Exercise 14.R

Answer 1CC.

(a)

A function of two variable is a rule that assigns to each ordered pair of real numbers (*x*, *y*) in a set *D* a unique real number denoted by *f*(*x*, *y*). The set *D* is the domain of *f* and its range is the set of values that *f* takes on.

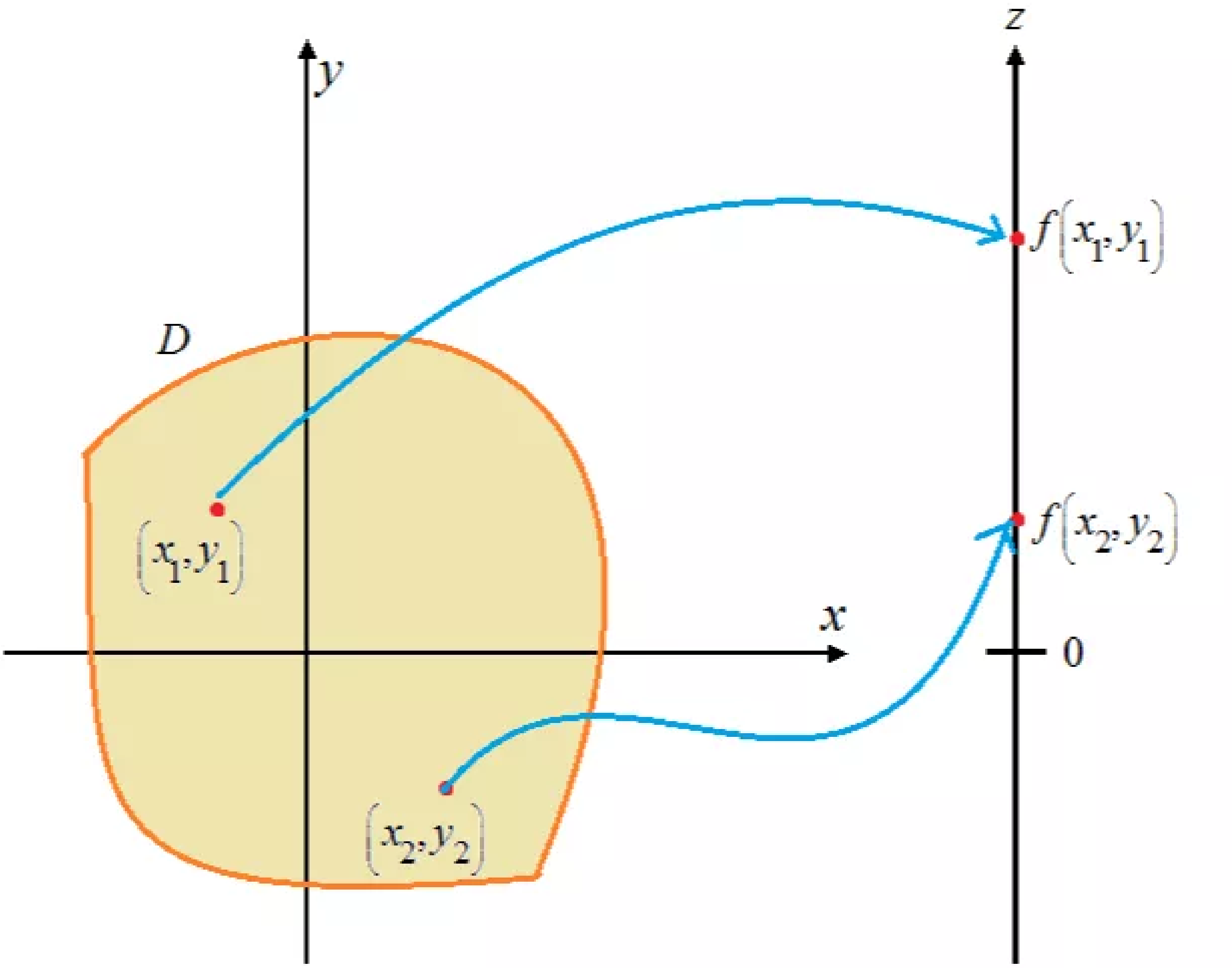
This means that $\{f(x, y) | (x, y) \in D\}$.

Here, the variables x and y are independent variables and z is the dependent variables.

(b)

One way of visualizing a function *f* is by an arrow diagram, where the domain *D* is represented as a subset of the *xy*-plane and the range is a set of numbers on a real line, shown as a *z*-axis.

Observe the below arrow diagram:



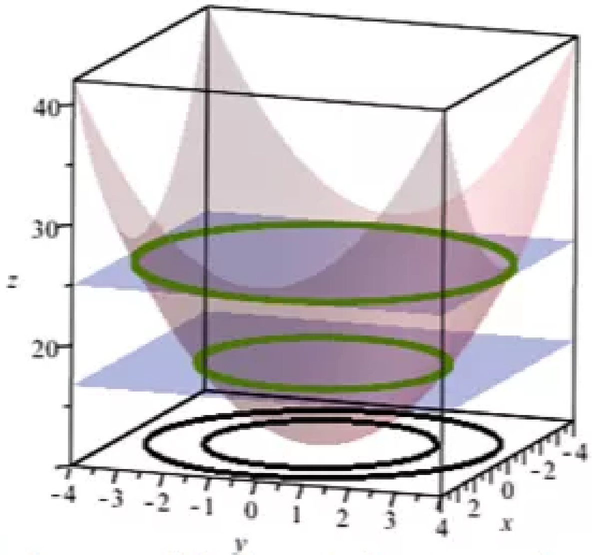
Another method of visualizing a function is by the use of graphs.

The graph of f is the set of all point (x, y, z) in $\mathbb{R}^3$ such that $z = f(x, y)$ and (x, y) is in D.

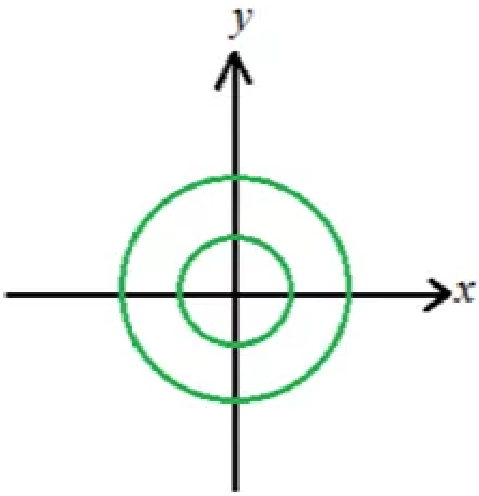
Observe the below graph:

The third method used to visualize a function is level curves. The level curves of f are the curves with equation $f(x, y) = k$, where k is a constant in the range of f.

Observe the below graph:



level curves lifted up to form a surface



level curves

Answer 1E.

Consider the function,

$$f(x,y) = \ln(x + y + 1).$$

The domain of the function f of two variables is the set of all pairs (x, y) for which the given expression is a well-defined real number.

In general, logarithmic functions are undefined for 0 and negative values of x.

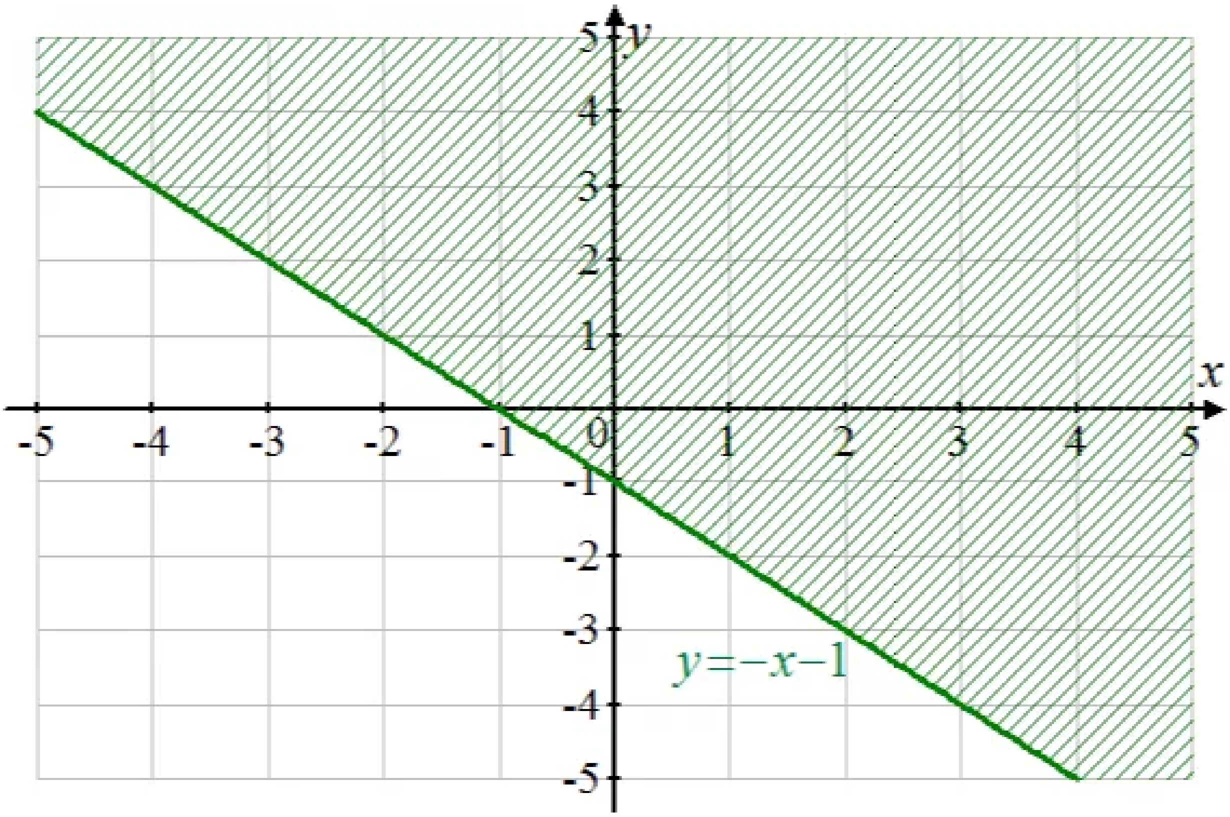
So, the expression for f makes sense if the expression inside the parentheses greater than 0.

Thus, the domain of f is

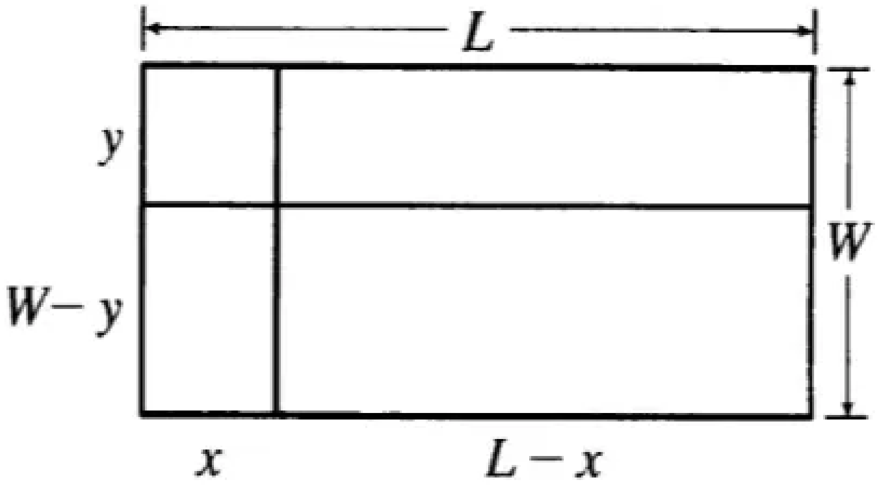
$$D = \{(x,y) \mid x + y + 1 > 0\}$$
$$= \boxed{\{(x,y) \mid y > -x - 1\}}$$

The inequality $y > -x - 1$ describes the points that lie above the line $y = -x - 1$.

The graph of the domain of the function $f(x,y)=\ln(x+y+1)$ as shown below:



Answer 1P.



The areas of the smaller rectangles are A1 = xy, A2 = (L - x)y, A3 = (L - x)(W - y), and

A4 = x(W - y). For 0 ≤ x ≤ L, 0 ≤ y ≤ W, let

$$f\left(x,y\right)= {A_1}^2+{A_2}^2+{A_3}^2+{A_4}^2$$

$$= x^2y^2+(L-x)^2y^2+(L-x)^2(W-y)^2+x^2(W-y)^2$$

$$= \left[x^2+(L-x)^2\right]\left[y^2+(W-y)^2\right]$$

Then we need to find the maximum and minimum values of f(x,y). So here we have

$f_x\left(x,y\right)=\left[2x-2(L-x)\right]\left[y^2+(W-y)^2\right]=0 \Rightarrow 4x-2L=0$ or $x=\frac{1}{2}L$, an

$f_y\left(x,y\right)=\left[x^2+(L-x)^2\right]\left[2y-2(W-y)\right]=0 \Rightarrow 4y-2W=0$ or $y=\frac{W}{2}$.

We also have that

$$f_{xx} = 4\left[y^2 + (W - y)^2\right], f_{yy} = 4\left[x^2 + (L - x)^2\right], \text{ and } f_{xy} = \left(4x - 2L\right)\left(4y - 2\right.$$

So then from this we know that

$$D = 16\left[y^2 + (W - y)^2\right]\left[x^2 + (L - x)^2\right] - (4x - 2L)^2 (4y - 2W)^2$$

Therefore, when x = (1/2)L and y = (1/2)W, D > 0 and fxx = 2W2 > 0. Therefore a minimum of f occurs at $\left(\frac{1}{2}L, \frac{1}{2}W\right)$ and this minimum value is $f\left(\frac{1}{2}L, \frac{1}{2}W\right) = \frac{1}{4}L^2 W^2$.

There are no other critical points, so the maximum must occur on the boundary.

Now along the width of the rectangle let g(y) = f(0,y) = f(L,y) = L2[y2 + (W - y)2], 0 ≤ y ≤ W. Then g'(y) = L2[2y - 2(W - y)] = 0 ↔ y = (1/2)W and

g(1/2) = (1/2)L2W2.

Checking the endpoints, we get g(0) = g(W) = L2W2. Along the length of the rectangle let h(x) = f(x,0) = f(x,W) = W2[x2 + (L - x)2], 0 ≤ x ≤ L.

By symmetry, h'(x) = 0 ↔ x = (1/2)L and h((1/2)L) = (1/2)L2W2. At the endpoints we have h(0) = h(L) = L2W2.

Therefore L2W2 is the maximum value of f. This maximum value of f occurs when the "cutting" lines correspond to sides of the rectangle.

Answer 1TFQ.

Given $f_y(a, b) = \lim_{h \to 0} \frac{f(a, b+h) - f(a, b)}{h}$.

Let $y = b + h$. Then, as $h \to 0$, $y \to b$.

Thus, we can write as

$$f_y(a, b) = \lim_{y \to b} \frac{f(a, y) - f(a, b)}{y - b}.$$

Therefore the given statement is **true**.

Answer 2CC.

(a) A function of three variable is a rule that assigns to each ordered pair of real numbers (x, y, z) in a domain $D \subset \mathbb{R}^3$ a unique real number denoted by $f(x, y, z)$. The set D is the domain of f and its range is the set of values that f takes on.

(b) A function f of three variables lies in a four-dimensional space. Such functions can be visualized by examining its level surfaces, with equation $f(x,y,z) = k$, where k is a constant. If the point (x, y, z) moves along a level surface, the value of $f(x, y, z)$ remains fixed.

Answer 2E.

Consider the function:

$$f(x,y) = \sqrt{4 - x^2 - y^2} + \sqrt{1 - x^2}$$

The domain of the function f of two variables is the set of all pairs (x, y) for which the given expression is a well-defined.

The expression for f is well defined if the expression under the square root is nonnegative.

Since the function has two expressions in the square root and both expression should be greater than zero fur the function as a whole to be well defined.

So, examine two conditions to determine the domain of the function.

First condition is given by:

$$\begin{aligned} 4 - x^2 - y^2 &\geq 0 \\ -x^2 - y^2 &\geq -4 \\ x^2 + y^2 &\leq 4 \end{aligned}$$

This indicates a disk with center $(0,0)$ and radius 2.

Second condition is:

$$\begin{aligned} 1 - x^2 &\geq 0 \\ 1 &\geq x^2 \\ x^2 &\leq 1 \\ |x| &\leq 1 \end{aligned}$$

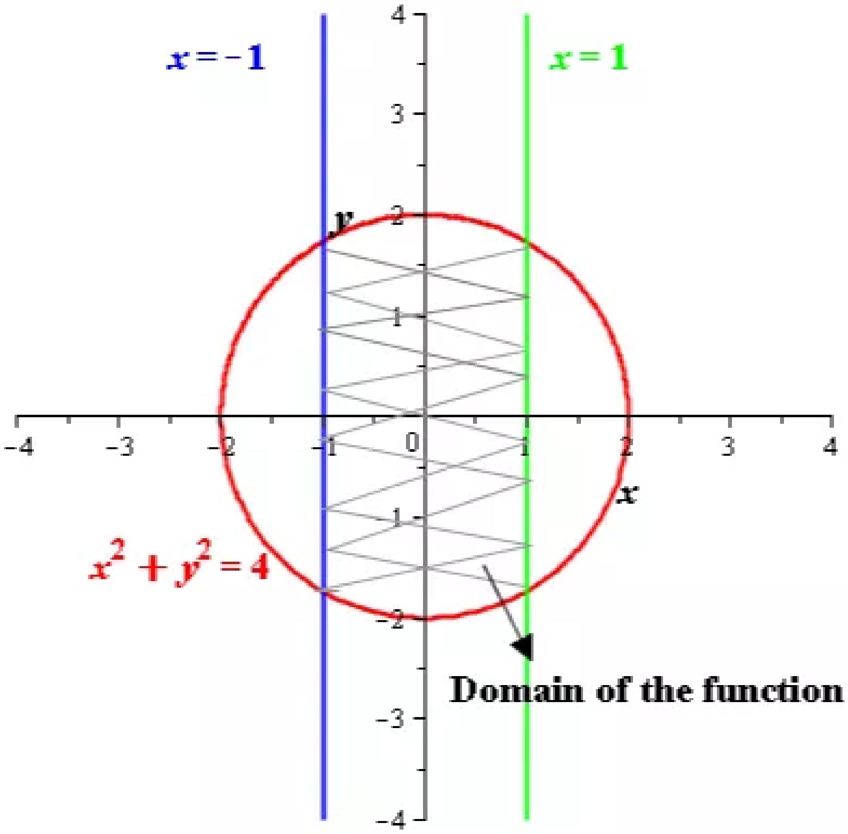
This gives:

$$-1 \leq x \leq 1$$

So, the domain of the given function is:

$$\boxed{D = \{(x, y) \mid x^2 + y^2 \leq 4, -1 \leq x \leq 1\}}.$$

The graph of the domain of the function $f(x,y)=\sqrt{4-x^2-y^2}+\sqrt{1-x^2}$ as shown below:



Answer 2P.

(a)

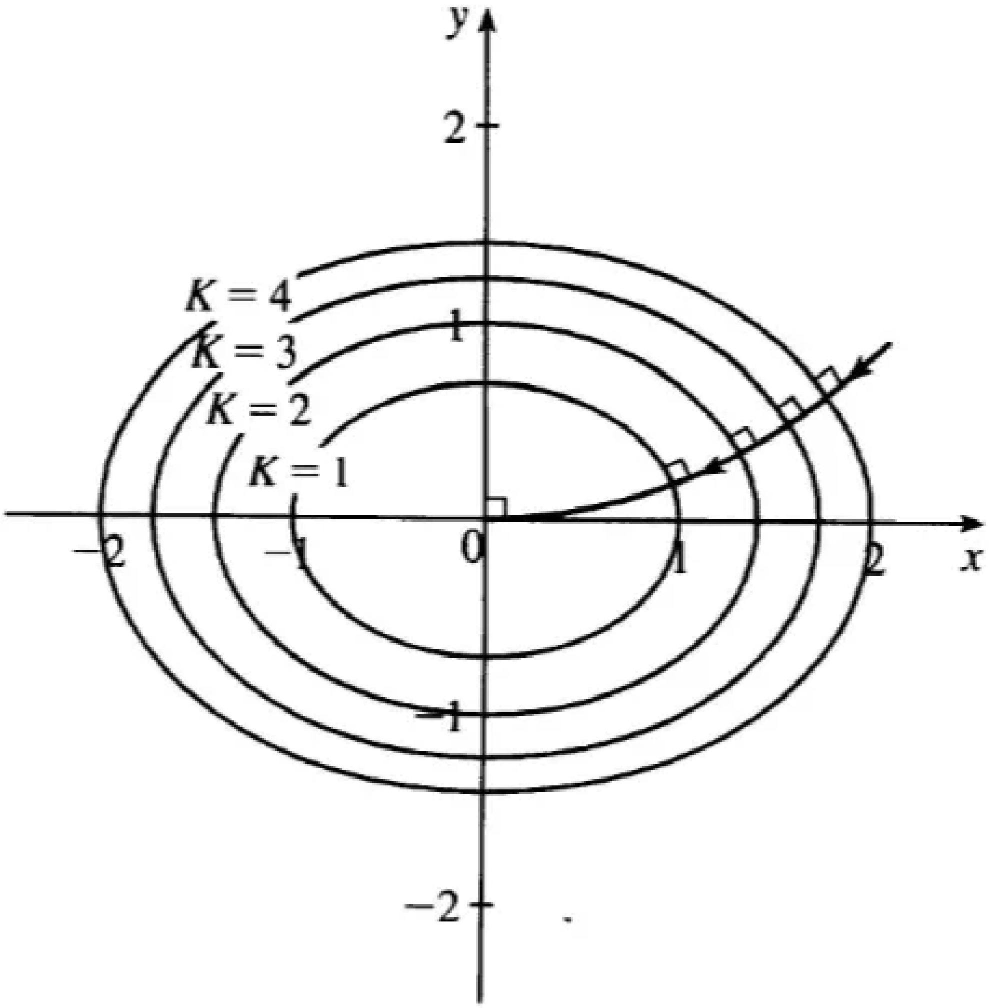
The level curves of the function $C(x,y)=e^{-(x^2+2y^2)/10^4}$ are actually the curves

$e^{-(x^2+2y^2)/10^4}=k$, where k is a positive constant. The equation is equivalent to

$x^2+2y^2=K \Rightarrow \frac{x^2}{(\sqrt{K})^2}+\frac{y^2}{(\sqrt{K/2})^2}=1$, where $K=-10^4\ln k$, which is a family of ellipses.

We sketch level curves for K = 1,2,3, and 4. If the shark always swims in the direction of maximum increase of blood concentration, its direction at any point would coincide with the gradient vector.

Then we know the shark's path is perpendicular to the level curves it intersects. We sketch one example of such a path.


y
2
1
K = 4
K = 3
K = 2
K = 1
−2
−1
0
1
2
x
−1
−2

(b)

$\nabla C = -\frac{2}{10^4} e^{-(x^2+2y^2)/10^4}\left(xi + 2yj\right)$, and ∇C points in the direction of most rapid increase in concentration, which means ∇C is tangent to the most rapid increase curve.

If r(t) = x(t) **i** + y(t) **j** is a parametrization of the most rapid increase curve, then

$\frac{dr}{dt} = \frac{dx}{dt} i + \frac{dy}{dt} j$ is tangent to the curve, so then we have that

$\frac{dr}{dt} = \lambda \nabla C \Rightarrow \frac{dx}{dt} = \lambda\left[-\frac{2}{10^4} e^{-(x^2+2y^2)/10^4}\right]x$ and $\frac{dy}{dt} = \lambda\left[-\frac{2}{10^4} e^{-(x^2+2y^2)/1}\right.$

Therefore, we get

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = 2\frac{y}{x} \Rightarrow \frac{dy}{y} = 2\frac{dx}{x} \Rightarrow \ln|y| = 2\ln|x|$ so that $y = kx^2$ for some constant k. But we know that $y\left(x_0\right) = y_0 \Rightarrow y_0 = kx_0^{\ 2} \Rightarrow k = \frac{y_0}{x_0^{\ 2}}$.

Therefore, the path the shark will follow is along the parabola $y = y_0\left(\frac{x}{x_0}\right)^2$.

Answer 2TFQ.

To determine that, whether the following statement is true or false:

"There exists a function f with continuous second-order partial derivatives such that

$f_x(x,y)=x+y^2$, and $f_y(x,y)=x-y^2$"

If there is continuous second-order partial derivatives for f then,

$$\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial^2 f}{\partial y\partial x}$$

Here $f_x(x,y)=x+y^2$

That is,

$$\begin{aligned}\frac{\partial f}{\partial x}&=x+y^2\\ \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)&=\frac{\partial}{\partial y}(x+y^2)\\ \frac{\partial^2 f}{\partial y\partial x}&=0+2y\\ &=2y\end{aligned}$$

And,

$f_y(x,y)=x-y^2$

That is,

$$\begin{aligned}\frac{\partial f}{\partial y}&=x-y^2\\ \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)&=\frac{\partial}{\partial x}(x-y^2)\\ \frac{\partial^2 f}{\partial x\partial y}&=1-0\\ &=1\end{aligned}$$

Therefore, $\frac{\partial^2 f}{\partial x\partial y}\neq\frac{\partial^2 f}{\partial y\partial x}$

So, there is no f with continuous second-order partial derivatives such that

$f_x(x,y)=x+y^2$, and $f_y(x,y)=x-y^2$

Hence the given statement is $\boxed{\text{false}}$

Answer 3CC.

(a) If f is a function of two variable whose domain D includes points arbitrarily close to (a, b), then we can say that $\lim_{(x,y)\to(a,b)} f(x,y) = L$.

This means as (x, y) approaches (a, b), $f(x, y)$ tends to L.

(b) If $f(x,y) \to L_1$ as $(x,y) \to (a,b)$ along a path C_1 and $f(x,y) \to L_2$ as $(x,y) \to (a,b)$ along a path C_2, where $L_1 \neq L_2$, then $\lim_{(x,y)\to(a,b)} f(x,y)$ does not exist.

Answer 3E.

Consider the following function:

$$f(x,y) = 1 - y^2$$

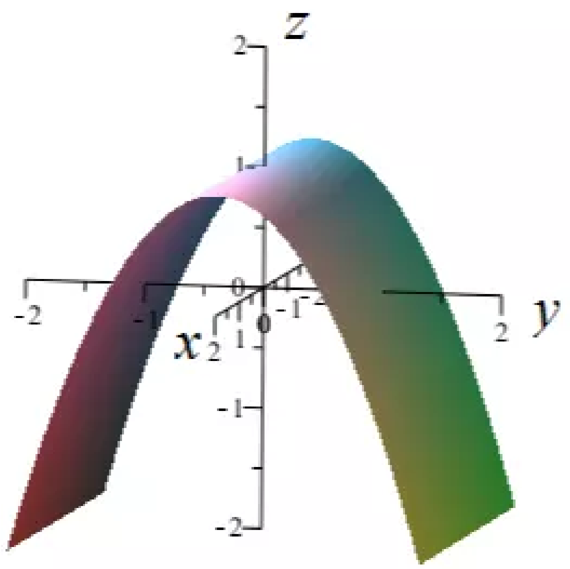
Rewrite as:

$$z = 1 - y^2$$

Where z is a function of x & y.

This is a function of three variables so the graph of this function will be a 3D graph.

The maple graph of $f(x,y) = 1 - y^2$ is as follows:



Answer 3P.

(a)

The area of the trapezoid is $\frac{1}{2}h\left(b_1 + b_2\right)$, where h is the height and b1, b2 are the lengths of the bases. From the figure in the text, we see that h = x sin θ, b1 = w - 2x, and

b2 = w - 2x + 2x cos θ. Therefore the cross-sectional area of the rain gutter is

$$A\left(x,\theta\right) = \frac{1}{2}x\sin\theta\left[(w-2x)+(w-2x+2x\cos\theta)\right] = \left(x\sin\theta\right)\left(w-2x+x\cos\theta\right)$$

$$= \text{wx}\sin\theta - 2x^2\sin\theta + x^2\sin\theta\cos\theta,\ 0 < x \le \frac{1}{2}w,\ 0 < \theta \le \frac{\pi}{2}$$

We now look for the critical points of A: ∂A/∂x = w sin θ - 4x sin θ + 2x sin θ cos θ and

∂A/∂θ = wx cos θ - 2x2 cos θ + x2(cos2θ - sin2θ), so ∂A/∂x = 0 ↔ sin θ(w - 4x + 2x cos θ) = 0 ↔ cos θ = (4x - w)/2x = 2 - (w/2x).

If, in addition, ∂A/∂θ = 0, then we have

$$0 = \text{wx}\cos\theta - 2x^2\cos\theta + x^2\left(2\cos^2\theta - 1\right)$$

$$= \text{wx}\left(2 - \frac{w}{2x}\right) - 2x^2\left(2 - \frac{w}{2x}\right) + x^2\left[2\left(2 - \frac{w}{2x}\right)^2 - 1\right]$$

$$= 2\text{wx} - \frac{1}{2}w^2 - 4x^2 + \text{wx} + x^2\left[8 - \frac{4w}{x} + \frac{w^2}{2x^2} - 1\right] = -\text{wx} + 3x^2 = x\left(3x - \right.$$

Since x > 0, we must have x = (1/3)w, in which case cos θ = 1/2, so θ = π/3, sin θ = √3/2, k = √3/6 w, b1 = (1/3)w, b2 = (2/3)w, and A = √3/12 w2.

We can argue from the physical nature of this problem that we have found a local maximum of A. Now checking the boundary of A, we let

$$g(\theta) = A\left(\frac{w}{2}, \theta\right) = \frac{1}{2}w^2\sin\theta - \frac{1}{2}w^2\sin\theta + \frac{1}{4}w^2\sin\theta\cos\theta = \frac{1}{8}w^2\sin 2\theta,\ 0 <$$

So clearly g is maximized when sin 2θ = 1 in which case A = (1/8)w2. Also along the line

θ = π/2, we let

$$h(x) = A\left(x, \frac{\pi}{2}\right) = wx - 2x^2,\ 0 < x < \frac{1}{2}w \Rightarrow h'(x) = w - 4x = 0 \Leftrightarrow x = \frac{1}{4}$$

$h\left(\frac{1}{4}w\right) = w\left(\frac{1}{4}w\right) - 2\left(\frac{1}{4}w\right)^2 = \frac{1}{8}w^2$. Since $\frac{1}{8}w^2 < \frac{\sqrt{3}}{12}w^2$, we conclude that the local maximum found earlier was an absolute maximum.

(b)

If the metal were bent into a semi-circular gutter of radius r, then we would have w = πr and $A = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi\left(\frac{w}{\pi}\right)^2 = \frac{w^2}{2\pi}$. Since $\frac{w^2}{2\pi} > \frac{\sqrt{3}w^2}{12}$,

it would be better to bend the metal into a gutter with a semi-circular cross-section.

