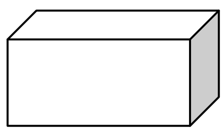
10. Geometry & Mensuration

10.1 Rectangular Solids & Cylinders

Cuboid:

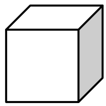
A cuboid is a three-dimensional figure formed by six rectangular surfaces, as shown below. Each rectangular surface is a face. Each solid line segment is an edge, and each point at which the edges meet is a vertex. A rectangular solid has six faces, twelve edges, and eight vertices. Edges mean sides and vertices mean corners. Opposite faces are parallel rectangles that have the same dimensions.
The surface area of a rectangular solid is



equal to the sum of the areas of all the faces. The volume is equal to (length) × (width) × (height); in other words, (area of base) × (height).
Body diagonal of a cuboid = Length of the longest rod that can be kept inside a rectangular room is $=\sqrt{L^2+B^2+H^2}$.

Cube: A rectangular solid in which all edges are of equal length is a cube. In a cube, just like cuboid, there are six faces, eight vertices & twelve edges.

Volume = a^3.



Surface Area = $6a^2$, where a is the side of a cube.

Body Diagonal = Length of the longest rod inside a cubical room = $a\sqrt{3}$

Right Prism:

A prism is a solid, whose vertical faces are rectangular and whose bases are parallel polygons of equal area.

A prism is said to be triangular prism, pentagonal prism, hexagonal prism, octagonal prism according to the number of sides of the polygon that form the base.

In a prism with a base of n sides, number of vertices = 2n, number of faces = n + 2.

Curved Surface Area of vertical faces of a prism = perimeter of base × height.

Total surface area of a prism = perimeter of base × height + 2 × area of base

Volume of a prism = area of base × height

Cylinder: Total Surface area of a right circular cylinder with a base of radius ‘r’ and height ‘h’ is equal to $2(\pi r^2) + 2\pi rh$ (the sum of the areas of the two bases plus the area of the curved surface).

The volume of a cylinder is equal to $\pi r^2 h$, that is (area of base) × (height).

Cone: A cone is having one circle on one of its ending & rest is the curved circle part with a corner on the other end.

Volume = 1/3 $\pi r^2 h$. Surface Area (curved) = $\pi r l$, where l = slant height.

As per the Pythagoras theorem, $l^2 = r^2 + h^2$.

Surface Area (total) = $\pi r l + \pi r^2$.

Frustum Of A Cone: A frustum is the lower part of a cone, containing the base, when it is cut by a plane parallel to the base of the cone.

Slant height, L = $\sqrt{h^2 + (R - r)^2}$

Curved Surface area of cone = $\pi (R + r) L$.

Total surface area of frustum = Base area + Area of upper circle + Area of lateral surface = $\pi (R^2 + r^2 + RL + rL)$.

Volume of frustum = $\frac{\pi h}{3}(R^2 + r^2 + Rr)$

Sphere: The set of all points in space, which are at a fixed distance from a fixed point, is called a sphere. The fixed point is the centre of the sphere and the fixed distance is the radius of the sphere.

Volume = $4/3\pi r^3$. Surface Area (curved and total) = $4 \pi r^2$.

Hemisphere: A sphere cut by a plane passing through its centre forms two hemispheres. The upper surface of a hemisphere is a circular region.

Volume = $2/3\pi r^3$. Surface Area (curved) $=2\pi r^2$.

Surface Area (Total) = $2\pi r^2 + \pi r^2 \Rightarrow 3\pi r^2$.

Spherical Shell: If R and r are the outer and inner radius of a hollow sphere, then volume of material in a spherical shell = $4/3\pi (R^3 - r^3)$.

Pyramid:

A pyramid is a solid, whose lateral faces are triangular with a common vertex and whose base is a polygon. A pyramid is said to be tetrahedron (triangular base), square pyramid, hexagonal pyramid etc, according to the number of sides of the polygon that form the base.

In a pyramid with a base of n sides, number of vertices = n + 1. Number of faces including the base = n + 1.

Surface area of lateral faces

= ½ × perimeter of base × slant height

Total surface area of pyramid

= Base area + ½ × perimeter of base × slant height

Volume of pyramid = 1/3 × Base area × height. A cone is also a pyramid.

10.2 Lines & Angles

Some Important Points:

(i) There is one and only one line passing through two distinct points.
(ii) Three or more points are said to be collinear if they lie on a line, otherwise they are said to be non-collinear.
(iii) Two or more lines are said to be coplanar if they lie in the same plane, otherwise they are said to be non-coplanar.
(iv) A line, which is perpendicular to a line segment i.e., intersects at 90° and passes through the midpoint of the segment is called the perpendicular bisector of the segment.
(v) Every point on the perpendicular bisector of a segment is equidistant

from the two endpoints of the segment.

