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Advanced Trigonometric Identities*

1. **Compound Angle :** Sum or difference of two angles is called compound angle. $A + B$, $A - B$, $A + B + C$ etc are compound angle.

Important formulae :

1.1 $\sin(A+B) = \sin A \cos B + \cos A \sin B$

1.2 $\cos(A+B) = \cos A \cos B - \sin A \sin B$

1.3 $\sin(A-B) = \sin A \cos B - \cos A \sin B$

1.4 $\cos(A-B) = \cos A \cos B + \sin A \sin B$

1.5 $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

1.6 $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

1.7 $\cot(A+B) = \dfrac{\cot A \cot B - 1}{\cot B + \cot A}$

1.8 $\cot(A-B) = \dfrac{\cot A \cot B + 1}{\cot B - \cot A}$

1.9.1 $\tan\left(\dfrac{\pi}{4}+\theta\right) = \dfrac{1+\tan\theta}{1-\tan\theta}$

1.9.2 $\tan\left(\dfrac{\pi}{4}-\theta\right) = \dfrac{1-\tan\theta}{1+\tan\theta}$ $\left(\because \tan\dfrac{\pi}{4} = 1\right)$

1.9.3 $\cot\left(\dfrac{\pi}{4}+\theta\right) = \dfrac{\cot\theta-1}{\cot\theta+1}$

1.9.4 $\cot\left(\dfrac{\pi}{4}-\theta\right) = \dfrac{\cot\theta+1}{\cot\theta-1}$ $\left(\because \cot\dfrac{\pi}{4} = 1\right)$

These results are obtained by putting $A = \dfrac{\pi}{4}$, $B = \theta$ respectively in 1.5, 1.6, 1.7, 1.8

1.10 (More results)

1.10.1 $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A$

1.10.2 $\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A$

1.10.2 $\tan(A+B+C) = \dfrac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}$

2. $\sin(A+B)$, $\sin(A-B)$, $\cos(A+B)$ and $\cos(A-B)$ can be transformed to get the following results.

2.1 $2\sin A \cos B = \sin(A+B) + \sin(A-B)$

2.2 $2\cos A \sin B = \sin(A+B) - \sin(A-B)$

2.3 $2\cos A \cos B = \cos(A+B) + \cos(A-B)$

2.4 $2\sin A \sin B = \cos(A-B) - \cos(A+B)$

2.5 $\sin C + \sin D = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2}$

2.6 $\sin C - \sin D = 2\cos\frac{C+D}{2}\sin\frac{C-D}{2}$

2.7 $\cos C + \cos D = 2\cos\frac{C+D}{2}\cos\frac{C-D}{2}$

2.8 $\cos C - \cos D = 2\sin\frac{C+D}{2}.\sin\frac{D-C}{2}$

2.9 $\tan C + \tan D = \frac{\sin(C+D)}{\cos C\cos D}$

2.10 $\tan C - \tan D = \frac{\sin(C-D)}{\cos C\cos D}$

2.11 $\cot C + \cot D = \frac{\sin(D+C)}{\sin C\sin D}$

2.12 $\cot C - \cot D = \frac{\sin(D-C)}{\sin C\sin D}$

3. **Multiple and submultiple angle :** Integral multiple of θ i.e. $2\theta, 3\theta, 4\theta$ etc. are called multiple angle, while $\frac{\theta}{2}, \frac{\theta}{3}, \frac{\theta}{4}$ etc. are called submultiple angle. Formulae for Multiple and submultiple angle can be derived, from compound angle formulae as follows

$\because$ $\sin(A+B) = \sin A\cos B + \cos A\sin B$

Putting, $A = B$

$\sin(A+A) = \sin A\cos A + \cos A\sin A$

or, $\sin 2A = 2\sin A\cos A$

Again replacing $2A$ by A and A by $\frac{A}{2}$

$\sin A = 2\sin\frac{A}{2}\cos\frac{A}{2}$ etc.

Important formulae :

3.1 $\sin 2A = 2\sin A\cos A = \frac{2\tan A}{1+\tan^2 A}$

3.2 $\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A = \frac{1-\tan^2 A}{1+\tan^2 A}$

Replacing $2A$ by A and A by $\frac{A}{2}$

$$\cos A = \cos^2\frac{A}{2} - \sin^2\frac{A}{2} = 2\cos^2\frac{A}{2} - 1 = 1 - 2\sin^2\frac{A}{2} = \frac{1-\tan^2\frac{A}{2}}{1+\tan^2\frac{A}{2}}$$

3.3 $\tan 2A = \frac{2\tan A}{1-\tan^2 A}$

Replacing A by $\frac{A}{2}$

$$\tan A = \frac{2\tan\frac{A}{2}}{1-\tan^2\frac{A}{2}}$$

3.4 $\sin 3A = 3\sin A - 4\sin^3 A$

3.5 $\cos 3A = 4\cos^3 A - 3\cos A$

3.6 $\tan 3A = \frac{3\tan A - \tan^3 A}{1-3\tan^2 A}$

3.7 Some special condition

3.7.1 $\because \cos 2A = 2\cos^2 A - 1$

$\therefore 1+\cos 2A = 2\cos^2 A$ or, $1+\cos A = 2\cos^2\frac{A}{2}$

3.7.2 $\because \cos 2A = 1 - 2\sin^2 A$

$\therefore 1-\cos 2A = 2\sin^2 A$ or, $1-\cos A = 2\sin^2\frac{A}{2}$

3.7.3 $\because \sin 3A = 3\sin A - 4\sin^3 A$

$\therefore \sin^3 A = \frac{3\sin A - \sin 3A}{4}$

3.7.4 $\because \cos 3A = 4\cos^3 A - 3\cos A$

$\therefore \cos^3 A = \frac{3\cos A + \cos 3A}{4}$

3.7.5 $\cot(A+B) = \frac{\cot A\cot B - 1}{\cot B + \cot A}$

$\Rightarrow \cot 2A = \frac{\cot^2 A - 1}{2\cot A}$

3.7.6 $\because \sin^2 A + \cos^2 A \pm 2\sin A\cos A = 1 \pm \sin 2A$

$\therefore 1 \pm \sin 2A = (\sin A \pm \cos A)^2$

4. Must learn it for shortcut

4.1 $\sin\theta \sin(60° - \theta)\sin(60° + \theta) = \frac{3}{4}\sin 3\theta$

4.2 $\cos\theta\cos(60° - \theta)\cos(60° + \theta) = \frac{3}{4}\cos 3\theta$

4.3 $\tan\theta\tan(60° - \theta)\tan(60° + \theta) = \tan 3\theta$

4.4 $\sin 18° = \frac{\sqrt{5}-1}{4}$

4.5 $\cos 36° = \frac{\sqrt{5}+1}{4}$

4.6 $\sin 15^\circ = \frac{\sqrt{3}-1}{2\sqrt{2}}$

4.7 $\cos 15^\circ = \frac{\sqrt{3}+1}{2\sqrt{2}}$

4.8 $\tan 15^\circ = \frac{\sqrt{3}-1}{\sqrt{3}+1} = 2-\sqrt{3}$

4.9 $\cot 15^\circ = \frac{\sqrt{3}+1}{\sqrt{3}-1} = 2+\sqrt{3}$

4.10 $\tan 22\frac{1}{2}^\circ = \sqrt{2}-1$

4.11 $\cot 22\frac{1}{2}^\circ = \sqrt{2}+1$

5. Quadrant Rule.

α equals in rad	$\sin\alpha$	$\cos\alpha$	$\tan\alpha$
$-\theta$	$-\sin\theta$	$\cos\theta$	$-\tan\theta$
$90-\theta$	$\cos\theta$	$\sin\theta$	$\cot\theta$
$90+\theta$	$\cos\theta$	$-\sin\theta$	$-\cot\theta$
$180-\theta$	$\sin\theta$	$-\cos\theta$	$-\tan\theta$
$180+\theta$	$-\sin\theta$	$-\cos\theta$	$\tan\theta$
$270-\theta$	$-\cos\theta$	$-\sin\theta$	$\cot\theta$
$270+\theta$	$-\cos\theta$	$\sin\theta$	$-\cot\theta$
$360-\theta$	$-\sin\theta$	$\cos\theta$	$-\tan\theta$
$360+\theta$	$\sin\theta$	$\cos\theta$	$\tan\theta$
$2n\pi-\theta$	$-\sin\theta$	$\cos\theta$	$-\tan\theta$
$2n\pi+\theta$	$\sin\theta$	$\cos\theta$	$\tan\theta$

To realise above result, we have following two steps—

Step–I. : In different quadrant, sign of *t*-ratio value are as follows.

Quadrant	I	II	III	IV
$\sin\theta$	+ ve	+ ve	– ve	– ve
$\cos\theta$	+ ve	– ve	– ve	+ ve
$\tan\theta$	+ ve	– ve	+ ve	– ve

Step–II.: (i) If angle is $(90^\circ \pm \theta)$ or $(270^\circ + \theta)$; change sin into cos, cos into sin, tan into cot, cot into tan etc.

(ii) If angle is $(180^\circ \pm \theta)$ or $(360^\circ \pm \theta)$ donot make any change i.e sin remains sin, cos remains cos etc.

Some selected *t*–ratio value between 0° and 360°

α	(in rad)	sinα	cosα	tanα
0°	0	0	1	0
30°	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
45°	$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
60°	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
90°	$\frac{\pi}{2}$	1	0	U.D.
120°	$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\sqrt{3}$
135°	$\frac{3\pi}{4}$	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	-1
150°	$\frac{5\pi}{6}$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{-1}{\sqrt{3}}$
180°	π	0	-1	0
210°	$\frac{7\pi}{6}$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
225°	$\frac{5\pi}{4}$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	1
240°	$\frac{4\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$\sqrt{3}$
270°	$\frac{3\pi}{2}$	-1	0	U.D.
300°	$\frac{5\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$-\sqrt{3}$
315°	$\frac{7\pi}{4}$	$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	-1
330°	$\frac{11\pi}{6}$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{3}}$
360°	2π	0	1	0

(ii) $\tan 75^\circ = \tan(45^\circ + 30^\circ)$

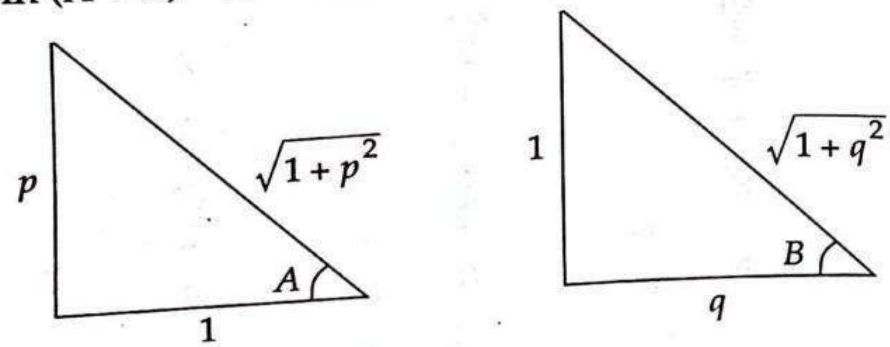
$$= \frac{\tan 45^\circ + \tan 30^\circ}{1 - \tan 45^\circ \tan 30^\circ}$$

$$= \frac{1 + \frac{1}{\sqrt{3}}}{1 - 1\cdot\frac{1}{\sqrt{3}}} = \frac{\sqrt{3}+1}{\sqrt{3}-1}\cdot\left(\frac{\sqrt{3}+1}{\sqrt{3}+1}\right)$$

$$= \frac{(\sqrt{3}+1)^2}{3-1} = \frac{3+1+2\sqrt{3}}{2} = \frac{4+2\sqrt{3}}{2} = 2+\sqrt{3}$$

2. If $\tan A = p$ and $\cot B = q$ then express $\sin(A+B)$ in terms of p and q

Solution : $\sin(A+B) = \sin A \cos B + \cos A \sin B$



$$= \frac{p}{\sqrt{1+p^2}}\cdot\frac{q}{\sqrt{1+q^2}} + \frac{1}{\sqrt{1+p^2}}\cdot\frac{1}{\sqrt{1+q^2}} = \frac{pq+1}{\sqrt{(1+p^2)(1+q^2)}}$$

3. If $\sin\theta = 3\sin(\theta + 2\alpha)$ then prove that $\tan(\theta+\alpha) + 2\tan\alpha = 0$

Solution : Given $\sin\theta = 3\sin(\theta + 2\alpha)$

$\sin\{(\theta+\alpha) - \alpha\} = 3\sin\{(\theta+\alpha)+\alpha\}$ ... $(\theta + \alpha)\cos\alpha + \cos(\theta+\alpha)\sin\alpha\}$

or, $\cot A \cot 2A - 1 = \cot 3A \cot 2A + \cot 3A \cot A$

or $\cot A \cot 2A - \cot 3A \cot 2A - \cot 3A \cot A = 1$, Proved.