Answer 3TFQ.

We know that

$$f_{xy} = (f_x)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$$

The given statement is **false**.

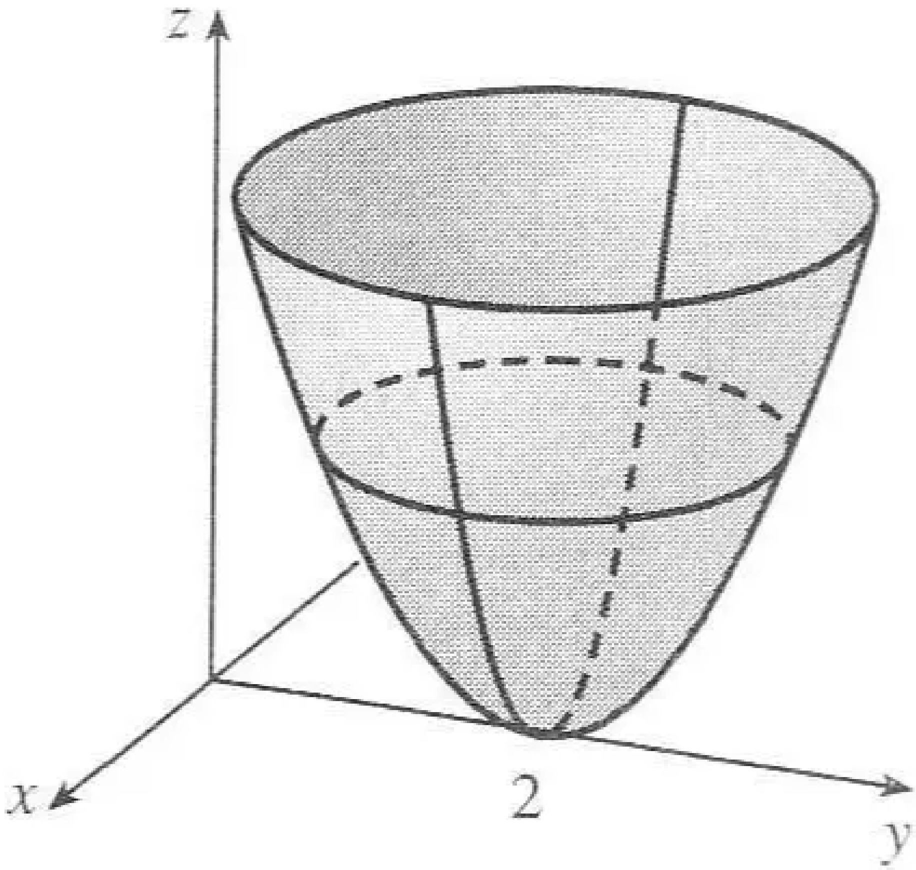
Answer 4CC.

(a) A function f of two variables is called continuous at (a, b) if $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$. We say f is continuous on D if f is continuous at every point (a, b) in D.

(b) The graph of a continuous function has no gaps or breaks in the curve.

Answer 4E.

We have $z = f(x,y) = x2 + (y - 2)2$, which is a circular paraboloid with vertex (0,2,0) and axis parallel to the z-axis.



Answer 4P.

Since $\frac{(x+y+z)^r}{x^2+y^2+z^2}$ is a rational function with domain $\{(x, y, z)|(x, y, z) \neq (0, 0, 0)\}$, then f is continuous on R3 if and only if

$$\lim_{(x, y, z)\to(0, 0, 0)} f(x, y, z) = f(0, 0, 0) = 0$$

Recall that $(a+b)^2 \leq 2a^2 + 2b^2$ and a double application of this inequality to

(x + y + z)2 gives $(x+y+z)^2 \leq 4x^2 + 4y^2 + 4z^2 \leq 4\left(x^2+y^2+z^2\right)$. Now for each r, we have

$$\left|(x+y+z)^r\right| = \left(|x+y+z|^2\right)^{r/2} = \left[(x+y+z)^2\right]^{r/2} \leq \left[4\left(x^2+y^2+z^2\right)\right]^{r/2} :$$

Therefore, from this we obtain

$$\left|f(x, y, z) - 0\right| = \left|\frac{(x+y+z)^r}{x^2+y^2+z^2}\right| = \frac{\left|(x+y+z)^r\right|}{x^2+y^2+z^2} \leq 2^r \frac{\left(x^2+y^2+z^2\right)^{r/2}}{x^2+y^2+z^2} = 2^r\left(x^2+y^2\right.$$

Thus, if (r/2) - 1 > 0, or r > 2, then 2r(x2 + y2 + z2)(r/2) - 1 → 0 as (x,y,z) → (0,0,0) and so

$\lim_{(x, y, z)\to(0, 0, 0)} \frac{(x+y+z)^r}{x^2+y^2+z^2} = 0$. Therefore, for r > 2, f is continuous on R3. Now if r ≤ 2, then as (x,y,z) → (0,0,0) along the x-axis, f(x,0,0) = xr/x2 = xr - 2 for x ≠ 0.

So when r = 2, f(x,y,z) → 1 ≠ 0 as (x,y,z) → (0,0,0) along the x-axis and when r < 2 the limit of

f(x,y,z) as (x,y,z) → (0,0,0) along the x-axis doesn't exist and therefore can't be zero. Therefore, for r ≤ 2, f isn't continuous at (0,0,0) and therefore is not continuous on R3.

Answer 4TFQ.

If f is a differentiable function of two variables x and y, then f has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle a, b\rangle$ and

$D_{\mathbf{u}} f(x, y) = f_x(x, y)\cos\theta + f_y(x, y)\sin\theta$.

We have to find $D_{\mathbf{k}} f(x, y, z)$.

This means that $\mathbf{u} = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}$.

Then, $D_{\mathbf{k}} f(x, y, z) = f_x(x, y, z)(0) + f_y(x, y, z)(0) + f_z(x, y, z)(1)$ or

$D_{\mathbf{k}} f(x, y, z) = f_z(x, y, z)(1)$.

Therefore the given statement is **true**.

Answer 5CC.

(a) If f is function of two variables, then the partial derivative of f with respect to x at (a, b) is given by

$$f_x(a, b) = \lim_{h\to 0} \frac{f(a+h, b) - f(a, b)}{h}.$$

Similarly, the partial derivative of f with respect to y at (a, b) is

$$f_y(a, b) = \lim_{h\to 0} \frac{f(a, b+h) - f(a, b)}{h}.$$

(b) Let $z = f(x, y)$ represent a surface S.

If $f(a, b) = c$, then the point $P(a, b, c)$ lies on the surface S.

Let C_1 be the trace of S in the plane $y = b$ and C_2 be the trace of S in the plane $x = a$.

The partial derivatives $f_x(a, b)$ is the slope of the tangent lines at $P(a, b, c)$ to C_1 in the plane $y = b$.

Similarly, $f_y(a, b)$ is the slope of the tangent lines at $P(a, b, c)$ to C_2 in the plane $x = a$.

Also, the partial derivative $f_x(x, y)$ or $\frac{\partial z}{\partial x}$ represents the rate of change of z with respect to x, when y is a constant and $\frac{\partial z}{\partial y}$ represents the rate of change of z with respect to y, when x is a constant.

(c) If $f(x, y)$ is given by a formula, then $f_x(x, y)$ is given by $\frac{\partial f}{\partial x}$ and $f_y(x, y)$ is given by $\frac{\partial f}{\partial y}$.

Answer 5E.

Consider the function:

$$f(x,y)=\sqrt{4x^2+y^2}$$

Write the above function in the form of the level curves.

$$\sqrt{4x^2+y^2}=c$$

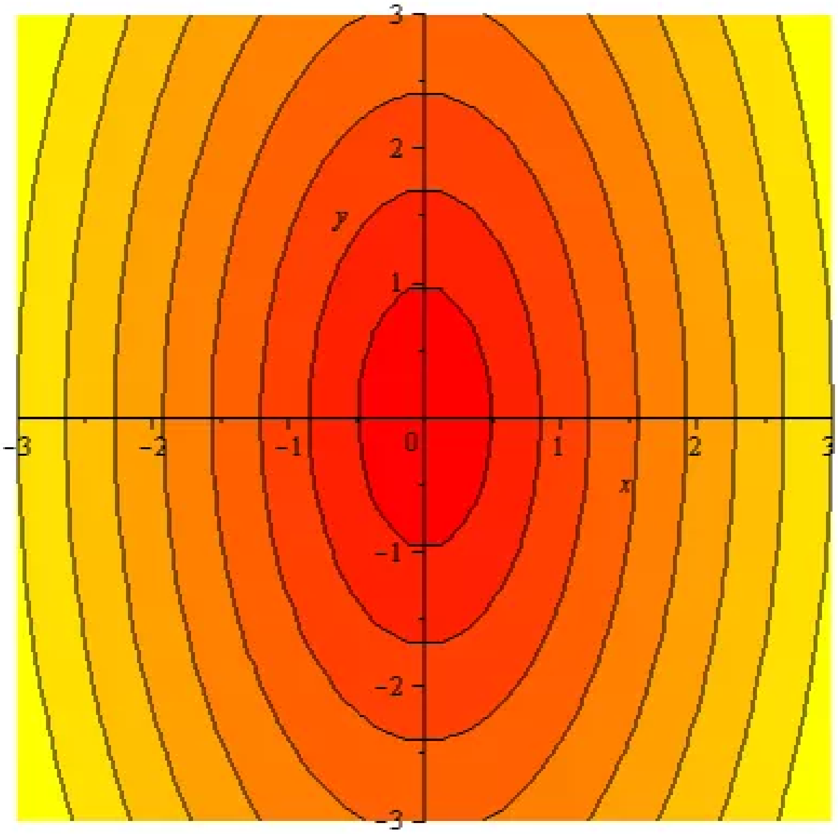
Take square root both sides.

$$\left(\sqrt{4x^2+y^2}\right)^2=c^2$$

$$4x^2+y^2=c^2$$

Thus, the level curve for the function is **ellipse**.

Use Maple to sketch level curves of the function for different values of *c*.



Answer 5P.

Let $g(x, y) = xf\left(\frac{y}{x}\right)$. Then $g_x(x, y) = f\left(\frac{y}{x}\right) + xf'\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right) = f\left(\frac{y}{x}\right) - \frac{y}{x}$

Therefore, the tangent plane at (x_0, y_0, z_0) on the surface has equation

$$z - x_0 f\left(\frac{y_0}{x_0}\right) = \left[f\left(\frac{y_0}{x_0}\right) - y_0 x_0^{-1} f'\left(\frac{y_0}{x_0}\right)\right](x - x_0) + f'\left(\frac{y_0}{x_0}\right)(y - y_0)$$

$$\Rightarrow \left[f\left(\frac{y_0}{x_0}\right) - y_0 x_0^{-1} f'\left(\frac{y_0}{x_0}\right)\right]x + \left[f'\left(\frac{y_0}{x_0}\right)\right]y - z = 0$$

But any plane whose equation is of the form $ax + by + cz = 0$ passes through the origin. Therefore, the origin is the common point of intersection.

Answer 5TFQ.

Given $f(x,y) \to L$ as $(x,y) \to (a,b)$ and $Lim_{(x,y)\to(a,b)} f(x,y) = L$

The limit as (x,y) tends to (a,b) should exist uniquely in every curve being traveled not only on straight lines.

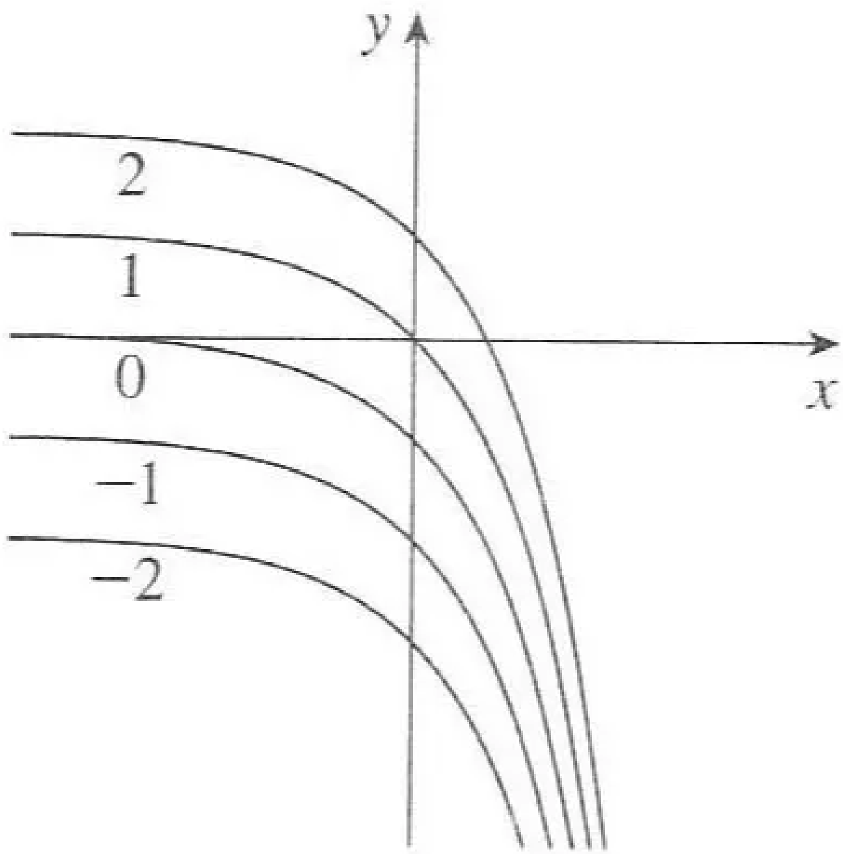
Therefore the given statement is **false**.

Answer 6CC.

The Clairaut's Theorem states that suppose f is defined on a disk D that contains the point (a, b) and if the functions f_{xy} and f_{yx} are both continuous on D, then $f_{xy}(a, b) = f_{yx}(a, b)$.

Answer 6E.

The level curves are ex + y = k or y = -ex + k, which is a family of exponential curves.



Answer 6P.

(a)

At (x1,y1,0) the equations of the tangent planes to z = f(x,y) and z = g(x,y) are

$$P_1 : z - f(x_1, y_1) = f_x(x_1, y_1)(x - x_1) + f_y(x_1, y_1)(y - y_1) \text{ and}$$

$$P_2 : z - g(x_1, y_1) = g_x(x_1, y_1)(x - x_1) + g_y(x_1, y_1)(y - y_1) \text{ respectively.}$$

P1 intersects the xy-plane in the line given by

$$f_x(x_1, y_1)(x - x_1) + f_y(x_1, y_1)(y - y_1) = -f(x_1, y_1),\ z = 0\ ; \text{ and}$$

P2 intersects the xy-plane in the line given by

$$g_x(x_1, y_1)(x - x_1) + g_y(x_1, y_1)(y - y_1) = -g(x_1, y_1),\ z = 0.$$

The point (x2,y2,0) is the point of intersection of these two lines since (x2,y2,0) is the point where the line of intersection of the two tangent planes intersects the xy-plane. Therefore,

(x2,y2) is the solution of the simultaneous equations

$$f_x(x_1, y_1)(x_2 - x_1) + f_y(x_1, y_1)(y_2 - y_1) = -f(x_1, y_1) \text{ and}$$

$$g_x(x_1, y_1)(x_2 - x_1) + g_y(x_1, y_1)(y_2 - y_1) = -g(x_1, y_1)$$

For simplicity, rewrite fx(x1,y1) as fx and similarly fy, gx, gy, f, and g and solve the equations

$$(f_x)(x_2 - x_1) + (f_y)(y_2 - y_1) = -f \text{ and } (g_x)(x_2 - x_1) + (g_y)(y_2 - y_1) = -,$$

$$y_2 - y_1 = \frac{gf_x - fg_x}{g_x f_y - f_x g_y} \text{ or } y_2 = y_1 - \frac{gf_x - fg_x}{f_x g_y - g_x f_y} \text{ and } \left(f_x\right)\left(x_2 - x_1\right) + \frac{(f_y)(gf_x -}{g_x f_y - f}$$

$$x_2 - x_1 = \frac{-f - [(f_y)(gf_x - fg_x)/(g_x f_y - f_x g_y)]}{f_x} = \frac{fg_y - f_y g}{g_x f_y - f_x g_y}$$. Therefore, we have

$$x_2 = x_1 - \frac{fg_y - f_y g}{f_x g_y - g_x f_y}$$.

(b)

Let $f\left(x, y\right) = x^x + y^y - 1000$ and $g\left(x, y\right) = x^y + y^x - 100$.

Then we wish to solve the system of equations f(x,y) = 0, g(x,y) = 0. Recall $\frac{d}{dx}\left[x^x\right] = x^x\left(1 + \ln x\right)$, so then from this we get

$$f_x\left(x, y\right) = x^x\left(1 + \ln x\right),\ f_y\left(x, y\right) = y^y\left(1 + \ln y\right),\ g_x\left(x, y\right) = yx^{y-1} + y^x \ln$$

Looking at the graph given in the text, we estimate the first point of intersection of the curves, and thus the solution to the system, to be approximately (2.5, 4.5).

Then following the method in part (a), x1 = 2.5, y1 = 4.5 and

$$x_2 = 2.5 - \frac{f(2.5, 4.5)g_y(2.5, 4.5) - f_y(2.5, 4.5)g(2.5, 4.5)}{f_x(2.5, 4.5)g_y(2.5, 4.5) - f_y(2.5, 4.5)g_x(2.5, 4.5)} \approx 2.447674117$$

$$y_2 = 4.5 - \frac{f_x(2.5, 4.5)g(2.5, 4.5) - f(2.5, 4.5)g_x(2.5, 4.5)}{f_x(2.5, 4.5)g_y(2.5, 4.5) - f_y(2.5, 4.5)g_x(2.5, 4.5)} \approx 4.555657467$$

Continuing this procedure, we arrive at the following values.

x1 = 2.5	y1 = 2.5
x2 = 2.447674117	y2 = 4.555657467
x3 = 2.449614877	y3 = 4.551969333
x4 = 2.449624628	y4 = 4.551951420
x5 = 2.449624628	y5 = 4.551951420

Therefore, to six decimal places, the point of intersection is (2.449625, 4.551951). The second point of intersection can be found similarly, or, by symmetry it is approximately

(4.551951, 2.449625).

Answer 6TFQ.

If the partial derivatives $f_x(a,b)$ and $f_y(a,b)$ exists near (a,b) and are continuous at (a,b), then f is differentiable at (a,b).

Therefore the given statement is **false**.

Answer 7CC.

(a) Assume f has continuous partial derivatives.

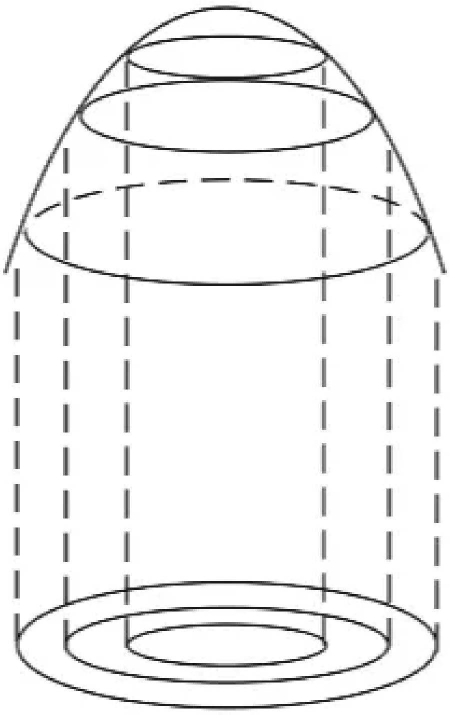
As equation of the tangent plane to the surface $z = f(x, y)$ at the point $P(x_0, y_0, z_0)$ is

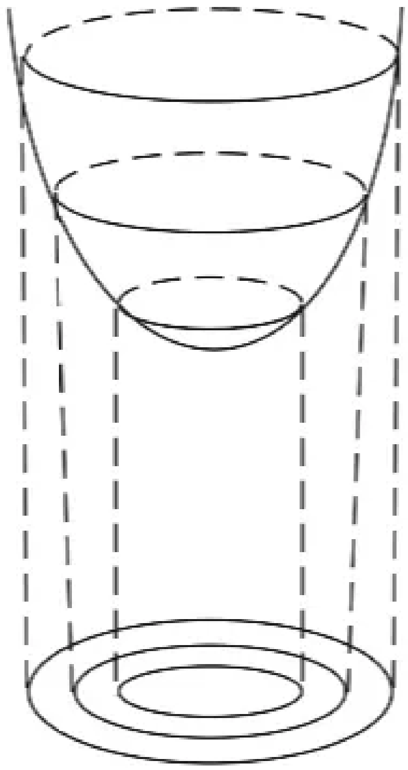
$z - z_0 = f_x(x - x_0) + f_y(y - y_0)$.

(b) The tangent plane to the level surface $F(x, y, z) = k$ at $P(x_0, y_0, z_0)$ is a plane that passes through P and has normal vector $\nabla F(x_0, y_0, z_0)$ and is given by the equation

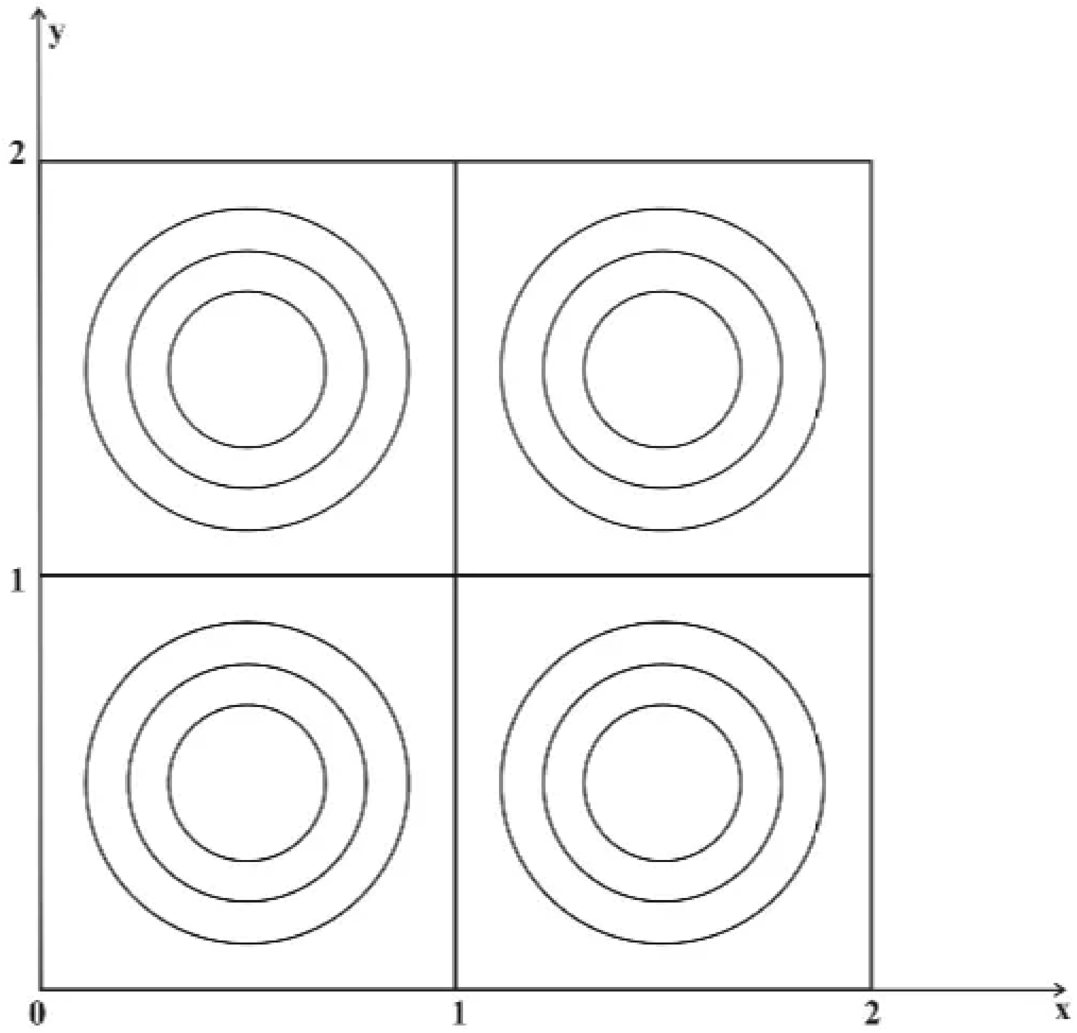
$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$

Answer 7E.





The contour map of function is: -



y
2
1
0
1
2
x

Answer 7P.

Given that the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is to enclose the circle $x^2+y^2=2y$

For the circle

$$x^2+y^2=2y$$

$$x^2+y^2-2y=0$$

$$x^2+y^2-2y+1=0+1$$

$$x^2+(y-1)^2=1$$

We have a circle with center (0,1) and radius 1, so the circle lies above the x-axis, and we can conclude that the ellipse will intersect the circle for only one value of y.

The value of y must satisfy both equations.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad (1)$$

$$x^2+(y-1)^2=1$$

$$\rightarrow x^2=1-(y-1)^2 \quad (2)$$

Substituting (2) into (1)

$$\frac{1-(y-1)^2}{a^2}+\frac{y^2}{b^2}=1$$

$$\frac{2y - y^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{-y^2}{a^2} + \frac{y^2}{b^2} + \frac{2y}{a^2} = 1$$

$$y^2\left(\frac{a^2 - b^2}{a^2b^2}\right) + \frac{2y}{a^2} - 1 = 0$$

From algebra we know that if the discriminant is zero, there is exactly one distinct root, and that root is a real number.

$$D = \left(\frac{2}{a^2}\right)^2 - 4\left(\frac{a^2 - b^2}{a^2b^2}\right)\left(-1\right)$$

$$\left(\frac{2}{a^2}\right)^2 + 4\left(\frac{a^2 - b^2}{a^2b^2}\right) = 0$$

$$\frac{4}{a^4} + \frac{4\left(a^2 - b^2\right)}{a^2b^2} = 0$$

$$\frac{1}{a^2} + \frac{a^2 - b^2}{b^2} = 0$$

$$\frac{b^2 + a^4 - a^2b^2}{a^2b^2} = 0$$

$$b^2 - a^2b^2 + a^4 = 0$$

The area of the ellipse is given by A = πab and we use g(a,b) = $b^2 - a^2b^2 + a^4 = 0$

$$\nabla A = \lambda \nabla g$$

$$\pi b = \lambda\left(4a^3 - 2ab^2\right) \quad \rightarrow \lambda = \frac{\pi b}{2a\left(2a^2 - b^2\right)} \text{ (3)}$$

$$\pi a = \lambda\left(2b - 2ba^2\right) \quad \rightarrow \lambda = \frac{\pi a}{2b\left(1 - a^2\right)} \text{ (4)}$$

Since λ = λ

$$\frac{\pi b}{2a\left(2a^2 - b^2\right)} = \frac{\pi a}{2b\left(1 - a^2\right)}$$

$$b^2\left(1 - a^2\right) = a^2\left(2a^2 - b^2\right)$$

$$b^2 - b^2 a^2 = 2a^4 - a^2 b^2$$

$$b^2 = 2a^4$$

Substituting into $b^2 - a^2 b^2 + a^4 = 0$

$$2a^4 - a^2 (2a^4) + a^4 = 0$$

$$3a^4 - 2a^6 = 0$$

$$a^4\left(3 - 2a^2\right) = 0$$

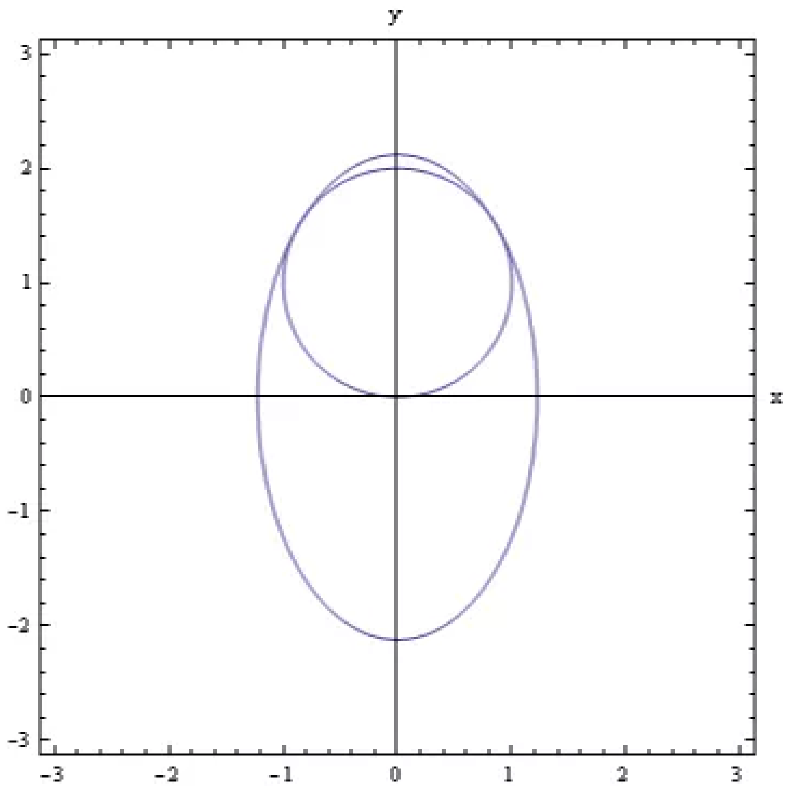
$a^4 = 0$ or $3 - 2a^2 = 0$

Since a > 0 the only solution that we need is

$$a = \sqrt{\frac{3}{2}}$$

substituting into $b^2 = 2a^4$

b = 3/√2



Answer 7TFQ.

If f is a function of two variables x and y, then the gradient of f is the vector function ∇f defined by $\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle$ or $\nabla f(x, y) = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$.

Since f has a local minimum at (a,b), we can say that (a,b) is the critical point.

This means that $f_x(a,b) = 0$ and $f_y(a,b) = 0$.

Then, $\nabla f(a, b) = \langle f_x(a, b), f_y(a, b) \rangle$ or $\nabla f(a, b) = \langle 0, 0 \rangle$.

Therefore the given statement is **true**.

Answer 8CC.

A linear function whose graph is a tangent plane given by

$L(x, y) = f(a, b) + f_x(a, b)(x-a) + f_y(a, b)(y-b)$

is called the linearization of f at (a, b).

The linear approximation or the tangent plane approximation of f at (a, b) is given by

$f(x, y) \approx f(a, b) + f_x(a, b)(x-a) + f_y(a, b)(y-b)$.

The linear approximation is the tangent plane approximation of f at (a, b).