Conversely, if any point is equidistant

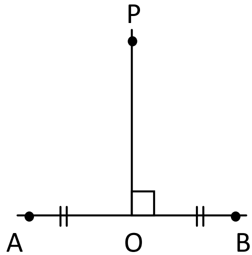
from the two endpoints of the segment,

then it must lie on the perpendicular

bisector of the segment.

If PO is the perpendicular bisector of segment AB, then, AP = PB.

Also, if AP = PB, then P lies on the perpendicular bisector of segment AB.

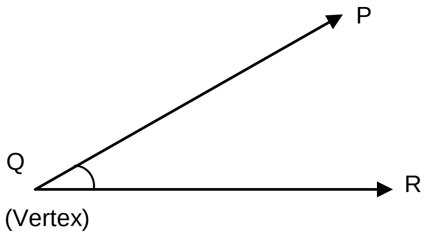


(vi) The ratio of intercepts made by three parallel lines on a transversal is equal to the ratio of the corresponding intercepts made on any other transversal by the same parallel lines.

Angles:

When two rays have the same starting or end points, they form an angle and the common end point is called vertex.

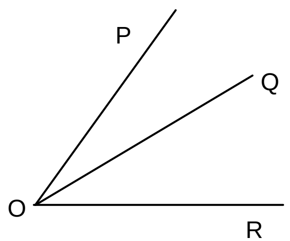
Angle is also defined as the measure of rotation of a ray. For the purpose of Trigonometry, the measure of rotation is termed positive if it is in the anticlockwise direction.



In the figure, ray PQ and QR from angle∠PQR

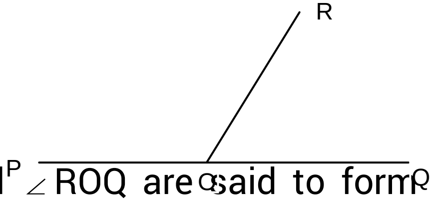
Types Of Angles:

(i) Adjacent Angles:



$\angle$POQ and $\angle$QOR are called adjacent angles, because they have a common side and their interiors are disjoint.

(ii) Linear Pair:

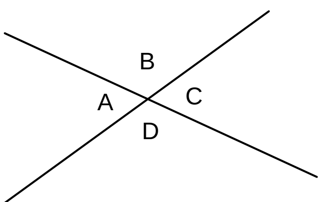


$\angle$POR and $\angle$ROQ are said to form a linear pair because they have a common side and other two sides are opposite rays $\angle$POR+$\angle$ROQ =180^0

(iii) $\angle$POR and $\angle$POQ form a linear pair
An angle greater than 180°, but less than 360° is called a reflex angle.

(iv) Two angles whose sum is 90° are called complementary angles.

(v) Two angles having a sum of 180° are



called supplementary angles.

(vi) When two lines intersect, two pairs of vertically opposite angles are equal. The sum of 2 adjacent angles is 180°. As given in the above diagram $\angle A = \angle C$ & $\angle B = \angle D$.

Secondly $\angle A + \angle B = 180°$ & $\angle C + \angle D = 180°$.

Two Lines Are Parallel To Each Other If

- They are parallel to a 3rd line.
- They are opposite sides of a rectangle/ square/ rhombus/ parallelogram.
- If they are perpendicular to a 3rd line.
- If one of them is a side of the triangle & other joins the midpoints of the remaining two sides.
- If one of them is a side of a triangle & other divides other 2 sides proportionately.

Two Lines Are Perpendicular To Each Other If

- They are arms of a right-angle triangle.
- If the adjacent angles formed by them are equal and supplementary.
- They are adjacent sides of a rectangle or a square.
- If they are diagonals of a rhombus.
- If one of them is a tangent & other is radius of the circle through the point of contact.

If the sum of their squares is equal to the square of line joining their ends.

Two Angles Are Said To Be Equal If

- They are vertically opposite angles.
- Their arms are parallel to each other.
- They are the corresponding angles of two congruent triangles.
- They are the opposite angles of a parallelogram.
- They are the angles of an equilateral triangle.

- They are the angles of a regular polygon.
- They are in the same segment of a circle.
- One of them lies between a tangent & a chord through the point of contact & the other is in the alternate segment, in a circle.