5. Prove that $\tan 40° + 2 \tan 10° = \tan 50°$

Solution : $\because$ $40° + 10° = 50°$

$\therefore$ $\tan(40° + 10°) = \tan 50°$

or, $\dfrac{\tan 40° + \tan 10°}{1 - \tan 40° + \tan 10°} = \tan 50°$

or, $\tan 40° + \tan 10° = \tan 50° (1 - \tan 40° \tan 10°)$

or, $\tan 40° + \tan 10° = \tan 50° - \tan 50° \tan 40° \tan 10°$

or, $\tan 40° + \tan 10° = \tan 50° - \cot 40° \tan 40° \tan 10°$

$(\because \tan(90° - \theta) = \cot \theta$, here $\theta = 40°)$

or, $\tan 40° + \tan 10° = \tan 50° - 1 \cdot \tan 10°$ $(\because \tan \theta \cot \theta = 1)$

or, $\tan 40° + 2 \tan 10° = \tan 50°$, Proved.

6. If $A + B = \frac{\pi}{4}$ then prove that $(\cot A - 1)(\cot B - 1) = 2$.

Solution : $A + B = \frac{\pi}{4} \Rightarrow \cot(A + B) = \cot \frac{\pi}{4}$

$\Rightarrow \dfrac{\cot A \cot B - 1}{\cot B + \cot A} = 1$

$\Rightarrow \cot A \cot B - 1 = \cot B + \cot A$

$\Rightarrow \cot A \cot B - \cot B - \cot A = 1$

Adding '1' both sides,

$\cot A \cot B - \cot B - \cot A + 1 = 1 + 1$

or, $\cot B (\cot A - 1) - 1 (\cot A - 1) = 2$

or, $(\cot A - 1)(\cot B - 1) = 2$, Proved.

7. Prove that $\dfrac{1}{\tan 3A - \tan A} - \dfrac{1}{\cot 3A - \cot A} = \cot 2A$.

Solution : L.H.S. $= \dfrac{1}{\tan 3A - \tan A} - \dfrac{1}{\cot 3A - \cot A}$

$= \dfrac{1}{\quad} \quad 1$

8. Prove that $\cot\left(\frac{\pi}{4}+\theta\right)\cot\left(\frac{\pi}{4}-\theta\right)=1$.

Solution : L.H.S. $=\cot\left(\frac{\pi}{4}+\theta\right)\cot\left(\frac{\pi}{4}-\theta\right)$

$$=\frac{\cot\theta-1}{\cot\theta+1}\cdot\frac{\cot\theta+1}{\cot\theta-1}=1$$

9. Prove that $\sin\frac{\pi}{9}\sin\frac{2\pi}{9}\sin\frac{3\pi}{9}\sin\frac{4\pi}{9}=\frac{3}{16}$

Solution : L.H.S. $=\sin\left(\frac{180°}{9}\right)\sin\left(2\cdot\frac{180°}{9}\right)\sin\left(3\cdot\frac{180°}{9}\right)\sin\left(4\cdot\frac{180°}{9}\right)$

$$=\sin 20°\sin 40°\sin 60°\sin 80°$$

$$=\frac{1}{2}(2\sin 20°\sin 40°)\frac{\sqrt{3}}{2}\cdot\sin 80° \qquad \left(\because \sin 60°=\frac{\sqrt{3}}{2}\right)$$

$$=\frac{\sqrt{3}}{4}\{\cos(20°-40°)-\cos(20°+40°)\}\sin 80°$$

$$(\because 2\sin A\sin B=\cos(A-B)-\cos(A+B))$$

$$=\frac{\sqrt{3}}{4}\{\cos 20°-\cos 60°\}\sin 80° \qquad (\because \cos(-\theta)=\cos\theta)$$

$$=\frac{\sqrt{3}}{4}\left(\sin 80°\cos 20°-\frac{1}{2}\sin 80°\right) \qquad \left(\because \cos 60°=\frac{1}{2}\right)$$

$$=\frac{\sqrt{3}}{4}\left(\frac{2\sin 80°\cos 20°-\sin 80°}{2}\right)$$

$$=\frac{\sqrt{3}}{8}\{\sin(80°+20°)+\sin(80°-20°)-\sin 80°\}$$

$$=\frac{\sqrt{3}}{8}(\sin 100°+\sin 60°-\sin 80°)$$

$$=\frac{\sqrt{3}}{8}\left\{\sin(180°-80°)+\frac{\sqrt{3}}{2}-\sin 80°\right\}$$

$$=\frac{\sqrt{3}}{8}\left(\sin 80°+\frac{\sqrt{3}}{2}-\sin 80°\right)=\frac{\sqrt{3}}{8}\cdot\frac{\sqrt{3}}{2}=\frac{3}{16}$$

10. Prove that $4\cos\theta\cos(60°+\theta)\cos(60°-\theta)=\cos 3\theta$.

Solution : L.H.S. $=4\cos\theta\cos(60°+\theta)\cos(60°-\theta)$

$$=2\cos\theta\{2\cos(60°+\theta)\cos(60°-\theta)\}$$

$$=2\cos\theta[2\cos\{(60°+\theta)-(60°-\theta)\}+\cos\{(60°+\theta)+(60°-\theta)\}]$$

$$(\because 2\cos A\cos B=\cos(A-B)+\cos(A+B);$$
(here $A=60°+\theta$ and $B=60°-\theta$)

$$=2\cos\theta[\cos 2\theta+\cos 120°]$$

$$=2\cos\theta\cos 2\theta+2\cos\theta\left(\frac{-1}{2}\right) \qquad \left(\because \cos 120°=\frac{-1}{2}\right)$$

$= [\cos(\theta - 2\theta) + \cos(\theta + 2\theta)] - \cos\theta$ (2 cos A cos B formula)

$= \cos\theta + \cos 3\theta - \cos\theta = \cos 3\theta.$ ($\because \cos(-\theta) = \cos\theta$)

11. Prove that the value of $\cos^2\theta + \cos^2(\alpha + \theta) - 2\cos\alpha\cos\theta\cos(\alpha + \theta)$ is independent of θ

Solution : Given expression $= \cos^2\theta + \cos^2(\alpha + \theta) - [\cos(\alpha - \theta) + \cos(\alpha + \theta)]\cos(\alpha + \theta)$

$= \cos^2\theta + \cos^2(\alpha + \theta) - \cos(\alpha - \theta)\cos(\alpha + \theta) - \cos^2(\alpha + \theta)$

$= \cos^2\theta - \cos(\alpha - \theta)\cos(\alpha + \theta)$

$= \cos^2\theta - (\cos^2\alpha - \sin^2\theta)$

$= \cos^2\theta + \sin^2\theta - \cos^2\alpha = 1 - \cos^2\alpha = \sin^2\alpha$

Which is independent of θ

12. Prove that $\dfrac{\sin\theta + \sin 3\theta + \sin 5\theta + \sin 7\theta}{\cos\theta + \cos 3\theta + \cos 5\theta + \cos 7\theta} = \dfrac{\tan 3\theta + \tan\theta}{1 - \tan 3\theta\tan\theta}$.

Solution : [**Key** : $\theta + 7\theta = 3\theta + 5\theta = 8\theta$]

$$\text{L.H.S.} = \frac{(\sin 7\theta + \sin\theta) + (\sin 5\theta + \sin 3\theta)}{(\cos 7\theta + \cos\theta) + (\cos 5\theta + \cos 3\theta)}$$

$$= \frac{2\sin\frac{7\theta+\theta}{2}\cos\frac{7\theta-\theta}{2} + 2\sin\frac{5\theta+3\theta}{2}\cos\frac{5\theta-3\theta}{2}}{2\cos\frac{7\theta+\theta}{2}.\cos\frac{7\theta-\theta}{2} + 2\cos\frac{5\theta+3\theta}{2}\cdot\cos\frac{5\theta-3\theta}{2}}$$

$$= \frac{2\sin 4\theta\cos 3\theta + 2\sin 4\theta\cos\theta}{2\cos 4\theta\cos 3\theta + 2\cos 4\theta\cos\theta} = \frac{2\sin 4\theta(\cos 3\theta + \cos\theta)}{2\cos 4\theta(\cos 3\theta + \cos\theta)}$$

$$= \tan 4\theta = \tan(3\theta + \theta) = \frac{\tan 3\theta + \tan\theta}{1 - \tan 3\theta\tan\theta}$$

13. Prove that $\cos\alpha + \cos\beta + \cos\gamma + \cos(\alpha + \beta + \gamma)$

$$= 4\cos\frac{\alpha+\beta}{2}\cos\frac{\beta+\gamma}{2}\cos\frac{\gamma+\alpha}{2}$$

Solution : L.H.S. $= (\cos\alpha + \cos\beta) + \cos(\alpha + \beta + \gamma) + \cos\gamma$

$$= 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} + 2\cos\frac{\alpha+\beta+\gamma+\gamma}{2}\cos\frac{\alpha+\beta+\gamma-\gamma}{2}$$

$$= 2\cos\frac{\alpha+\beta}{2}\left\{\cos\frac{\alpha-\beta}{2} + \cos\frac{\alpha+\beta+2\gamma}{2}\right\}$$

$$\frac{\alpha+\beta}{} \left\{ \quad \frac{\alpha-\beta}{2} + \frac{\alpha+\beta+2\gamma}{2} \quad \cos\frac{\frac{\alpha-\beta}{2} - \frac{\alpha+\beta+2\gamma}{2}}{} \right\}$$

14. If cosec A + sec A = cosec B + sec B then prove that

$\tan A \tan B = \cot \frac{A+B}{2}$.

Solution : Given that cosec A + sec A = cosec B + sec B

or, cosec A − cosec B = sec B − sec A

or, $\frac{1}{\sin A} - \frac{1}{\sin B} = \frac{1}{\cos B} - \frac{1}{\cos A}$

or, $\frac{\sin B - \sin A}{\sin A \sin B} = \frac{\cos A - \cos B}{\cos B \cos A}$

or, $\frac{2\cos\frac{B+A}{2}\sin\frac{B-A}{2}}{\sin A \sin B} = \frac{2\sin\frac{A+B}{2}\sin\frac{B-A}{2}}{\cos B \cos A}$

or, $\frac{\cos\frac{B+A}{2}}{\sin A \sin B} = \frac{\sin\frac{A+B}{2}}{\cos B \cos A}$

or, $\frac{\cos\frac{A+B}{2}}{\sin\frac{A+B}{2}} = \frac{\sin A \sin B}{\cos A \cos B}$

or, $\cot\frac{A+B}{2} = \tan A \tan B$

or, $\tan A \tan B = \cot\frac{A+B}{2}$.