Answer 8E.

When the three dimensional images are not fully able to visualize, then contour map is an important tool that depict the function with a two dimensional input and a one dimensional output.

Consider the contour map of the function f given in the problem,

Observe that the level curves of the function centered at y- axis and radius of the circle decreases as its centered moves near the origin.

So the level curve satisfies the equation,

$$x^2+\left(y-\frac{1}{c}\right)^2=\frac{1}{c^2}$$

Or,

$$x^2+y^2-\frac{2}{c}y+\frac{1}{c^2}=\frac{1}{c^2}$$

Simplify this equation,

$$x^2+y^2=\frac{2}{c}y$$

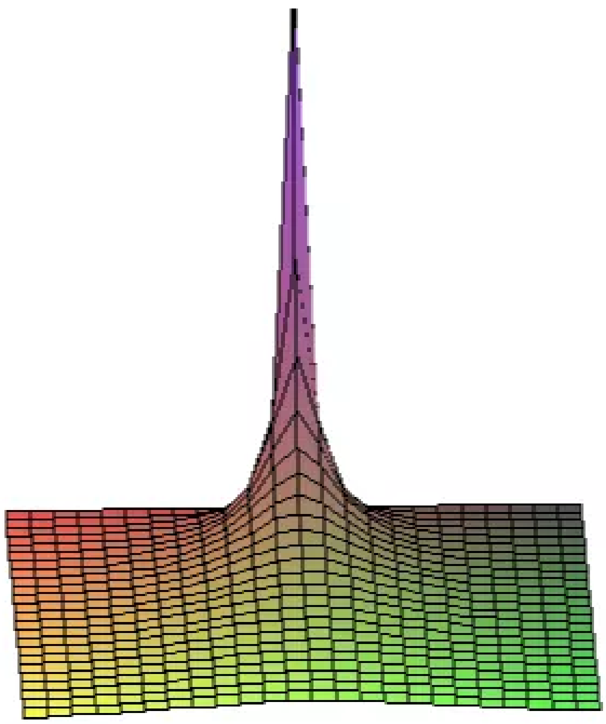
Or,

$$\frac{2y}{x^2+y^2}=c$$

Hence, the required function is,

$$\boxed{f(x,y)=\frac{2y}{x^2+y^2}}$$

The rough sketch of the graph of the function is shown below,



Answer 8P.

Among all planes that are tangent to the surface $xy^2z^2 = 1$, we need to find the ones that are farthest from the origin.

We can write the equation of the tangent plane to the level surface F(x,y,z) = k at the point (x_0, y_0, z_0) as

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$$

The tangent plane to the surface $xy^2z^2 = 1$, at the point (x_0, y_0, z_0) is

$$y_0^2 z_0^2 \left(x - x_0\right) + 2x_0 y_0 z_0^2 \left(y - y_0\right) + 2x_0 y_0^2 z_0 \left(z - z_0\right) = 0$$

$$y_0^2 z_0^2 x + 2x_0 y_0 z_0^2 y + 2x_0 y_0^2 z_0 z = 5x_0 y_0^2 z_0^2$$

Since $xy^2z^2 = 1$ then $x_0 y_0^2 z_0^2 = 1$

$$y_0^2 z_0^2 x + 2x_0 y_0 z_0^2 y + 2x_0 y_0^2 z_0 z = 5$$

The distance from (0,0,0) to this tangent plane is given by

$$D\left(x_0, y_0, z_0\right) = \frac{\left|5x_0 y_0^2 z_0^2\right|}{\sqrt{\left(y_0^2 z_0^2\right)^2 + \left(2x_0 y_0 z_0^2\right)^2 + \left(2x_0 y_0^2 z_0\right)^2}}$$

$$D\left(x_0, y_0, z_0\right) = \frac{5}{\sqrt{\left(y_0^2 z_0^2\right)^2 + \left(2x_0 y_0 z_0^2\right)^2 + \left(2x_0 y_0^2 z_0\right)^2}}$$

When D is a maximum, then D2 is a maximum and $\nabla D^2 = 0$

Now we use

$$D^2\left(x_0,\, y_0,\, z_0\right) = \frac{25}{\left(y_0^2 z_0^2\right)^2 + \left(4x_0^2 y_0^2 z_0^4\right) + \left(4x_0^2 y_0^4 z_0^2\right)}$$

$$D^2\left(x_0,\, y_0,\, z_0\right) = \frac{25}{\left(y_0^2 z_0^2\right)^2 + \left(y_0^2 z_0^2\right)\left(4x_0^2\ z_0^2\right) + \left(y_0^2 z_0^2\right)\left(4x_0^2 y_0^2\ \right)}$$

$$D^2\left(x_0,\, y_0,\, z_0\right) = \frac{25}{\left(y_0^2 z_0^2\right)\left\{y_0^2 z_0^2 + 4x_0^2 z_0^2\ + 4x_0^2 y_0^2\ \right\}}$$

$$x_0\, y_0{}^2\, z_0{}^2 = 1\ \longrightarrow z_0^2 = \frac{1}{x_0 y_0^2}$$

$$D^2\left(x_0,\, y_0,\, z_0\right) = \frac{25}{\frac{1}{x_0}\left[\frac{1}{x_0} + \frac{4x_0}{y_0^2} + 4x_0^2 y_0^2\right]}$$

$$D^2\left(x_0,\, y_0,\, z_0\right) = \frac{25\ x_0^2 y_0^2}{1 + 4x_0^2 + 4x_0^3 y_0^4}$$

$$\nabla D^2 = 0\ \longrightarrow\ D^2_{x_0} = 0 \text{ and } D^2_{y_0} = 0$$

$$0\ = \frac{50x_0 y_0^2\left(y_0{}^2 + 4x_0^2 + 4x_0^3 y_0^4\right) - \left(8x_0 + 12x_0^2 y_0^4\right)\left(25x_0^2 y_0^2\right)}{\left(y_0^2 + 4x_0^2 + 4x_0^3 y_0^4\right)^2}$$

$$x_0\, y_0{}^4\ - 2x_0^4 y_0^6 = 0$$

$$x_0 y_0{}^4\left(1-2x_0^3 y_0^2\right)=0$$

$$x_0^3 y_0^2 = \frac{1}{2}$$

$$D^2_{y_0} = \frac{50y_0x_0^2\left(y_0{}^2+4x_0^2+4x_0^3y_0^4\right)-\left(2y_0+16x_0^3y_0^3\right)\left(25x_0^2y_0^2\right)}{\left(y_0^2+4x_0^2+4x_0^3y_0^4\right)^2}$$

$$0 = \frac{50y_0x_0^2\left(y_0{}^2+4x_0^2+4x_0^3y_0^4\right)-\left(2y_0+16x_0^3y_0^3\right)\left(25x_0^2y_0^2\right)}{\left(y_0^2+4x_0^2+4x_0^3y_0^4\right)^2}$$

$$4x^4{}_0\, y_0 \; -4x_0^5 y_0^5 = 0$$

$$4\,x_0{}^4 y_0 \;\left(1-x_0 y_0^4\right)=0$$

$$x_0 y_0^4 = 1$$

Substituting $x_0 = \frac{1}{y_0^4}$ into $x_0^3 y_0^2 = \frac{1}{2}$

$$\frac{1}{y_0^{10}} = \frac{1}{2}$$

$$y_0^{10} = 2$$

$y_0 = \pm\sqrt[10]{2}$ and $x_0 = \frac{1}{\sqrt[5]{4}}$

Substituting into $x_0 y_0{}^2 z_0{}^2 = 1$

$$z^2{}_0 = \frac{1}{2^{-2/5} \cdot 2^{2/10}}$$

$$z_0^2 = \frac{1}{2^{-1/5}}$$

$$z_0 = \pm \frac{1}{2^{-1/10}}$$

$$z_0 = \pm \sqrt[10]{2}$$

So, the tangent planes that are farthest from the origin are at the four points

$\left(2^{-2/5}, \pm 2^{1/10}, \pm 2^{1/10}\right)$, the minimum distance occurs when $x_0 = 0$ or $y_0 = 0$, that is when D = 0

Substituting the values $\left(2^{-2/5}, \pm 2^{1/10}, \pm 2^{1/10}\right)$ into

$$y_0{}^2 z_0{}^2 x \ + 2x_0 y_0 z_0{}^2 y \ + 2x_0 y_0{}^2 z_0 z \ = 5$$

The equations are

$$2^{2/5} x + 2^{9/10} y + 2^{9/10} z = 5$$

$$2^{2/5} x - 2^{9/10} y - 2^{9/10} z = 5$$

$$2^{2/5} x - 2^{9/10} y - 2^{9/10} z = 5$$

$$2^{2/5} x + 2^{9/10} y - 2^{9/10} z = 5$$

$$2^{2/5} x - 2^{9/10} y + 2^{9/10} z = 5$$

Answer 8TFQ.

We know that for a function f of two variables if $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$, then f is continuous at (a,b). But, it is also possible that f is discontinuous at some interval.
Therefore, there are cases in which the given statement does not hold true.
Therefore the given statement is **false**.

Answer 9CC.

(a) If $z = f(x,y)$ then f is differentiable at (a, b) if Δz can be expressed in the form $\Delta z = f_x(a,b)\Delta x + f_y(a,b)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$ where ε_1 and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$.

(b) If the partial derivatives f_x and f_y exist near (a, b) and are continuous at (a, b), then f is differentiable at (a, b).

Answer 9E.

Consider the following limit:

$$\lim_{(x,y)\to(1,1)} \frac{2xy}{x^2+2y^2}$$

To evaluate the above limit note that if the function of two variables whose limit has to be evaluated is defined on (a,b) then the limit can be evaluated by direct substitution.

Since, the given function is defined on $(1,1)$, so apply the limits directly:

$$\lim_{(x,y)\to(1,1)} \frac{2xy}{x^2+2y^2} = \frac{2(1)(1)}{(1)^2+2(1)^2}$$
$$= \frac{2}{1+2}$$
$$= \frac{2}{3}$$

Therefore, the final answer is:

$$\boxed{\lim_{(x,y)\to(1,1)} \frac{2xy}{x^2+2y^2} = \frac{2}{3}}$$

Answer 9TFQ.

Given $f(x,y)=\ln y$ and $\nabla f(x,y)=\frac{1}{y}$.

If f is a function of two variables x and y, then the gradient of f is the vector function ∇f defined by $\nabla f(x,y)=\langle f_x(x,y), f_y(x,y)\rangle$ or $\nabla f(x,y)=\frac{\partial f}{\partial x}\mathbf{i}+\frac{\partial f}{\partial y}\mathbf{j}$.

Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

$$\frac{\partial f}{\partial x}=\frac{\partial}{\partial x}[\ln y] = 0$$

$$\frac{\partial f}{\partial y}=\frac{\partial}{\partial y}[\ln y] = \frac{1}{y}$$

Thus, we get $\nabla f(x,y)=(0)\mathbf{i}+\frac{1}{y}\mathbf{j}$ or $\nabla f(x,y)=\frac{1}{y}\mathbf{j}$.

The given statement is **false**.

Answer 10CC.

If $z=f(x,y)$ is a differentiable function, then the differentials dx and dy are defined to be two independent variables that can be given values of any real number.
Then, dz is called the total differential and is defined by

$$dz = f_x(x,y)dx + f_y(x,y)dy \text{ or}$$

$$dz = \frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy.$$

Answer 10E.

Consider the function,

$$\lim_{(x,y)\to(0,0)}\frac{2xy}{x^2+2y^2}$$

Show that the limit does not exist.

Let $f(x,y)=\frac{2xy}{x^2+2y^2}$.

First let's approach $(0,0)$ along the *x*-axis.

Then $y=0$ gives

$$f(x,0)=\frac{2x(0)}{x^2+2(0)^2} = \frac{0}{x^2} = 0$$

Thus, $f(x,y)\to 0$ as $(x,y)\to(0,0)$ along the *x*-axis

Now, approach along the *y*-axis by putting $x = 0$.

Then

$$f(0, y) = \frac{2(0)y}{(0)^2 + 2y^2}$$
$$= \frac{0}{2y^2}$$
$$= 0$$

Thus, $f(x, y) \to 0$ as $(x, y) \to (0,0)$ along the *y*-axis

Here, obtained identical limits along the axes, that does not show that the given limit is 0.

The value of the limit must be same from all paths that approach the origin.

Now, approach $(0,0)$ along another line, say $y = x$.

For all $x \neq 0$,

$$f(x,x) = \frac{2x^2}{x^2 + 2x^2}$$
$$= \frac{2x^2}{3x^2}$$
$$= \frac{2}{3}$$

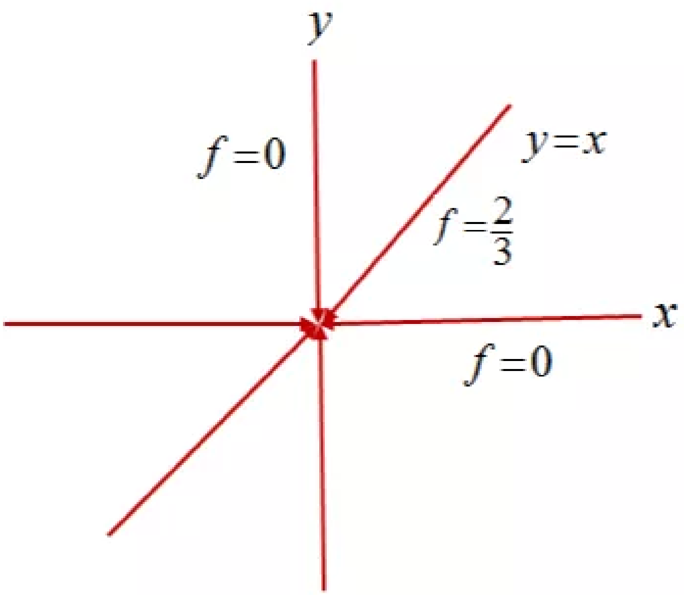
Thus, $f(x, y) \to \frac{2}{3}$ as $(x, y) \to (0,0)$ along $y = x$.

Recall that,

If $f(x,y)\to L_1$ as $(x,y)\to(a,b)$ along a path C_1 and $f(x,y)\to L_2$ as $(x,y)\to(a,b)$ along a path C_2, where $L_1 \neq L_2$, then $\lim_{(x,y)\to(a,b)} f(x,y)$ does not exist.

In present case, different limits are obtained along different paths.

The graphical representation as shown below



Therefore, the limit $\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+2y^2}$ $\boxed{\text{does not exist}}$

Need to sketch the graph of the above function and also find the $\lim_{(x,y)\to(0,0)} f(x,y)$.

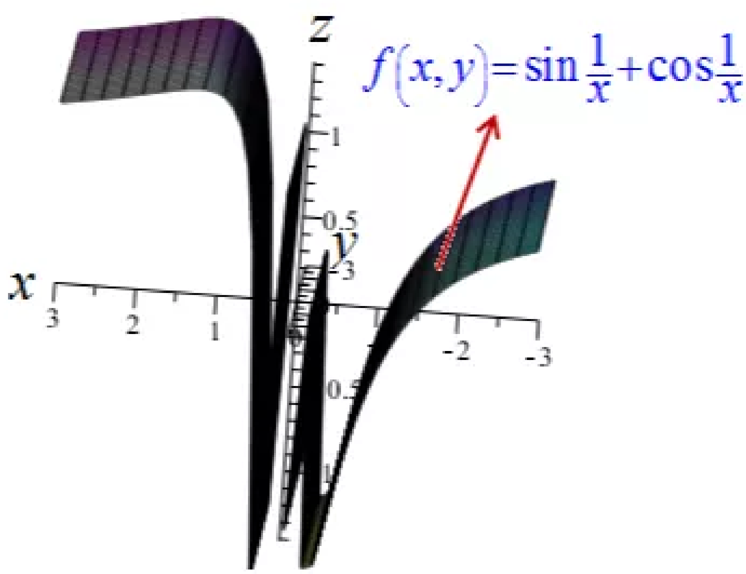
To graph $f(x,y)$, use computer algebraic system.

First load the package, $with(plots)$;

After that use the following command, to plot the function

> $plot3d\left(\sin\left(\frac{1}{x}\right)+\cos\left(\frac{1}{x}\right), x=-3..3, y=-3..3\right);$

The graph of $f(x,y)$ as shown in below:



Definition:

Let f be a function of two variables whose domain D includes points arbitrary close to (a,b).
Then say that the limit of $f(x,y)$ as (x,y) approaches (a,b) is L and write

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

However, from the graph of $f(x,y)$, it seems reasonable that the limit does not exist at 0.

Verify:

First, note that

$$\lim_{(x,y)\to(0,0)}\left(\sin\frac{1}{x}\right)=\infty \text{ and } \lim_{(x,y)\to(0,0)}\left(\cos\frac{1}{x}\right)=\infty$$

So, the limit laws can't be applied to infinite limit because ∞ is not a number
$((\infty+\infty)\text{ can't be defined})$

The limit can be written as,

$$\lim_{(x,y)\to(0,0)}\left(\sin\frac{1}{x}+\cos\frac{1}{x}\right)=\lim_{(x,y)\to(0,0)}\left[\sin\frac{1}{x}\left(1+\cot\frac{1}{x}\right)\right]$$
$$=\infty$$

Because both $\sin\frac{1}{x}$ and $1+\cot\frac{1}{x}$ becomes arbitrarily large

Thus, the values of $f(x,y)$ becomes larger and larger (or "increase without bound) as (x,y) approaches $(0,0)$.

Therefore, $f(x,y)=\sin\frac{1}{x}+\cos\frac{1}{x}$ is $\boxed{\text{does not exists}}$

Answer 10TFQ.

Given that $f_{xx}(2,1)f_{yy}(2,1) < [f_{xy}(2,1)]^2$

We know that if $D(a,b)<0$, then $f(a,b)$ is not a local maximum or minimum and (a,b) is called the saddle point.

Also, $D(a,b)$ is given by $f_{xx}(a,b)f_{yy}(a,b)-[f_{xy}(a,b)]^2$.

It is given that $f_{xx}(2,1)f_{yy}(2,1) < [f_{xy}(2,1)]^2$.

This means that $D(2,1)<0$.

Therefore, $f(2,1)$ is not a local maximum or minimum and has a saddle point at $(2,1)$.

The given statement is **true**.

Answer 11CC.

If $z = f(x,y)$ is a differentiable function of x and y, where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t, then z is a differentiable function of t given by

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

Now, consider $z = f(x,y)$ is a differentiable function of x and y, where $x = g(s,t)$ and $y = h(s,t)$ are both differentiable functions of s and t, then z is a differentiable function of s and t given by

$$\frac{\partial z}{ds} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} \text{ and}$$

$$\frac{\partial z}{dt} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}.$$

Answer 11E.

$$T = T(x,y)$$

(A)

Then $T_x(x,y) = \lim_{h\to 0}\dfrac{T(x+h,y)-T(x,y)}{h}$

Then $T_x(6,4) = \lim_{h\to 0}\dfrac{T(6+h,4)-T(6,4)}{h}$

By taking h = 2 and -2 and using given table,

$$T_x(6,4) = \frac{T(8,4)-T(6,4)}{2} = \frac{86-80}{2} = 3$$

$$T_x(6,4) = \frac{T(4,4)-T(6,4)}{-2} = \frac{72-80}{-2} = 4$$

On averaging we find; $T_x(6,4) = 3.5$

Now $T_y(6,4) = \lim_{k \to 0} \frac{T(6,4+h) - T(6,4)}{h}$

By taking h = 2, -2 and using table

$$T_y(6,4) = \frac{T(6,6) - T(6,4)}{2} = \frac{75-80}{2} = -2.5$$

$$T_y(6,4) = \frac{T(6,2) - T(6,4)}{-2} = \frac{87-80}{-2} = -3.5$$

On averaging we find $T_y(6,4) = -3.0$

Hence $T_x(6,4) = \boxed{3.5^{o} c/m}$

$T_y(6,4) = \boxed{-3.0^{o} c/m}$

(B)

$$\vec{u} = \frac{\hat{i}}{\sqrt{2}} + \frac{\hat{j}}{\sqrt{2}}$$

Then $D_{\vec{u}} T(6,4) = \langle T_x(6,4), T_y(6,4) \rangle \, \vec{u}$

$$= \langle 3.5, -3 \rangle . \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$$

$$= \frac{3.5}{\sqrt{2}} - \frac{3}{\sqrt{2}}$$

$$= \frac{0.5}{\sqrt{2}}$$

$$= \frac{1}{2\sqrt{2}} = 0.35$$

Hence $D_u T(6,4) = \boxed{0.35^{o} c/m}$

(C)

Now $T_{xy}(6,4) = \lim_{k \to 0} \frac{T_x(6,4+h) - T_x(6,4)}{h}$

By taking h = 2, -2

$$T_{xy}(6,4) = \frac{T_x(6,6) - T_x(6,4)}{2}$$

And $T_{xy}(6,4) = \frac{T_x(6,2) - T_x(6,4)}{-2}$

Now $T_x(6,6) = \frac{T(8,6) - T(6,6)}{2} = \frac{80-75}{2} = 2.5$

And $T_x(6,6) = \frac{T(4,6) - T(6,6)}{-2} = \frac{68-75}{-2} = 3.5$

On averaging $T_x(6,6) = 3.0^{o} c/m$

Also $T_x(6,2) = \frac{T(4,2)-T(6,2)}{-2} = \frac{74-87}{-2} = 6.5$

And $T_x(6,2) = \frac{T(8,2)-T(6,2)}{2} = \frac{90-87}{2} = 1.5$

On averaging $T_x(6,2) = 4.0^{o} c/m$

Hence $T_{xy}(6,4) = \frac{T_x(6,6)-T_x(6,4)}{2} = \frac{3-3.5}{2} = -0.25$

$$T_{xy}(6,4) = \frac{T_x(6,2)-T_x(6,4)}{-2} = \frac{4.0-3.5}{-2} = -0.25$$

On averaging we find

$$T_{xy}(6,4) = \boxed{-0.25}$$

Answer 11TFQ.

Let $D_u f(x,y) = f_x(x,y)a + f_y(x,y)b$, where $u = (a,b)$ is a unit vector.

We have $\frac{\partial f}{\partial x} = \cos x$ and $\frac{\partial f}{\partial y} = \cos y$.

Then, $D_u f(x, y) = \cos x \cos\theta + \cos y \sin\theta$.

Now, $D_u f(x, y)$ is maximum at $|\nabla f(x, y)|$ and it occurs when $u = (\cos\theta, \sin\theta)$ and has the same direction as $\nabla f(x, y)$.

In the given case $D_u f(x, y)$ is maximum at $\theta = \frac{\pi}{4}$.

Thus, we get $D_u f(x, y) = \frac{1}{2} + \frac{1}{2}$ or $D_u f(x,y) = 1$.

We also have $|\nabla f(x, y)| = \sqrt{2}$.

Therefore the given statement is **true**.

Answer 12CC.

If z is given implicitly as a function $z = f(x,y)$ by an equation of the form $F(x,y,z) = 0$,

then $\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}$ and $\frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}$.

Answer 12E.

We know the linear approximation of "f" at (a, b) is

$$f(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

Then linear approximation of $T(x, y)$ near $(6, 4)$ is

$$T(x,y) = T(6,4) + T_x(6,4)(x-6) + T_y(6,4)(y-4)$$

Using results obtained in exercise 11

$$T_x(6,4) = 3.5° \text{ c/m}$$

$$T_y(6,4) = -3.0° \text{ c/m}$$

Then $T(x,y) = 80 + 3.5(x-6) - 3(y-4)$

Then
$$\begin{aligned} T(5,3.8) &= 80 + 3.5(5-6) - 3(3.8-4) \\ &= 80 + 3.5(-1) - 5(-0.2) \\ &= 80 - 3.5 + 0.6 \\ &= 77.1 \end{aligned}$$

Hence $T(5,3.8) = \boxed{77.1}$

Answer 12TFQ.

Consider the function given by $f(x,y) = 10x^2y - 5x^2 - 4y^2 - x^4 - 2y^4$.
We note that the function has four local maxima and no local minima.
Its not necessary that it should have local minima if it has local maxima.
The given statement is **false**.

Answer 13CC.

(a) Let $z = f(x,y)$. The directional derivative of f at (x_0, y_0) in the direction of a unit vector $\mathbf{u} = \langle a, b \rangle$ is

$$D_{\mathbf{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}.$$

The directional derivative of f in the direction of $\mathbf{u}$ is the rate of change of z in the direction of $\mathbf{u}$.

A geometric interpretation of a directional derivative of $z = f(x,y)$ is a tangent to the surface at point P.

(b) If f is a differentiable function of x and y, then f has a directional derivative in the direction of a unit vector $\mathbf{u} = \langle a, b \rangle$ and $D_{\mathbf{u}} f(x, y) = f_x(x, y)a + f_y(x, y)b$.

Answer 13E.

Let us start by finding f_x.

We know that $f_x = \frac{\partial}{\partial x} f(x, y)$.

$$f_x = \frac{\partial}{\partial x}\left(5y^3 + 2x^2y\right)^8$$
$$= 8\left(5y^3 + 2x^2y\right)^7 (0 + 4xy)$$
$$= 32xy\left(5y^3 + 2x^2y\right)^7$$

We get $\boxed{f_x = 32xy\left(5y^3 + 2x^2y\right)^7}$

Now, we have $f_y = \frac{\partial}{\partial y} f(x, y)$.

$$f_y = \frac{\partial}{\partial y}\left(5y^3 + 2x^2y\right)^8$$
$$= 8\left(5y^3 + 2x^2y\right)^7\left(15y^2 + 2x^2\right)$$
$$= \left(5y^3 + 2x^2y\right)^7\left(120y^2 + 16x^2\right)$$

Thus, we get $\boxed{f_y = \left(5y^3 + 2x^2y\right)^7\left(120y^2 + 16x^2\right)}$

Answer 14CC.

a.

Consider a function *f* of two variables *x* and *y*.

The gradient of *f* is the vector function ∇f defined as shown below:

$$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y)\rangle$$
$$= \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

b.

Consider a function *f* of two variables *x* and *y*.

Consider a unit vector: $\mathbf{u} = \langle a, b\rangle$

The directional derivative in the direction of a unit vector $\mathbf{u} = \langle a, b\rangle$ is expressed as the scalar projection of the gradient vector on to u.

$$D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u}$$

c.

Consider a function *f* of two variables *x* and *y*.

The gradient vector $\nabla f(x, y) = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$ is the normal vector to the tangent plane.

Answer 14E.

Let us start by finding g_u.

We know that $f_x = \frac{\partial}{\partial x} f(x, y)$.

$$g_u = \frac{\partial}{\partial u}\left(\frac{u+2v}{u^2+v^2}\right)$$
$$= \frac{1}{u^2+v^2} - \frac{2u(u+2v)}{(u^2+v^2)^2}$$
$$= -\frac{u^2-v^2+4uv}{(u^2+v^2)^2}$$

We get $\boxed{g_u = -\frac{u^2-v^2+4uv}{(u^2+v^2)^2}}$

Now, we have $g_v = \frac{\partial}{\partial v} g(u, v)$.

$$g_v = \frac{\partial}{\partial v}\left(\frac{u+2v}{u^2+v^2}\right)$$
$$= \frac{2}{u^2+v^2} - \frac{2v(u+2v)}{(u^2+v^2)^2}$$
$$= \frac{2(u^2-v^2-uv)}{(u^2+v^2)^2}$$

Thus, we get $\boxed{g_v = \frac{2(u^2-v^2-uv)}{(u^2+v^2)^2}}$

Answer 15CC.

(a) A function of two variables has a local maximum at (a, b) if $f(x, y) \le f(a, b)$ when (x, y) is near (a, b). This means that $f(x, y) \le f(a, b)$ for all points (x, y) in some disk with center (a, b). The number $f(a, b)$ is the maximum value.

(b) A function of two variables is said to have absolute maximum at (a,b) if $f(x,y) \le f(a,b)$ for all (x,y) in the domain of f.

(c) If $f(x,y) \ge f(a,b)$ when (x,y) is near (a,b), then f has a local minimum at (a,b) and $f(a,b)$ is the local minimum value.

(d) If $f(x,y) \ge f(a,b)$ for all point (x,y) in the domain of f, then f has a absolute minimum at (a,b).

(e) If $f(a,b)$ is not a local maximum or minimum, then (a,b) is called a saddle point of f and the graph of f crosses its tangent plane at (a,b).

Answer 15E.

Let us start by finding F_α.

We know that $f_x = \dfrac{\partial}{\partial x} f(x,y)$.

$$F_\alpha = \frac{\partial}{\partial \alpha}\left[\alpha^2 \ln\left(\alpha^2 + \beta^2\right)\right]$$
$$= 2\alpha \ln\left(\alpha^2 + \beta^2\right) + \frac{2\alpha^3}{\alpha^2 + \beta^2}$$

We get $\boxed{F_\alpha = 2\alpha \ln\left(\alpha^2 + \beta^2\right) + \dfrac{2\alpha^3}{\alpha^2 + \beta^2}}$.

Now, we have $F_\beta = \dfrac{\partial}{\partial \beta} F(\alpha, \beta)$.

$$F_\beta = \frac{\partial}{\partial \beta}\left[\alpha^2 \ln\left(\alpha^2 + \beta^2\right)\right]$$
$$= \frac{2\alpha^2 \beta}{\alpha^2 + \beta^2}$$

Thus, we get $\boxed{F_\beta = \dfrac{2\alpha^2 \beta}{\alpha^2 + \beta^2}}$.

Answer 16CC.

(a) According to the Second Derivative Test, if f has a local maximum at (a, b), then $D > 0$ and $f_{xx}(a, b) < 0$, where

$$D = D(a,b) = f_{xx}(a,b)f_{yy}(a,b) - \left[f_{xy}(a,b)\right]^2.$$

(b) Suppose that the second partial derivatives of f are continuous on a disk with center (a, b) and $f_x(a, b) = 0$ and $f_y(a, b) = 0$, then (a, b) is the critical point of f.

Answer 16E.

Let us start by finding G_x.

We know that $f_x = \dfrac{\partial}{\partial x} f(x, y)$.

$$G_x = \frac{\partial}{\partial x}\left[e^{xz}\sin\left(\frac{y}{z}\right)\right]$$

$$= ze^{xz}\sin\left(\frac{y}{z}\right)$$

We get $\boxed{G_x = ze^{xz}\sin\left(\frac{y}{z}\right)}$.

Now, we have $G_y = \dfrac{\partial}{\partial y} G(x, y, z)$.

$$G_y = \frac{\partial}{\partial y}\left[e^{xz}\sin\left(\frac{y}{z}\right)\right]$$

$$= \frac{e^{xz}}{z}\cos\left(\frac{y}{z}\right)$$

Thus, we get $\boxed{G_y = \dfrac{e^{xz}}{z}\cos\left(\dfrac{y}{z}\right)}$.