Two Sides Are Equal To Each Other If

- They are corresponding sides of two congruent triangles.
- They are sides of an equilateral triangle.
- They are opposite sides of a parallelogram.
- They are the sides of a regular polygon.
- They are radii of the same circle.
- They are chords equidistant from centre of circle.
- They are tangents to a circle from an external point.

10.3 Triangles

Types Of Triangles

With regard to their sides, triangles are of three types:

(i) **Scalene Triangle:** A triangle in which none of the three sides is equal is called a scalene triangle.

(ii) **Isosceles Triangle:** A triangle in which at least two sides are equal is called an isosceles triangle. In an isosceles triangle, the angles opposite to the congruent sides are congruent.

Conversely, if two angles of a triangle are congruent, then the sides opposite to them are congruent.

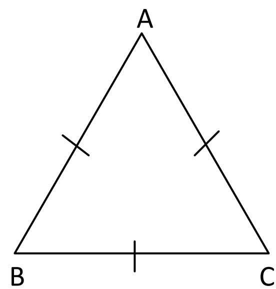
In Δ ABC, AB = AC,

$\angle$ ABC = $\angle$ ACB

(iii) **Equilateral Triangle:** A triangle in which all the three sides are equal is called an equilateral triangle. In an equilateral triangle, all the angles are congruent and equal to 60°.

In Δ ABC, AB = BC = AC.

∠ ABC = ∠ BCA = ∠ CAB = 60°



With Regard To Their Angles, Triangles Are Of Five Types:

(i) **Acute Triangle:** If all the three angles of a triangle are acute i.e., less than 90°, then the triangle is an acute-angled triangle.

(ii) **Obtuse Triangle:** If any one angle of a triangle is obtuse i.e., greater than

90°, then the triangle is an obtuse-angled triangle. The other two angles of the obtuse triangle will be acute.

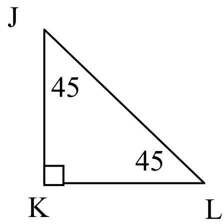
(iii) **Right Triangle:** A triangle that has a right angle is a right triangle. In a right triangle, the side opposite the right angle is the hypotenuse, and the other two sides are the legs. An important theorem concerning right triangles is the Pythagorean theorem, which states: In a right triangle, the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the legs.

There are some standard Pythagorean triplets, which are repeatedly used in the questions. It is better to remember these triplets by heart.

- 3, 4, 5
- 5, 12, 13
- 7, 24, 25
- 8, 15, 17
- 9, 40, 41
- 11, 60, 61
- 12, 35, 37
- 16, 63, 65
- 20, 21, 29
- 28, 45, 53.

Any multiple of these triplets will also be a triplet i.e. when we say 3, 4, 5 is a triplet, if we multiply all the numbers by 2, it will also be a triplet i.e. 6, 8, 10 will also be a triplet.

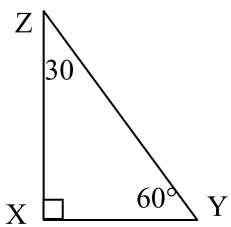
(iv) **45°- 45° - 90° Triangle:** If the angles of a triangle are 45°, 45° and 90°, then the perpendicular sides are $1/\sqrt{2}$ times the hypotenuse. In a 45°- 45°- 90° triangle, the lengths of the sides are in the ratio 1 : 1 : √2. For example, in ∆JKL, if JL = 2, then JK = √2 and KL = √2.



(v) **30°- 60° - 90° Triangle:** In 30°- 60° - 90° triangle, the lengths of the sides are in the ratio 1: √3: 2. For example,

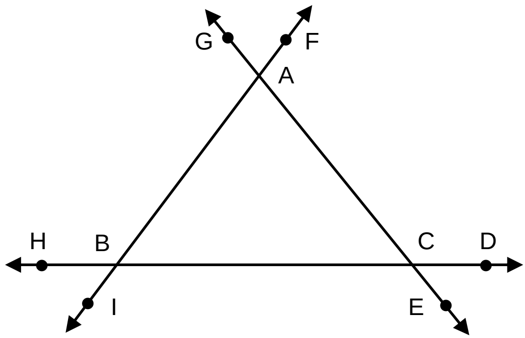
in ∆XYZ, if XZ = 3, then XY = 3√3 and YZ = 6. In short, the following formulas can be applied to calculate the two sides of a 30°-60°-90° triangle, when the third side is given.

Side opposite to 30° = ½ of hypotenuse.



Side opposite to $60^\circ = \sqrt{3}/2$ of hypotenuse.

