x+1} = 4^{2x-1}$

(a) 1 (b) 2

(c) $\frac{3}{2}$ (d) $\frac{1}{2}$

(e) None of these

Ans. (c)

Explanation: We have $2^{2x+1} = 2^{2(2x-1)} \Leftrightarrow 2^{2x+1} = 2^{4x-2}$

$\therefore 2x + 1 = 4x - 2 \Rightarrow 2x - 4x = -2 - 1$

$= -2x = -3 \quad \Rightarrow 2x = 3 \Rightarrow x = \frac{3}{2}$