Now, find G_z.

$$G_z = \frac{\partial}{\partial z}\left[e^{xz}\sin\left(\frac{y}{z}\right)\right]$$

$$= xe^{xz}\sin\left(\frac{y}{z}\right) - \frac{ye^{xz}}{z^2}\cos\left(\frac{y}{z}\right)$$

$$= e^{xz}\left[x\sin\left(\frac{y}{z}\right) - \frac{y}{z^2}\cos\left(\frac{y}{z}\right)\right]$$

Therefore, we get $\boxed{G_z = e^{xz}\left[x\sin\left(\dfrac{y}{z}\right) - \dfrac{y}{z^2}\cos\left(\dfrac{y}{z}\right)\right]}$.

Answer 17CC.

Suppose that the second partial derivatives of f are continuous on a disk with center (a, b)(critical point), and suppose that $f_x(a, b) = 0$ and $f_y(a, b) = 0$. Let

$$D = D(a, b) = f_{xx}(a, b) f_{yy}(a, b) - [f_{xy}(a, b)]^2.$$

Then, the Second Derivative Test states that

(a) If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.

(b) If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.

(c) If $D < 0$, then $f(a, b)$ is not a local maximum or minimum. The point (a, b) is called the saddle point of f and the graph of f crosses its tangent plane at (a, b).

Answer 17E.

Let us start by finding S_u.

We know that $f_x = \frac{\partial}{\partial x} f(x, y)$.

$$S_u = \frac{\partial}{\partial u}\left[u \arctan\left(v\sqrt{w}\right)\right]$$
$$= \arctan\left(v\sqrt{w}\right)$$

We get $\boxed{S_u = \arctan\left(v\sqrt{w}\right)}$.

Now, we have $S_v = \frac{\partial}{\partial v} S(u, v, w)$.

$$S_v = \frac{\partial}{\partial v}\left[u \arctan\left(v\sqrt{w}\right)\right]$$
$$= \frac{u\sqrt{w}}{1 + \left(v\sqrt{w}\right)^2}$$
$$= \frac{u\sqrt{w}}{1 + v^2 w}$$

Thus, we get $\boxed{S_v = \frac{u\sqrt{w}}{1+v^2w}}$

Now, find S_w.

$$S_w = \frac{\partial}{\partial w}\left[u\arctan\left(v\sqrt{w}\right)\right]$$

$$= \frac{1}{2\sqrt{w}}\left(\frac{uv}{1+\left(v\sqrt{w}\right)^2}\right)$$

$$= \frac{1}{2\sqrt{w}}\left(\frac{uv}{1+v^2w}\right)$$

Therefore, we get $\boxed{S_w = \frac{1}{2\sqrt{w}}\left(\frac{uv}{1+v^2w}\right)}$

Answer 18CC.

(a) A boundary point of D is a point (a, b) such that every disk with center (a,b) contains point in D and also points not in D.
A closed set in $\mathbb{R}^2$ is one that contains all its boundary points.

(b) The extreme value theorem for functions of two variables states that if f is continuous on a closed, bounded set D in $\mathbb{R}^2$, then f attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2) in D.

(c) In order to find the absolute maximum and minimum values of a continuous function f on a closed bounded set D

(i) Find the values of f at the critical points of f in D.

(ii) Find the extreme values of f on the boundary of D.

(iii) The largest of the values from steps (i) and (ii) is the absolute maximum value; the smallest of these values is the absolute minimum value

Answer 18E.

$$C = 1449.2 + 4.6T - 0.055T^2 + 0.00029T^3$$
$$+ (1.34 - 0.01T)(3.35) + 0.016D$$

Then $\dfrac{\partial C}{\partial T} = 4.6 - (0.055T)2 + 3(0.00029)T^2$

$$+ (S - 35)(-0.01)$$

$$\frac{\partial C}{\partial S} = (1.34 - 0.01T)(1)$$

$$\frac{\partial C}{\partial D} = 0.016$$

When $T = 10^{\circ}C$, $S = 35$ parts per thousand

$D = 100$ m

Then $\dfrac{\partial C}{\partial T} = 4.6 - 2(0.055)(10) + 3(0.00029)(10)^2$

$$+ (35 - 35)(-0.01)$$
$$= 4.6 - 1.1 + 0.087$$
$$= 3.587$$

$$\frac{\partial C}{\partial S} = (1.34 - 0.01 \times 10)$$
$$= 1.34 - 0.1$$
$$= 1.24$$

$$\frac{\partial C}{\partial D} = 0.016$$

Now $\dfrac{\partial C}{\partial T}$ is the rate of change of speed of sound with respect to temperature T when the salinity of water S and depth of ocean below water D remains constant.

Similarly $\dfrac{\partial C}{\partial S}$ is the rate of change of speed of sound with respect to salinity of water S when the temperature and depth D remains constant

And $\dfrac{\partial C}{\partial D}$ is the rate of change of speed of sound with respect to depth of ocean under water when the salinity S and temperature T remains constant.

Since $\dfrac{\partial C}{\partial T}$, $\dfrac{\partial C}{\partial S}$ and $\dfrac{\partial C}{\partial D}$ are all positive, then we see that the speed of the sound increases with the increase in T, S or D when the other two quantities remain constant.

Answer 19CC.

Let us assume that the extreme values of $f(x,y,z)$ exist and $\nabla g \neq 0$ on the surface $g(x,y,z)=k$. In order to find the maximum and minimum values of $f(x,y,z)$ subject to the constraint $g(x,y,z)=k$, we have two methods.

(a) Find all values of x, y, z, and λ such that $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$ and $g(x,y,z)=k$.

(b) Evaluate f at all points (x,y,z) that result from step **(a)**. The largest of these values is the maximum value of f; the smallest is the minimum value of f.

Answer 19E.

$$f(x,y)=4x^3-xy^2 \quad \text{------------ (1)}$$

$$f_x=12x^2-y^2 \text{ ------------ (2)}$$

$$f_{xx}=24x$$

Differentiating (1) partially with respect to y

$$f_y=-2xy \quad \text{-------------- (3)}$$

$$f_{yy}=-2x$$

Differentiating (2) partially with respect to y

$$f_{xy}=-2y$$

Differentiating (3) partially with respect to x

$$f_{yx}=-2y$$

Hence $\boxed{f_{xx}=24x,\ f_{yy}=-2x,\ f_{xy}=f_{yx}=-2y}$

Answer 20E.

$$z = xe^{-2y} \quad \text{------------ (1)}$$

Differentiating partially with respect to x

$$\frac{\partial z}{\partial x} = e^{-2y} \quad \text{------------ (2)}$$

$$\frac{\partial^2 z}{\partial x^2} = 0$$

Differentiating (1) partially with respect to y

$$\frac{\partial z}{\partial y} = -2xe^{-2y} \quad \text{------------- (3)}$$

$$\frac{\partial^2 z}{\partial y^2} = 4xe^{-2y}$$

Differentiating (2) partially with respect to y

$$\frac{\partial^2 z}{\partial y\,\partial x} = -2e^{-2y}$$

Differentiating (3) partially with respect to x

$$\frac{\partial^2 z}{\partial x\,\partial y} = -2e^{-2y}$$

Hence $\boxed{\frac{\partial^2 z}{\partial x^2} = 0,\ \frac{\partial^2 z}{\partial y^2} = 4xe^{-2y},\ \frac{\partial^2 z}{\partial x\,\partial y} = \frac{\partial^2 z}{\partial y\,\partial x} = -2e^{-2y}}$

Answer 21E.

$$f(x,y,z) = x^k y^l z^m \quad \text{---------------- (1)}$$

Then $\frac{\partial f}{\partial x} = k\,x^{k-1} y^l z^m \quad \text{---------------- (2)}$

$$\frac{\partial^2 f}{\partial x^2} = k(k-1)\,x^{k-2} y^l z^m$$

Differentiating (1) partially with respect to y

$$\frac{\partial f}{\partial y} = l\,x^k y^{l-1} z^m \quad \text{---------------- (3)}$$

$$\frac{\partial^2 f}{\partial y^2} = l(l-1)\,x^k y^{l-2} z^m$$

Differentiating (1) partially with respect to z

$$\frac{\partial f}{\partial z} = m x^k y^l z^{m-1} \qquad \text{----------------- (4)}$$

$$\frac{\partial^2 f}{\partial z^2} = m(m-1) x^k y^l z^{m-2}$$

Differentiating (2) partially with respect to y

$$\frac{\partial^2 f}{\partial y \, \partial x} = k l x^{k-1} y^{l-1} z^m$$

Differentiating (2) partially with respect to z

$$\frac{\partial^2 f}{\partial z \, \partial x} = k m x^{k-1} y^l z^{m-1}$$

Differentiating (3) partially with respect to x

$$\frac{\partial^2 f}{\partial x \, \partial y} = l k x^{k-1} y^{l-1} z^m$$

Differentiating (3) partially with respect to z

$$\frac{\partial^2 f}{\partial z \, \partial y} = l m x^k y^{l-1} z^{m-1}$$

Differentiating (4) partially with respect to x

$$\frac{\partial^2 f}{\partial x \, \partial z} = m k x^{k-1} y^l z^{m-1}$$

Differentiating (4) partially with respect to y

$$\frac{\partial^2 f}{\partial y \, \partial z} = m l x^k y^{l-1} z^{m-1}$$

Hence $f_{xx} = k(k-1) x^{k-2} y^l z^m$, $f_{yy} = l(l-1) x^k y^{l-2} z^m$

$f_{zz} = m(m-1) x^k y^l z^{m-2}$, $f_{xy} = kl x^{k-1} y^{l-1} z^m = f_{yx}$

$f_{xz} = km x^{k-1} y^l z^{m-1} = f_{zx}$, $f_{yz} = lm x^k y^{l-1} z^{m-1} = f_{zy}$

Answer 22E.

$$v = r\cos(s+2t) \qquad \text{-------------- (1)}$$

Then $v_r = \cos(s+2t)$ -------------- (2)

$v_{rr} = 0$

Differentiating (1) partially with respect to s

$$v_s = -r\sin(s+2t) \qquad \text{-------------- (3)}$$

$$v_{ss} = -r\cos(s+2t)$$

Differentiating (1) partially with respect to t

$$v_t = -2r\sin(s+2t) \qquad \text{-------------- (4)}$$

$$v_{tt} = -4r\cos(s+2t)$$

Differentiating (2) partially with respect to s and t,

$$v_{rs} = -\sin(s+2t)$$

$$v_{rt} = -2\sin(s+2t)$$

Differentiating (3) partially with respect to r and t,

$$v_{sr} = -\sin(s+2t)$$

$$v_{st} = -2r\cos(s+2t)$$

Differentiating (4) partially with respect to r and s

$$v_{tr} = -2\sin(s+2t)$$

$$v_{ts} = -2r\cos(s+2t)$$

Hence $v_{rr} = 0$, $v_{ss} = -r\cos(s+2t)$

$$v_{tt} = -4r\cos(s+2t)$$

$$v_{rs} = v_{sr} = -\sin(s+2t)$$

$$v_{rt} = v_{tr} = -2\sin(s+2t)$$

$$v_{st} = v_{ts} = -2r\cos(s+2t)$$

Answer 23E.

Consider the equation

$$z = xy + xe^{\frac{y}{x}}$$

Now, show that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = xy + z.$

The partial derivative of z with respect to x is

$$\begin{aligned}\frac{\partial z}{\partial x} &= \frac{\partial}{\partial x}\left(xy + xe^{\frac{y}{x}}\right)\\ &= \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial x}\left(xe^{\frac{y}{x}}\right)\\ &= y + e^{\frac{y}{x}} + xe^{\frac{y}{x}}\cdot\left(-\frac{y}{x^2}\right)\\ &= y - \frac{ye^{\frac{y}{x}}}{x} + e^{\frac{y}{x}}\end{aligned}$$

Multiply both sides by x.

$$x\frac{\partial z}{\partial x} = x\left(y - \frac{ye^{\frac{y}{x}}}{x} + e^{\frac{y}{x}}\right)$$

$$x\frac{\partial z}{\partial x} = xy - ye^{\frac{y}{x}} + xe^{\frac{y}{x}} \quad \text{......} \ (1)$$

The partial derivative of z with respect to y is

$$\frac{\partial z}{\partial y}=\frac{\partial}{\partial y}\left(xy+xe^{\frac{y}{x}}\right)$$

$$=\frac{\partial}{\partial y}(xy)+\frac{\partial}{\partial y}\left(xe^{\frac{y}{x}}\right)$$

$$=x+xe^{\frac{y}{x}}\cdot\left(\frac{1}{x}\right)$$

$$=x+e^{\frac{y}{x}}$$

Multiply both sides by y.

$$y\frac{\partial z}{\partial y}=y\left(x+e^{\frac{y}{x}}\right)$$

$$y\frac{\partial z}{\partial y}=xy+ye^{\frac{y}{x}} \text{ ...... (2)}$$

Add equations (1) and (2).

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=xy-ye^{\frac{y}{x}}+xe^{\frac{y}{x}}+xy+ye^{\frac{y}{x}}$$

$$=xy+xy+xe^{\frac{y}{x}}$$

$$=xy+z$$

Therefore,

$$\boxed{x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=xy+z}$$

Answer 24E.

Consider the equation

$$z=\sin(x+\sin t)$$

Now, show that $\frac{\partial z}{\partial x}\frac{\partial^2 z}{\partial x\partial t}=\frac{\partial z}{\partial t}\frac{\partial^2 z}{\partial x^2}$.

The partial derivative of z with respect to x is

$$\frac{\partial z}{\partial x}=\frac{\partial}{\partial x}(\sin(x+\sin t))$$

$$=\cos(x+\sin t)\cdot(1+0)$$

$$=\cos(x+\sin t) \text{ ...... (1)}$$

The second partial derivative of z with respect to x is

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)$$
$$= \frac{\partial}{\partial x}\cos(x+\sin t)$$
$$= -\sin(x+\sin t)\cdot(1+0)$$
$$= -\sin(x+\sin t) \quad \ldots\ldots (2)$$

The partial derivative of z with respect to t is

$$\frac{\partial z}{\partial t} = \frac{\partial}{\partial t}(\sin(x+\sin t))$$
$$= \cos(x+\sin t)\cdot(0+\cos t)$$
$$= \cos(x+\sin t)\cos t \quad \ldots\ldots (3)$$

The partial derivative of $\frac{\partial z}{\partial t}$ with respect to x is

$$\frac{\partial^2 z}{\partial x \partial t} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial t}\right)$$
$$= \frac{\partial}{\partial x}(\cos(x+\sin t)\cos t)$$
$$= \cos t(-\sin(x+\sin t))\cdot(1+0)$$
$$= -\cos t\cdot\sin(x+\sin t) \quad \ldots\ldots (4)$$

Multiply equation (1) and (4).

$$\frac{\partial z}{\partial x}\cdot\frac{\partial^2 z}{\partial x \partial t} = [\cos(x+\sin t)]\cdot[-\cos t\cdot\sin(x+\sin t)]$$
$$= -\cos t\sin(x+\sin t)\cdot\cos(x+\sin t) \quad \ldots\ldots (5)$$

Multiply equation (2) and (3).

$$\frac{\partial z}{\partial t}\cdot\frac{\partial^2 z}{\partial x^2} = [\cos(x+\sin t)\cos t]\cdot[-\sin(x+\sin t)]$$
$$= -\cos t\sin(x+\sin t)\cdot\cos(x+\sin t) \quad \ldots\ldots (6)$$

From equations (5) and (6), conclude that

$$\boxed{\frac{\partial z}{\partial x}\frac{\partial^2 z}{\partial x \partial t} = \frac{\partial z}{\partial t}\frac{\partial^2 z}{\partial x^2}}$$

Answer 25E.

Give that $z = 3x^2 - y^2 + 2x$

Take $f(x,y,z) = 3x^2 - y^2 + 2x - z$

Then $f_x = 6x + 2$

$f_y = -2y$

$f_z = -1$

(a). The equation of tangent plane at (x_1, y_1, z_1) is

$$f_x(x_1,y_1,z_1)(x-x_1) + f_y(x_1,y_1,z_1)(y+y_1) + f_z(x_1,y_1,z_1)(z-z_1) = 0$$

So the equation of tangent plane at (1, -2, 1) is

$$f_x(1,-2,1)(x-1) + f_y(1,-2,1)(y+2) + f_z(1,-2,1)(z-1) = 0$$

i.e. $(6+2)(x-1) + (4)(y+2) + (-1)(z-1) = 0$

i.e. $8x - 8 + 4y + 8 - z + 1 = 0$

i.e. $8x + 4y - z + 1 = 0$

Or $\boxed{z = 8x + 4y + 1}$

(b). The equation of normal line at (x_1, y_1, z_1) is

$$\frac{x - x_1}{f_x(x_1,y_1,z_1)} = \frac{y + y_1}{f_y(x_1,y_1,z_1)} = \frac{z - z_1}{f_z(x_1,y_1,z_1)}$$

So the equation of normal line is

$$\frac{x-1}{f_x(1,-2,1)} = \frac{y+2}{f_y(1,-2,1)} = \frac{z-1}{f_z(1,-2,1)}$$

i.e. $\frac{x-1}{8} = \frac{y+2}{4} = \frac{z-1}{-1}$

Or $\boxed{\frac{x-1}{8} = \frac{y+2}{4} = 1 - z}$

(B)

The equation of normal line is

$$\frac{x-1}{f_x(1,-2,1)} = \frac{y+2}{f_y(1,-2,1)} = \frac{z-1}{f_z(1,-2,1)}$$

i.e. $\frac{x-1}{8} = \frac{y+2}{4} = \frac{z-1}{-1}$

Or $\boxed{\frac{x-1}{8} = \frac{y+2}{4} = 1 - z}$

Answer 26E.

$$z = e^x \cos y$$

Take $f(x,y,z) = e^x \cos y - z$

Then $f_x = e^x \cos y$

$f_y = -e^x \sin y$

$f_z = -1$

(A)

The equation of tangent plane at (0, 0, 1) is

$$f_x(0,0,1)(x-0) + f_y(0,0,1)(y-0) + f_z(0,0,1)(z-1) = 0$$

$$1(x-0) + (0)(y-0) + (-1)(z-1) = 0$$

$$x - z + 1 = 0$$

Or $\boxed{z = x + 1}$

(B)

The equation of normal line is

$$\frac{x-0}{f_x(0,0,1)} = \frac{y-0}{f_y(0,0,1)} = \frac{z-0}{f_z(0,0,1)}$$

i.e. $\frac{x}{1} = \frac{y}{0} = \frac{z-1}{-1}$

i.e. $\boxed{\frac{x}{1} = 1 - z,\ \ y = 0}$

Answer 27E.

$$x^2 + 2y^2 - 3z^2 = 3$$

Take $f(x,y,z) = x^2 + 2y^2 - 3z^2 - 3$

Then $f_x = 2x$

$f_y = 4y$

$f_z = -6z$

(A)

The equation of tangent plane at (2, -1, 1) is

$$f_x(2,-1,1)(x-2) + f_y(2,-1,1)(y+1) + f_z(2,-1,1)(z-1) = 0$$

i.e. $4(x-2) + (-4)(y+1) + (-6)(z-1) = 0$

i.e. $4x - 4y - 6z - 8 - 4 + 6 = 0$

i.e. $4x - 4y - 6z - 6 = 0$

i.e. $\boxed{2x - 2y - 3z = 3}$

(B)

The equation of normal line is

$$\frac{x-2}{f_x(2,-1,1)}=\frac{y+1}{f_y(2,-1,1)}=\frac{z-1}{f_z(2,-1,1)}$$

i.e. $\boxed{\frac{x-2}{4}=\frac{y+1}{-4}=\frac{z-1}{-6}}$

Answer 28E.

$$xy+yz+zx=3$$

Take $f(x,y,z)=xy+yz+zx-3$

Then $f_x=y+z$

$f_y=x+z$

$f_z=x+y$

(A)

The equation of tangent plane at (1, 1, 1) is

$$f_x(1,1,1)(x-1)+f_y(1,1,1)(y-1)+f_z(1,1,1)(z-1)=0$$

i.e. $2(x-1)+2(y-1)+2(z-1)=0$

i.e. $x-1+y-1+z-1=0$

Or $\boxed{x+y+z=3}$

(B)

The equation of normal line is

$$\frac{x-1}{f_x(1,1,1)}=\frac{y-1}{f_y(1,1,1)}=\frac{z-1}{f_z(1,1,1)}$$

i.e. $\boxed{\frac{x-1}{2}=\frac{y-1}{2}=\frac{z-1}{2}}$

Answer 29E.

$$\sin(xyz)=x+2y+3z$$

Take $f(x,y,z)=\sin(xyz)-x-2y-3z$

Then $f_x=yz\cos(xyz)-1$

$f_y=xz\cos(xyz)-2$

$f_z=xy\cos(xyz)-3$

(A)

The equation of tangent plane at (2, -1, 0) is

$$f_x(2,-1,0)(x-2)+f_y(2,-1,0)(y+1)+f_z(2,-1,0)(z-0)=0$$

i.e. $(-1)(x-2)+(-2)(y+1)+(-2-3)(z-0)=0$

i.e. $-x+2-y-2-5z=0$

i.e. $\boxed{x+2y+5z=0}$

(B)

The equation of normal line is

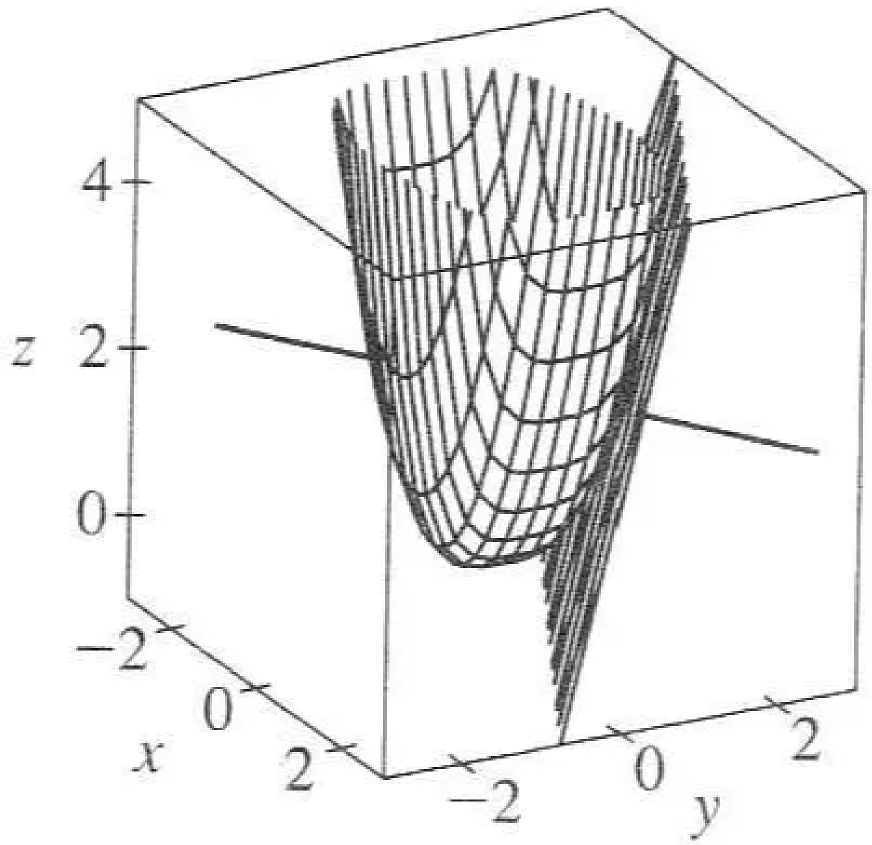
$$\frac{x-2}{f_x(2,-1,0)}=\frac{y+1}{f_y(2,-1,0)}=\frac{z-0}{f_z(2,-1,0)}$$

$$\boxed{\frac{x-2}{-1}=\frac{y+1}{-2}=\frac{z}{-5}}$$

Answer 30E.

Let f(x,y) = x2 + y4. Then fx(x,y) = 2x and fy(x,y) = 4y3, so fx(1,1) = 2, fy(1,1) = 4 and an equation of the tangent plane is z - 2 = 2(x - 1) + 4(y - 1) or 2x + 4y - z = 4. A normal vector to the tangent plane is <2, 4, -1> so the normal line is given by

$\frac{x-1}{2}=\frac{y-1}{4}=\frac{z-2}{-1}$ or x = 1 + 2t, y = 1 + 4t, z = 2 - t.



Answer 31E.

Consider the hyperboloid

$$x^2+4y^2-z^2=4$$

And the plane

$$2x+2y+z=5$$

Recall that,

Let S be a surface with equation $F(x,y,z)=k$, and $p(x_0,y_0,z_0)$ be a point of S.

The tangent plane to S at (x_0,y_0,z_0) is

$$F_x(x_0,y_0,z_0)(x-x_0)+F_y(x_0,y_0,z_0)(y-y_0)+F_z(x_0,y_0,z_0)(z-z_0)=0$$

In present case,

$$F(x,y,z)=x^2+4y^2-z^2$$

The partial derivatives of the function $F(x,y,z)=x^2+4y^2-z^2$ with respect to x is

$$\begin{aligned}F_x(x,y,z)&=\frac{\partial}{\partial x}(x^2+4y^2-z^2)\\&=2x\end{aligned}$$

The partial derivatives of the function $F(x,y,z)=x^2+4y^2-z^2$ with respect to y is

$$\begin{aligned}F_y(x,y,z)&=\frac{\partial}{\partial y}(x^2+4y^2-z^2)\\&=4(2y)\\&=8y\end{aligned}$$

The partial derivatives of the function $F(x,y,z)=x^2+4y^2-z^2$ with respect to z is

$$\begin{aligned}F_z(x,y,z)&=\frac{\partial}{\partial z}(x^2+4y^2-z^2)\\&=-2z\end{aligned}$$

The direction of the normal line is the gradient vector $\nabla F(x,y,z)$.

For a function F of three variables, the gradient vector is

$$\nabla F(x,y,z)=\langle F_x(x,y,z),F_y(x,y,z),F_z(x,y,z)\rangle$$

Thus, the normal vector of the tangent plane to the hyperboloid at (x,y,z) is

$$\langle 2x,8y,-2z\rangle$$

The normal vector of the plane $2x+2y+z=5$ is

$$\langle 2,2,1\rangle$$

It is given that; the tangent plane to the hyperboloid is parallel to the plane

$$2x+2y+z=5$$

So, the normal vector of the tangent plane to the hyperboloid is parallel to the normal vector of the given plane.

Note that, two non-zero vectors are parallel if they are scalar multiple of one another

Thus,

$$\langle 2x,8y,-2z\rangle = k\langle 2,2,1\rangle$$
$$\langle 2x,8y,-2z\rangle = \langle 2k,2k,k\rangle$$

Where *k* is any arbitrary scalar

$$2x=2k$$
$$8y=2k$$
$$-2z=k$$

From first equation,

$$2x=2k$$
$$x=k$$

From second equation,

$$8y=2k$$
$$y=\frac{2}{8}k$$
$$y=\frac{1}{4}k$$

From third equation,

$$-2z=k$$
$$z=-\frac{1}{2}k$$

Substitute $x = k, y = \frac{1}{4}k$ and $z = -\frac{1}{2}k$ into $x^2 + 4y^2 - z^2 = 4$.

$$(k)^2 + 4\left(\frac{k}{4}\right)^2 - \left(-\frac{k}{2}\right)^2 = 4$$

$$k^2 + 4\left(\frac{k^2}{16}\right) - \frac{k^2}{4} = 4$$

$$k^2 + \frac{k^2}{4} - \frac{k^2}{4} = 4$$

$$k^2 = 4$$

$$k = \pm 2$$

Substitute $k = 2$ in $x = k, y = \frac{1}{4}k$ and $z = -\frac{1}{2}k$. Then

$$(x, y, z) = \left(2, \frac{1}{2}, -1\right)$$

Substitute $k = -2$ in $x = k, y = \frac{1}{4}k$ and $z = -\frac{1}{2}k$. Then

$$(x, y, z) = \left(-2, -\frac{1}{2}, 1\right)$$

Therefore, the points on the hyperboloid $x^2 + 4y^2 - z^2 = 4$ where the tangent plane is parallel to the plane $2x + 2y + z = 5$ are $\boxed{\left(2, \frac{1}{2}, -1\right) \text{ and } \left(-2, -\frac{1}{2}, 1\right)}$

Answer 32E.

Consider the following function:

$$u = \ln\left(1 + se^{2t}\right)$$

Recall that, for a differential function of two variables, $z = f(x, y)$, the differential dz, also called the total differential is defined by the following equation:

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy \quad \text{...... (1)}$$

In present case, u is a function of two variables s and t.

By using (1), the differential du is defined by the following equation:

$$du = \frac{\partial u}{\partial s}ds + \frac{\partial u}{\partial t}dt \quad \text{...... (2)}$$

The partial derivative of u with respect to s is as follows:

$$\begin{aligned}\frac{\partial u}{\partial s} &= \frac{\partial}{\partial s}(u) \\ &= \frac{\partial}{\partial s}\left(\ln\left(1+se^{2t}\right)\right) \\ &= \frac{1}{1+se^{2t}}\left(e^{2t}\right) \qquad \text{Since: } \frac{d}{dx}(\ln x)=\frac{1}{x}\end{aligned}$$

The partial derivative of u with respect to t is as shown below:

$$\begin{aligned}\frac{\partial u}{\partial t} &= \frac{\partial}{\partial t}(u) \\ &= \frac{\partial}{\partial t}\left(\ln\left(1+se^{2t}\right)\right) \\ &= \frac{1}{1+se^{2t}}\left(2se^{2t}\right) \qquad \text{Since: } \frac{d}{dx}(\ln x)=\frac{1}{x}\end{aligned}$$

Substitute $\frac{\partial u}{\partial s}=\frac{e^{2t}}{1+se^{2t}}$ and $\frac{\partial u}{\partial t}=\frac{2se^{2t}}{1+se^{2t}}$ in equation (1).

$$\begin{aligned}du &= \frac{\partial u}{\partial s}ds+\frac{\partial u}{\partial t}dt \\ &= \left(\frac{e^{2t}}{1+se^{2t}}\right)ds+\left(\frac{2se^{2t}}{1+se^{2t}}\right)dt \\ &= \left(\frac{1}{1+se^{2t}}\right)\left(e^{2t}ds+2se^{2t}dt\right)\end{aligned}$$

Therefore, $\boxed{du=\left(\frac{1}{1+se^{2t}}\right)\left(e^{2t}ds+2se^{2t}dt\right)}$.

Answer 33E.