15. Prove that $\left(\frac{\sin 2A - \sin 2B}{\cos 2A + \cos 2B}\right)^n + \left(\frac{\cos 2A - \cos 2B}{\sin 2A + \sin 2B}\right)^n = 2\tan^n(A-B)$ or 0

According as n is even or n is odd.

Solution : L.H.S. $= \left(\frac{\sin 2A - \sin 2B}{\cos 2A + \cos 2B}\right)^n + \left(\frac{\cos 2A - \cos 2B}{\sin 2A + \sin 2B}\right)^n$

$$= \left[\frac{2\cos\frac{2A+2B}{2}\sin\frac{2A-2B}{2}}{2\cos\frac{2A+2B}{2}\cos\frac{2A-2B}{2}}\right]^n + \left[\frac{2\sin\frac{2A+2B}{2}\sin\frac{2B-2A}{2}}{2\sin\frac{2A+2B}{2}\cos\frac{2A-2B}{2}}\right]^n$$

$$= \left(\frac{\sin(A-B)}{\cos(A-B)}\right)^n + \left(\frac{-\sin(A-B)}{\cos(A-B)}\right)^n$$

$$= (\tan(A-B))^n + (-\tan(A-B))^n$$

$$= \tan^n(A-B) + (-1)^n \tan^n(A-B)$$

When n is an even number $(-1)^n = 1$

$\therefore$ L.H.S $= \tan^n(A-B) + \tan^n(A-B)$

$= 2\tan^n(A-B)$

When n is an odd number $(-1)^n = -1$

$\therefore$ L.H.S $= \tan^n(A-B) - \tan^n(A-B) = 0$.

16. If $x \cot(\theta + 120°) = y \cot(\theta - 30°)$ then prove that $\frac{x+y}{x-y} = 2\cos 2\theta$

Solution : Given that $x \cot(\theta + 120°) = y \cot(\theta - 30°)$

or, $\frac{x}{y} = \frac{\cot(\theta - 30°)}{\cot(\theta + 120°)}$

By componendo–dividendo

$$\frac{x+y}{x-y} = \frac{\cot(\theta - 30°) + \cot(\theta + 120°)}{\cot(\theta - 30°) - \cot(\theta + 120°)}$$

We know that $\cot C + \cot D = \frac{\sin(D+C)}{\sin C \sin D}$ & $\cot C - \cot D = \frac{\sin(D-C)}{\sin C \sin D}$

Thus, we have $\frac{x+y}{x-y} = \frac{\left(\frac{\sin(\theta + 120° + \theta - 30°)}{\sin(\theta + 120°)\sin(\theta - 30°)}\right)}{\frac{\sin(\theta + 120° - \theta + 30°)}{\sin(\theta + 120°)\sin(\theta - 30°)}}$

or, $\frac{x+y}{x-y} = \frac{\sin(90° + 2\theta)}{\sin 150°} = \frac{\cos 2\theta}{\left(\frac{1}{2}\right)}$

or, $\frac{x+y}{x-y} = 2\cos 2\theta$. Proved.

17. If $2\tan A = 3\tan B$ then prove that $\sin(A+B) = 5\sin(A-B)$.

Solution : Given $2\tan A = 3\tan B$

$\therefore \frac{\tan A}{\tan B} = \frac{3}{2}$

Using componendo-dividendo,

$$\frac{\tan A + \tan B}{\tan A - \tan B} = \frac{3+2}{3-2}$$

or, $\frac{\left(\frac{\sin(A+B)}{\cos A \cos B}\right)}{\left(\frac{\sin(A-B)}{\cos A \cos B}\right)} = \frac{5}{1}$

$[(\because \tan C + \tan D = \frac{\sin(C+D)}{\cos C \cos D})$ and $\tan C - \tan D = \frac{\sin(C-D)}{\cos C \cos D})]$

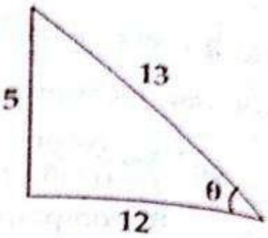
or, $\frac{\sin(A+B)}{\sin(A-B)} = 5$

or, $\sin(A+B) = 5\sin(A-B)$

18. If $\sin\theta = \frac{5}{13}$, $0 < \theta < \frac{\pi}{2}$ then find the value of following.

(i) $\sin 2\theta$ (ii) $\cos 2\theta$ (iii) $\tan 2\theta$ (iv) $\sin 3\theta$

(v) $\cos 3\theta$ (vi) $\sin 4\theta$



Solution : $\because \sin\theta = \frac{5}{13}$

$\therefore \cos\theta = \frac{12}{13}$

and $\tan\theta = \frac{5}{12}$

Now, (i) $\sin 2\theta = 2\sin\theta\cos\theta = 2\cdot\frac{5}{13}\cdot\frac{12}{13} = \frac{120}{169}$

(ii) $\cos 2\theta = 2\cos^2\theta - 1 = 2\cdot\left(\frac{12}{13}\right)^2 - 1 = \frac{288}{169} - 1 = \frac{119}{169}$

(iii) $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{2\cdot\frac{5}{12}}{1-\left(\frac{5}{12}\right)^2} = \frac{\frac{10}{12}}{\frac{144-25}{144}} = \frac{10\times 12}{119} = \frac{120}{119}$

(iv) $\sin 3\theta = 3\sin\theta - 4\sin^3\theta = 3\cdot\frac{5}{13} - 4\left(\frac{5}{13}\right)^3$

$$= \frac{15\cdot 13^2 - 500}{13^3} = \frac{2035}{2197}$$

(v) $\cos 3\theta = 4\cos^3\theta - 3\cos\theta = 4\left(\frac{12}{13}\right)^3 - 3\cdot\frac{12}{13}$

$$= \frac{4\cdot 12^3 - 36\cdot 13^2}{13^3} = \frac{828}{2197}$$

(vi) $\sin 4\theta = \sin\left(2(2\theta)\right) = 2\sin 2\theta\cos 2\theta = 2\cdot\frac{120}{169}\cdot\frac{119}{169} = \frac{28560}{28561}$

19. Prove that $\frac{1+\tan\theta}{1-\tan\theta} = \frac{1+\sin 2\theta}{\cos 2\theta}$

Solution : $\text{L.H.S} = \frac{1+\frac{\sin\theta}{\cos\theta}}{1-\frac{\sin\theta}{\cos\theta}} = \frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\times\frac{\cos\theta+\sin\theta}{\cos\theta+\sin\theta}$

$$= \frac{\cos^2\theta+\sin^2\theta+2\cos\theta\sin\theta}{\cos^2\theta-\sin^2\theta} = \frac{1+\sin 2\theta}{\cos 2\theta}$$

20. Prove that $\cot^2\frac{\theta}{2} - \tan^2\frac{\theta}{2} = 4\cot\theta\,\text{cosec}\,\theta$

Solution : $\text{L.H.S} = \left(\cot\frac{\theta}{2} - \tan\frac{\theta}{2}\right)\left(\cot\frac{\theta}{2} + \tan\frac{\theta}{2}\right)$

$$= \left(\frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} - \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\right)\left(\frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} + \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\right)$$

$$= \left(\frac{\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}}{\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right)\left(\frac{\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}}{\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right)$$

$$= \left(\frac{2\cdot\cos\theta}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right)\left(\frac{2\cdot 1}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right)$$

$$= \frac{2\cos\theta}{\sin\theta}\cdot\frac{2}{\sin\theta} = 4\cot\theta\,\mathrm{cosec}\,\theta = \text{R.H.S.}$$

21. If $\sin\alpha + \sin\beta = a$ and $\cos\alpha + \cos\beta = b$ then proved that

$$\cos(\alpha-\beta) = \frac{a^2+b^2-2}{2} \text{ and } \tan\frac{\alpha-\beta}{2} = \pm\sqrt{\frac{4-a^2-b^2}{a^2+b^2}}$$

Solution : R.H.S. $= \frac{a^2+b^2-2}{2} = \frac{(\sin\alpha+\sin\beta)^2+(\cos\alpha+\cos\beta)^2-2}{2}$

$$= \frac{\sin^2\alpha+\sin^2\beta+2\sin\alpha\sin\beta+\cos^2\alpha+\cos^2\beta+2\cos\alpha\cos\beta-2}{2}$$

$$= \frac{\left(\sin^2\alpha+\cos^2\alpha\right)+\left(\sin^2\beta+\cos^2\beta\right)+2(\cos\alpha\cos\beta+\sin\alpha\sin\beta)-2}{2}$$

$$= \frac{1+1+2\cos(\alpha-\beta)-2}{2} = \frac{2\cos(\alpha-\beta)}{2} = \cos(\alpha-\beta) = \text{L.H.S.}$$

Second part : $\because 1-\cos\theta = 2\sin^2\frac{\theta}{2}$ and $1+\cos\theta = 2\cos^2\frac{\theta}{2}$

$$\therefore\ \frac{1-\cos\theta}{1+\cos\theta} = \frac{2\sin^2\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}} = \tan^2\frac{\theta}{2}$$ [Note]

Hence, $\tan^2\frac{\alpha-\beta}{2} = \frac{1-\cos(\alpha-\beta)}{1+\cos(\alpha-\beta)}$

$$= \frac{1-\frac{a^2+b^2-2}{2}}{1+\frac{a^2+b^2-2}{2}} = \frac{2-a^2-b^2+2}{2+a^2+b^2-2} = \frac{4-a^2-b^2}{a^2+b^2}$$

$$\therefore\ \tan\frac{\alpha-\beta}{2} = \pm\sqrt{\frac{4-a^2-b^2}{a^2+b^2}}$$

22. Express $\sin^6 x + \cos^6 x$ in terms of $\sin 2x$ and hence find the maximum and minimum value of $\sin^6 x + \cos^6 x$

Solution : $\sin^6 x + \cos^6 x = (\sin^2 x)^3 + (\cos^2 x)^3$

$$= (\sin^2 x + \cos^2 x)(\sin^4 x - \sin^2 x\cos^2 x + \cos^4 x)$$

$$(\because a^3 + b^3 = (a+b)(a^2 - ab + b^2))$$

$$= 1\cdot(\sin^4 x + \cos^4 x + 2\sin^2 x\cos^2 x - 3\sin^2 x\cos^2 x)$$

$$= (\sin^2 x + \cos^2 x)^2 - \frac{3}{4}(2\sin x\cos x)^2$$

$$= 1 - \frac{3}{4}\sin^2 2x$$

Now, $-1 \le \sin 2x \le 1 \Rightarrow 0 \le \sin^2 2x \le 1$.

at $\sin^2 2x = 0$, $\sin^6 x + \cos^6 x = 1 - \frac{3}{4}\cdot 0 = 1$

at $\sin^2 2x = 1$, $\sin^6 x + \cos^6 x = 1 - \frac{3}{4}\cdot 1 = \frac{1}{4}$

Hence, Maximum and Minimum value of $\sin^6 x + \cos^6 x$ are respectively 1 and $\frac{1}{4}$.