Some Important Properties Of Triangles:



(i) In Δ ABC, $\angle ABC + \angle BAC + \angle ACB = 180^\circ$

(ii) The sum of an interior angle and the adjacent exterior angle is 180°.
In figure, $\angle ABC + \angle ABH = 180^\circ$
$\angle ABC + \angle CBI = 180^\circ$

(iii) Two exterior angles having the same vertex are congruent.

In figure, $\angle GAB \cong \angle FAC$

(iv) The measure of an exterior angle is equal to the sum of the measures of the two interior angles (called remote interior angles) of the triangle, not adjacent to it.

(v) The sum of any two sides of a triangle is always greater than the third side.

In Δ ABC, AB + BC > AC, also AB + AC > BC and AC + BC > AB.

(vi) The difference of any two sides is always less than the third side.

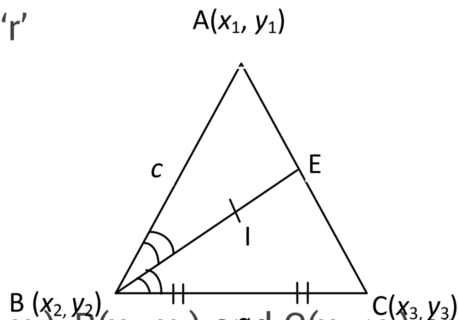
Area Of A Triangle

Area of a triangle = $\sqrt{s(s-a)(s-b)(s-c)} = r \times s = \frac{abc}{4R}$

where, a, b and c are the sides of the triangle,

s = semi perimeter, r = in-radius, R = circumradius.

In-centre: Point of intersection of angles bisectors of a triangle is known as the in-centre of triangle. The circle drawn from this point, which touches all the three sides of the triangle, is known as in-circle & its radius is called as in-radius. The in-radius is denoted by the letter ‘r’



(i) If A(x_1, y_1), B(x_2, y_2) and C(x_3, y_3) are the co-ordinates of a triangle and a, b, c are the lengths of sides BC, AC and AB respectively then co-ordinates of the in-centre are:

$$\left(\frac{ax_1+bx_2+cx_3}{a+b+c}, \frac{ay_1+by_2+cy_3}{a+b+c}\right)$$

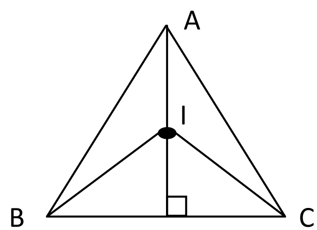
(ii) In- centre is equidistant for all the three sides, in-circle can be drawn touching all the three sides.

The radius of this 'circle' is termed as 'in-radius'

i.e. r = $\frac{\Delta}{s} = \frac{\text{Area of the triangle}}{\text{Semiperimeter of the triangle}}$

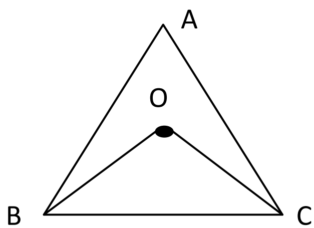
(iii) $\frac{AB}{BC} = \frac{AE}{EC}$ (Angle Bisector Theorem)

(iv) If I is the in-centre, then $\angle BIC = 90^\circ + \frac{1}{2}\angle A$



Circumcentre:

It is the point of intersection of perpendicular side bisectors of the triangle. The circle drawn with this point as centre and passing through the vertices is known as circumcircle and its radius is called as circumradius. The circumradius is denoted by the letter 'R'.



Circumradius = $\frac{abc}{4\Delta}$

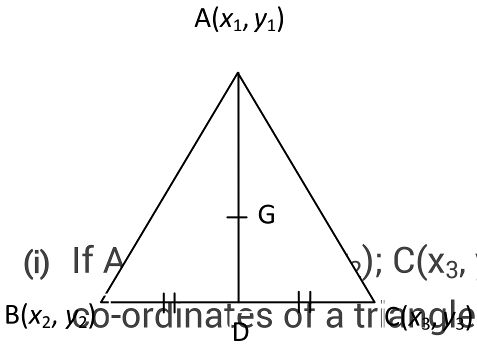
R = $\frac{\text{Product of three sides}}{4 \times \text{Area of the triangle}}$

$\angle BOC = 2\angle A$

Centroid: The segment joining a vertex & midpoint of the opposite side is called median of a triangle. There are three medians

& they meet in a single point called centroid of the triangle. The centroid divides medians in the ratio of 2 : 1.
The following formula is applied to calculate

the length of the median. The sum of the squares of two sides = 2[median^2 + (½ 3rd side)^2].