We know the linear approximation at point (a, b, c) is given by

$$f(x,y,z)=f(a,b,c)+f_x(a,b,c)(x-a)+f_y(a,b,c)(y-b)+f_z(a,b,c)(z-c)$$

Now $f(x,y,z)=x^3\sqrt{y^2+z^2}$

Then $f_x=3x^2\sqrt{y^2+z^2}$

$$f_y=\frac{x^3y}{\sqrt{y^2+z^2}}$$

$$f_z=\frac{x^3z}{\sqrt{y^2+z^2}}$$

The given point is $(2,3,4)$

Then $f_x(2,3,4) = 3(2)^2\sqrt{9+16}$

$= 60$

$f_y(2,3,4) = \dfrac{(2)^3(3)}{\sqrt{9+16}}$

$= \dfrac{24}{5}$

$f_z(2,3,4) = \dfrac{(2)^3(4)}{\sqrt{9+16}}$

$= \dfrac{32}{5}$

Then the linear approximation at $(2,3,4)$ is

$$f(x,y,z) = f(2,3,4) + 60(x-2) + \frac{24}{5}(y-3) + \frac{32}{5}(z-4)$$

$$= 40 + 60x - 120 + \frac{24}{5}y - \frac{72}{5} + \frac{32}{5}z - \frac{128}{5}$$

$$f(x,y,z) = 60x + \frac{24}{5}y + \frac{32}{5}z - 120$$

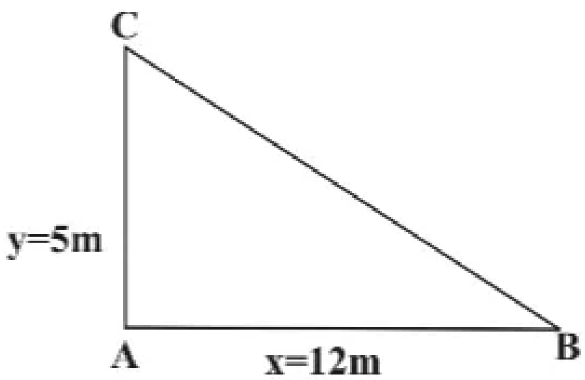
Then $(1.98)^3\sqrt{(3.01)^2 + (3.97)^2}$

$= f(1.98,\ 3.01,\ 3.97)$

$= 60(1.98) + \dfrac{24}{5}(3.01) + \dfrac{32}{5}(3.97) - 120$

$= \boxed{38.656}$

Answer 34E.



(A)

The area of the triangle is

$$A = \frac{1}{2}(x)(y)$$

Then $dA = \frac{\partial A}{\partial x}dx + \frac{\partial A}{\partial y}dy$

i.e. $dA = \frac{1}{2}y\,dx + \frac{1}{2}x\,dy$

Now $dx = dy = 0.2\text{ cm}$

And $x = 1200\text{cm}\ ,\ y = 500\text{ cm}$

Then $dA = \frac{1}{2}(500)(0.2) + \frac{1}{2}(1200)(0.2)$

$= 50 + 120$

$= 170\text{ cm}^2$

Therefore the error in area of the triangle is

$\boxed{170\text{ cm}^2}$

(B)

The length of the hypotenuse is

$$L = \sqrt{x^2 + y^2}$$

Then $dL = \frac{\partial L}{\partial x}dx + \frac{\partial L}{\partial y}dy$

$= \frac{x}{\sqrt{x^2+y^2}}dx + \frac{y}{\sqrt{x^2+y^2}}dy$

Now $x = 1200\text{cm}\ ,\ y = 500\text{cm}\ ,\ dx = dy = 0.2\text{ cm}$

Then $dL = \frac{1200}{1300}(0.2) + \frac{500}{1300}(0.2)$

$= \frac{12}{13}(0.2) + \frac{5}{13}(0.2)$

$= \frac{2.4+1}{13}$

$= \frac{3.4}{13} = 0.2615\text{ cm}$

Then the error in the length of hypotenuse is

0.2615 cm

Or $\boxed{0.002615\text{ m}}$

Answer 35E.

Consider the function

$u = x^2 y^3 + z^4$

Where $x = p + 3p^2$, $y = pe^p$ and $z = p\sin p$

Recall that, the chain rule

Suppose that $z = f(x,y)$ is differentiable function of x and y, where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t. Then z is differentiable function of t and

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

By using the chain rule, the derivative of the function $u = x^2 y^3 + z^4$ with respect to p

$$\frac{du}{dp} = \frac{\partial u}{\partial x}\frac{dx}{dp} + \frac{\partial u}{\partial y}\frac{dy}{dp} + \frac{\partial u}{\partial z}\frac{dz}{dp} \quad \text{...... (1)}$$

The partial derivative of $u = x^2 y^3 + z^4$ with respect to x is

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial}{\partial x}(u)\\ &= \frac{\partial}{\partial x}(x^2 y^3 + z^4)\\ &= 2xy^3\end{aligned}$$

The partial derivative of $u = x^2 y^3 + z^4$ with respect to y is

$$\begin{aligned}\frac{\partial u}{\partial y} &= \frac{\partial}{\partial y}(u)\\ &= \frac{\partial}{\partial y}(x^2 y^3 + z^4)\\ &= 3x^2 y^2\end{aligned}$$

The partial derivative of $u = x^2 y^3 + z^4$ with respect to z is

$$\begin{aligned}\frac{\partial u}{\partial z} &= \frac{\partial}{\partial z}(u)\\ &= \frac{\partial}{\partial z}(x^2 y^3 + z^4)\\ &= 4z^3\end{aligned}$$

The derivative of $x = p + 3p^2$ with respect to p is

$$\begin{aligned}\frac{dx}{dp} &= \frac{d}{dp}(x) \\ &= \frac{d}{dp}\left(p + 3p^2\right) \\ &= 1 + 6p\end{aligned}$$

The derivative of $y = pe^p$ with respect to p is

$$\begin{aligned}\frac{dy}{dp} &= \frac{d}{dp}(y) \\ &= \frac{d}{dp}\left(pe^p\right) \\ &= pe^p + e^p\end{aligned}$$

The derivative of $z = p\sin p$ with respect to p

$$\begin{aligned}\frac{dz}{dp} &= \frac{d}{dp}(z) \\ &= \frac{d}{dp}(p\sin p) \\ &= p\cos p + \sin p\end{aligned}$$

Substitute the values of $\frac{dx}{dp}, \frac{dy}{dp}, \frac{dz}{dp}, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$, and $\frac{\partial u}{\partial z}$ into equation (1)

$$\begin{aligned}\frac{du}{dp} &= \frac{\partial u}{\partial x}\frac{dx}{dp} + \frac{\partial u}{\partial y}\frac{dy}{dp} + \frac{\partial u}{\partial z}\frac{dz}{dp} \\ &= \left(2xy^3\right)(1 + 6p) + \left(3x^2y^2\right)\left(pe^p + e^p\right) + \left(4z^3\right)(p\cos p + \sin p) \\ &= 2xy^3(1 + 6p) + 3x^2y^2\left(pe^p + e^p\right) + 4z^3(p\cos p + \sin p)\end{aligned}$$

Therefore, the derivative of the function u with respect to p

$$\boxed{\frac{du}{dp} = 2xy^3(1 + 6p) + 3x^2y^2\left(pe^p + e^p\right) + 4z^3(p\cos p + \sin p)}$$

Answer 36E.

Consider the function:

$$v = x^2 \sin y + ye^{xy}$$

Where $x = s + 2t, \text{and } y = st$

The partial derivative of $v = x^2 \sin y + ye^{xy}$ with respect to x is:

$$\begin{aligned}\frac{\partial v}{\partial x} &= \frac{\partial}{\partial x}(v) \\ &= \frac{\partial}{\partial x}\left(x^2 \sin y + ye^{xy}\right) \\ &= 2x \sin y + ye^{xy} \cdot y \\ &= 2x \sin y + y^2 e^{xy}\end{aligned}$$

The partial derivative of $v = x^2 \sin y + ye^{xy}$ with respect to y is:

$$\begin{aligned}\frac{\partial v}{\partial y} &= \frac{\partial}{\partial y}(v) \\ &= \frac{\partial}{\partial y}\left(x^2 \sin y + ye^{xy}\right) \\ &= x^2 \cos y + 1 \cdot e^{xy} + ye^{xy} \cdot x \\ &= x^2 \cos y + e^{xy} + yxe^{xy}\end{aligned}$$

The partial derivative of $x = s + 2t$ with respect to s is:

$$\begin{aligned}\frac{\partial x}{\partial s} &= \frac{\partial}{\partial s}(x) \\ &= \frac{\partial}{\partial s}(s + 2t) \\ &= 1\end{aligned}$$

The partial derivative of $x = s + 2t$ with respect to t is

$$\begin{aligned}\frac{\partial x}{\partial t} &= \frac{\partial}{\partial t}(x) \\ &= \frac{\partial}{\partial t}(s + 2t) \\ &= 2\end{aligned}$$

The partial derivative of $y = st$ with respect to s is:

$$\begin{aligned}\frac{\partial y}{\partial s} &= \frac{\partial}{\partial s}(y) \\ &= \frac{\partial}{\partial s}(st) \\ &= t\end{aligned}$$

The partial derivative of $y = st$ with respect to t is:

$$\begin{aligned}\frac{\partial y}{\partial t} &= \frac{\partial}{\partial t}(y) \\ &= \frac{\partial}{\partial t}(st) \\ &= s\end{aligned}$$

Now, by the chain rule:

$$\frac{\partial v}{\partial s} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial s}$$

Substitute the values of $\frac{\partial x}{\partial s}, \frac{\partial y}{\partial s}, \frac{\partial v}{\partial x},$ and $\frac{\partial v}{\partial y}$ as evaluated above:

$$\frac{\partial v}{\partial s} = (2x\sin y)(1) + \left(x^2\cos y + e^{xy} + yxe^{xy}\right)(t)$$

$$\frac{\partial v}{\partial s} = 2x\sin y + t\left(x^2\cos y + e^{xy} + yxe^{xy}\right)$$

Therefore, the partial derivative of the function v with respect to s

$$\boxed{\frac{\partial v}{\partial s} = 2x\sin y + t\left(x^2\cos y + e^{xy} + yxe^{xy}\right)}$$

Again, by the chain rule:

$$\frac{\partial v}{\partial t}=\frac{\partial v}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial v}{\partial y}\frac{\partial y}{\partial t}$$

Substitute the values of $\frac{\partial x}{\partial t},\frac{\partial y}{\partial t},\frac{\partial v}{\partial x},$ and $\frac{\partial v}{\partial y}$ into the above equation:

$$\frac{\partial v}{\partial t}=(2x\sin y)(2)+\left(x^2\cos y+e^{xy}+yxe^{xy}\right)(s)$$
$$\frac{\partial v}{\partial t}=4x\sin y+s\left(x^2\cos y+e^{xy}+yxe^{xy}\right)$$

Therefore, the partial derivative of the function v with respect to t

$$\boxed{\frac{\partial v}{\partial t}=4x\sin y+s\left(x^2\cos y+e^{xy}+yxe^{xy}\right)}$$

Now, as the partial derivative has been obtained, find the value of $\frac{\partial v}{\partial s}$ and $\frac{\partial v}{\partial t}$ at $t=1$ and $s=0$

:

To do this, find the values of x and y at $t=1$ and $s=0$:

$$\begin{aligned}x&=s+2t\\&=0+2(1)\\&=2\end{aligned}$$

Also,

$$\begin{aligned}y&=st\\&=(0)(1)\\&=0\end{aligned}$$

Substitute $x=2, y=0, s=0,$ and $t=1$ into $\frac{\partial v}{\partial s}$

$$\begin{aligned}\left.\frac{\partial v}{\partial s}\right|_{s=0,t=1}&=2(2)\sin(0)+(1)\left((2)^2\cos(0)+e^{(2)(0)}+(0)(2)e^{(2)(0)}\right)\\&=0+(1)\left(4(1)+e^0+0\right)\\&=0+(1)\left(4(1)+1+0\right)\\&=5\end{aligned}$$

Therefore,

$$\boxed{\left.\frac{\partial v}{\partial s}\right|_{s=0,t=1}=5}$$

Similarly, Substitute $x=2, y=0, s=0,$ and $t=1$ into $\frac{\partial v}{\partial t}$.

$$\left.\frac{\partial v}{\partial t}\right|_{s=0,t=1} = 4(2)\sin(0)+(0)\left((2)^2\cos(0)+e^{(2)(0)}+(0)(2)e^{(2)(0)}\right)$$

$$\left.\frac{\partial v}{\partial t}\right|_{s=0,t=1} = 0+0$$

$$\left.\frac{\partial v}{\partial t}\right|_{s=0,t=1} = 0$$

Therefore,

$$\boxed{\left.\frac{\partial v}{\partial t}\right|_{s=0,t=1} = 0}$$

Answer 37E.

$$z = f(x,y)$$

Where $x = g(s,t),\ y = h(s,t)$

Then by chain rule

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

And $$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

Now when $s=1, t=2$

$$\begin{aligned}\frac{\partial z}{\partial s} &= f_x(1,2)g_s(1,2)+f_y(1,2)h_s(1,2)\\ &= f_x\left(g(1,2),h(1,2)\right)g_s(1,2)+f_y\left(g(1,2),h(1,2)\right)h_s(1,2)\\ &= f_x(3,6)g_s(1,2)+f_y(3,6)h_s(1,2)\\ &= 7(-1)+8(-5)\\ &= -7-40=-47\end{aligned}$$

And $\frac{\partial z}{\partial t} = f_x(1,2)g_t(1,2) + f_y(1,2)h_t(1,2)$

$= f_x(g(1,2), h(1,2))g_t(1,2) + f_y(g(1,2), h(1,2))h_t(1,2)$

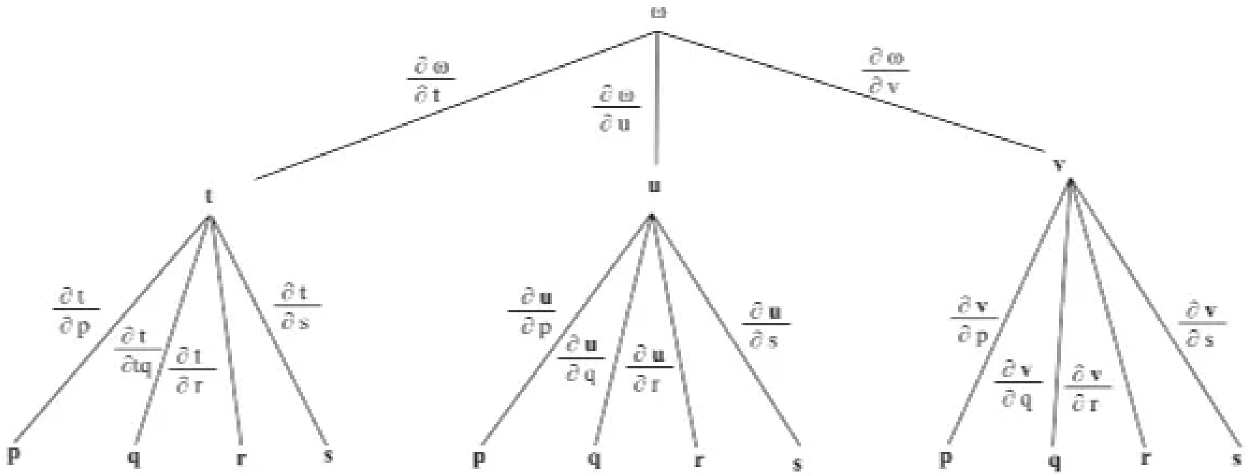
$= f_x(3,6)g_t(1,2) + f_y(3,6)h_t(1,2)$

$= 7(4) + 8(10)$

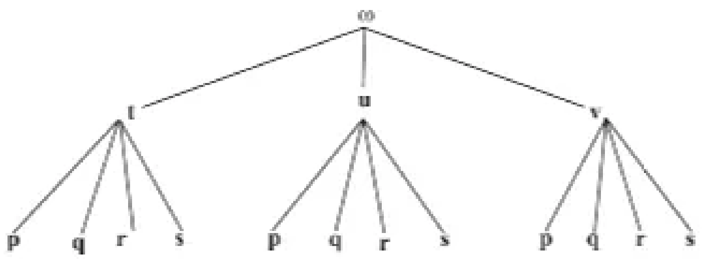
$= 28 + 80 = 108$

Hence $\boxed{\frac{\partial z}{\partial s} = -47, \quad \frac{\partial z}{\partial t} = 108}$

Answer 38E.



This is the tree diagram for the chain rule for the case:



On using tree diagram we have

$$\frac{\partial w}{\partial p} = \frac{\partial w}{\partial t}\frac{\partial t}{\partial p} + \frac{\partial w}{\partial u}\frac{\partial u}{\partial p} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial p}$$

$$\frac{\partial w}{\partial q} = \frac{\partial w}{\partial t}\frac{\partial t}{\partial q} + \frac{\partial w}{\partial u}\frac{\partial u}{\partial q} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial q}$$

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial t}\frac{\partial t}{\partial r} + \frac{\partial w}{\partial u}\frac{\partial u}{\partial r} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial r}$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial t}\frac{\partial t}{\partial s} + \frac{\partial w}{\partial u}\frac{\partial u}{\partial s} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial s}$$

Answer 39E.

$$z = y + f\left(x^2 - y^2\right)$$

Then $\dfrac{\partial z}{\partial x} = 0 + \dfrac{\partial}{\partial x} f\left(x^2 - y^2\right)$ ------------- (1)

And $\dfrac{\partial z}{\partial y} = 1 + \dfrac{\partial}{\partial y} f\left(x^2 - y^2\right)$ ------------- (2)

Consider $f\left(x^2 - y^2\right)$

Put $x^2 - y^2 = t$

Then $\dfrac{dx}{dt} = \dfrac{1}{2x}, \dfrac{dy}{dt} = \dfrac{-1}{2y}$

Therefore $\dfrac{\partial f}{\partial x} = \dfrac{\partial f}{\partial t}\dfrac{dt}{dx}$

(As f is a function of t only then $\dfrac{\partial f}{\partial t} = \dfrac{df}{dt} = f'(t)$)

$= f'(t)(2x)$

$= f'\left(x^2 - y^2\right)(2x)$ ----------- (3)

And $\dfrac{\partial f}{\partial y} = \dfrac{\partial f}{\partial t}\dfrac{dt}{dy}$

$= f'(t)(-2y)$

$= -2y\, f'\left(x^2 - y^2\right)$ ------------ (4)

Then $\dfrac{\partial z}{\partial x} = 2x\, f'\left(x^2 - y^2\right)$ {From (1) and (3)}

And $\dfrac{\partial z}{\partial y} = 1 - 2y\, f'\left(x^2 - y^2\right)$ {From (2) and (4)}

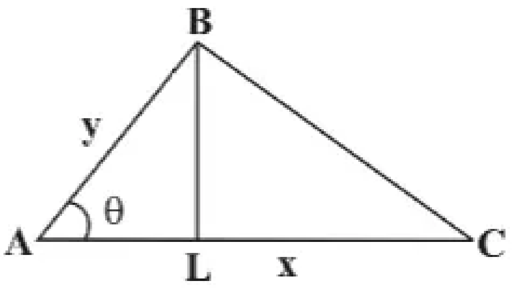
Consider $y\dfrac{\partial z}{\partial x} + x\dfrac{\partial z}{\partial y}$

$= 2xy\, f'\left(x^2 - y^2\right) - 2xy\, f'\left(x^2 - y^2\right) + x$

$= x$

Hence $y\dfrac{\partial z}{\partial x} + x\dfrac{\partial z}{\partial y} = x$

Answer 40E.



Area of triangle is $A=\frac{1}{2}xy\sin\theta$

Now $A=A(x,y,\theta)$

Then $\frac{dA}{dt}=\frac{\partial A}{\partial x}\frac{dx}{dt}+\frac{\partial A}{\partial y}\frac{dy}{dt}+\frac{\partial A}{\partial \theta}\frac{d\theta}{dt}$

Now $\frac{dx}{dt}=3,\ \frac{dy}{dt}=-2,\ \frac{d\theta}{dt}=0.05$

And $\frac{\partial A}{\partial x}=\frac{1}{2}y\sin\theta,\ \frac{\partial A}{\partial y}=\frac{1}{2}x\sin\theta,\ \frac{\partial A}{\partial \theta}=\frac{1}{2}xy=\theta$

Then $\frac{dA}{dt}=\frac{1}{2}y\sin\theta(3)+\frac{1}{2}x\sin\theta(-2)+\frac{1}{2}xy\cos\theta(0.05)$

When $x=40,\ y=50$ and $\theta=\frac{\pi}{6}$

Then $\frac{dA}{dt}=\frac{1}{2}(50)(3)\left(\frac{1}{2}\right)-\frac{1}{2}(40)\left(\frac{1}{2}\right)(2)$

$=\frac{1}{2}(40)(50)\frac{\sqrt{3}}{2}(0.05)$

$=37.5-20+43.30$

$=\boxed{60.8\text{in}^2/\text{s}}$

Answer 41E.

$$z = f(u,v)$$

Where $u = xy$, $v = y/x$

i.e. $u = u(x,y)$, $v = v(x,y)$

Then $$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}$$
$$= \frac{\partial z}{\partial u}y + \frac{\partial z}{\partial v}\left(\frac{-y}{x^2}\right)$$
$$= y\frac{\partial z}{\partial u} - \frac{y}{x^2}\frac{\partial z}{\partial v}$$

Again differentiating partially with respect to x

$$\frac{\partial^2 z}{\partial x^2} = y\left[\frac{\partial^2 z}{\partial u^2}\frac{\partial u}{\partial x} + \frac{\partial^2 z}{\partial u\,\partial v}\frac{\partial v}{\partial x}\right] + \frac{2y}{x^3}\frac{\partial z}{\partial v}$$
$$- \frac{y}{x^2}\left[\frac{\partial^2 z}{\partial u\,\partial v}\frac{\partial u}{\partial x} + \frac{\partial^2 z}{\partial v^2}\frac{\partial v}{\partial x}\right]$$
$$= y\left[\frac{\partial^2 z}{\partial u^2}y + \frac{\partial^2 z}{\partial u\,\partial v}\left(\frac{-y}{x^2}\right)\right] + \frac{y}{x^3}\frac{\partial z}{\partial v}$$
$$- \frac{y}{x^2}\left[\frac{\partial^2 z}{\partial u\,\partial v}y + \frac{\partial^2 z}{\partial u^2}\left(\frac{-y}{x^2}\right)\right]$$
$$= y^2\frac{\partial^2 z}{\partial u^2} - \frac{y^2}{x^2}\frac{\partial^2 z}{\partial u\,\partial v} + \frac{2y}{x^3}\frac{\partial z}{\partial v} - \frac{y^2}{z^2}\frac{\partial^2 z}{\partial u\,\partial v}$$
$$+ \frac{y^2}{x^4}\frac{\partial^2 z}{\partial v^2}$$

Or $$\frac{\partial^2 z}{\partial x^2} = y^2\frac{\partial^2 z}{\partial u^2} - 2\frac{y^2}{x^2}\frac{\partial^2 z}{\partial u\,\partial v} + \frac{y^2}{x^4}\frac{\partial^2 z}{\partial v^2} + 2\frac{y}{x^3}\frac{\partial z}{\partial v}$$

Now $$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$$
$$= \frac{\partial z}{\partial u}x + \frac{\partial z}{\partial v}\left(\frac{1}{x}\right)$$

Again differentiating partially with respect to y

$$\frac{\partial^2 z}{\partial y^2} = x\left[\frac{\partial^2 z}{\partial u^2}\frac{\partial u}{\partial y} + \frac{\partial^2 z}{\partial u\,\partial v}\frac{\partial v}{\partial y}\right]$$

$$+\frac{1}{x}\left[\frac{\partial^2 z}{\partial u\,\partial v}\frac{\partial u}{\partial y} + \frac{\partial^2 z}{\partial v^2}\frac{\partial v}{\partial y}\right]$$

$$= x\left[\frac{\partial^2 z}{\partial u^2}x + \frac{\partial^2 z}{\partial u\,\partial v}\left(\frac{1}{x}\right)\right]$$

$$+\frac{1}{x}\left[\frac{\partial^2 z}{\partial u\,\partial v}x + \frac{\partial^2 z}{\partial v^2}\left(\frac{1}{x}\right)\right]$$

$$= x^2\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial u\,\partial v} + \frac{\partial^2 z}{\partial u\,\partial v} + \frac{1}{x^2}\frac{\partial^2 z}{\partial v^2}$$

$$= x^2\frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u\,\partial v} + \frac{1}{x^2}\frac{\partial^2 z}{\partial v^2}$$

Consider $x^2\frac{\partial^2 z}{\partial x^2} - y^2\frac{\partial^2 z}{\partial y^2}$

$$= x^2y^2\frac{\partial^2 z}{\partial u^2} - 2y^2\frac{\partial^2 z}{\partial u\,\partial v} + \frac{y^2}{x^2}\frac{\partial^2 z}{\partial v^2} + \frac{2y}{x}\frac{\partial z}{\partial v}$$

$$-2y^2\frac{\partial^2 z}{\partial u\,\partial v} - \frac{y^2}{x^2}\frac{\partial^2 z}{\partial v^2} - x^2y^2\frac{\partial^2 z}{\partial u^2}$$

$$= -4y^2\frac{\partial^2 z}{\partial u\,\partial v} + \frac{2y}{x}\frac{\partial z}{\partial v}$$

$$= -4\left(\frac{y}{x}\right)xy\frac{\partial^2 z}{\partial u\,\partial v} + 2\left(\frac{y}{x}\right)\frac{\partial z}{\partial v}$$

$$= -4uv\frac{\partial^2 z}{\partial u\,\partial v} + 2v\frac{\partial z}{\partial v}$$

Hence $x^2\frac{\partial^2 z}{\partial x^2} - y^2\frac{\partial^2 z}{\partial y^2} = -4uv\frac{\partial^2 z}{\partial u\,\partial v} + 2v\frac{\partial z}{\partial v}$

Answer 42E.

The partial derivative of a function in two or more variables with respect to a variable is determined by keeping the other variables as a constant in the function.

Consider the function:

$$\cos(xyz) = 1 + x^2y^2 + z^2$$

Differentiate both sides of the equation with respect *x* (keeping *y* as constant):

$$\cos(xyz) = 1 + x^2y^2 + z^2$$

$$\frac{\partial}{\partial x}\left[\cos(xyz)\right] = \frac{\partial}{\partial x}\left[1 + x^2y^2 + z^2\right]$$

$$-\sin(xyz)\left[\frac{\partial}{\partial x}(xyz)\right] = 0 + y^2\frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial x}(z^2)$$

$$-\sin(xyz)\left[y\frac{\partial}{\partial x}(xz)\right] = y^2(2x) + 2z\frac{\partial z}{\partial x}$$

Continue further to evaluate the partial derivative:

$$-\sin(xyz)\left[y\left(x\frac{\partial z}{\partial x} + z\frac{\partial x}{\partial x}\right)\right] = 2xy^2 + 2z\frac{\partial z}{\partial x}$$

$$-\sin(xyz)\left[xy\frac{\partial z}{\partial x} + yz\right] = 2xy^2 + 2z\frac{\partial z}{\partial x}$$

$$-\left[xy\sin(xyz)\right]\frac{\partial z}{\partial x} - yz\sin(xyz) = 2xy^2 + 2z\frac{\partial z}{\partial x}$$

$$-yz\sin(xyz) - 2xy^2 = 2z\frac{\partial z}{\partial x} + \left[xy\sin(xyz)\right]\frac{\partial z}{\partial x}$$

Determine the value of the partial derivative from the above equation:

$$-yz\sin(xyz) - 2xy^2 = 2z\frac{\partial z}{\partial x} + \left[xy\sin(xyz)\right]\frac{\partial z}{\partial x}$$

$$-\left[yz\sin(xyz) + 2xy^2\right] = \left[2z + xy\sin(xyz)\right]\frac{\partial z}{\partial x}$$

$$\frac{\partial z}{\partial x} = -\frac{\left(2xy^2 + yz\sin(xyz)\right)}{2z + xy\sin(xyz)}$$

Differentiate both sides of the equation with respect y (keeping x as constant):

$$\cos(xyz) = 1 + x^2y^2 + z^2$$

$$\frac{\partial}{\partial y}[\cos(xyz)] = \frac{\partial}{\partial y}[1 + x^2y^2 + z^2]$$

$$-\sin(xyz)\left[\frac{\partial}{\partial y}(xyz)\right] = 0 + x^2\frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial y}(z^2)$$

$$-\sin(xyz)\left[x\frac{\partial}{\partial y}(yz)\right] = x^2(2y) + 2z\frac{\partial z}{\partial y}$$

Continue further to evaluate the partial derivative:

$$-\sin(xyz)\left[x\left(y\frac{\partial z}{\partial y} + z\right)\right] = 2x^2y + 2z\frac{\partial z}{\partial y}$$

$$-xy\sin(xyz)\frac{\partial z}{\partial y} - xz\sin(xyz) = 2x^2y + 2z\frac{\partial z}{\partial y}$$

$$-2x^2y - xz\sin(xyz) = [2z + xy\sin(xyz)]\frac{\partial z}{\partial y}$$

$$\frac{\partial z}{\partial y} = -\frac{(2x^2y + xz\sin(xyz))}{2z + xy\sin(xyz)}$$

Hence, the final value of the partial derivatives is:

$$\boxed{\begin{aligned}\frac{\partial z}{\partial x} &= -\frac{(2xy^2 + yz\sin(xyz))}{2z + xy\sin(xyz)} \\ \frac{\partial z}{\partial y} &= -\frac{(2x^2y + xz\sin(xyz))}{2z + xy\sin(xyz)}\end{aligned}}$$

Answer 43E.

If f is a function of two variables x and y, then the gradient of f is the vector function ∇f is defined by $\nabla f(x, y) = \langle f_x(x, y), f_y(x, y)\rangle$ or $\nabla f(x, y) = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$.

Find $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, and $\frac{\partial f}{\partial z}$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left[x^2 e^{yz^2}\right]$$
$$= 2xe^{yz^2}$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left[x^2 e^{yz^2}\right]$$
$$= x^2 z^2 e^{yz^2}$$

$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\left[x^2 e^{yz^2}\right]$$
$$= 2x^2 yze^{yz^2}$$

Thus, we get $\boxed{\nabla f(x, y, z) = \left\langle 2xe^{yz^2}, x^2 z^2 e^{yz^2}, 2x^2 yze^{yz^2}\right\rangle}$

Answer 44E.

The directional derivative of a function "f" in the direction of a unit vector is given by

$$D_{\vec{u}} f = \vec{\nabla} f \vec{u}$$
$$= \left|\vec{\nabla} f\right|\left|\vec{u}\right|\cos\theta$$

(Where θ is the angle between $\vec{\nabla} f$ and $\vec{u}$)

$$= \left|\vec{\nabla} f\right|\cos\theta \qquad \{\text{As } \left|\vec{u}\right| = 1\}$$

i.e. $\quad D_{\vec{u}} f = \left|\vec{\nabla} f\right|\cos\theta$

That is the value of directional derivative of function "f" in the direction of unit vector $\vec{u}$ varies with θ the angle between $\vec{\nabla} f$ and u.