23. Prove that $\tan\alpha + 2\tan 2\alpha + 4\tan 4\alpha + 8\cot 8\alpha = \cot\alpha$

Solution : L.H.S. $= \tan\alpha + 2\tan 2\alpha + 4\tan 4\alpha + \frac{8}{\tan 8\alpha}$

$$= \tan\alpha + 2\tan 2\alpha + 4\tan 4\alpha + \frac{8}{\tan 2(4\alpha)}$$

$$= \tan\alpha + 2\tan 2\alpha + 4\tan 4\alpha + \frac{8(1-\tan^2 4\alpha)}{2\tan 4\alpha}$$

$$= \tan\alpha + 2\tan 2\alpha + 4\tan 4\alpha + \frac{4(1-\tan^2 4\alpha)}{\tan 4\alpha}$$

$$= \tan\alpha + 2\tan 2\alpha + \frac{4\tan^2 4\alpha + 4 - 4\tan^2 4\alpha}{\tan 4\alpha}$$

$$= \tan\alpha + 2\tan 2\alpha + \frac{4}{\tan 2(2\alpha)}$$

$$= \tan\alpha + 2\tan 2\alpha + 4\cdot\frac{1-\tan^2 2\alpha}{2\tan 2\alpha}$$

$$= \tan\alpha + 2\tan 2\alpha + \frac{2(1-\tan^2 2\alpha)}{\tan 2\alpha}$$

$$= \tan\alpha + \frac{2\tan^2 2\alpha + 2 - 2\tan^2 2\alpha}{\tan 2\alpha}$$

$$= \tan\alpha + \frac{2}{\tan 2\alpha} = \tan\alpha + \frac{2(1-\tan^2\alpha)}{2\tan\alpha}$$

$$= \tan\alpha + \frac{1-\tan^2\alpha}{\tan\alpha} = \tan\alpha + \frac{1}{\tan\alpha} - \tan\alpha$$

$$= \frac{1}{\tan\alpha} = \cot\alpha = \text{R.H.S}$$

Second Method :

We know that $\cot\theta - \tan\theta = \frac{\cos\theta}{\sin\theta} - \frac{\sin\theta}{\cos\theta}$

$$= \frac{\cos^2\theta - \sin^2\theta}{\sin\theta\cos\theta} = \frac{\cos 2\theta}{\left(\frac{\sin 2\theta}{2}\right)} = \frac{2\cos 2\theta}{\sin 2\theta}$$

or, $\cot\theta - \tan\theta = 2\cot 2\theta$ ... (i)

Now, L.H.S. $= \tan\alpha + 2\tan 2\alpha + 4\tan 4\alpha + 8\cot 8\alpha$

$= \tan\alpha + 2\tan 2\alpha + 4\tan 4\alpha + 4(2\cot 8\alpha)$

$= \tan\alpha + 2\tan 2\alpha + 4\tan 4\alpha + 4(\cot 4\alpha - \tan 4\alpha)$

$= \tan\alpha + 2\tan 2\alpha + 4\cot 4\alpha.$

$= \tan\alpha + 2\tan 2\alpha + 2(\cot 2\alpha - \tan 2\alpha)$; ((i) $\theta = 2\alpha$ in (i))

$= \tan\alpha + 2\cot 2\alpha$

$= \tan\alpha + \cot\alpha - \tan\alpha = \cot\alpha =$ R.H.S.

4. Prove that $\sin^3\theta + \sin^3(120^\circ + \theta) + \sin^3(240^\circ + \theta) = \frac{-3}{4}\sin 3\theta.$

Solution : $\because \sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha$

$$\therefore \sin^3\alpha = \frac{3\sin\alpha - \sin 3\alpha}{4}$$

$$\text{Now L.H.S} = \frac{3\sin\theta - \sin 3\theta}{4} + \frac{3\sin(120^\circ+\theta) - \sin 3(120^\circ+\theta)}{4} + \frac{3\sin(240^\circ+\theta) - \sin 3(240^\circ+\theta)}{4}$$

$$= \frac{3}{4}\{\sin\theta + \sin(120^\circ+\theta) + \sin(240^\circ+\theta)\} - \frac{1}{4}\{\sin 3\theta + \sin(360^\circ+3\theta) + \sin(720^\circ+3\theta)\}$$

$$= \frac{3}{4}\left\{\sin\theta + 2\sin\frac{120^\circ+\theta+240+\theta}{2}\cos\frac{120^\circ+\theta-240^\circ-\theta}{2}\right\} - \frac{1}{4}\{\sin 3\theta + \sin 3\theta + \sin 3\theta\}$$

$$= \frac{3}{4}\{\sin\theta + 2\sin(180^\circ+\theta)\cos 60^\circ\} - \frac{1}{4}\{3\sin 3\theta\}$$

$$= \frac{3}{4}\left\{\sin\theta - 2\sin\theta\cdot\frac{1}{2}\right\} - \frac{3}{4}\sin 3\theta$$

$$= -\frac{3}{4}\sin 3\theta = \text{RHS}$$

5. Prove that $\cot\theta + \cot(60^\circ + \theta) + \cot(120^\circ + \theta) = 3\cot 3\theta.$

Solution : L.H.S. $= \cot\theta + \cot(60^\circ + \theta) + \cot(120^\circ + \theta)$

$$= \cot\theta + \frac{\cot 60^\circ \cot\theta - 1}{\cot\theta + \cot 60^\circ} + \frac{\cot 120^\circ \cot\theta - 1}{\cot\theta + \cot 120^\circ}$$

$$= \cot\theta + \frac{\frac{1}{\sqrt{3}}\cot\theta - 1}{\cot\theta + \frac{1}{\sqrt{3}}} + \frac{\frac{-1}{\sqrt{3}}\cot\theta - 1}{\cot\theta - \frac{1}{\sqrt{3}}}$$

$$= \cot\theta + \frac{\cot\theta - \sqrt{3}}{\sqrt{3}\cot\theta + 1} - \frac{\cot\theta + \sqrt{3}}{\sqrt{3}\cot\theta - 1}$$

$$= \cot\theta + \frac{(\cot\theta - \sqrt{3})(\sqrt{3}\cot\theta - 1) - (\cot\theta + \sqrt{3})(\sqrt{3}\cot\theta + 1)}{3\cot^2\theta - 1}$$

$$\theta + \frac{(\sqrt{3}\cot^2\theta - \cot\theta - 3\cot\theta + \sqrt{3}) - (\sqrt{3}\cot^2\theta + \cot\theta + 3\cot\theta + \sqrt{3})}{3\cot^2\theta - 1}$$

$$= \cot\theta + \left(\frac{-8\cot\theta}{3\cdot\cot^2\theta - 1}\right) = \frac{3\cot^3\theta - \cot\theta - 8\cot\theta}{3\cot^2\theta - 1}$$

$$= \frac{3\cot^3\theta - 9\cot\theta}{3\cot^2\theta - 1} = 3\left(\frac{\cot^3\theta - 3\cot\theta}{3\cot^2\theta - 1}\right) = 3\cot 3\theta.$$

26. Prove that $\sin\theta\cos^3\theta - \cos\theta\sin^3\theta = \frac{1}{4}\sin 4\theta$.

Solution : L.H.S. $= \sin\theta\left(\frac{3\cos\theta + \cos 3\theta}{4}\right) - \cos\theta\left(\frac{3\sin\theta - \sin 3\theta}{4}\right)$

$$= \frac{1}{4}\{\sin\theta\cos 3\theta + \cos\theta\sin 3\theta\} = \frac{1}{4}\sin(\theta + 3\theta) = \frac{\sin 4\theta}{4}$$

27. If α and β are two distinct roots of $a\cos\theta + b\sin\theta = c$, prove that

(i) $\sin(\alpha + \beta) = \frac{2ab}{a^2 + b^2}$ (ii) $\cos\alpha + \cos\beta = \frac{2ac}{a^2 + b^2}$

Solution : $\because$ α, β are roots of $a\cos\theta + b\sin\theta = c$

$\therefore$ $a\cos\alpha + b\sin\alpha = c$ ... (i)

and $a\cos\beta + b\sin\beta = c$ ... (ii)

equation (i) – (ii) $a(\cos\alpha - \cos\beta) + b(\sin\alpha - \sin\beta) = 0$

or, $a\cdot 2\sin\frac{\alpha+\beta}{2}\sin\frac{\beta-\alpha}{2} + b\,2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2} = 0$

or, $2\sin\frac{\alpha-\beta}{2}\left(-a\sin\frac{\alpha+\beta}{2} + b\cos\frac{\alpha+\beta}{2}\right) = 0$

or, $-a\sin\frac{\alpha+\beta}{2} + b\cos\frac{\alpha+\beta}{2} = 0$

or, $b\cos\frac{\alpha+\beta}{2} = a\sin\frac{\alpha+\beta}{2}$

or, $\tan\frac{\alpha+\beta}{2} = \frac{b}{a}$

Now, $\sin(\alpha+\beta) = \dfrac{2\tan\frac{\alpha+\beta}{2}}{1 + \tan^2\frac{\alpha+\beta}{2}} = \dfrac{2\frac{b}{a}}{1 + \frac{b^2}{a^2}} = \dfrac{2ab}{a^2 + b^2}$

Second part : Given equation is $b\sin\theta = c - a\cos\theta$

squaring both sides $b^2\sin^2\theta = c^2 + a^2\cos^2\theta - 2ac\cos\theta$

or, $b^2(1 - \cos^2\theta) = c^2 + a^2\cos^2\theta - 2ac\cos\theta$

or, $(a^2 + b^2)\cos^2\theta - 2ac\cos\theta + c^2 - b^2 = 0$

$\because$ two values of θ are α, β

$\therefore$ Roots of above equation are α, β

Hence sum of roots $= \cos\alpha + \cos\beta = \frac{-B}{A} = \frac{2ac}{a^2 + b^2}$

28. Find the value of $\sin^4\frac{\pi}{8}+\sin^4\frac{3\pi}{8}+\sin^4\frac{5\pi}{8}+\sin^4\frac{7\pi}{8}$

Solution : Given $=\sin^4\frac{\pi}{8}+\sin^4\frac{3\pi}{8}+\sin^4\left(\pi-\frac{3\pi}{8}\right)+\sin^4\left(\pi-\frac{\pi}{8}\right)$

$$\left(\because \frac{5\pi}{8}=\pi-\frac{3\pi}{8}, \frac{7\pi}{8}=\pi-\frac{3}{8}\right)$$

$$=\sin^4\frac{\pi}{8}+\sin^4\frac{3\pi}{8}+\sin^4\frac{3\pi}{8}+\sin^4\frac{\pi}{8} \qquad (\because \sin(\pi-\theta)=\sin\theta)$$

$$=2\left\{\left(\sin^2\frac{\pi}{8}\right)^2+\left(\sin^2\frac{3\pi}{8}\right)^2\right\}$$

$$=2\left\{\left(\frac{1-\cos\frac{\pi}{4}}{2}\right)^2+\left(\frac{1-\cos\frac{3\pi}{4}}{2}\right)^2\right\} \qquad \left(\because \sin^2\frac{\theta}{2}=\frac{1-\cos\theta}{2}\right)$$

$$=\frac{2}{2^2}\left\{\left(1-\frac{1}{\sqrt{2}}\right)^2+\left(1+\frac{1}{\sqrt{2}}\right)^2\right\}$$

$$=\frac{1}{2}\left\{2\left(1+\frac{1}{2}\right)\right\}=1+\frac{1}{2}=\frac{3}{2} \qquad \left(\because (a+b)^2+(a-b)^2=2(a^2+b^2)\right)$$