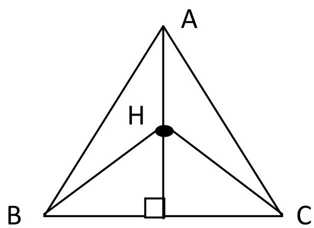
(i) If A [illegible]); C(x_3, y_3) are the co-ordinates of a triangle , then co-ordinates of centroid G are

$$\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right).$$

(ii) The median divides the triangle in two equal parts (not necessarily congruent).

(iii) The centroid divides the median in the ratio 2 : 1 with the larger part towards the vertex. So AG : GD = 2:1.

Orthocentre: The line drawn from any vertex, perpendicular to the opposite side is called the altitude/height. The three altitudes meet in a single point called the orthocentre. The angle made by any side at orthocentre is equal to (180° – vertical angle).



- If Δ ABC, is an isosceles triangle with AB ≅ AC, then the angle bisector of ∠ BAC is the perpendicular bisector of the base BC and is also the median to the base.

Area of an isosceles triangle $=\frac{c}{4}\sqrt{4a^2 - c^2}$

, where c is the unequal side and a is one of the equal sides.

The altitudes on the congruent sides are equal i.e., BE = CF.

- For an equilateral triangle,

height $=\frac{\sqrt{3}}{2}\times \text{side};$

$\text{area} =\frac{\sqrt{3}}{4}\times(\text{side})^2$, $\text{inradius} =\frac{1}{3}\times \text{height};$

$\text{circumradius} =\frac{2}{3}\times \text{height}$, perimeter=3×sides

Also, the altitude, median, angle bisector, perpendicular bisector of each base is the same and the ortho-centre, centroid, in-centre and circum-centre is the same.

Congruency Of Triangles:

If the sides and angles of one triangle are equal to the corresponding sides and angles of the other triangle, then the two triangles are said to be congruent.

Two Triangles Are Congruent If

- Two sides & the included angle of a triangle are respectively equal to two sides & included angle of other triangle (SAS).
- 2 angles & 1 side of a triangle are respectively equal to two angles & the corresponding side of the other triangle (AAS).
- Three sides of a triangle are respectively congruent to three sides of the other triangle (SSS).
- 1 side & hypotenuse of a right-triangle are respectively congruent to 1 side & hypotenuse of other rt. triangle (RHS).

Similarity Of Triangles:

- Two triangles are similar if they alike in shape only. The corresponding angles are congruent, but corresponding sides are only proportional. All congruent triangles are similar but all similar triangles are not necessarily congruent.

Two Triangles Are Similar If

- Three sides of a triangle are proportional to the three sides of the other triangle (SSS).
- Two angles of a triangle are respectively equal to the two angles of the other triangle (AA).
- Two sides of a triangle are proportional to two sides of the other triangle & the included angles are equal (SAS).

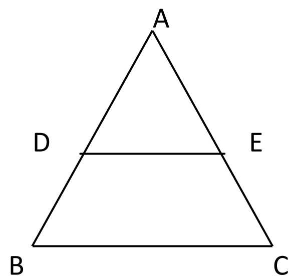
Properties Of Similar Triangles:

- If two triangles are similar, ratios of sides = ratio of heights = ratio of medians = ratio of angle bisectors = ratio of in-radii = ratio of circumradii.

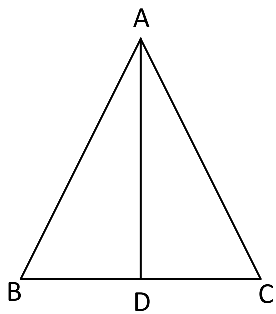
- Ratio of areas = b_1h_1/b_2h_2 = $(s_1)^2/(s_2)^2$, where b_1 & h_1 are the base & height of first triangle and b_2 & h_2 are the base & height of second triangle. s_1 & s_2 are the corresponding sides of first and second triangle respectively.

Some Important Theorems:

(i) **Basic Proportionality Theorem:** If a line is drawn parallel to one side of a triangle and intersects the other sides in two distinct points, then the other sides are divided in the same ratio by it. If DE is parallel to BC, then, $\frac{AD}{DB} = \frac{AE}{EC}$

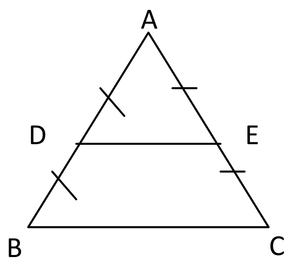


(ii) **The Angle Bisector Theorem:** The angle bisector divides the opposite side in ratio of the lengths of its adjacent arms.