(A)

Cos θ is maximum when $\theta = 0 \qquad \{\cos\theta = 1\}$

Hence $D_{\vec{u}} f$ is maximum when $\boxed{\theta = 0}$

(B)

Minimum value of $\cos\theta = -1$ and it occurs when $\theta = \pi$

Hence $D_{\mathbf{u}} f$ is minimum when $\boxed{\theta = \pi}$

(C)

$\text{Cos}\,\theta$ is 0 when $\theta = \frac{\pi}{2}$

Hence $D_{\mathbf{u}} f$ is 0 when $\boxed{\theta = \frac{\pi}{2}}$

(D)

The maximum value of $\cos\theta = 1$ and half of the maximum value is $\frac{1}{2}$ which occurs at $\theta = \frac{\pi}{3}$

Hence $D_{\mathbf{u}} f$ is half of its maximum value when $\boxed{\theta = \frac{\pi}{3}}$

Answer 45E.

We know that $D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u}$, where $\mathbf{u}$ is the unit vector and $\nabla f(x, y) = \langle f_x(x, y), f_y(x, y)\rangle$.

Find $f_x(x, y)$ and $f_y(x, y)$.

$$f_x(x, y) = \frac{\partial}{\partial x}\left(x^2 e^{-y}\right)$$
$$= 2xe^{-y}$$

$$f_y(x, y) = \frac{\partial}{\partial y}\left(x^2 e^{-y}\right)$$
$$= -x^2 e^{-y}$$

Now, compute $f_x(-2, 0)$ and $f_y(-2, 0)$.

$$f_x(-2, 0) = 2(-2)e^{-(0)}$$
$$= -4$$

$$f_y(-2, 0) = -(-2)^2 e^{-(0)}$$
$$= -4$$

Then, $\nabla f(-2, 0) = \langle -4, -4 \rangle$.

Find $\mathbf{v}$ in the direction towards $(2, -3)$ from $(-2, 0)$.

$$\mathbf{v} = \langle 2+2, -3-0 \rangle$$
$$= \langle 4, -3 \rangle$$

We have to find the unit vector $\mathbf{u}$ in the direction of $\mathbf{v}$ given by $\mathbf{u} = \dfrac{\mathbf{v}}{|\mathbf{v}|}$.

$$\mathbf{u} = \frac{\langle 4, -3 \rangle}{\sqrt{4^2 + (-3)^2}}$$
$$= \frac{\langle 4, -3 \rangle}{\sqrt{16+9}}$$
$$= \frac{\langle 4, -3 \rangle}{5}$$
$$= \left\langle \frac{4}{5}, -\frac{3}{5} \right\rangle$$

Substitute the known values in $D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u}$.

$$D_{\mathbf{u}} f(-2, 0) = \langle -4, -4 \rangle \cdot \left\langle \frac{4}{5}, -\frac{3}{5} \right\rangle$$
$$= -\frac{16}{5} + \frac{12}{5}$$
$$= -\frac{4}{5}$$

Thus, we get $\boxed{D_{\mathbf{u}} f(-2, 0) = -\frac{4}{5}}$.

Answer 46E.

$$f(x, y, z) = x^2 y + x\sqrt{1+z}$$

Then
$$f_x = 2xy + \sqrt{1+z}$$
$$f_y = x^2$$
$$f_z = \frac{x}{2\sqrt{1+z}}$$

Then gradient of "f" $\vec{\nabla} f(x,y,z) = < f_x, f_y, f_z >$

$$= < 2xy + \sqrt{1+z}, x^2, \frac{x}{2\sqrt{1+z}} >$$

At $(1,2,3)$, $\vec{\nabla} f(1,2,3) = < 6, 1, \frac{1}{4} >$

The given vector $\vec{v}$ is not a unit vector then we will find a unit vector $\vec{u}$ in the direction of $\vec{v}$ where $u = \frac{\vec{v}}{|\vec{v}|}$

Now $|\vec{v}| = \sqrt{4+1+4}$

$= \sqrt{9}$

$= 3$

Then $\vec{u} = \frac{1}{3} < 2, 1, -2 >$

Hence the directional derivative of "f" in the direction of $\vec{u}$ is

$$D_{\vec{u}} f = \vec{\nabla} f(1,2,3) . \vec{u}$$

$$= \frac{1}{3} < 6, 1, \frac{1}{4} > . < 2, 1, -2 >$$

$$= \frac{1}{3}\left(12 + 1 - \frac{1}{2}\right)$$

$$= \boxed{\frac{25}{6}}$$

Answer 47E.

Given function is $f(x,y) = x^2 y + \sqrt{y}$

Then $f_x = 2xy$, $f_y = x^2 + \frac{1}{2\sqrt{y}}$

The gradient of "f" is; $\vec{\nabla} f(x,y) = < f_x, f_y >$

$$= < 2xy, x^2 + \frac{1}{2\sqrt{y}} >$$

Then gradient f at (2,1) is $\vec{\nabla} f(2,1) = < 4, \frac{9}{2} >$

The rate of change of "f" at (2, 1) has maximum value $\left|\vec{\nabla} f(2, 1)\right|$ and it takes place in the direction of $\vec{\nabla} f$.

That is maximum rate of change of "f" is

$$\left|\vec{\nabla} f(2, 1)\right| = \sqrt{16+\frac{81}{4}}$$

$$= \boxed{\frac{\sqrt{145}}{2}}$$

And it takes place in the direction of $\boxed{<4, \frac{9}{2}>}$.

Answer 48E.

$$f(x,y,z) = ze^{xy}$$

Then $f_x = zye^{xy}$

$f_y = zxe^{xy}$

$f_z = e^{xy}$

The gradient of "f" is

$$\vec{\nabla} f(x,y,z) = < f_x, f_y, f_z >$$

$$= < zye^{xy}, zxe^{xy}, e^{xy} >$$

$$= e^{xy} < zy, zx, 1 >$$

At point $(0,1,2)$

$$\vec{\nabla} f(0,1,2) = <2, 0, 1>$$

We know "f" increases most rapidly in the direction of $\vec{\nabla} f$ and the maximum rate of change of "f" is $\left|\vec{\nabla} f\right|$

That is "f" increases most rapidly in the direction

$$\boxed{<2,0,1>}$$

And the maximum rate of change of "f" is

$$\left|\vec{\nabla} f(2,0,1)\right| = \sqrt{4+0+1}$$

$$= \boxed{\sqrt{5}}$$

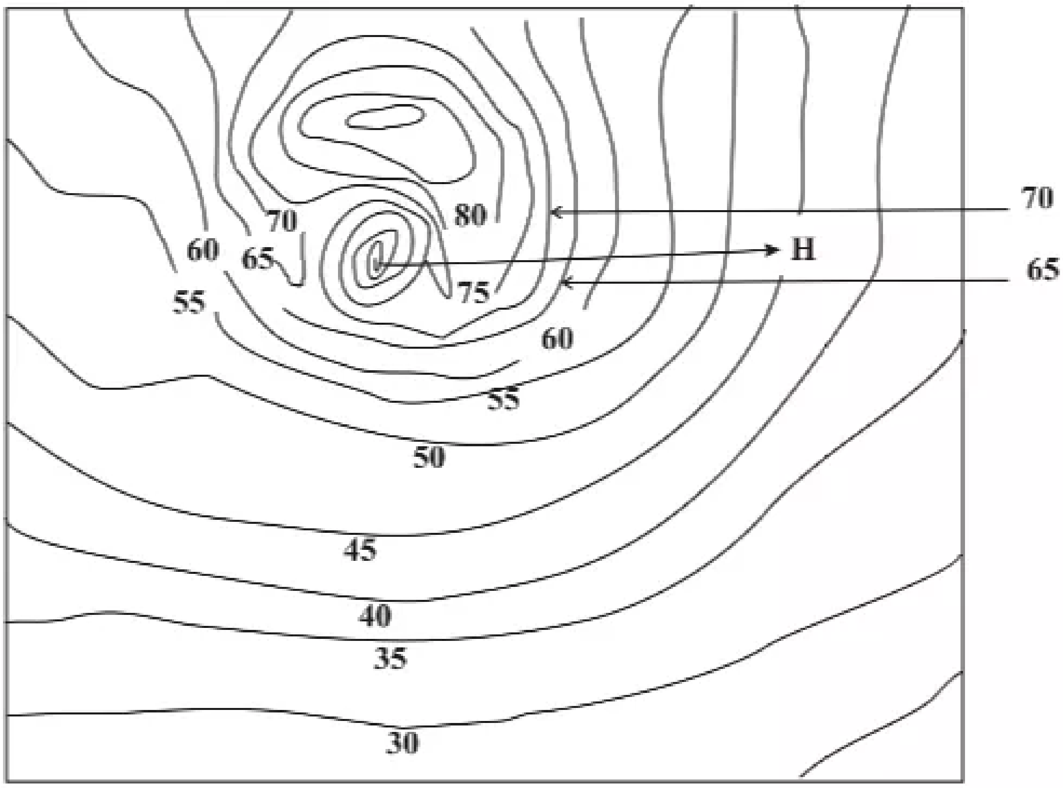
Answer 49E.

First of all we draw a line through place H in the direction of the eye of hurricane. We approximate the directional derivative of wind speed (S) by the average rate of change of the wind speed between the points where this line intersects the contour lines $S = 45$ and $S = 50$

The wind – speed at the point east of place H is $S = 45$ knots and the wind – speed at the point west of place h is $S = 50$ knots. The distance between these two points looks to be about 8 miles. So the rate of change of wind speed in the direction of hurricane is

$$D_u S = \frac{50-45}{8}$$

$$= \boxed{\frac{5}{8} \text{ knot/mi}}$$



Answer 50E.

The given surface is $z = 2x^2 - y^2$

Take $f(x,y,z) = z - 2x^2 + y^2$

Then $f_x = -4x$, $f_y = 2y$, $f_z = 1$

And thus the gradient of "f" is

$$\vec{\nabla} f(x,y,z) = < f_x,\ f_y,\ f_z >$$

i.e. $\vec{\nabla} f(x,y,z) = < -4x,\ 2y,\ 1 >$

And then $\vec{\nabla} f(-2,2,4) = < 8,\ 4,1 >$

Now the given plane is $z = 4$

Take $g(x,y,z) = z - 4$

Then $g_x = 0,\ g_y = 0,\ g_z = 1$

And then $\vec{\nabla} g(x,y,z) = < g_x,\ g_y, g_z >$

$$= < 0,\ 0,\ 1 >$$

And $\vec{\nabla} g(-2,\ 2,\ 4) = < 0,\ 0,1 >$

The tangent at the point of intersection of two curves is perpendicular to both $\vec{\nabla} f$ and $\vec{\nabla} g$ at $(-2,\ 2,\ 4)$ and therefore the vector $\vec{v} = \vec{\nabla} f \times \vec{\nabla} g$ will be parallel to the tangent line

$$\vec{v} = \vec{\nabla} f \times \vec{\nabla} g = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 8 & 4 & 1 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= 4\hat{i} - 8\hat{j}$$

$$= < 4,\ -8,\ 0 >$$

Then the direction numbers of the tangent line are same as of "$\vec{v}$" i.e. $< 4,\ -8, 0 >$

Now the direction numbers of tangent line are $< 4, -8,\ 0 >$ and it passes through $(-2,\ 2,\ 4)$

And therefore its parametric equation is

$$\frac{x+2}{4} = \frac{y-2}{-8} = \frac{z-4}{0}$$

Or $\boxed{x = 4t - 2,\ y = 2 - 8t,\ z = 4}$

Answer 51E.

$$f(x,y) = x^2 - xy + y^2 + 9x - 6y + 10$$

Then $f_x = 2x - y + 9$

$$f_y = -x + 2y - 6$$

First we find critical points by setting $f_x = 0$, $f_y = 0$

i.e. $2x - y + 9 = 0$

And $-x + 2y - 6 = 0$

On solving these equations we find; $y = 1$, $x = -4$

Then the only critical point is (-4, 1)

Now $f_{xx} = 2$, $f_{yy} = 2$, $f_{xy} = -1$

Then $D = f_{xx}f_{yy} - f_{xy}^2$

$= 2(2) - 1$

$= 3$

Now $f_{xx} = 2$, $f_{yy} = 2$, $f_{xy} = -1$

Then $D = f_{xx}f_{yy} - f_{xy}^2$

$= 2(2) - 1$

$= 3$

Answer 52E.

$$f(x,y) = x^3 - 6xy + 8y^3$$

Then $f_x(x,y) = 3x^2 - 6y$

$f_y(x,y) = -6x + 24y^2$

First we find the critical points by setting $f_x = 0$, $f_y = 0$

i.e. $3x^2 - 6y = 0$

And $-6x + 24y^2 = 0$

i.e. $x^2 - 2y = 0$

And $x - 4y^2 = 0$

On solving these equations we find the critical points are; $(0,0)$, $\left(1, \frac{1}{2}\right)$

Now $f_{xx} = 6x$, $f_{yy} = 48y$, $f_{xy} = -6$

Then $D = f_{xx}\, f_{yy} - f_{xy}^2$

$= 288xy - 36$

At (0, 0); $D = -36 < 0$

That is $\boxed{(0, 0)}$ is a saddle point

At $\left(1, \frac{1}{2}\right)$; $D = 108 > 0$ and $f_{xx} = 6 > 0$

That is $\left(1, \frac{1}{2}\right)$, is a point of local minima and the minimum value of "f" is

$$f\left(1, \frac{1}{2}\right) = (1)^3 - 6(1)(1/2) + 8(1/8)$$
$$= \boxed{-1}$$

Answer 53E.

$$f(x, y) = 3xy - x^2y - xy^2$$

Then $f_x = 3y - 2xy - y^2$

$f_y = 3x - x^2 - 2xy$

First we find the critical points by setting $f_x = 0$, $f_y = 0$

i.e. $3y - 2xy - y^2 = 0$

And $3x - x^2 - 2xy = 0$

i.e. $y(3 - 2x - y) = 0$

And $x(3 - x - 2y) = 0$

On solving we find the critical points

$$(0,0),\ (3,0),\ (0,3),\ (1,1)$$

Now $f_{xx} = -2y$, $f_{yy} = -2x$, $f_{xy} = 3 - 2x - 2y$

Then $D = f_{xx} f_{yy} - f_{xy}^2$

$= 4xy - (3 - 2x - 2y)^2$

At (0, 0); $D = 0 - (3)^2$

$= -9 < 0$

That is (0, 0), is a saddle point

At (3, 0); $D = 0 - (3 - 6)^2$

$= -9 < 0$

That is (3, 0), is a saddle point

At (0, 3); $D = 0 - (3 - 6)$

$= -9 < 0$

That is (0, 3), is a saddle point

At (1, 1); $D = 4 - (3-2-2)^2$

$= 4 - 1$

$= 3 > 0$

And $f_{xx} = -2 < 0$

That is (1, 1), is a point of local maximum

And the maximum value of "f" is

$f(1,1) = 1$

Hence saddle points; $\boxed{(0,0), (3,0), (0,3)}$

And point of local maximum is (1, 1) with f(1, 1) = $\boxed{1}$

Answer 54E.

Find the local maximum and local minimum values and saddle points of the Function $f(x,y) = (x^2+y)e^{y/2}$

First, find the partial derivatives of the function $f(x,y) = (x^2+y)e^{y/2}$.

$$f_x = \frac{\partial}{\partial x}\left((x^2+y)e^{y/2}\right) \quad \text{Treat } y \text{ constant}$$

$$= e^{y/2}\frac{\partial}{\partial x}(x^2+y) \quad \text{Use the formula } \frac{d}{dx}(k \cdot f(x)) = k \cdot \frac{d}{dx}(f(x))$$

$$= e^{y/2}(2x+0) \quad \left\{\text{Use the formula } \frac{d}{dx}(x^n) = n \cdot x^{n-1}\right.$$

$$= 2xe^{y/2} \quad \text{Simplify}$$

Hence, $f_x = 2xe^{y/2}$.

$$f_y = \frac{\partial}{\partial y}\left((x^2+y)e^{y/2}\right) \quad \text{Treat } x \text{ constant}$$

$$= e^{y/2}\frac{\partial}{\partial y}(x^2+y) + (x^2+y)\frac{\partial}{\partial y}\left(e^{y/2}\right) \quad \left\{\begin{array}{l}\text{Use product rule} \\ \frac{d}{dx}(f \cdot g) = \begin{cases} g \cdot \frac{d}{dx}(f) \\ + f \cdot \frac{d}{dx}(g) \end{cases}\end{array}\right.$$

$$= e^{y/2}(0+1) + (x^2+y)e^{y/2}\left(\frac{y}{2}\right)' \quad \left\{\begin{array}{l}\text{Use chain rule} \\ \frac{d}{dx}\left(e^{f(x)}\right) = e^{f(x)} f'(x)\end{array}\right.$$

$$= e^{y/2} + \left(\frac{x^2+y}{2}\right)e^{y/2} \quad \text{Simplify}$$

$$= \left(1 + \frac{y+x^2}{2}\right)e^{y/2}$$

Hence, $f_y = \left(1+\dfrac{y+x^2}{2}\right)e^{y/2}$.

Find the critical points of the function $f(x,y) = (x^2+y)e^{y/2}$.

Solve the equations $f_x = 0$ and $f_y = 0$ to get the coordinates of the critical points.

$$f_x = 0$$
$$2xe^{y/2} = 0$$
$$x = 0 \quad \text{Since } e^{y/2} \neq 0$$
$$f_y = 0$$
$$\left(1+\frac{x^2+y}{2}\right)e^{y/2} = 0$$
$$1+\frac{x^2+y}{2} = 0 \quad \text{Since } e^{y/2} \neq 0$$
$$1+\frac{0^2+y}{2} = 0 \quad \text{Put } x = 0$$
$$y = -2 \quad \text{Solve for } y$$

Hence, the critical point is $(0,-2)$.

Find the second derivatives of the function $f(x,y) = (x^2+y)e^{y/2}$.

$$f_{xx} = \frac{\partial}{\partial x}\left(2xe^{y/2}\right) \quad \text{Treat } y \text{ constant}$$
$$= 2e^{y/2}\frac{\partial}{\partial x}(x) \quad \begin{cases}\text{Use the formula} \\ \dfrac{d}{dx}(k\cdot f(x)) = k\cdot\dfrac{d}{dx}(f(x))\end{cases}$$
$$= 2e^{y/2} \quad \begin{cases}\text{Use the formula} \\ \dfrac{d}{dx}(x^n) = n\cdot x^{n-1}\end{cases}$$

Hence, $f_{xx} = 2e^{y/2}$.

Continue the above

$$f_{yy} = \frac{\partial}{\partial y}\left[\left(1+\frac{y+x^2}{2}\right)e^{y/2}\right]$$

$$= e^{\frac{y}{2}}\frac{\partial}{\partial y}\left[1+\frac{y+x^2}{2}\right]+\left(1+\frac{y+x^2}{2}\right)\frac{\partial}{\partial y}\left[e^{\frac{y}{2}}\right] \quad \left\{\text{Use product rule of differentiation}\right.$$

$$= e^{\frac{y}{2}}\left(\frac{1}{2}\right)+\left(1+\frac{y+x^2}{2}\right)\left(\frac{1}{2}e^{\frac{y}{2}}\right) \quad \left\{\begin{array}{l}\text{Use chain rule}\\ \frac{d}{dx}\left(e^{f(x)}\right)=e^{f(x)}f'(x)\\ \text{for the last term}\end{array}\right.$$

$$= \frac{\left(4+y+x^2\right)e^{y/2}}{4} \quad \text{Simplify}$$

Hence $f_{yy} = \frac{\left(4+y+x^2\right)e^{y/2}}{4}$.

$$f_{xy} = \frac{\partial}{\partial y}\left(2xe^{y/2}\right) \quad \text{Treat } x \text{ constant}$$

$$= 2x\frac{\partial}{\partial y}\left(e^{y/2}\right) \quad \left\{\begin{array}{l}\text{Use the formula}\\ \frac{d}{dx}\left(k\cdot f(x)\right)=k\cdot\frac{d}{dx}\left(f(x)\right)\end{array}\right.$$

$$= 2xe^{y/2}\left(\frac{y}{2}\right)' \quad \left\{\begin{array}{l}\text{Use chain rule}\\ \frac{d}{dx}\left(e^{f(x)}\right)=e^{f(x)}f'(x)\end{array}\right.$$

$$= 2xe^{y/2}\left(\frac{1}{2}\right) \quad \left\{\begin{array}{l}\text{Use the formula}\\ \frac{d}{dx}\left(x^n\right)=n\cdot x^{n-1}\end{array}\right.$$

$$= xe^{y/2} \quad \text{Simplify}$$

Hence $f_{xy} = xe^{y/2}$.

Hence, $f_{xx} = 2e^{y/2}$.

Continue the above

$$f_{yy} = \frac{\partial}{\partial y}\left[\left(1+\frac{y+x^2}{2}\right)e^{y/2}\right]$$

$$= e^{\frac{y}{2}}\frac{\partial}{\partial y}\left[1+\frac{y+x^2}{2}\right]+\left(1+\frac{y+x^2}{2}\right)\frac{\partial}{\partial y}\left[e^{\frac{y}{2}}\right] \quad \begin{cases}\text{Use product rule of}\\ \text{differentiation}\end{cases}$$

$$= e^{\frac{y}{2}}\left(\frac{1}{2}\right)+\left(1+\frac{y+x^2}{2}\right)\left(\frac{1}{2}e^{\frac{y}{2}}\right) \quad \begin{cases}\text{Use chain rule}\\ \frac{d}{dx}\left(e^{f(x)}\right)=e^{f(x)}f'(x)\\ \text{for the last term}\end{cases}$$

$$= \frac{\left(4+y+x^2\right)e^{y/2}}{4} \quad \text{Simplify}$$

Hence $f_{yy} = \dfrac{\left(4+y+x^2\right)e^{y/2}}{4}$.

$$f_{xy} = \frac{\partial}{\partial y}\left(2xe^{y/2}\right) \quad \text{Treat } x \text{ constant}$$

$$= 2x\frac{\partial}{\partial y}\left(e^{y/2}\right) \quad \begin{cases}\text{Use the formula}\\ \frac{d}{dx}\left(k\cdot f(x)\right)=k\cdot\frac{d}{dx}\left(f(x)\right)\end{cases}$$

$$= 2xe^{y/2}\left(\frac{y}{2}\right)' \quad \begin{cases}\text{Use chain rule}\\ \frac{d}{dx}\left(e^{f(x)}\right)=e^{f(x)}f'(x)\end{cases}$$

$$= 2xe^{y/2}\left(\frac{1}{2}\right) \quad \begin{cases}\text{Use the formula}\\ \frac{d}{dx}\left(x^n\right)=n\cdot x^{n-1}\end{cases}$$

$$= xe^{y/2} \quad \text{Simplify}$$

Hence $f_{xy} = xe^{y/2}$.

Use the second derivative test.

According to the second derivative test, if $f_{xx} > 0$ and $D > 0$ at the critical point then the function has local minimum at that point.

So the given function $f(x,y) = (x^2 + y)e^{y/2}$ has its local minimum at the critical point $(0,-2)$.

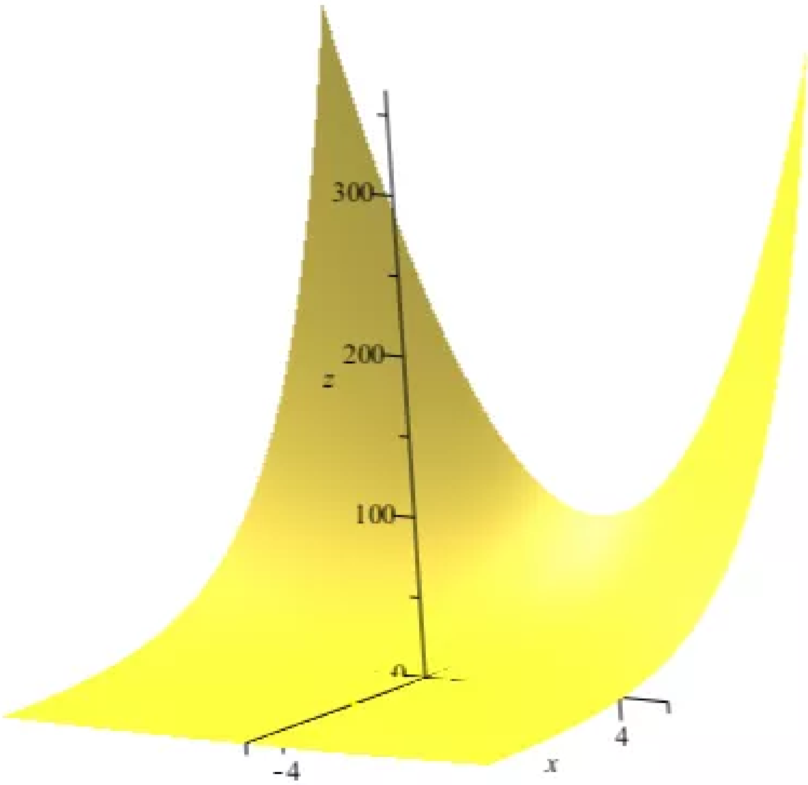
Substitute 0 for *x* and -2 for *y* in $f(x,y) = (x^2 + y)e^{y/2}$ to obtain its local minimum.

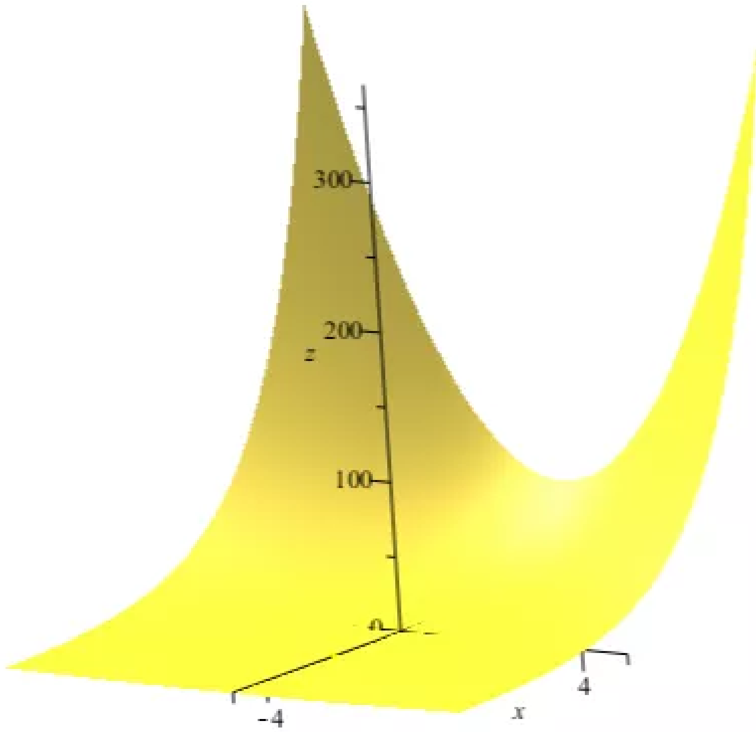
$$f(0,-2) = (0^2 - 2)e^{-2/2}$$
$$= \frac{-2}{e}$$

Hence the local minimum of the given function is $\boxed{\frac{-2}{e} \approx -0.74}$.

Moreover the function has **no local maximum.**

The following sketch supports the solution.





Answer 55E.

Consider the function:

$$f(x,y)=4xy^2-x^2y^2-xy^3$$

Since the function f is a polynomial function, so it is continuous on the closed bounded region D, then there exist both an absolute maximum and an absolute minimum of the function on this region.

Find the partial derivatives of the function:

$$f_x=4y^2-2xy^2-y^3$$

And

$$f_y=8xy-2x^2y-3xy^2$$

First of all, find critical points by of the function, for that set $f_x=0$ & $f_y=0$, then:

$$y^2(4-2x-y)=0$$

Also,

$$xy(8-2x-3y)=0$$

Solve the above equations simultaneously, and then the critical points are:

$(0,0)$, $(0,4)$, $(1,2)$

The value of the given function on its critical points is:

$$f(0,0)=0$$
$$f(0,4)=0 \quad \text{...... (1)}$$
$$f(1,2)=4$$

Now, look at the value of the function f on the boundary of D, which consists of the line segments as L_1, L_2, L_3 shown in the figure.

On the L_1, $x=0$ and $0 \le y \le 6$ then the function on this line is given by:

$$f(0,y)=0, 0 \le y \le 6$$

This is a constant function. Hence, on the line L_1 function f has zero value.

On L_2, $y=0$ and $0 \le x \le 6$ then function on this line is given by:

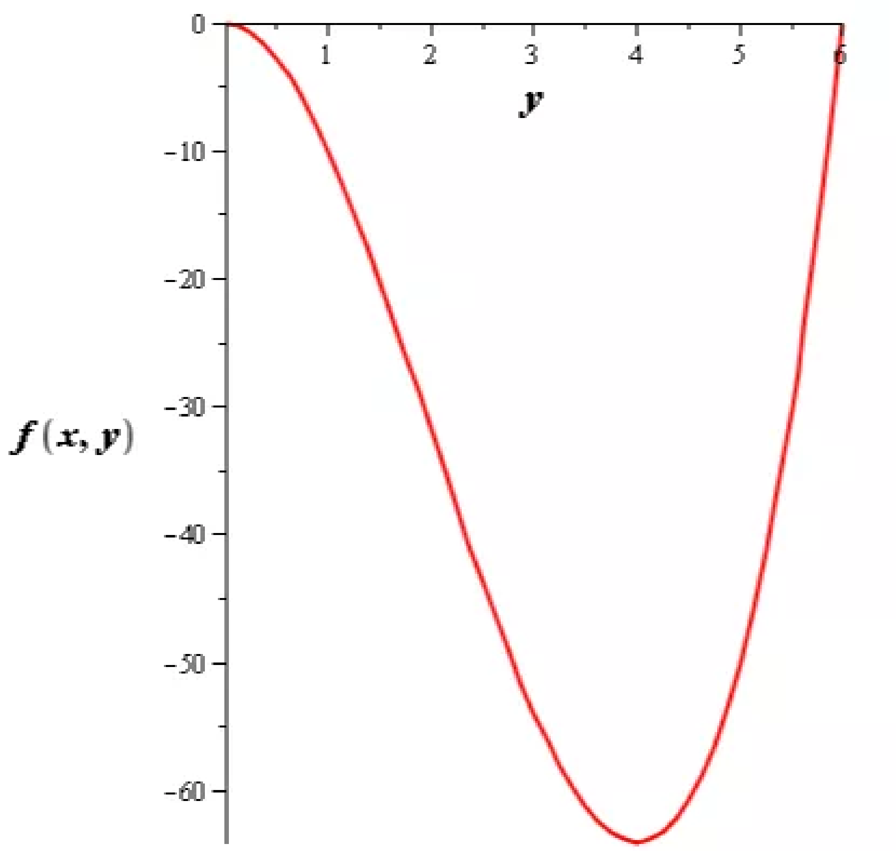
$$f(x,0)=0, 0 \le x \le 6$$

This is a constant function. Hence, on the line L_1 function f has zero value.