29. Find the value of cosec $10° - \sqrt{3}$ sec $10°$.

Solution : Given expression = cosec $10° - \sqrt{3}$ sec $10°$

$$=\frac{1}{\sin 10°}-\frac{\sqrt{3}}{\cos 10°}$$

$$=\frac{\cos 10°-\sqrt{3}\sin 10°}{\sin 10°\cos 10°}=\frac{2\left(\frac{1}{2}\cos 10°-\frac{\sqrt{3}}{2}\sin 10°\right)}{\sin 10°\cos 10°}$$

$$=\frac{2\times 2(\sin 30°\cos 10°-\cos 30°\sin 10°)}{2\sin 10°\cos 10°}$$

$$=\frac{4\sin(30°-10°)}{\sin(2\times 10°)}=\frac{4\sin 20°}{\sin 20°}=4$$

30. Prove that $\sin 6°\sin 42°\sin 66°\sin 78°=\frac{1}{16}$.

Solution : L.H.S. $=\frac{1}{2}(2\sin 78°\sin 42°)\frac{1}{2}(2\sin 66°\sin 6°)$

$$=\frac{1}{2}\cdot\frac{1}{2}\{\cos(78°-42°)-\cos(78°+42°)\}$$
$$\{\cos(66°-6°)-\cos(66°+6°)\}$$

$$=\frac{1}{4}(\cos 36°-\cos 120°)(\cos 60°-\cos 72°)$$

$$=\frac{1}{4}\left\{\frac{\sqrt{5}+1}{4}-\left(\frac{-1}{2}\right)\right\}\left\{\frac{1}{2}-\frac{\sqrt{5}-1}{4}\right\}$$

$\left(\because \cos 36^\circ = \frac{\sqrt{5}+1}{4}\right.$ तथा $\left.\cos 72^\circ = \sin 18^\circ = \frac{\sqrt{5}-1}{4}\right)$

$$= \frac{1}{4}\left(\frac{\sqrt{5}+1+2}{4}\right)\left(\frac{2-\sqrt{5}+1}{4}\right)$$

$$= \frac{1}{4}\cdot\frac{1}{4}\cdot\frac{1}{4}\left(3+\sqrt{5}\right)\left(3-\sqrt{5}\right)$$

$$= \frac{1}{16}\cdot\frac{1}{4}(9-5) = \frac{1}{16}\cdot\frac{1}{4}\cdot 4 = \frac{1}{16}$$

31. If $\alpha = \frac{\pi}{13}$, Prove that $\cos\alpha\cos 2\alpha\cos 3\alpha\cos 4\alpha\ \cos 5\alpha\cos 6\alpha = \frac{1}{64}$.

Solution : $\text{L.H.S} = \frac{1}{2\sin\alpha}(2\sin\alpha\cos\alpha)\cos 2\alpha\cos 3\alpha\cos 4\alpha\cos 5\alpha\cos 6\alpha$

$$= \frac{(2\sin 2\alpha\cos 2\alpha)}{2\times 2\sin\alpha}\cos 3\alpha\cos 4\alpha\cos 5\alpha\cos 6\alpha$$

$$= \frac{(2\sin 4\alpha\cos 4\alpha)}{2\times 4\sin\alpha}\cos 3\alpha\cos 5\alpha\cos 6\alpha$$

$$= \frac{\sin 8\alpha\cos 3\alpha\cos 5\alpha\cos 6\alpha}{8\sin\alpha}$$

But, $\alpha = \frac{\pi}{13} \Rightarrow 13\alpha = \pi \Rightarrow 8\alpha = \pi - 5\alpha$

$$\therefore\ \text{L.H.S.} = \frac{\sin(\pi-5\alpha)\cos 3\alpha\cos 5\alpha\cos 6\alpha}{8\sin\alpha}$$

$$= \frac{(2\sin 5\alpha\cos 5\alpha)\cos 3\alpha\cos 6\alpha}{2\times 8\sin\alpha}a$$

$$= \frac{\sin 10\alpha\cos 3\alpha\cos 6\alpha}{16\sin\alpha}$$

$$= \frac{\sin(\pi-3\alpha)\cos 3\alpha\cos 6\alpha}{16\sin\alpha} \qquad (\because\ 10\alpha + 3\alpha = \pi)$$

$$= \frac{(2\sin 3\alpha\cos 3\alpha)\cos 6\alpha}{2\times 16\sin\alpha} = \frac{2\sin 6\alpha\cos 6\alpha}{2\times 32\sin\alpha}$$

$$= \frac{\sin 12\alpha}{64\sin\alpha} = \frac{\sin(\pi-\alpha)}{64\sin\alpha} \qquad (\because\ 12\alpha + \alpha = \pi)$$

$$= \frac{\sin\alpha}{64\sin\alpha} = \frac{1}{64} = \text{R.H.S.}$$

32. Prove that $\sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14}\sin\frac{7\pi}{14}\sin\frac{9\pi}{14}\sin\frac{11\pi}{14}\sin\frac{13\pi}{14} = \frac{1}{64}$.

Solution : $\text{L.H.S.} = \sin\frac{\pi}{14}\cdot\sin\frac{3\pi}{14}\cdot\sin\frac{5\pi}{14}\cdot\sin\frac{\pi}{2}\cdot\sin\left(\pi-\frac{5\pi}{14}\right)$

$$\sin\left(\pi-\frac{3\pi}{14}\right)\sin\left(\pi-\frac{\pi}{14}\right)$$

$$= \sin\frac{\pi}{14}\cdot\sin\frac{3\pi}{14}\cdot\sin\frac{5\pi}{14}\cdot 1\cdot\sin\frac{5\pi}{14}\cdot\sin\frac{3\pi}{14}\cdot\sin\frac{\pi}{14}$$

3:

$$= \left(\sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14}\right)^2$$

$$= \left\{\cos\left(\frac{\pi}{2}-\frac{\pi}{14}\right)\cos\left(\frac{\pi}{2}-\frac{3\pi}{14}\right)\cos\left(\frac{\pi}{2}-\frac{5\pi}{14}\right)\right\}^2$$

$$= \left(\cos\frac{7\pi-\pi}{14}\cdot\cos\frac{7\pi-3\pi}{14}\cdot\cos\frac{7\pi-5\pi}{14}\right)^2$$

$$= \left(\cos\frac{3\pi}{7}\cdot\cos\frac{2\pi}{7}\cdot\cos\frac{\pi}{7}\right)^2$$

$$= \left(\frac{2\sin\frac{\pi}{7}}{2\sin\frac{\pi}{7}}\cdot\cos\frac{\pi}{7}\cdot\cos\frac{2\pi}{7}\cos\frac{3\pi}{7}\right)^2$$

$$= \left(\frac{2}{2}\cdot\frac{\sin\frac{2\pi}{7}\cdot\cos\frac{2\pi}{7}\cos\frac{3\pi}{7}}{2\sin\frac{\pi}{7}}\right)^2$$

$$= \left\{\frac{\sin\frac{4\pi}{7}\cdot\cos\left(\pi-\frac{4\pi}{7}\right)}{4\sin\frac{\pi}{7}}\right\}^2 \qquad \left(\because \frac{3\pi}{7}+\frac{4\pi}{7}=\pi\right)$$

$$= \left\{\frac{-\sin\frac{4\pi}{7}\cos\frac{4\pi}{7}}{4\sin\frac{\pi}{7}}\right\}^2 \qquad (\because \cos(\pi-\theta)=\cos\theta)$$

$$= \left\{\frac{-2\sin\frac{4\pi}{7}\cos\frac{4\pi}{7}}{2\cdot 4\sin\frac{\pi}{7}}\right\}^2 = \left\{\frac{-\sin\frac{8\pi}{7}}{8\sin\frac{\pi}{7}}\right\}^2$$

$$= \left\{\frac{-\sin\left(\pi+\frac{\pi}{7}\right)}{8\sin\frac{\pi}{7}}\right\}^2$$

$$= \left(\frac{\sin\frac{\pi}{7}}{8\sin\frac{\pi}{7}}\right)^2 = \frac{1}{8^2} = \frac{1}{64}$$

33. For positive integer n if

$f_n(\theta) = \tan\frac{\theta}{2}(1+\sec\theta)(1+\sec 2\theta)(1+\sec 4\theta)\ldots\ldots(1+\sec 2^n\theta)$ then find

the value of $f_2\left(\frac{\pi}{16}\right)$ and $f_5\left(\frac{\pi}{128}\right)$

Solution : Given that

$f_n(\theta) = \tan(1 + \sec\theta)(1 + \sec 2\theta)(1 + \sec 2^2\theta)(1 + \sec 2^n\theta)$.

here, $\tan\frac{\theta}{2}(1+\sec\theta) = \tan\frac{\theta}{2}\left(1+\frac{1+\tan^2\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}\right)$

$\left[\because \cos\theta = \frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}} \Rightarrow \sec\theta = \frac{1+\tan^2\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}\right]$

$$= \tan\frac{\theta}{2}\left(\frac{1-\tan^2\frac{\theta}{2}+1+\tan^2\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}\right)$$

$$\therefore \quad \tan\frac{\theta}{2}(1+\sec\theta) = \frac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}} = \tan\theta$$

Replacing θ by $\theta, 2\theta, 2^2\theta\ \frac{\theta}{2}$ .... one by one

$\tan\theta\,(1 + \sec 2\theta) = \tan 2\theta$

$\tan 2\theta(1 + \sec 2^2\theta) = \tan 2^2\theta$ …… etc.

Using these identities

$$f_n(\theta) = \left\{\tan\frac{\theta}{2}(1+\sec\theta)\right\}(1+\sec 2\theta)(1+\sec 2^2\theta)\ldots(1+\sec 2^n\theta)$$

$$= \{\tan\theta(1+\sec 2\theta)\}(1+\sec 2^2\theta)\ldots\ldots(1+\sec 2^n\theta)$$

$$= \{\tan 2\theta(1+\sec 2^2\theta)\}(1+\sec 2^3\theta)\ldots\ldots(1+\sec 2^n\theta)$$

Proceeding in the same manner.

$$f_n(\theta) = \tan 2^{n-1}(1 + \sec 2^n\theta) = \tan 2^n\theta.$$

$$\therefore \quad f_2\left(\frac{\pi}{16}\right) = \tan\left(2^2\cdot\frac{\pi}{16}\right) = \tan\left(\frac{\pi}{4}\right) = 1$$

$$f_5\left(\frac{\pi}{128}\right) = \tan\left(2^5\cdot\frac{\pi}{128}\right) = \tan\left(\frac{32\pi}{128}\right) = \tan\frac{\pi}{4} = 1.$$

34. Find the value of following.

(i) $\cot 82\frac{1^\circ}{2}$ (ii) $\tan 142\frac{1^\circ}{2}$

Solution : (i) We know that $1 + \cos\theta = 2\cos^2\frac{\theta}{2}$ and $\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$

$$\therefore \quad \frac{1+\cos\theta}{\sin\theta} = \frac{2\cos^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} = \cot\frac{\theta}{2}$$

or, $\cot\frac{\theta}{2} = \frac{1+\cos\theta}{\sin\theta}$

[Note]

putting $\theta = 165°$ $\quad \cot\frac{165°}{2} = \frac{1+\cos 165°}{\sin 165°}$

or, $\cot 82\frac{1}{2}° = \frac{1-\cos 15°}{\sin 15°} = \dfrac{1-\frac{\sqrt{3}+1}{2\sqrt{2}}}{\frac{\sqrt{3}-1}{2\sqrt{2}}}$

$$= \frac{2\sqrt{2}-\sqrt{3}-1}{\sqrt{3}-1} \times \frac{\sqrt{3}+1}{\sqrt{3}+1} = \frac{2\sqrt{6}+2\sqrt{2}-3-\sqrt{3}-\sqrt{3}-1}{3-1}$$

$$= \frac{2\sqrt{6}+2\sqrt{2}-4-2\sqrt{3}}{2} = \sqrt{6}+\sqrt{2}-2-\sqrt{3}$$

(ii) $\because 1-\cos\theta = 2\sin^2\frac{\theta}{2}$ and $\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$