If AD is the angle bisector, then AB/AC = BD/DC.

(iii) **Midpoint Theorem:** The segment joining the midpoints of any two sides



of a triangle is parallel to the third side and is half of the third side.
If AD = DB, AE = EC, then DE is parallel to BC and DE = $\frac{1}{2}$BC.

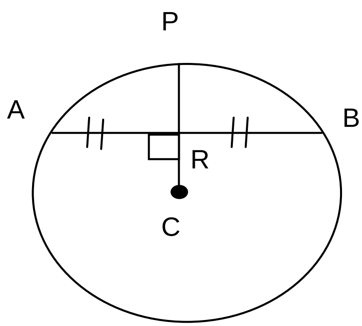
10.4 Circles

If r is the radius of the circle, then the circumference = $2\pi r$,
The area of a circle of radius r is = πr^2.

Congruent Circles: Circles with equal radii are called congruent circles.
Concentric Circles: Circles lying in the same plane with a common centre are called concentric circles.

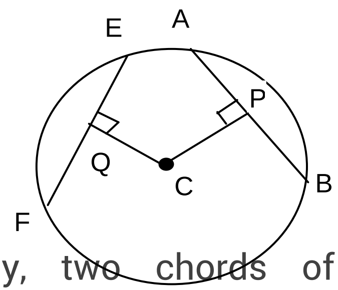
Some Important Properties Of Circles

(i) The perpendicular from the centre of a circle to a chord of the circle bisects the chord.

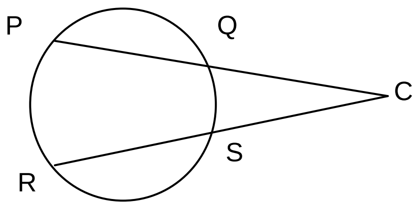


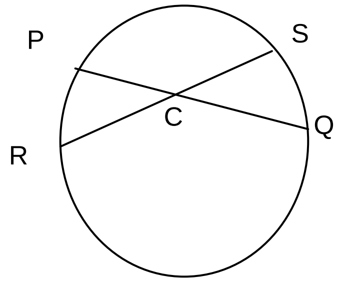
Conversely, the line joining the centre of the circle and the midpoint of a chord is perpendicular to the chord.

(ii) Equal chords of a circle or congruent circles are equidistant from the centre.

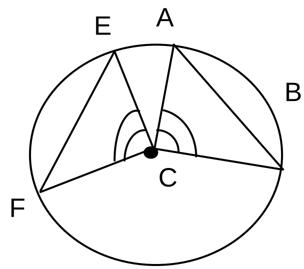


Conversely, two chords of a circle or congruent circles that are equidistant from the centre are equal. Two chords PQ, RS intersect at a point then CP × CQ = CR× CS



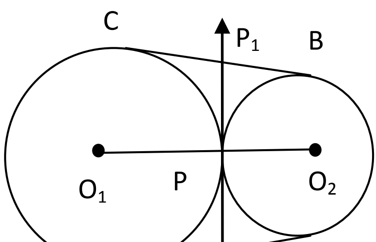


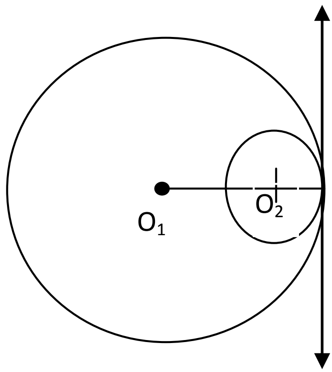
(i) In a circle or congruent circles, equal chords subtend equal angles at the centre.



Conversely, chords, which subtend equal angles at the centre of the same or congruent circles, are equal.

(ii) If the two circles touch each other externally, distance between their centres = sum of their radii.

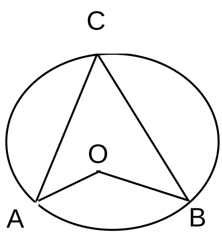




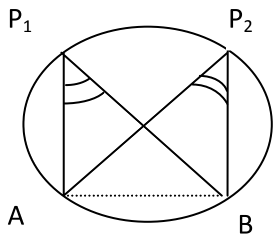
If the two circles touch each other internally, distance between their centres

= difference of their radii.