Now, on L_3, $x=6-y$ and $0 \le y \le 6$:

$$\begin{aligned} f(x,y) &= 4(6-y)y^2-(6-y)^2 y^2-(6-y)y^3 \\ &= 24y^2-4y^3-y^4+12y^3-36y^2-6y^3+y^4 \\ &= 2y^3-12y^2, 0 \le y \le 6 \end{aligned}$$

Draw the graph of this function:



From the graph function is decreasing for $0 \le y \le 4$ and increasing for $4 \le y \le 6$.

Then for $0 \le y \le 4$, the maximum value of function is $f(6,0)=0$ and minimum value is $f(2,4)=-64$.

And for $4 \le y \le 6$, the minimum value of function is $f(2,4)=-64$ and maximum value is $f(0,6)=0$.

Now, compare these values of f and those values obtained on its critical points, then absolute maximum of the function is $\boxed{f(1,2)=4}$ and the absolute minimum is $\boxed{f(2,4)=-64}$.

Answer 56E.

Consider the following function:

$$f(x,y)=e^{-x^2-y^2}\left(x^2+2y^2\right)$$

To get the absolute maximum and absolute minimum of the above function, first determine the partial derivatives:

$$\begin{aligned} f_x &= -2xe^{-x^2-y^2}\left(x^2+2y^2\right)+2xe^{-x^2-y^2} \\ &= -2xe^{-x^2-y^2}\left[x^2+2y^2-1\right] \end{aligned}$$

And

$$\begin{aligned} f_y &= -2ye^{-x^2-y^2}\left(x^2+2y^2\right)+4ye^{-x^2-y^2} \\ &= -2ye^{-x^2-y^2}\left[x^2+2y^2-2\right] \end{aligned}$$

Find the critical points:

$f_x,\ f_y = 0$

That is,

$x\left[x^2 + 2y^2 - 1\right] = 0$

And

$y\left[x^2 + 2y^2 - 2\right] = 0$

Solve the above equations simultaneously for $x \,\&\, y$:

$(0,0)$, $(0,\pm 1)$, $(\pm 1,0)$

Which are in region D.

The constraint equation on the boundary of D is:

$$\begin{aligned} g(x,y) &= x^2 + y^2 \\ &= 4 \end{aligned}$$

Then by Lagrange's method of multipliers find all $x, y \,\&\, \lambda$ such that:

$\vec{\nabla} f(x,y) = \lambda \vec{\nabla} g(x,y)$

Also, note that

$g(x,y) = 4$

Hence, by the above equation following set of equation is obtained:

$-2x\left(x^2 + 2y^2 - 1\right)e^{-x^2-y^2} = 2\lambda y$ ...... (1)

$-2y\left(x^2 + 2y^2 - 2\right)e^{-x^2-y^2} = 2\lambda y$ ...... (2)

$x^2 + y^2 = 4$ ...... (3)

If $x, y \neq 0$ then from (1) and (2):

$\left(x^2 + 2y^2 - 1\right)e^{-x^2-y^2} = \left(x^2 + 2y^2 - 2\right)e^{-x^2-y^2}$

Compare the coefficients:

$x^2 + 2y^2 - 1 = x^2 + 2y^2 - 2$

But the above condition is not possible.

So, either $x = 0$ or $y = 0$:

If $x = 0$, $\lambda = 0$ then $y = \pm 2$.

Similarly, if $y = 0$, $\lambda = 0$ then $x = \pm 2$.

Then the extreme points are:

$(\pm 2,\ 0)$, $(0,\ \pm 2)$

Hence, by the above equation following set of equation is obtained:

$$-2x\left(x^2+2y^2-1\right)e^{-x^2-y^2}=2\lambda y \text{ ...... (1)}$$

$$-2y\left(x^2+2y^2-2\right)e^{-x^2-y^2}=2\lambda y \text{ ...... (2)}$$

$$x^2+y^2=4 \text{ ...... (3)}$$

If $x, y \neq 0$ then from (1) and (2):

$$\left(x^2+2y^2-1\right)e^{-x^2-y^2}=\left(x^2+2y^2-2\right)e^{-x^2-y^2}$$

Compare the coefficients:

$$x^2+2y^2-1=x^2+2y^2-2$$

But the above condition is not possible.

So, either $x=0$ or $y=0$:

If $x=0$, $\lambda=0$ then $y=\pm 2$,

Similarly, if $y=0$, $\lambda=0$ then $x=\pm 2$.

Then the extreme points are:

$(\pm 2, 0)$, $(0, \pm 2)$

Answer 57E.

Consider the function

$$f(x,y)=x^3-3x+y^4-2y^2$$

Find the local maximum, minimum values and saddle points of the function.

Let's use maple to estimate the local maximum, minimum and saddle points.

First load the package with(plots);

Then enter the function as follows.

f:=(x,y)and then select right arrowx^3=3*x+y^4-2*y^2;

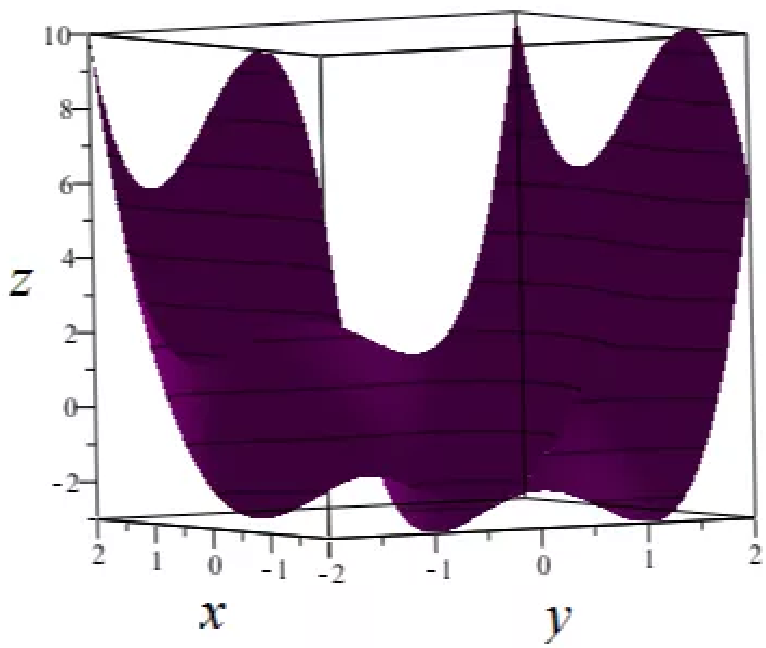
> $f:=(x,y)\rightarrow x^3-3x+y^4-2y^2$

$f:=(x,y)\rightarrow x^3-3x+y^4-2y^2$

To plot the graph, enter the following command.

>plot3d(f(x,y),x=-2..2,y=-2..2,style=patchcontour,color=purple,axes=boxed);

> $plot3d(f(x,y), x=-2..2, y=-2..2, style=patchcontour, color=purple, axes=boxed);$

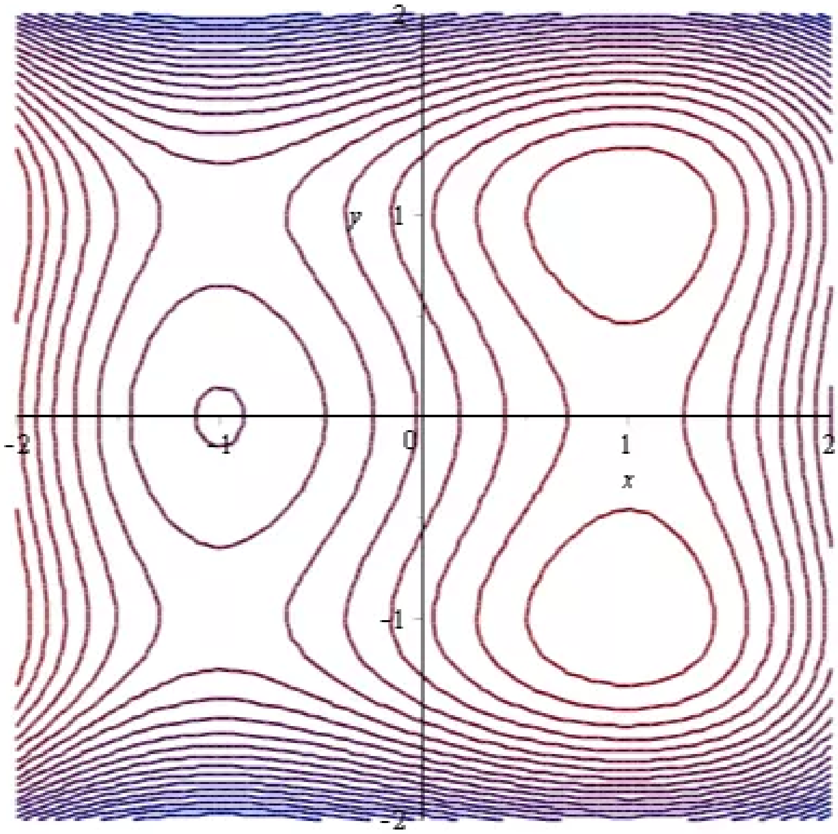


Find the contour plot so that it can found the points at which the function has maximum or minimum.

For contour plot, enter the following maple command.

plots[contourplot](f(x,y),x=-2..2,y=-2..2,contours=20);

> *plots*[*contourplot*]($f(x,y)$, $x = -2..2$, $y = -2..2$, *contours* = 20);



From the graph, it seems that the function has a local maximum at about $(-1,0)$, local minimum at about $(1,\pm 1)$, saddle point at about $(-1,\pm 1)$ and saddle point at $(1,0)$.

The partial derivative of $f(x,y)$ with respect to x is

$$f_x(x,y) = \frac{\partial}{\partial x}(f(x,y))$$
$$= \frac{\partial}{\partial x}(x^3 - 3x + y^4 - 2y^2)$$
$$= 3x^2 - 3 \quad \text{...... (1)}$$

The partial derivative of $f(x,y)$ with respect to y,

$$f_y(x,y) = \frac{\partial}{\partial y}(f(x,y))$$
$$= \frac{\partial}{\partial y}(x^3 - 3x + y^4 - 2y^2)$$
$$= 4y^3 - 4y \quad \text{...... (2)}$$

Equate equation (1) to zero.

$$3x^2 - 3 = 0$$
$$3(x^2 - 1) = 0$$
$$x^2 - 1 = 0$$
$$x^2 = 1$$
$$x = \pm 1$$

Equate equation (2) to zero.

$$4y^3 - 4y = 0$$
$$4y(y^2 - 1) = 0$$
$$y = 0 \text{ or } y = \pm 1$$

Thus, the critical points of the function are $(1,0)$, $(1,\pm 1)$, $(-1,0)$ and $(-1,\pm 1)$.

The second partial derivative of $f(x,y)$ with respect to x is

$$\begin{aligned} f_{xx}(x,y) &= \frac{\partial}{\partial x}\left(f_x(x,y)\right) \\ &= \frac{\partial}{\partial x}\left(3x^2-3\right) \\ &= 6x \end{aligned}$$

The second partial derivative of $f(x,y)$ with respect to y is

$$\begin{aligned} f_{yy}(x,y) &= \frac{\partial}{\partial y}\left(f_y(x,y)\right) \\ &= \frac{\partial}{\partial y}\left(4y^3-4y\right) \\ &= 12y^2-4 \end{aligned}$$

The partial derivative of $f_y(x,y)$ with respect to x is

$$\begin{aligned} f_{xy}(x,y) &= \frac{\partial}{\partial x}\left(f_y(x,y)\right) \\ &= \frac{\partial}{\partial x}\left(4y^3-4y\right) \\ &= 0 \end{aligned}$$

Recall that the theorem,

Suppose the second partial derivatives of f are continuous on a disk with center (a,b) and suppose that $f_x(a,b)=0$ and $f_y(a,b)=0$ [(a,b) is a critical point of f].

Let $D(a,b)=f_{xx}(a,b)f_{yy}(a,b)-\left(f_{xy}(a,b)\right)^2$.

(a) If $D>0$ and $f_{xx}(a,b)>0$ then $f(a,b)$ is a local minimum.

(b) If $D>0$ and $f_{xx}(a,b)<0$ then $f(a,b)$ is a local maximum.

(c) If $D<0$ then $f(a,b)$ is not a local maximum or local minimum.

Use this theorem to find behavior of f at the critical points.

At the critical point $(1,0)$.

$$\begin{aligned} D(a,b) &= f_{xx}(a,b) f_{yy}(a,b) - (f_{xy}(a,b))^2 \\ &= (6x)(12y^2 - 4) - [0]^2 \\ D(1,0) &= (6)(-4) \\ &= -24 \end{aligned}$$

Since $D < 0$, the point $(1,0)$ is a saddle point.

At the critical point $(1,\pm 1)$.

$$\begin{aligned} D(a,b) &= f_{xx}(a,b) f_{yy}(a,b) - (f_{xy}(a,b))^2 \\ &= (6x)(12y^2 - 4) - [0]^2 \\ D(1,\pm 1) &= (6)(8) \\ &= 24 \end{aligned}$$

> 0

And

$$\begin{aligned} f_{xx}(x,y) &= 6x \\ f_{xx}(1,\pm 1) &= 6 \end{aligned}$$

Since $D > 0$ and $f_{xx} > 0$, the function has local minimum at $(1,\pm 1)$.

$$\begin{aligned} f(x,y) &= x^3 - 3x + y^4 - 2y^2 \\ f(1,\pm 1) &= (1)^3 - 3(1) + (\pm 1)^4 - 2(\pm 1)^2 \\ &= 1 - 3 + 1 - 2 \\ &= -3 \end{aligned}$$

At the critical point $(-1,\pm1)$,

$$D(a,b) = f_{xx}(a,b)f_{yy}(a,b) - (f_{xy}(a,b))^2$$
$$= (6x)(12y^2 - 4) - [0]^2$$
$$D(-1,\pm1) = (-6)(8)$$
$$= -24$$

< 0

Since $D < 0$, the point $(-1,\pm1)$is a saddle point.

At the critical point $(1,0)$,

$$D(a,b) = f_{xx}(a,b)f_{yy}(a,b) - (f_{xy}(a,b))^2$$
$$= (6x)(12y^2 - 4) - [0]^2$$
$$\mathrm{D}(1,0) = (6)(0)$$
$$= 6$$

And

$$f_{xx}(x,y) = 6x$$
$$f_{xx}(1,0) = 6$$

Since $D > 0$ and $f_{xx} < 0$, the function has local maximum at $(1,0)$.

$$f(x,y) = x^3 - 3x + y^4 - 2y^2$$
$$f(1,0) = (1)^3 - 3(1) + (0)^4 - 2(0)^2$$
$$= 1 - 3 + 0 - 0$$
$$= -2$$

Thus, conclude that function $f(x,y) = x^3 - 3x + y^4 - 2y^2$ has a local maximum at about $\boxed{(-1,0)}$, local minimum at about $\boxed{(1,\pm1)}$, saddle point at about $\boxed{(-1,\pm1)}$ and saddle point at $\boxed{(1,0)}$.

Answer 58E.

Consider the function

$$f(x,y)=12+10y-2x^2-8xy-y^4$$

Use maple to find the critical points of the above function.

Enter the function as follows.

f:=(x, y) and then click right arrow (12+10*y-2*x^2-8*x*y-y^4);

The maple output as follows.

> $f := (x, y) \rightarrow (12 + 10*y - 2*x^{\wedge}2 - 8*x*y - y^{\wedge}4);$

$$f := (x, y) \rightarrow 12 + 10y - 2x^2 - 8xy - y^4$$

To find the partial derivative of the function with respect to *x*, use the following maple command

fx:=diff (f(x, y),x);

The maple output as follows.

> $fx := diff(f(x, y), x);$

$$fx := -4x - 8y$$

To find the partial derivative of the function with respect to *y*, use the following maple command

fy:=diff (f(x, y),y);

> $fy := diff(f(x, y), y);$

$$fy := -4y^3 - 8x + 10$$

To find the critical points, solve the equations $f_x = 0$ and $f_y = 0$.

For this apply the below commands.

>solve({fx=0,fy=0},{x,y}):

>map(allvalues,{%}):

>evalf(%);

The maple input followed by output as shown below:

> *solve*({*fx* = 0, *fy* = 0}, {*x*, *y*}) :

> *map*(*allvalues*, {%}) :

> *evalf*(%);

$$\{\{\{x = -4.519438914 + 2.\,10^{-10}\,I, y = 2.259719457 - 1.\,10^{-10}\,I\}, \{x = 1.434489835 - 1.832050808\,10^{-9}\,I, y = -0.7172449172 + 8.660254040\,10^{-10}\,I\}, \{x = 3.084949079 + 1.632050808\,10^{-9}\,I, y = -1.542474539 - 8.660254040\,10^{-10}\,I\}\}\}$$

Round the answer to three decimal places

Thus, the critical points of the function $f(x,y) = 12 + 10y - 2x^2 - 8xy - y^4$ are $\boxed{(-4.519, 2.260), (1.434, -0.717)}$ and $\boxed{(3.085 - 1.542)}$.

To find the critical points, solve the equations $f_x = 0$ and $f_y = 0$.

For this apply the below commands.

>solve({fx=0,fy=0},{x,y}):

>map(allvalues,{%}):

>evalf(%);

The maple input followed by output as shown below:

> *solve*({ *fx* = 0, *fy* = 0}, {*x*, *y*}) :

> *map*(*allvalues*, {%}) :

> *evalf*(%);

$$\{\{\{x = -4.519438914 + 2.\,10^{-10}\,\mathrm{I}, y = 2.259719457 - 1.\,10^{-10}\,\mathrm{I}\}, \{x = 1.434489835 - 1.832050808\,10^{-9}\,\mathrm{I}, y = -0.7172449172 + 8.660254040\,10^{-10}\,\mathrm{I}\}, \{x = 3.084949079 + 1.632050808\,10^{-9}\,\mathrm{I}, y = -1.542474539 - 8.660254040\,10^{-10}\,\mathrm{I}\}\}\}$$

Round the answer to three decimal places

Thus, the critical points of the function $f(x,y) = 12 + 10y - 2x^2 - 8xy - y^4$ are $\boxed{(-4.519, 2.260), (1.434, -0.717)}$ and $\boxed{(3.085 - 1.542)}$.

The partial derivative of $f_y(x,y)$ with respect to x is

$$\begin{aligned} f_{xy}(x,y) &= \frac{\partial}{\partial x}\left(f_y(x,y)\right) \\ &= \frac{\partial}{\partial x}\left(10-8x-4y^3\right) \\ &= -8 \end{aligned}$$

Recall the theorem,

Suppose the second partial derivatives of f are continuous on a disk with center (a,b) and suppose that $f_x(a,b)=0$ and $f_y(a,b)=0$ [(a,b) is a critical point of f].

Let $D(a,b)=f_{xx}(a,b)f_{yy}(a,b)-\left(f_{xy}(a,b)\right)^2$.

(a) If $D>0$ and $f_{xx}(a,b)>0$ then $f(a,b)$ is a local minimum.

(b) If $D>0$ and $f_{xx}(a,b)<0$ then $f(a,b)$ is a local maximum.

(c) If $D<0$ then $f(a,b)$ is not a local maximum or local minimum.

Now, use this theorem, to find the local maximum or minimum or saddle point at the critical points.

At the critical point $(-4.519,2.260)$.

$$\begin{aligned} D(a,b) &= f_{xx}(a,b)f_{yy}(a,b)-\left(f_{xy}(a,b)\right)^2 \\ &= (-4)\left(-12y^2\right)-[-8]^2 \end{aligned}$$

$$\begin{aligned} D(-4.519,2.260) &= 48(2.260)^2-64 \\ &\approx 181 \\ &> 0 \end{aligned}$$

And

$$\begin{aligned} f_{xx}(x,y) &= -4 \\ f_{xx}(-4.519,2.260) &= -4 \\ &< 0 \end{aligned}$$

Since $D>0$ and $f_{xx}<0$, the function has local maximum at $(-4.519,2.260)$.

The corresponding value of f is

$$\begin{aligned} f(x,y) &= 12+10y-2x^2-8xy-y^4 \\ f(-4.519,2.260) &= 12+10(2.260)-2(-4.519)^2-8(-4.519)(2.260)-(2.260)^4 \\ &= 49.37 \end{aligned}$$

At the critical point $(3.085, -1.542)$.

$$\begin{aligned} D(a,b) &= f_{xx}(a,b) f_{yy}(a,b) - (f_{xy}(a,b))^2 \\ &= (-4)(-12y^2) - [-8]^2 \end{aligned}$$

$$\begin{aligned} D(3.085, -1.542) &= 48(-1.542)^2 - 64 \\ &= 50.13 \\ &> 0 \end{aligned}$$

And

$$\begin{aligned} f_{xx}(x,y) &= -4 \\ f_{xx}(3.085, -1.542) &= -4 \\ &< 0 \end{aligned}$$

Since $D > 0$ and $f_{xx} < 0$, the function has local maximum at $(3.085, -1.542)$.

The corresponding value of f is

$$\begin{aligned} f(x,y) &= 12 + 10y - 2x^2 - 8xy - y^4 \\ f(3.08, -1.54) &= 12 + 10(-1.542) - 2(3.085)^2 - 8(3.085)(-1.542) - (-1.542)^4 \\ &= 9.95 \end{aligned}$$

At the critical point $(1.43, -0.72)$.

$$\begin{aligned} D(a,b) &= f_{xx}(a,b) f_{yy}(a,b) - (f_{xy}(a,b))^2 \\ &= (-4)(-12y^2) - [-8]^2 \end{aligned}$$

$$\begin{aligned} D(1.43, -0.72) &= 48(-0.72)^2 - 64 \\ &= -39.1168 \\ &< 0 \end{aligned}$$

Since $D < 0$, the point $(1.43, -0.72)$ is a saddle point.

Therefore, conclude that the local maxima exist at the critical points $(-4.52, 2.26)$, $(3.08, -1.54)$ and corresponding values of f are

$$\boxed{f(-4.52, 2.26) = 49.37, f(3.08, -1.54) = 9.95}.$$

Thus, the highest point on the graph of $f(x,y) = 12 + 10y - 2x^2 - 8xy - y^4$ is

$$\boxed{(-4.52, 2.26, 49.37)}.$$

The graph of $f(x,y)=12+10y-2x^2-8xy-y^4$ is as shown below:

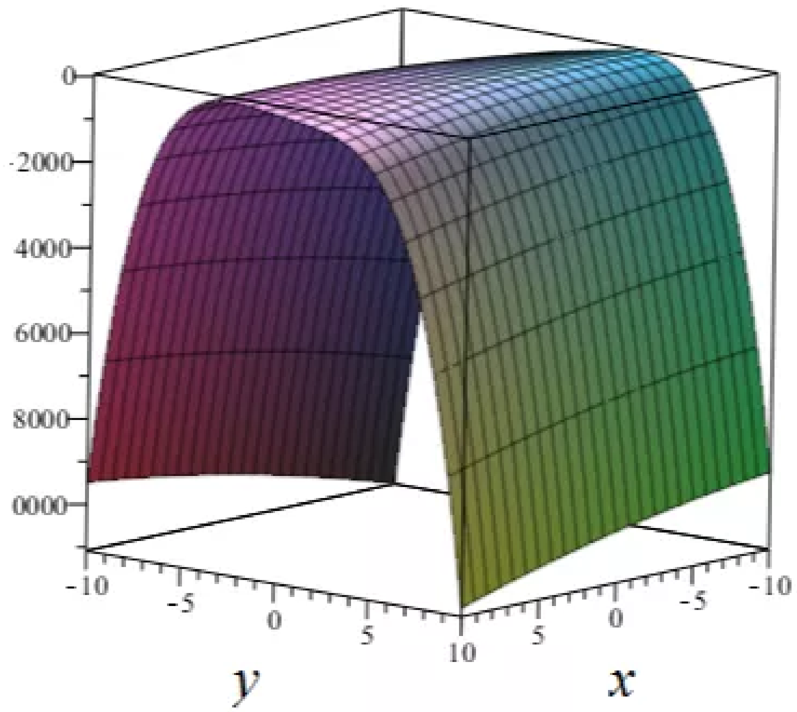
First load the package with(plots);

Then enter the following maple command.

plot3d(f(x,y),x=-10..10,y=-10..10,style=patchcontour,color=purple,axes=boxed);

The maple input followed by output as shown below:

> *plot3d*($f(x, y)$, $x=-10..10$, $y=-10..10$, *style* = *patchcontour*, *color* = *purple*, *axes* = *boxed*);



Answer 59E.

$$f(x,y)=x^2y$$

Then $f_x=2xy$, $f_y=x^2$

The given constraint is $g(x,y)=x^2+y^2=1$

By Lagrange's method of multipliers we find all x, y and λ such that

$$\vec{\nabla} f(x,y)=\lambda\vec{\nabla} g(x,y)$$

And $g(x,y)=1$

i.e. $f_x=\lambda g_x$, $f_y=\lambda g_y$, $g(x,y)=1$

i.e. $2xy=\lambda 2x$ ------------- (1)

$x^2=\lambda 2y$ ------------- (2)

$x^2+y^2=1$ ------------- (3)

If $\lambda = 0$ and $x = 0$
Using this in equation (3), $y = \pm 1$

If $\lambda \neq 0$, $x, y \neq 0$
From equation (1); $y = \lambda$
From equation (2); $x^2 = \lambda 2y = 2\lambda^2$
Using this in equation (3); $3\lambda^2 = 1$
i.e. $\lambda = \pm \frac{1}{\sqrt{3}}$
Then $y = \pm \frac{1}{\sqrt{3}}$, $x = \pm \sqrt{\frac{2}{3}}$

Then the extreme points of "f" are;
$$(0, \pm 1), \left(\pm\sqrt{\frac{2}{3}}, \frac{1}{\sqrt{3}}\right), \left(\pm\sqrt{\frac{2}{3}}, \frac{-1}{\sqrt{3}}\right)$$

Now evaluating "f" at these extreme points
$$f(0, \pm 2) = 0$$
$$f\left(\pm\sqrt{\frac{2}{3}}, \frac{1}{\sqrt{3}}\right) = \frac{2}{3} \times \frac{1}{\sqrt{3}} = \frac{2}{3\sqrt{3}}$$
$$f\left(\pm\sqrt{\frac{2}{3}}, \frac{-1}{\sqrt{3}}\right) = \frac{-2}{3\sqrt{3}}$$
Hence the maximum value of "f" is
$$f\left(\pm\sqrt{\frac{2}{3}}, \frac{1}{\sqrt{3}}\right) = \boxed{\frac{2}{3\sqrt{3}}}$$
And the minimum value of "f" is
$$f\left(\pm\sqrt{\frac{2}{3}}, \frac{-1}{\sqrt{3}}\right) = \boxed{\frac{-2}{3\sqrt{3}}}$$

Answer 60E.

$$f(x,y) = \frac{1}{x} + \frac{1}{y}$$

Then $f_x = \frac{-1}{x^2}$, $f_y = \frac{-1}{y^2}$

The given constraint is $g(x,y) = \frac{1}{x^2} + \frac{1}{y^2} = 1$

By Lagrange's method of multipliers we find all x, y and λ such that

$$\vec{\nabla} f(x,y) = \lambda \vec{\nabla} g(x,y)$$

$$g(x,y) = 1$$

i.e. $$-\frac{1}{x^2} = -\frac{\lambda 2}{x^3} \quad \text{----------- (1)}$$

$$-\frac{1}{y^2} = -\frac{\lambda 2}{y^3} \quad \text{----------- (2)}$$

$$\frac{1}{x^2} + \frac{1}{y^2} = 1 \quad \text{----------- (3)}$$

From equation (1) and (2) $\quad x = y = 2\lambda$

Using these values in equation (3)

$$\frac{1}{4\lambda^2} + \frac{1}{4\lambda^2} = 1$$

i.e. $$\lambda = \pm\frac{1}{\sqrt{2}}$$

Then $$x = y = \pm\frac{2}{\sqrt{2}} = \pm\sqrt{2}$$

Then the extreme points of "f" are;

$$\left(\sqrt{2}, \sqrt{2}\right), \left(-\sqrt{2}, -\sqrt{2}\right)$$

Evaluating "f" at these extreme points

$$f\left(\sqrt{2},\sqrt{2}\right) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}$$

$$= \frac{2}{\sqrt{2}}$$

$$= \sqrt{2}$$

$$f\left(-\sqrt{2}, -\sqrt{2}\right) = \frac{-1}{\sqrt{2}} - \frac{1}{\sqrt{2}}$$

$$= -\frac{2}{\sqrt{2}}$$

$$= -\sqrt{2}$$

Hence the maximum value of "f" is

$$f\left(\sqrt{2}, \sqrt{2}\right) = \boxed{\sqrt{2}}$$

And the minimum value of "f" is

$$f\left(-\sqrt{2}, -\sqrt{2}\right) = \boxed{-\sqrt{2}}$$

Answer 61E.

Given that $f(x,y,z) = xyz$

Then $f_x = yz$

$f_y = xz$

$f_z = xy$

The given constraint is $g(x,y,z) = x^2 + y^2 + z^2 = 3$

By Lagrange's method of multipliers we find all x, y, z and λ such that

$\vec{\nabla} f(x,y,z) = \lambda \vec{\nabla} g(x,y,z)$

And $g(x,y,z) = 3$

i.e $f_x = \lambda g_x$, $f_y = \lambda g_y$, $f_z = \lambda g_z$

And $g(x,y,z) = 3$

i.e. $yz = 2\lambda x$ ------------ (1)

$xz = 2\lambda y$ ------------ (2)

$xy = 2\lambda z$ ------------ (3)

$x^2 + y^2 + z^2 = 3$ ------------ (4)

If $\lambda = 0$, $y,z = 0$ from equation (4) $x = \pm\sqrt{3}$

Similarly if $\lambda = 0$, $x,z = 0$, $y = \pm\sqrt{3}$

If $\lambda = 0$, $x, y = 0$, $z = \pm\sqrt{3}$

If $\lambda \neq 0$, from equations (1), (2) and (3)

$xyz = 2\lambda x^2 = 2\lambda y^2 = 2\lambda z^2$

i.e. $x^2 = y^2 = z^2$

Using this in equation (4)

$x^2 = y^2 = z^2 = 1$

Then the possible extreme points of "f" are

$\left(\pm\sqrt{3}, 0, 0\right)$, $\left(0, \pm\sqrt{3}, 0\right)$, $\left(0, 0, \pm\sqrt{3}\right)$

$(1, 1, 1)$, $(1, -1, -1)$, $(-1, 1, 1)$, $(-1, -1, 1)$

Evaluating "f" at these extreme points

$f\left(\pm\sqrt{3}, 0, 0\right) = f\left(0, \pm\sqrt{3}, 0\right) = f\left(0, 0, \pm\sqrt{3}\right) = 0$

$f(1, 1, 1) = 1$

$f(1, -1, -1) = f(-1, 1, -1) = f(-1, -1, 1) = -1$

Hence the maximum value of "f" is $\boxed{1}$ and the minimum value is $\boxed{-1}$

Answer 62E.