$\therefore \tan\frac{\theta}{2} = \frac{1-\cos\theta}{\sin\theta}$

putting $\theta = 285°$

$$\tan\frac{285°}{2} = \frac{1-\cos 285°}{\sin 285°} = \frac{1-\sin 15°}{-\cos 15°}$$

$$= \dfrac{1-\frac{\sqrt{3}-1}{2\sqrt{2}}}{-\frac{\sqrt{3}+1}{2\sqrt{2}}} = -\frac{2\sqrt{2}-\sqrt{3}+1}{\sqrt{3}+1} \times \frac{\sqrt{3}-1}{\sqrt{3}-1}$$

$$= -\left(\frac{2\sqrt{6}-2\sqrt{2}-3+\sqrt{3}+\sqrt{3}-1}{3-1}\right)$$

$$= -\left(\frac{2\sqrt{6}-2\sqrt{2}-4+2\sqrt{3}}{2}\right) = -\left(\sqrt{6}-\sqrt{2}-2+\sqrt{3}\right)$$

Exercise—13A

1. If p and q are two quantities such that $p^2 + q^2 = 1$, then maximum value of $p+q$ is

(a) $\frac{1}{2}$ (b) $\frac{1}{\sqrt{2}}$ (c) $\sqrt{2}$ (d) 2

2. If θ is real then $3 - \cos\theta + \cos\left(\theta + \frac{\pi}{3}\right)$ lies in the interval

(a) $[-2, 3]$ (b) $[-2, 1]$ (c) $[2, 4]$ (d) $[1, 5]$

3. If $\sin A = \frac{1}{\sqrt{10}}$ and $\sin B = \frac{1}{\sqrt{5}}$, where A and B are acute angle then $A+B$ is

(a) π (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{4}$

4. If $\cos(\theta-\alpha) = a$, $\cos(\theta-\beta) = b$, then the value of $\sin^2(\alpha-\beta) + 2ab\cos(\alpha-\beta)$ is

(a) a^2+b^2 (b) a^2-b^2 (c) b^2-a^2 (d) $-a^2-b^2$

5. A positive angle is divided into two parts such that their tangents are respectively $\frac{1}{2}$ and $\frac{1}{3}$. The measure of the angles is
(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{5}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{6}$

6. In ΔPQR, $\angle R = \frac{\pi}{2}$. If $\tan\left(\frac{P}{2}\right)$ and $\tan\left(\frac{Q}{2}\right)$ are roots of equation, $ax^2 + bx + c = 0$, then which of the following is true.
(a) $c = a + b$ (b) $a = b + c$ (c) $b = a + c$ (d) $b = c$

7. The value of $\frac{\sin 55° - \cos 55°}{\sin 10°}$ is
(a) $\frac{1}{\sqrt{2}}$ (b) 2 (c) 1 (d) $\sqrt{2}$

8. The value of $\cos 15° - \sin 15°$ is
(a) 0 (b) $\frac{1}{\sqrt{2}}$ (c) $-\frac{1}{\sqrt{2}}$ (d) $\frac{1}{2\sqrt{2}}$

9. Minimum value of $27^{\cos 2x} 81^{\sin 2x}$ is
(a) $\frac{1}{243}$ (b) $\frac{1}{27}$ (c) -5 (d) $\frac{1}{5}$

10. $3(\sin x - \cos x)^4 + 6(\sin x + \cos x)^2 + 4(\sin^6 x + \cos^6 x) =$
(a) 14 (b) 11 (c) 12 (d) 13

11. If $\sin\theta = \sin 15° + \sin 45°$, where $0° < \theta < 90°$, then value of θ is
(a) 45° (b) 54° (c) 60°
(d) 72° (e) 75°

12. If $\cos\alpha + \cos\beta = 0 = \sin\alpha + \sin\beta$, then value of $\cos 2\alpha + \cos 2\beta$ is
(a) $-2\sin(\alpha+\beta)$ (b) $2\cos(\alpha+\beta)$ (c) $2\sin(\alpha-\beta)$
(d) $-2\cos(\alpha+\beta)$ (e) $-2\cos(\alpha-\beta)$

13. If $A + B = 45°$, then $(\tan A - 1)(\tan B - 1)$ is
(a) 1 (b) $\frac{1}{2}$ (c) -1
(d) -2 (e) 2

14. If $\alpha, \beta \in \left(0, \frac{\pi}{2}\right)$, $\sin\alpha = \frac{4}{5}$ and $\cos(\alpha+\beta) = -\frac{12}{13}$, then the value of $\sin\beta$ is
(a) $\frac{63}{65}$ (b) $\frac{61}{65}$ (c) $\frac{3}{5}$
(d) $\frac{5}{13}$ (e) $\frac{8}{65}$

15. The value of $\tan 40° + \tan 20° + \sqrt{3}\tan 20° \tan 40°$ is
(a) $\sqrt{12}$ (b) $\frac{1}{\sqrt{3}}$ (c) 1
(d) $\frac{\sqrt{3}}{2}$ (e) $\sqrt{3}$

16. The value of $\cos 20° + \cos 100° + \cos 140°$ is

(a) $\frac{1}{2}$ (b) $\frac{1}{\sqrt{3}}$ (c) $\sqrt{3}$

(d) 0 (e) 1

17. If $\frac{-\pi}{2} < \theta < \frac{\pi}{2}$ and $\theta \neq \pm\frac{\pi}{4}$, then the value of $\cot\left(\frac{\pi}{4}+\theta\right)\cot\left(\frac{\pi}{4}-\theta\right)$ is

(a) 0 (b) -1 (c) 1

(d) -2 (e) 2

18. If $\sin\theta = 3\sin(\theta + 2\alpha)$, then the value of $\tan(\theta + \alpha) + 2\tan\alpha$ is

(a) 3 (b) 2 (c) -1

(d) 0 (e) 1

19. If $x = y\cos\frac{2\pi}{3} = z\cos\frac{4\pi}{3}$ then the value of $xy + yz + zx$ is

(a) -1 (b) 0 (c) 1 (d) 2

20. $\frac{\cos 9° + \sin 9°}{\cos 9° - \sin 9°}$ equals

(a) $\tan 26°$ (b) $\tan 81°$ (c) $\tan 51°$ (d) $\tan 54°$

21. The value of $\sin 50° - \sin 70° + \sin 10°$ is

(a) 0 (b) 1 (c) $\frac{1}{2}$ (d) $\frac{1}{\sqrt{2}}$

22. Maximum value of $3\cos\theta + 4\sin\theta$ is

(a) 3 (b) 4 (c) 5 (d) 7

23. If $\tan\alpha = \frac{n}{n+1}$ and $\tan\beta = \frac{1}{2n+1}$, then the value of $\alpha + \beta$ is

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{5}$

24. If $\frac{\sin(x+y)}{\sin(x-y)} = \frac{a+b}{a-b}$, then the value of $\frac{\tan x}{\tan y}$ is

(a) 0 (b) ab (c) $\frac{b}{a}$ (d) $\frac{a}{b}$

25. If $\tan\alpha = \frac{5}{6}$, $\tan\beta = \frac{1}{11}$ then the value of $\alpha + \beta$ is

(a) $-\frac{\pi}{4}$ (b) $\frac{\pi}{3}$ (c) $-\frac{\pi}{3}$ (d) $\frac{\pi}{4}$

26. Maximum value of $\sin\left(x+\frac{\pi}{6}\right) + \cos\left(x+\frac{\pi}{6}\right)$ is

(a) $\sqrt{2}$ (b) $\sqrt{7}$ (c) 2 (d) 8

27. Let $\cos(\alpha + \beta) = \frac{4}{5}$ and $\sin(\alpha - \beta) = \frac{5}{13}$, where $0 \le \alpha, \beta \le \frac{\pi}{4}$, then $\tan 2\alpha =$

(a) $\frac{25}{16}$ (b) $\frac{56}{33}$ (c) $\frac{19}{12}$ (d) $\frac{20}{17}$

28. If $A = \sin^2 x + \cos^4 x$, where x is a real number

(a) $1 \le A \le 2$ (b) $\frac{3}{4} \le A \le \frac{13}{16}$ (c) $\frac{3}{4} \le A \le 1$ (d) $\frac{13}{16} \le A \le 1$

29. If $x = \tan 15^\circ$, $y = \text{cosec } 75^\circ$ and $z = 4 \sin 18^\circ$, then

(a) $x < y < z$ (b) $y < z < x$ (c) $z < x < y$ (d) $x < z < y$

30. If $\sin(x + 3\alpha) = 3 \sin(\alpha - x)$, then

(a) $\tan x = \tan \alpha$ (b) $\tan x = \tan^2 \alpha$

(c) $\tan x = \tan^3 \alpha$ (d) $\tan x = 3 \tan \alpha$

31. Expression $\frac{1}{\cos 290^\circ} + \frac{1}{\sqrt{3} \sin 250^\circ}$ equals

(a) $\frac{\sqrt{3}}{4}$ (b) $\frac{4}{\sqrt{3}}$ (c) $\frac{2}{\sqrt{3}}$ (d) $\frac{\sqrt{3}}{2}$

...he value of $\left(1+\cos\frac{\pi}{8}\right)\left(1+\cos\frac{3\pi}{8}\right)\left(1+\cos\frac{5\pi}{8}\right)\left(1+\cos\frac{7\pi}{8}\right)$ is

(b) $\frac{1}{6}$ (c) $\frac{1}{8}$ (d) $\frac{1}{2}$

...lue of $\cos^4\frac{\pi}{8} + \cos^4\frac{3\pi}{8} + \cos^4\frac{5\pi}{8} + \cos^4\frac{7\pi}{8}$ is

(a) $\frac{3}{2}$ (b) $-\frac{2}{3}$ (c) -1 (d) 1

34. The value of $\cos\frac{\pi}{15} \cos\frac{2\pi}{15} \cos\frac{4\pi}{15} \cos\frac{8\pi}{15}$ is

(a) $\frac{1}{16}$ (b) $-\frac{1}{16}$ (c) 1 (d) 0

35. $\frac{2}{\sqrt{2+\sqrt{2+\sqrt{2+2\cos 4x}}}}$ equals

(a) $\sec\frac{x}{2}$ (b) $\sec x$ (c) $\text{cosec } x$ (d) 1

36. The value of $\cos 15^\circ \cos 7\frac{1}{2}^\circ \sin 7\frac{1}{2}^\circ$ is

(a) $\frac{1}{2}$ (b) $\frac{1}{8}$ (c) $\frac{1}{4}$ (d) $\frac{1}{16}$

37. $\frac{\cot x - \tan x}{\cot 2x}$ equals

(a) 1 (b) 2 (c) -1 (d) 4

38. $\tan 67\frac{1^\circ}{2} + \cot 67\frac{1^\circ}{2}$ equals

(a) $2\sqrt{2}$ (b) $2-\sqrt{2}$ (c) $\sqrt{2}$ (d) $3\sqrt{2}$

39. If $\sin 4A - \cos 2A = \cos 4A - \sin 2A$, $\left(0 < A < \frac{\pi}{4}\right)$ then the value of $\tan 4A$ is

(a) 1 (b) $\frac{1}{\sqrt{3}}$ (c) $\sqrt{3}$

(d) $\frac{\sqrt{3}-1}{\sqrt{3}+1}$ (e) $\frac{\sqrt{3}+1}{\sqrt{3}-1}$

60. $\frac{1}{8}(3 - 4\cos 2\theta + \cos 4\theta)$ equals

(a) $\cos 4\theta$ (b) $\sin 4\theta$ (c) $\sin^4\theta$

(d) $\cos^4\theta$ (e) $\sin^4(\theta/2)$

61. If $8\cos 2\theta + 8\sec 2\theta = 65$, $0 < \theta < \frac{\pi}{2}$ then the value of $4\cos 4\theta$ is

(a) $\frac{-23}{8}$ (b) $\frac{-31}{8}$ (c) $\frac{-31}{32}$

(d) $\frac{-33}{32}$ (e) $\frac{-32}{4}$

62. $\cos A \cos 2A \cos 4A \ldots \cos 2^{n-1}A$ equals

(a) $\frac{\sin 2^n A}{2^n \sin A}$ (b) $\frac{2^n \sin 2^n A}{\sin A}$ (c) $\frac{2^n \sin A}{\sin 2^n A}$ (d) $\frac{\sin A}{2^n \sin 2^n A}$