(iii) The measure of an inscribed angle is half the measure of its intercepted arc. (Inscribed angle theorem). $\angle ACB = \frac{1}{2} \angle AOB$

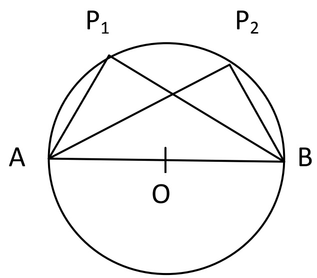


(iii) Angles subtended is the same segment are equal



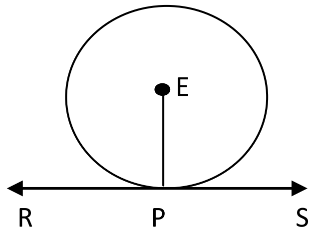
$\angle AP_1B = \angle AP_2B$.

(iv) Angle subtended in a semicircle is 90^0

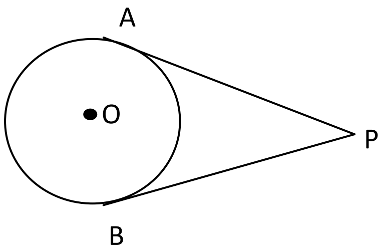


i.e. $\angle AP_1B = 90^0 = \angle AP_2B$

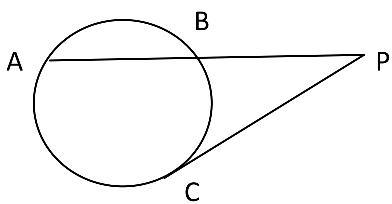
(v) Tangent is always perpendicular to the line joining the centre and the point of tangency.



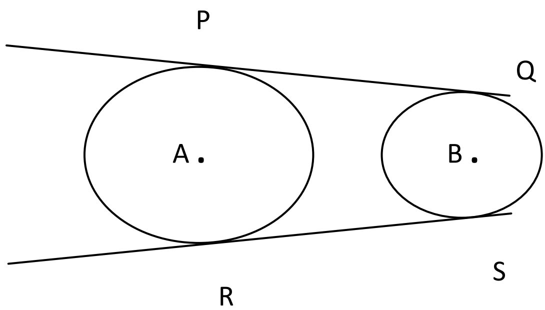
(vi) Two tangents from the same external point are equal in length. PA = PB



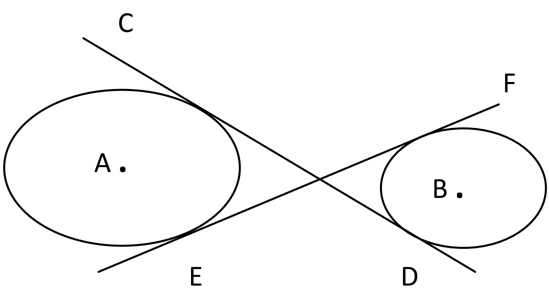
(vii) If AB is any chord of a circle and PC is the tangent (both for a external point P) than PA × PB = PC^2

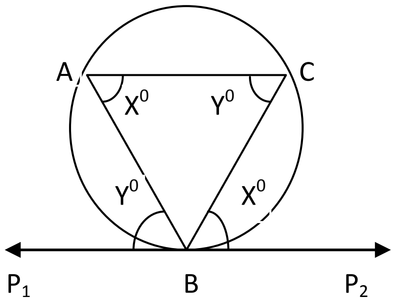


(viii) Length of direct common tangent = $\sqrt{(\text{distance between centres})^2 - (r_1 - r_2)^2}$



(ix) Length of transverse common tangent = $\sqrt{(\text{distance between centres})^2 - (r_1 + r_2)^2}$



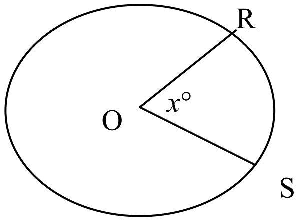


(x) Alternate Segment Theorem:

Angle between any chord (at the point of tangency) and the tangent is equal to the angle subtended by the chord to any point on the other side of the segment (alternate segment). In the fig $\angle CBP_2 = \angle CAB = X^0$ & $\angle P_1BA = \angle ACB = Y^0$

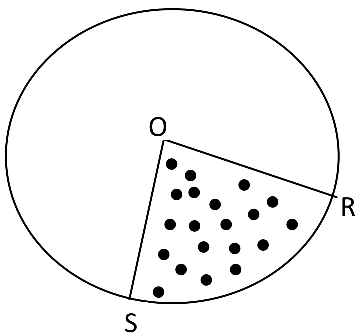
Sectors Of A Circle

The number of degrees of arc in a circle (or the number of degrees in a complete revolution) is 360.