63. $\tan 9° - \tan 27° - \tan 63° + \tan 81°$ equals

(a) 2 (b) 3

(c) 4 (d) Non of these

64. $\frac{\cos\theta}{1+\sin\theta}$ equals

(a) $\tan\left(\frac{\theta}{2}-\frac{\pi}{4}\right)$ (b) $\tan\left(-\frac{\pi}{4}-\frac{\theta}{2}\right)$ (c) $\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)$ (d) $\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$

65. Suppose α, β are such that $\pi < \alpha - \beta < 3\pi$. if $\sin\alpha + \sin\beta = -\frac{21}{65}$ and $\cos\alpha + \cos\beta = -\frac{27}{65}$ then the value of $\cos\frac{\alpha-\beta}{2}$ is

(a) $-\frac{3}{\sqrt{130}}$ (b) $\frac{3}{\sqrt{130}}$ (c) $\frac{6}{65}$ (d) $-\frac{6}{65}$

66. The value of $\cos^2(A-B) + \cos^2 B - 2\cos(A-B)\cos A\cos B$ is

(a) $\cos^2 A$ (b) $\sin^2 A$

(c) $\tan^2 A$ (d) $\cot^2 A$

67. $\cos^2\frac{\pi}{16} + \cos^2\frac{3\pi}{16} + \cos^2\frac{5\pi}{16} + \cos^2\frac{7\pi}{16}$ equals

(a) 0 (b) 1 (c) 2 (d) 3

68. If $\sin\theta - \cos\theta = \frac{\sqrt{3}-1}{2}$, then θ equals

(a) 30° (b) 45° (c) 60° (d) 90°

69. Which of the following is true about ... sin x ... (1 ...

51. If $1 + \cos x \cos y + \sin x \sin y = 0$, then which of the following is/are true
1. $\cos x + \cos y = 0$
2. $\sin x + \sin y = 0$
3. $\sin x + \cos y = 0$
chose the correct option from the code given below
(a) only 1 (b) only 2 (c) only 3 (d) 1 and 2

Answers—13A

1. (c)	2. (c)	3. (d)	4. (a)	5. (a)	6. (a)	7. (d)	8. (b)
9. (a)	10. (d)	11. (e)	12. (d)	13. (e)	14. (a)	15. (e)	16. (d)
17. (c)	18. (d)	19. (b)	20. (d)	21. (a)	22. (c)	23. (a)	24. (d)
25. (d)	26. (a)	27. (b)	28. (c)	29. (a)	30. (c)	31. (b)	32. (c)
33. (a)	34. (b)	35. (a)	36. (a)	37. (c)	38. (a)	39. (c)	40. (c)
41. (b)	42. (a)	43. (c)	44. (c)	45. (a)	46. (b)	47. (c)	48. (c)
49. (c)	50. (c)	51. (d)					

Explanation

1. (c) $\because$ $p^2 + q^2 = 1$, let $p = \sin\theta$, $q = \cos\theta$
$\therefore$ $p + q = \sin\theta + \cos\theta$
Its maximum value $= \sqrt{1^2+1^2} = \sqrt{2}$
(Recall that $-\sqrt{a^2+b^2} \le a\cos\theta + b\sin\theta \le \sqrt{a^2+b^2}$)

2. (c) $3 - \cos\theta - \cos(\theta + \frac{\pi}{3}) = 3 - \cos\theta + (\cos\theta \cos\frac{\pi}{3} - \sin\theta \sin\frac{\pi}{3})$
$= 3 - \cos\theta + \frac{1}{2}\cos\theta - \frac{\sqrt{3}}{2}\sin\theta$
$= 3 - \frac{1}{2}\cos\theta - \frac{\sqrt{3}}{2}\sin\theta$
here, $\sqrt{a^2+b^2} = \sqrt{\frac{1}{4}+\frac{3}{4}} = 1$
$\therefore$ Maximum value $= 3 + 1 = 4$, Minimum value $= 3 - 1 = 2$

3. (d) $\sin A = \frac{1}{\sqrt{10}} \Rightarrow \tan A = \frac{1}{3}$ and $\sin B = \frac{1}{\sqrt{5}} \Rightarrow \tan B = \frac{1}{2}$
Now, $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} = \frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\times\frac{1}{3}} = 1 \Rightarrow A + B = 45^\circ$

4. (a) $\cos(\alpha - \beta) = \cos(\alpha - \theta) - (\beta - \theta)$
$= \cos(\alpha-\theta)\cos(\beta-\theta) + \sin(\alpha-\theta)\sin(\beta-\theta)$
$= ab + \sqrt{1-a^2}\sqrt{1-b^2}$ $\quad(\because \sin(\alpha-\theta) = \sqrt{1-\cos^2(\alpha-\theta)})$

Now,

$\sin^2(\alpha-\beta)+2ab\cos(\alpha-\beta)=1-\cos^2(\alpha-\beta)+2ab\cos(\alpha-\beta)$

$=1-(ab+\sqrt{1-a^2}\ \sqrt{1-b^2})^2+2ab\,(ab+\sqrt{1-a^2}\ \sqrt{1-b^2})$

Simplify it

5. (a) tangent of angle means $\tan\theta$

here, $\tan A=\frac{1}{2}$, $\tan B=\frac{1}{3}$ as in question number (3) $A+B=45^\circ$

6. (a) $P+Q+R=180^\circ \Rightarrow P+Q=90^\circ$ $(\because R=90^\circ)$

$$\Rightarrow \frac{P}{2}+\frac{Q}{2}=45^\circ$$

$$\Rightarrow \tan\left(\frac{P}{2}+\frac{Q}{2}\right)=\tan 45^\circ=1$$

$$\Rightarrow \frac{\tan\frac{P}{2}+\tan\frac{Q}{2}}{1-\tan\frac{P}{2}\tan\frac{Q}{2}}=1$$

$$\Rightarrow \frac{-\frac{b}{a}}{1-\frac{c}{a}}=1$$

$$\Rightarrow \frac{-b}{a-c}=1$$

$$\Rightarrow -b=a-c$$

$$\Rightarrow a+b=c$$

7. (d) $\frac{\sin 55^\circ-\cos 55^\circ}{\sin 10^\circ}=\frac{\sin 55^\circ-\sin 35^\circ}{\sin 10^\circ}$

$$=\frac{2\cos\frac{55^\circ+35^\circ}{2}\sin\frac{55^\circ-35^\circ}{2}}{\sin 10^\circ}=2\cos 45^\circ=\sqrt{2}$$

8. (b) Recall that $\cos 15^\circ=\frac{\sqrt{3}+1}{2\sqrt{2}}$ and $\sin 15^\circ=\frac{\sqrt{3}-1}{2\sqrt{2}}$

9. (a) $27^{\cos 2x}81^{\sin 2x}=3^{3\cos 2x}\,3^{4\sin 2x}$

$=3^{3\cos 2x+4\sin 2x}$

$\because$ minimum value of $a\cos\theta+b\sin\theta=-\sqrt{a^2+b^2}$

$\therefore$ Minimum value of given expression $=3^{-\sqrt{3^2+4^2}}=3^{-5}=\frac{1}{243}$

10. (d) $(\sin x+\cos x)^4=((\sin x-\cos x)^2)^2=(1-2\sin x\cos x)^2$

$\sin^6 x+\cos^6 x=(\sin^2 x+\cos^2 x)(\sin^4 x-\sin^2 x\cos^2 x+\cos^4 x)$

$=1\cdot\{(\sin^2 x+\cos^2 x)^2-3\sin^2 x\cos^2 x\}$

$=1-3\sin^2 x\cos^2 x$ etc.

Shortcut : Since all the options are free from x, putting $x = 0$

Required value $= 3(0-1)^4 + 6(0+1)^2 + 4(0+1^6) = 13$

11. (e) $\sin\theta = 2\sin\frac{15^\circ+45^\circ}{2}\cdot\cos\frac{15^\circ-45^\circ}{2}$

$= 2\sin30^\circ\cos15^\circ = \cos15^\circ = \sin75^\circ$

12. (d) $\cos\alpha + \cos\beta = 0 \Rightarrow \cos^2\alpha + \cos^2\beta + 2\cos\alpha\cos\beta = 0$

$\sin\alpha + \sin\beta = 0 \Rightarrow \sin^2\alpha + \sin^2\beta + 2\sin\alpha\sin\beta = 0$

adding $2 + 2\cos(\alpha-\beta) = 0$

$\Rightarrow \cos(\alpha-\beta) = -1$

Now, $\cos2\alpha + \cos2\beta = 2\cos\left(\frac{2\alpha+2\beta}{2}\right)\cos\left(\frac{2\alpha-2\beta}{2}\right)$

$= 2\cos(\alpha+\beta)\cos(\alpha-\beta) = -2\cos(\alpha+\beta)$

13. (e) put $B = 45^\circ - A$, and simplify

14. (a) $\sin\beta = \sin\{(\alpha+\beta)-\alpha\}$

$= \sin(\alpha+\beta)\cos\alpha - \cos(\alpha+\beta)\sin\alpha$

$= \frac{5}{13}\cdot\frac{3}{5} + \frac{12}{13}\cdot\frac{4}{5} = \frac{63}{65}$

[$\because \cos(\alpha+\beta) = \frac{-12}{13} \Rightarrow \sin(\alpha+\beta) = \frac{5}{13}$ because $(\alpha+\beta)$ lies in II quadrant so $\sin(\alpha+\beta)$ is positive]

15. (e) $\tan(20^\circ+40^\circ) = \tan60^\circ$

$\frac{\tan20^\circ+\tan40^\circ}{1-\tan20^\circ\tan40^\circ} = \sqrt{3}$

$\Rightarrow \tan20^\circ + \tan40^\circ = \sqrt{3} - \sqrt{3}\tan20^\circ\tan40^\circ$

$\Rightarrow \tan20^\circ + \tan40^\circ + \sqrt{3}\tan20^\circ\tan40^\circ = \sqrt{3}$.

16. (d) For $\cos100^\circ + \cos140^\circ$ apply $\cos C + \cos D$.

17. (c) Apply $\cot(A+B)$ and $\cot(A-B)$ and simplify.

18. (d) $\frac{\sin(\theta+2\alpha)}{\sin\theta} = \frac{1}{3}$, Apply componendo-dividendo

19. (b) Let $x = y\cos\frac{2\pi}{3} = z\cos\frac{4\pi}{3} = k$

$\therefore x = \frac{-1}{2}y = \frac{-1}{2} = k \Rightarrow x = k, y = -2k, z = -2k$

$\therefore xy + yz + zx = -2k^2 + 4k^2 - 2k^2 = 0.$

20. (d) $\frac{\cos9^\circ+\sin9^\circ}{\cos9^\circ-\sin9^\circ} = \frac{1+\tan9^\circ}{1-\tan9^\circ}$ (divide N^r & D^r by $\cos9^\circ$)

$= \tan(45^\circ+9^\circ) = \tan54^\circ$

24. (d) Apply Componendo and dividendo

26. (a)

27. (b)

28. (c

29. (

30.