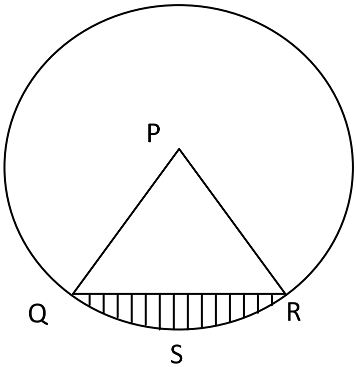


In the circle with center O above, the length of arc RS is x/360 of the circumference of the circle; for example, if x = 60°, then arc RS has length 1/6 of the circumference of the circle. We can remember the following formulas:

- Length of arc RS = $2\pi r \times x/360$. ∵ the complete circle is having 360 degrees & any part of that shall be equal to x/360.



- Area of Sector ORS = $\pi r^2 \times x/360$. ∵ the complete circle is having 360 degrees & any part of that shall be equal to x/360.



- Area of the segment of a circle (QSR) = $r^2\left[\frac{\pi\theta}{360°} - \frac{\sin\theta}{2}\right]$

Perimeter Of The Segment = $2\pi\ r\left(\frac{\theta}{360°}\right) + 2r\sin\frac{\theta}{2}$

10.5 Quadrilaterals & Polygons

Quadrilaterals

(i) Of all the quadrilaterals of the same perimeter, the one with the maximum area is the square.

(ii) The quadrilateral formed by joining the midpoints of the sides of any quadrilateral is always a parallelogram (rhombus in case of a rectangle, rectangle in case of a rhombus and square in case of a square).

(iii) The quadrilateral formed by the angle bisectors of the angles of a parallelogram is a rectangle.

(iv) For a rhombus □ABCD, if the diagonals are AC and BD, then $AC^2 + BD^2 = 4 \times AB^2$.

(v) If a square is formed by joining the midpoints of a square, then the side of the smaller square = side of the bigger square/√2.

(vi) If P is a point inside a rectangle □ABCD, then $AP^2 + CP^2 = BP^2 + DP^2$.

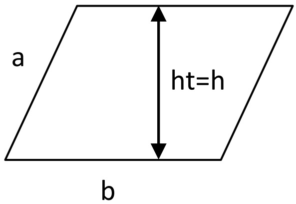
(vii) The segment joining the midpoints of the non-parallel sides of a trapezium is parallel to the two parallel sides and is half the sum of the parallel sides.

(viii) For a trapezium □ABCD, if the diagonals are AC and BD, and AB and CD are the parallel sides, then $AC^2 + BD^2 = AD^2 + BC^2 + 2 \times AB \times BD$.

(ix) If the length of the sides of a cyclic quadrilaterals area, b, c and d, then it's area = $\sqrt{(s-a)(s-b)(s-c)(s-d)}$, where s is the semi-perimeter.

(x) The opposite angles of a cyclic quadrilateral are supplementary.

Quadrilateral Areas



(i) Square = a^2

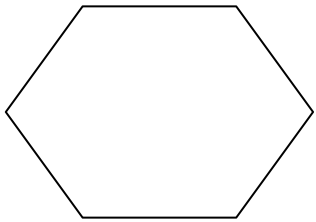
(ii) Rectangle = a $\times$ b, where a & b are the length and breadth of the rectangle.

(iii) Parallelogram = b $\times$ h, where b is the base and h is the height of the parallelogram.

(iv) For Rhombus = ½ × d_1 × d_2, where d_1 and d_2 are the diagonals of the rhombus.

(v) For Trapezium with a and b parallel sides and height h, Area = $\frac{1}{2}(a+b)h$

Polygons



(i) The sides a_n of regular inscribed polygons, where R is the radius of the circumscribed circle

$= a_n = 2R \sin\frac{180^o}{n}$

(ii) Area of a polygon of perimeter P and radius of in-circle r = ½ $\times$ p $\times$ r

(iii) The sum of the interior angles of a convex POLYGON, having n sides is 180^o (n - 2).

(iv) The sum of the exterior angles of a convex polygon, taken one at each vertex, is 360°.

(v) The measure of an exterior angle of a regular n- sided polygon is $\frac{360^{0}}{n}$.

(vi) The measure of the interior angle of a regular n-sided polygon is $\frac{(n-2)180^{\circ}}{n}$.

(vii) The number of diagonals of in an n-sided polygon is $\frac{n(n-3)}{2}$.