31.

26. (a) If $x + \frac{\pi}{6} = \theta$

then $\sin(x + \frac{\pi}{6}) + \cos(x + \frac{\pi}{6}) = \sin\theta + \cos\theta$

Its maximum value $= \sqrt{1^2+1^2} = \sqrt{2}$.

27. (b) $\cos(\alpha+\beta) = \frac{4}{5} \Rightarrow \tan(\alpha+\beta) = \frac{3}{4}$

$\sin(\alpha-\beta) = \frac{5}{13} \Rightarrow \tan(\alpha-\beta) = \frac{5}{12}$

$\therefore \tan 2\alpha = \tan((\alpha+\beta)+(\alpha-\beta))$

$$= \frac{\tan(\alpha+\beta)+\tan(\alpha-\beta)}{1-\tan(\alpha+\beta)\tan(\alpha-\beta)}$$

$$= \frac{\frac{3}{4}+\frac{5}{12}}{1-\frac{3}{4}\frac{5}{12}} = \frac{56}{33}$$

28. (c) $A = \sin^2 x + \cos^4 x$

$= 1 - \cos^2 x + \cos^4 x$

$= 1 - \cos^2 x(1-\cos^2 x)$

$= 1 - \cos^2 x \sin^2 x = 1 - \frac{\sin^2 2x}{4}$

$\because 0 \le \sin^2 2x \le 1$

$\therefore$ when $\sin^2 2x = 0, A = 1$

when $\sin^2 2x = 1, A = 1 - \frac{1}{4} = \frac{3}{4}$

29. (a) $x \tan 15° = 2 - \sqrt{3} = 2 - 1{\cdot}73 = 0{\cdot}27$

$y = \operatorname{cosec} 75° = \frac{1}{\sin 15°} = \frac{2\sqrt{2}}{\sqrt{3}-1} = \frac{2\times 1{\cdot}414}{1{\cdot}73-1} = \frac{2{\cdot}8}{0{\cdot}73} = 0{\cdot}4$ (Approximate)

$z = 4\sin 8° = 4\left(\frac{\sqrt{5}-1}{4}\right) = 2{\cdot}23 - 1 = 1{\cdot}23$

$\therefore x < y < z$

30. (c) $\sin(x+3\alpha) = 3\sin(\alpha - x)$

$\Rightarrow \sin x\cos 3\alpha + \cos x \sin 3\alpha = 3(\sin\alpha\cos x - \cos\alpha\sin x)$

$\Rightarrow \sin x(\cos 3\alpha + 3\cos\alpha) = \cos x(3\sin\alpha - \sin 3\alpha)$

$\Rightarrow \sin x\, 4\cos^3\alpha = \cos x\, 4\sin^3 x$ $(\because \sin 3\theta = 3\sin\theta - 4\sin^3\theta)$

$\Rightarrow \tan x = \tan^3\alpha.$

31. (b) $\frac{1}{\cos 290°} + \frac{1}{\sqrt{3}\sin 250°} = \frac{1}{\sin 20°} - \frac{1}{\sqrt{3}\cos 20°}$

$$= \frac{\sqrt{3}\cos 20° - \sin 20°}{\sqrt{3}\sin 20°\cos 20°}$$

$$= 2\frac{\left(\frac{\sqrt{3}}{2}\cos 20^\circ - \frac{1}{2}\sin 20^\circ\right)}{\sqrt{3}\left(\frac{2\sin 20^\circ\cos 20^\circ}{2}\right)}$$

$$= \frac{4(\sin 60^\circ\cos 20^\circ - \cos 60^\circ\sin 20^\circ)}{\sqrt{3}\sin 40^\circ}$$

$$= \frac{4\sin(60^\circ - 20^\circ)}{\sqrt{3}\sin 40^\circ} = \frac{4}{\sqrt{3}}$$

32. (c) here $\cos\frac{5\pi}{8} = \cos\left(\pi - \frac{3\pi}{8}\right) = -\cos\frac{3\pi}{8}$

and $\cos\frac{7\pi}{8} = \cos\left(\pi - \frac{\pi}{8}\right) = -\cos\frac{\pi}{8}$ etc.

33. (a) Apply $\cos^2\theta = \frac{1+\cos 2\theta}{2}$

34. (b) $\cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{8\pi}{15}$

$$\frac{\ldots\text{in}\frac{\pi}{15}\cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{8\pi}{15}}{2\sin\frac{\pi}{15}}$$

$$= \frac{\sin\frac{2\pi}{15}\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{8\pi}{15}}{2\sin\frac{\pi}{15}} \text{ etc.}$$

Now take the help of solved example 31.

35. (a) Applying $1 + \cos 2\theta = 2\cos^2\theta$

we have, $\sqrt{2+2\cos 4x} = \sqrt{2(1+\cos 4x)}$

$$= \sqrt{2\cdot 2\cos^2 2x} = 2\cos 2x$$

2 2

$$= 2\frac{\left(\frac{\sqrt{3}}{2}\cos 20^\circ - \frac{1}{2}\sin 20^\circ\right)}{\sqrt{3}\left(\frac{2\sin 20^\circ \cos 20^\circ}{2}\right)}$$

$$= \frac{4(\sin 60^\circ \cos 20^\circ - \cos 60^\circ \sin 20^\circ)}{\sqrt{3}\sin 40^\circ}$$

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33. (a) Apply $\cos^2\theta = \frac{1+\cos 2\theta}{2}$

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$$= \frac{2\sin\frac{\pi}{15}\cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{8\pi}{15}}{2\sin\frac{\pi}{15}}$$

$$= \frac{\sin\frac{2\pi}{15}\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{8\pi}{15}}{2\sin\frac{\pi}{15}} \text{ etc.}$$

Now take the help of solved example 31.

35. (a) Applying $1 + \cos 2\theta = 2\cos^2\theta$

we have, $\sqrt{2+2\cos 4x} = \sqrt{2(1+\cos 4x)}$

$$= \sqrt{2\cdot 2\cos^2 2x} = 2\cos 2x$$

$$\therefore \frac{2}{\sqrt{2+2\sqrt{2+2\cos 4x}}} = \frac{2}{\sqrt{2+\sqrt{2+2\cos 2x}}} \text{ etc.}$$

36. (a) $\tan 67\frac{1}{2}^\circ + \cot 67\frac{1}{2}^\circ = \cot 22\frac{1}{2}^\circ + \tan 22\frac{1}{2}^\circ$

$$= (\sqrt{2}+1) + (\sqrt{2}-1) = 2\sqrt{2}$$

37. (c) $\sin 4A + \sin 2A = \cos 4A + \cos 2A$

$\Rightarrow 2\sin 3A \cos A = 2\cos 3A \cos A$

[use formula of sinC + sin*D* and cosC + cos*D*]

$\Rightarrow \tan 3A = 1 \Rightarrow 3A = 45^\circ \Rightarrow A = 15^\circ$

$\therefore \tan 4A = \tan 60^\circ = \sqrt{3}$

44. (c) $\frac{\cos\theta}{1+\sin\theta} = \frac{\sin\left(\frac{\pi}{2}-\theta\right)}{1+\cos\left(\frac{\pi}{2}-\theta\right)}$

$$= \frac{2\sin\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right)}{2\cos^2\left(\frac{\pi}{4}--\frac{\theta}{2}\right)}$$

$$= \tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)$$

45. (a) $\frac{\sin\alpha+\sin\beta}{\cos\alpha+\cos\beta} = \frac{21}{27}$

$$= \frac{2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}}{2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}} = \frac{7}{9}$$

$$\Rightarrow \tan\left(\frac{\alpha+\beta}{2}\right) = \frac{7}{9}$$

$\sin\alpha + \sin\beta = \frac{-21}{65}$ for,

$$2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} = \frac{-21}{65}$$

$$2\frac{7}{\sqrt{7^2+9^2}}\cdot\cos\left(\frac{\alpha-\beta}{2}\right) = \frac{-21}{65}$$

$$\cos\left(\frac{\alpha-\beta}{2}\right) = \frac{-21}{65}\times\frac{\sqrt{130}}{14}$$

$$= \frac{-3\times\sqrt{130}}{65\times 2} = \frac{-3}{\sqrt{130}}$$

46. (b) $\cos^2(A-B) + \cos^2 B - \cos(A-B)\{2\cos A\cos B\}$

$= \cos^2(A-B) + \cos^2 B - \cos(A-B)\ \{\cos(A-B) + \cos(A+B)\}$

$= \cos^2(A-B) + \cos^2 B - \cos^2(A-B) - \cos(A-B)\cos(A+B)$

$= \cos^2 B - \cos^2 A + \sin^2 B = 1 - \cos^2 A = \sin^2 A$

47. (c) $\cos^2\frac{7\pi}{16} = \cos^2\left(\frac{\pi}{2}-\frac{\pi}{16}\right) = \sin^2\frac{\pi}{16}$

$$= \cos^2\frac{5\pi}{16}$$

$$= \cos^2\left(\frac{\pi}{2}-\frac{3\pi}{16}\right) = \sin^2\frac{3\pi}{16}\text{etc.}$$

48. (c) $\sin\theta - \cos\theta = \frac{\sqrt{3}-1}{2}$

Squaring

$(\sin\theta - \cos\theta)^2 = \left(\frac{\sqrt{3}-1}{2}\right)^2$

$\Rightarrow \sin^2\theta + \cos^2\theta - 2\sin\theta\cos\theta = \frac{3+1-2\sqrt{3}}{4}$

$\Rightarrow 1 - \sin 2\theta = 1 - \frac{\sqrt{3}}{2}$

$\Rightarrow \sin 2\theta = \frac{\sqrt{3}}{2} = \sin 120°$

$\Rightarrow 2\theta = 120°$

$\Rightarrow \theta = 60°$

49. (c) $\frac{\sin x}{1+\cos x} = \frac{1-\cos x}{\sin x}$

$\Rightarrow \sin^2 x = (1-\cos x)(1+\cos x)$

$\Rightarrow (1-\cos^2 x) = (1-\cos^2 x)$

$\Rightarrow \sin^2 x = \sin^2 x$

but at $x = 180°$, $\sin x = 0$, and $1 + \cos x = 0$, so denominator of given identity is not defirnd when $x = 180°$

$\therefore$ options (c) is correct.

50. (c) $\cos 0° + \cos 1° + \ldots\ldots + \cos 179° + \cos 180°$

$= \cos 0° + \cos 1° + \cos 2° + \ldots + \cos (180° - 2°) + \cos (180° - 1°) + \cos 180°$

$= 1 + \cos 1° + \cos 2° + \ldots\ldots - \cos 2° - \cos 1° - 1 = 0$

51. (d) $\because$ $1 + \cos x \cos y + \sin x \cdot \sin y = 0$

$\Rightarrow 1 + \cos(x-y) = 0$

$\Rightarrow \cos(x-y) = -1 = \cos 180°$

$\Rightarrow x - y = 180° \Rightarrow x = 180° + y$

1. $\cos x + \cos y = \cos(180° + y) + \cos y = -\cos y + \cos y = 0$

$\therefore$ Statement (1) is correct.

2. $\sin x + \sin y = \sin(180° + y) + \sin y = -\sin y + \sin y = 0$

$\therefore$ Statement (2) is correct

3. **$\sin x + \cos y = \sin(180° + y) + \cos y = -\sin y + \cos y \neq 0$**

$\therefore$ **Statement (3) is not correct.**

Hence option (d) is correct.

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