Consider the function

$$f(x,y,z) = x^2 + 2y^2 + 3z^2 \quad \text{...... (1)}$$

Subject to the constraints

$$g(x,y,z) = x + y + z = 1$$
$$h(x,y,z) = x - y + 2z = 2$$

Now, use the Lagrange multipliers to find the maximum and minimum values of *f* subject the constraints *g* and *h*.

Recall that, the method of Lagrange multipliers,

To find the maximum or minimum of a function subject to the constraints $g(x,y,z) = k$, $\nabla g \neq 0$, $h(x,y,z) = c, \nabla h \neq 0$

(a) find all the values of *x*, *y*, *z* and λ such that

$$\nabla f(x,y,z) = \lambda \nabla g(x,y,z) + \mu \nabla h(x,y,z)$$
$$\text{and } g(x,y,z) = k, h(x,y,z) = c$$

(b) Evaluate the value of *f* at these points. The largest of these values is the maximum value of *f* and the smallest values is the minimum value of *f*.

Differentiate f partially on both sides with respect to $x, y,$ and z.

$$f_x = \frac{\partial}{\partial x}\left(x^2 + 2y^2 + 3z^2\right) \quad f_y = \frac{\partial}{\partial y}\left(x^2 + 2y^2 + 3z^2\right) \quad f_z = \frac{\partial}{\partial z}\left(x^2 + 2y^2 + 3z^2\right)$$
$$= 2x \qquad\qquad = 4y \qquad\qquad = 6z$$

Differentiate g partially on both sides with respect to $x, y,$ and z.

$$g_x = \frac{\partial}{\partial x}(x+y+z-1) \quad g_y = \frac{\partial}{\partial y}(x+y+z-1) \quad g_z = \frac{\partial}{\partial z}(x+y+z-1)$$

$$= 1 \qquad = 1 \qquad = 1$$

Differentiate h partially on both sides with respect to $x, y,$ and z.

$$h_x = \frac{\partial}{\partial x}(x-y+2z-2) \quad h_y = \frac{\partial}{\partial y}(x-y+2z-2) \quad h_z = \frac{\partial}{\partial z}(x-y+2z-2)$$

$$= 1 \qquad = -1 \qquad = 2$$

The vector equation $\nabla f(x,y,z)=\lambda\nabla g(x,y,z)$ in terms of its components and using the constraint equations are

$$f_x=\lambda g_x+\mu h_x$$
$$f_y=\lambda g_y+\mu h_y$$
$$f_z=\lambda g_z+\mu h_z$$
$$g(x,y,z)=k$$
$$h(x,y,z)=c$$

Substitute the values of $f_x, f_y, f_z, g_x, g_y, g_z, g,$ **and** h into the Lagrange system of equations.

Substitution yields the equations,

$2x=\lambda+\mu$ ...... (2)

$4y=\lambda-\mu$ ...... (3)

$6z=\lambda+2\mu$ ...... (4)

$x+y+z=1$ ...... (5)

$x-y+2z=2$ ...... (6)

Solve the equation (2), (3), and (4) for *x*, *y* and *z*.

$$x=\frac{\lambda+\mu}{2}$$
$$y=\frac{\lambda-\mu}{4}$$
$$z=\frac{\lambda+2\mu}{6}$$

Substitute these values of *x*, *y* and *z* into equation (5)

$$\frac{\lambda+\mu}{2}+\frac{\lambda-\mu}{4}+\frac{\lambda+2\mu}{6}=1$$
$$\frac{6\lambda+6\mu+3\lambda-3\mu+2\lambda+4\mu}{12}=1$$

$11\lambda+7\mu=12$ ...... (7)

Similarly, substitute these values of *x*, *y* and *z* into equation (6)

$$\frac{\lambda+\mu}{2}-\frac{\lambda-\mu}{4}+\frac{2(\lambda+2\mu)}{6}=2$$

$$\frac{6\lambda+6\mu-3\lambda+3\mu+4\lambda+8\mu}{12}=2$$

$$\frac{7\lambda+17\mu}{12}=2$$

$7\lambda+17\mu=24$ ...... (8)

Multiply equation (8) by 11 and equation (7) by 7 and subtract them.

$$77\lambda+187\mu-77\lambda-49\mu=180$$

$$138\mu=180$$

$$\mu=\frac{30}{23}$$

Substitute $\mu=\frac{30}{23}$ into equation (8).

$$7\lambda+17\left(\frac{30}{23}\right)=24$$

$$7\lambda=24-\frac{510}{23}$$

$$7\lambda=\frac{42}{23}$$

$$\lambda=\frac{6}{23}$$

Substitute the values of and μ in the equation $x=\frac{\lambda+\mu}{2}$.

$$x=\frac{\left(\frac{30}{23}\right)+\left(\frac{6}{23}\right)}{2}$$

$$=\frac{36}{46}$$

$$=\frac{18}{23}$$

Substitute the values of and μ in the equation $y=\frac{\lambda-\mu}{4}$.

$$y=\frac{\left(\frac{6}{23}\right)-\left(\frac{30}{23}\right)}{4}$$

$$=\frac{-24}{23\cdot 4}$$

$$y=\frac{\lambda-\mu}{4} \text{ and } z=\frac{\lambda+2\mu}{6}$$

$$=\frac{-6}{23}$$

Substitute the values of and μ in the equation $z=\frac{\lambda+2\mu}{6}$

$$z=\frac{\left(\frac{6}{23}\right)+2\left(\frac{30}{23}\right)}{6}$$

$$=\frac{66}{23\cdot 6}$$

$$=\frac{11}{23}$$

Thus,

$$(x,y,z)=\left(\frac{18}{23},-\frac{6}{23},\frac{11}{23}\right)$$

The value of f at $\left(\frac{18}{23},-\frac{6}{23},\frac{11}{23}\right)$ is:

$$\begin{aligned}f\left(\frac{18}{23},-\frac{6}{23},\frac{11}{23}\right)&=\left(\frac{18}{23}\right)^2+2\left(-\frac{6}{23}\right)^2+3\left(\frac{11}{23}\right)^2\\&=\frac{324}{529}+\frac{72}{529}+\frac{363}{529}\\&=\frac{759}{529}\\&=\frac{33}{23}\end{aligned}$$

Therefore, the maximum value of the function $f(x,y,z)=x^2+2y^2+3z^2$ is $\boxed{\frac{33}{23}}$ at the point

$\boxed{\left(\frac{18}{23},-\frac{6}{23},\frac{11}{23}\right)}$.

Answer 63E.

Let the point (x, y, z) is closest to the origin and which lies on surface $xy^2z^3 = 2$

Then $d = \left[(x-0)^2 + (y-0)^2 + (z-0)^2\right]^{1/2}$

$= \left(x^2 + y^2 + z^2\right)^{1/2}$

Then $d^2 = x^2 + y^2 + z^2$

Take $f(x,y,z) = x^2 + y^2 + z^2$

And the constraint $g(x,y,z) = x\,y^2z^3 = 2$

By Lagrange's method of multipliers we find all x, y, z and λ such that

$$\vec{\nabla} f(x,y,z) = \lambda \vec{\nabla} g(x,y,z)$$

And $g(x,y,z) = 2$

Take $f(x,y,z) = x^2 + y^2 + z^2$

And the constraint $g(x,y,z) = x\,y^2z^3 = 2$

By Lagrange's method of multipliers we find all x, y, z and λ such that

$$\vec{\nabla} f(x,y,z) = \lambda \vec{\nabla} g(x,y,z)$$

And $g(x,y,z) = 2$

i.e. $2x = \lambda y^2 z^3$ ------------ (1)

$2y = 2\lambda xyz^3$ ------------ (2)

$2z = 3\lambda xy^2z^2$ ------------ (3)

$xy^2z^3 = 2$ ------------ (4)

Using these in equation (4)

$$(\lambda)^{1/2}\, 2\lambda (3\lambda)^{3/2} = 2$$

i.e. $\lambda = \dfrac{1}{(3)^{1/2}}$

Then $x = \pm(3)^{-1/4}$, $y = \sqrt{2}(3)^{-1/4}$, $z = \pm 3^{1/4}$

Then the possible extreme points of "f" are

$$\left(\pm 3^{-1/4}, \sqrt{2}(3)^{-1/4}, \pm 3^{1/4}\right)$$

And $\left(\pm 3^{-1/4}, -\sqrt{2}(3)^{-1/4}, \pm 3^{1/4}\right)$

Now the physical nature of the problem says that there must be an absolute minimum of "f" which has to occur at the extreme point

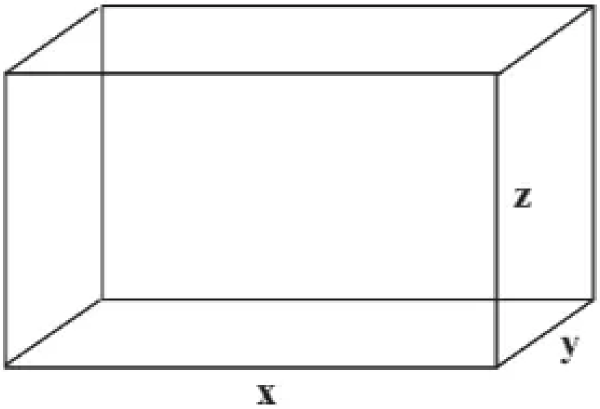
Now $f\left(\pm 3^{-1/4}, \sqrt{2}(3)^{-1/4}, \pm(3)^{1/4}\right)$

$$= f\left(\pm 3^{-1/4}, -3^{-1/4}\sqrt{2}, \pm 3^{1/4}\right)$$

Then the minimum value of "f" exists at both these extreme points. Now when "f" is minimum, d is also minimum. Hence the required points are

$$\boxed{\left(\pm 3^{-1/4}, 3^{-1/4}\sqrt{2}, \pm 3^{1/4}\right)} \text{ and } \boxed{\left(\pm 3^{-1/4}, -3^{-1/4}\sqrt{2}, \pm 3^{1/4}\right)}$$

Answer 64E.



Let the length of the box is x in and the dimensions of the cross – section are y in and z in

Then $x + 2(y+z) = 108\text{in}$

The volume of the box is $v = xyz$

Take $f(x,y,z) = xyz$

And the constraint $g(x,y,z) = x + 2y + 2z = 108$

By Lagrange's method of multipliers we find all x, y, z and λ such that

$$\vec{\nabla} f(x,y,z) = \lambda \vec{\nabla} g(x,y,z)$$

And $g(x,y,z) = 108$

i.e. $yz = \lambda$ ------------ (1)

$xz = 2\lambda$ ------------ (2)

$xy = 2\lambda$ ------------ (3)

$x + 2y + 2z = 108s$ ------------ (4)

From equation (1), (2) and (3)

$$xyz = \lambda x = 2\lambda y = 2\lambda z$$

i.e. $x = 2y = 2z$

{Because $\lambda \neq 0$ as if $\lambda = 0$, $x, y, z = 0$ which is not possible because of (4)}

Using these values in equation (4) we find

$$x = 36,\ y = 18,\ z = 18$$

Then the only extreme point of "f" is (36, 18, 18)

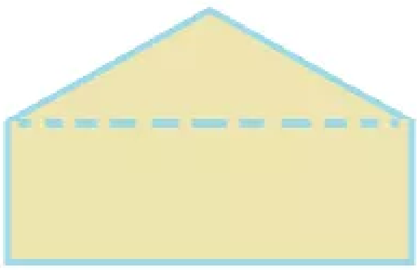
Now the physical nature of the problem says that there must be an absolute maximum of f which has to occur at the extreme point. Then at the extreme point (36, 18, 18) "f" is maximum.
Hence the volume of the box is maximum when its length is 36in and the dimensions of the cross – section perpendicular to length are 18 in and 18 in
The maximum volume will be:

$$v = (36)(18)(18)$$

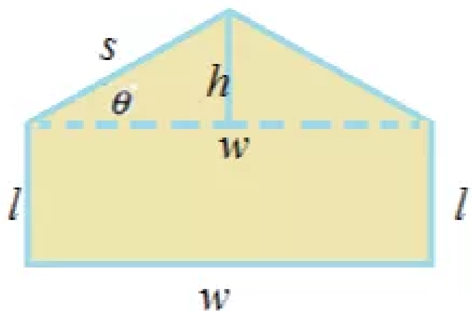
$$= \boxed{11664 \text{ in}^3}$$

Answer 65E.

Consider the pentagon with a constant perimeter *P*.



Draw an altitude to the base of the triangle. And note the pentagon is a combination of isosceles triangle and rectangle. Assign variables to the sides and draw the figure as follows.



To find the formula for the area of this pentagon, find the area of the rectangle and area of the isosceles triangle. To find the area of triangle, use trigonometric identities.

Since the triangle is isosceles, it bisects the base.

$$\tan\theta = \frac{h}{\left(\frac{w}{2}\right)}$$

$$h = \frac{w\tan\theta}{2}$$

The area of the triangle is given by:

$$A = \frac{1}{2}\times\text{Base}\times\text{height}$$

$$A = \frac{1}{2}\times(w)\times\left(\frac{w\tan\theta}{2}\right)$$

$$= \frac{1}{4}w^2\tan\theta$$

The area of the rectangle is $A = lw$

Thus, the area of the pentagon is given by the sum of the area of the triangle and the area of the rectangle

$$A = lw + \frac{1}{4}w^2\tan\theta \quad \ldots\ldots (1)$$

To find the formula for the area of this pentagon, find the area of the rectangle and area of the isosceles triangle. To find the area of triangle, use trigonometric identities.

Since the triangle is isosceles, it bisects the base.

$$\tan\theta = \frac{h}{\left(\frac{w}{2}\right)}$$
$$h = \frac{w\tan\theta}{2}$$

The area of the triangle is given by:

$$A = \frac{1}{2} \times \text{Base} \times \text{height}$$
$$A = \frac{1}{2} \times (w) \times \left(\frac{w\tan\theta}{2}\right)$$
$$= \frac{1}{4} w^2 \tan\theta$$

The area of the rectangle is $A = lw$

Thus, the area of the pentagon is given by the sum of the area of the triangle and the area of the rectangle

$$A = lw + \frac{1}{4} w^2 \tan\theta \text{ ...... (1)}$$

To find the critical points, equate equation (3) to zero:

$$\frac{-w^2\sec\theta\tan\theta}{2} + \frac{w^2\sec^2\theta}{4} = 0$$
$$\frac{w^2\sec^2\theta}{4} = \frac{w^2\sec\theta\tan\theta}{2}$$
$$\frac{\sec\theta}{\tan\theta} = 2$$
$$\text{cosec}(\theta) = 2$$

This gives $\theta = 30^\circ$.

To find the critical points, equate equation (3) to zero:

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$$\frac{\sec\theta}{\tan\theta} = 2$$

$$\operatorname{cosec}(\theta) = 2$$

This gives $\theta = 30^\circ$.

Substitute the value of *w* and θ in $l = \frac{1}{2}[P - w - w\sec\theta]$.

$$l = \frac{1}{2}\left[P - P\left(2-\sqrt{3}\right) - P\left(2-\sqrt{3}\right)\frac{2}{\sqrt{3}}\right]$$

$$= \frac{1}{2}\left[P - 2P + P\sqrt{3} - \frac{4P}{\sqrt{3}} + 2P\right]$$

$$= \frac{P}{2}\left[1 + \sqrt{3} - \frac{4}{\sqrt{3}}\right]$$

$$= \frac{P\left(\sqrt{3}-1\right)}{2\sqrt{3}}$$

$$l = \frac{P\left(\sqrt{3}-1\right)}{2\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}}$$ Rationalize the denominator

$$= \frac{P\left(3-\sqrt{3}\right)}{6}$$

Finally, the side of the triangle is given by:

$$s = \frac{w\sec\theta}{2}$$

$$= \frac{P\left(2-\sqrt{3}\right)}{2}\sec 30$$

$$= \frac{P\left(2-\sqrt{3}\right)}{2} \cdot \frac{2}{\sqrt{3}}$$

$$= \frac{P\left(2-\sqrt{3}\right)}{\sqrt{3}}$$

$$= \frac{P\left(2\sqrt{3}-3\right)}{3}$$

Classify the behavior of the critical points of f.

Recall the second derivative test,

"A function *f* has continuous partial derivatives on disk (a,b) and

$f_x(a,b)=0, f_y(a,b)=0$.

Let

$$D=D(a,b)=f_{xx}(a,b)f_{yy}(a,b)-\left[f_{xy}(a,b)\right]^2$$

a. If $D>0$ and $f_{xx}(a,b)>0$ then $f(a,b)$ is a local minimum

b. If $D>0$ and $f_{xx}(a,b)<0$ then $f(a,b)$ is a local maximum

c. If $D<0$ then $f(a,b)$ is not a local minimum or local maximum."

To find $D(x,y)$, find $A_{ww}, A_{\theta\theta}$ and $A_{w\theta}$.

$$A_{ww}=\frac{d}{dw}\left[\frac{1}{2}[P-2w-2w\sec\theta]+\frac{1}{2}w\tan\theta\right]$$

$$=\frac{1}{2}[-2-2\sec\theta]+\frac{1}{2}\tan\theta$$

$$A_{\theta\theta}=\frac{d}{d\theta}\left[\frac{-w^2\sec\theta\tan\theta}{2}+\frac{w^2\sec^2\theta}{4}\right]$$

$$=\frac{-w^2}{2}\frac{d}{d\theta}[\sec\theta\tan\theta]+\frac{w^2}{4}\frac{d}{d\theta}\left[\sec^2\theta\right]$$

$$=\frac{-w^2}{2}\left[\sec(\theta)\tan(\theta)^2+\sec(\theta)\left(1+\tan(\theta)^2\right)\right]+\frac{w^2}{4}\left[2\sec(\theta)^2\tan(\theta)\right]$$

$$A_{w\theta}=\frac{d}{d\theta}\left[\frac{1}{2}[P-2w-2w\sec\theta]+\frac{1}{2}w\tan\theta\right]$$

$$=-2w\sec\theta\tan\theta+\frac{1}{2}w\left(\tan^2\theta\right)$$

Substitute $A_{ww}, A_{\theta\theta}$ and $A_{w\theta}$ in D.

$$D = A_{ww} \cdot A_{\theta\theta} - \left[A_{w\theta}\right]^2$$

$$= \left[\frac{1}{2}[-2 - 2\sec\theta] + \frac{1}{2}\tan\theta\right]$$

$$\left[\begin{array}{l} \frac{-w^2}{2}\left[\sec(\theta)\tan(\theta)^2 + \sec(\theta)\left(1 + \tan(\theta)^2\right)\right] \\ + \frac{w^2}{4}\left[2\sec(\theta)^2\tan(\theta)\right] \end{array}\right] - \left[2w\sec\theta\tan\theta + \frac{1}{2}w\left(\tan^2\theta\right)\right]^2$$

Find D and A_{ww} at the critical point $\left(P\left(2-\sqrt{3}\right), 30°\right)$.

$$A_{ww}(w,\theta) = \frac{1}{2}[-2 - 2\sec\theta] + \frac{1}{2}\tan\theta$$

$$A\left(P\left(2-\sqrt{3}\right), 30°\right) = \frac{1}{2}[-2 - 2\sec\theta] + \frac{1}{2}\tan\theta$$

$$= \frac{1}{2}[-2 - 2\sec 30°] + \frac{1}{2}\tan 30°$$

$$= \frac{1}{2}\left[-2 - 2\left(\frac{2\sqrt{3}}{3}\right)\right] + \frac{1}{2}\left(\frac{\sqrt{3}}{3}\right)$$

$$= \frac{1}{2}\left[-2 - \frac{4\sqrt{3}}{3}\right] + \frac{\sqrt{3}}{6}$$

$$= \frac{1}{2}\left[\frac{-6 - 4\sqrt{3}}{3}\right] + \frac{\sqrt{3}}{6}$$

$$= \frac{-6 - 4\sqrt{3} + \sqrt{3}}{6}$$

$$= \frac{-6 - 3\sqrt{3}}{6}$$

$$= \frac{-3\left(\sqrt{3} + 2\right)}{6}$$

$$= \frac{-\left(\sqrt{3} + 2\right)}{2}$$

$$< 0$$

The value of D at the critical point $\left(P\left(2-\sqrt{3}\right), 30^\circ\right)$ is,

$$D\left(P\left(2-\sqrt{3}\right), 30^\circ\right)$$

$$=\left[\frac{1}{2}\left[-2-2\sec 30^\circ\right]+\frac{1}{2}\tan(30)\right]$$

$$\left[\begin{array}{l}\frac{-\left(P\left(2-\sqrt{3}\right)\right)^2}{2}\left[\sec(30^\circ)\tan(30^\circ)^2+\sec(30^\circ)\left(1+\tan(30^\circ)^2\right)\right] \\ +\frac{\left(P\left(2-\sqrt{3}\right)\right)^2}{4}\left[2\sec(30^\circ)^2\tan(30^\circ)\right]\end{array}\right]$$

$$-\left[2P\left(2-\sqrt{3}\right)\sec(30^\circ)\tan\theta+\frac{1}{2}\left(P\left(2-\sqrt{3}\right)\right)\left(\tan^2 30^\circ\right)\right]^2$$

$$=\left[\frac{1}{2}\left[-2-2\left(\frac{2\sqrt{3}}{3}\right)\right]+\frac{1}{2}\left(\frac{\sqrt{3}}{3}\right)\right]$$

$$\left[\begin{array}{l}\frac{-\left(P\left(2-\sqrt{3}\right)\right)^2}{2}\left[\frac{2\sqrt{3}}{3}\cdot\left(\frac{\sqrt{3}}{3}\right)^2+\frac{2\sqrt{3}}{3}\left(1+\left(\frac{\sqrt{3}}{3}\right)^2\right)\right] \\ +\frac{\left(P\left(2-\sqrt{3}\right)\right)^2}{4}\left[2\left(\frac{2\sqrt{3}}{3}\right)^2\left(\frac{\sqrt{3}}{3}\right)\right]\end{array}\right]$$

$$-\left[2P\left(2-\sqrt{3}\right)\left(\frac{2\sqrt{3}}{3}\right)\left(\frac{\sqrt{3}}{3}\right)+\frac{1}{2}\left(P\left(2-\sqrt{3}\right)\right)\left(\frac{\sqrt{3}}{3}\right)^2\right]^2$$

$$=\left[\frac{-6-4\sqrt{3}+\sqrt{3}}{6}\right]\left[\frac{-\left(P\left(2-\sqrt{3}\right)\right)^2}{2}\left[\frac{2\sqrt{3}}{9}+\frac{8\sqrt{3}}{3}\right]+\frac{\left(P\left(2-\sqrt{3}\right)\right)^2}{4}\left[\frac{8\sqrt{3}}{9}\right]\right]$$

$$-\left[2P\left(2-\sqrt{3}\right)\left(\frac{2}{3}\right)+\frac{1}{2}\left(P\left(2-\sqrt{3}\right)\right)\left(\frac{1}{3}\right)\right]^2$$

$$=\left[\frac{-\left(\sqrt{3}+2\right)}{2}\right]\left[-\left(P\left(2-\sqrt{3}\right)\right)^2\left[\frac{13\sqrt{3}}{9}\right]+\left(P\left(2-\sqrt{3}\right)\right)^2\left[\frac{2\sqrt{3}}{9}\right]\right]$$

$$-\frac{16}{9}P\left(2-\sqrt{3}\right)^2-\frac{4}{9}P\left(2-\sqrt{3}\right)^2-\frac{1}{36}P\left(2-\sqrt{3}\right)^2$$

$$=\frac{13}{6}\left(P\left(2-\sqrt{3}\right)\right)^2+\frac{13\sqrt{3}}{9}\left(P\left(2-\sqrt{3}\right)\right)^2-\frac{1}{3}\left(P\left(2-\sqrt{3}\right)\right)^2-\frac{2\sqrt{3}}{9}\left(P\left(2-\sqrt{3}\right)\right)^2$$

$$-\frac{16}{9}P\left(2-\sqrt{3}\right)^2-\frac{4}{9}P\left(2-\sqrt{3}\right)^2-\frac{1}{36}P\left(2-\sqrt{3}\right)^2$$

$$=\left(P\left(2-\sqrt{3}\right)\right)^2\left[\frac{13}{6}+\frac{13\sqrt{3}}{9}-\frac{1}{3}-\frac{2\sqrt{3}}{9}-\frac{16}{9}-\frac{4}{9}-\frac{1}{36}\right]$$

$$=\left(P\left(2-\sqrt{3}\right)\right)^2\left[\frac{11\sqrt{3}}{9}-\frac{5}{12}\right]$$

$$>0$$

By the second derivative test, the area of the pentagon maximum at $\left(P\left(2-\sqrt{3}\right),30^\circ\right)$

Thus, the dimensions of the pentagon that maximize the area of the pentagon are

$$\boxed{\begin{aligned} w &= P\left(2-\sqrt{3}\right) \\ l &= \frac{P\left(3-\sqrt{3}\right)}{6} \\ s &= \frac{P\left(2\sqrt{3}-3\right)}{3} \end{aligned}}$$

Answer 66E.

Consider the function for the particle

$$z = f(x, y)$$

And *x* and *y* coordinates of the function at a time *t* are

$$x = x(t)$$
$$y = y(t)$$

(a)

Find the velocity vector **v** and the kinetic energy *K* of the particle.

The velocity of the particle is the rate of change of position vector.

$$\mathbf{v}(t) = \mathbf{r}'(t)$$

The position vector of the object moving on a surface is

$$\begin{aligned}\mathbf{r}(t) &= x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k} \\ &= x(t)\mathbf{i} + y(t)\mathbf{j} + f(x(t), y(t))\mathbf{k}\end{aligned}$$

Differentiate $\mathbf{r}(t)$ with respect to *t*.

$$\mathbf{r}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + f'(x(t), y(t))\mathbf{k}$$

For the given function, the rate of change is $\dfrac{df(x,y)}{dt}$

$$\frac{df(x,y)}{dt} = \frac{df}{dx}\frac{dx}{dt} + \frac{df}{dy}\frac{dy}{dt}$$

$$= f_x x'(t) + f_y y'(t)$$

Then,

$$\begin{aligned}\mathbf{v}(t) &= \mathbf{r}'(t) \\ &= x'(t)\mathbf{i} + y'(t)\mathbf{j} + \left[f_x x'(t) + f_y y'(t)\right]\mathbf{k}\end{aligned}$$

Therefore, the velocity vector of the particle with *x* and *y* components are given by:

$$\boxed{\mathbf{v}(t) = \left\langle x'(t), y'(t), \left[f_x x'(t) + f_y y'(t)\right]\right\rangle}$$

The velocity of the particle is

$$\begin{aligned}\mathbf{v} &= \left|\mathbf{v}(t)\right| \\ &= \sqrt{\left[x'(t)\right]^2 + \left[y'(t)\right]^2 + \left[f_x x'(t) + f_y y'(t)\right]^2} \\ &= \sqrt{\left[x'(t)\right]^2 + \left[y'(t)\right]^2 + f_x^2\left[x'(t)\right]^2 + f_y^2\left[y'(t)\right]^2 + 2 f_x f_y x'(t) y'(t)} \\ &= \sqrt{\left(1 + f_x^2\right)\left[x'(t)\right]^2 + 2 f_x f_y x'(t) y'(t) + \left(1 + f_y^2\right)\left[y'(t)\right]^2}\end{aligned}$$

Now, the kinetic energy is given by:

$$\begin{aligned}K &= \frac{1}{2} m v^2 \\ &= \frac{1}{2} m \left(\sqrt{\left(1 + f_x^2\right)\left[x'(t)\right]^2 + 2 f_x f_y x'(t) y'(t) + \left(1 + f_y^2\right)\left[y'(t)\right]^2}\right)^2 \\ &= \boxed{\frac{1}{2} m \left(\left(1 + f_x^2\right)\left[x'(t)\right]^2 + 2 f_x f_y x'(t) y'(t) + \left(1 + f_y^2\right)\left[y'(t)\right]^2\right)}\end{aligned}$$

(b)

The acceleration of the particle is defined as the derivative of the velocity.

That is,

$$\mathbf{a}(t)=\mathbf{v}'(t)$$

$$\begin{aligned}\mathbf{a}(t)&=\frac{d\mathbf{v}(t)}{dt}\\&=\frac{d}{dt}\left[x'(t)\mathbf{i}+y'(t)\mathbf{j}+\left[f_x x'(t)+f_y y'(t)\right]\mathbf{k}\right]\\&=\frac{d}{dt}(x'(t))\mathbf{i}+\frac{d}{dt}(y'(t))\mathbf{j}+\frac{d}{dt}\left[f_x x'(t)+f_y y'(t)\right]\mathbf{k}\\&=x''(t)\mathbf{i}+y''(t)\mathbf{j}+\left[\frac{d}{dt}(f_x)\cdot x'(t)\cdot x'(t)+f_x x''(t)+\frac{d}{dt}(f_y)\cdot y'(t)\cdot y'(t)+f_y y''(t)\right]\mathbf{k}\end{aligned}$$

$$=x''(t)\mathbf{i}+y''(t)\mathbf{j}+\left[\frac{d}{dt}(f_x)\cdot\left[x'(t)\right]^2+f_x x''(t)+\frac{d}{dt}(f_y)\cdot\left[y'(t)\right]^2+f_y y''(t)\right]\mathbf{k}$$

Therefore, the acceleration vector is $\boxed{\left\langle x''(t),y''(t),\begin{bmatrix}\frac{d}{dt}(f_x)\cdot\left[x'(t)\right]^2+f_x x''(t)\\+\frac{d}{dt}(f_y)\cdot\left[y'(t)\right]^2+f_y y''(t)\end{bmatrix}\right\rangle}$.

(c)

Consider $z=f(x,y)=x^2+y^2$, $x(t)=t\cos t$ and $y(t)=t\sin t$

Find the velocity vector, kinetic energy and acceleration vector for the given function.

From part (a), the velocity vector is

$$\mathbf{v}(t)=\left\langle x'(t),y'(t),\left[f_x x'(t)+f_y y'(t)\right]\right\rangle$$

For this find $x'(t),y'(t),f_x$ and f_y.

$$\begin{aligned}x'(t)&=\frac{d}{dt}(t\cos t)\\&=\cos t-t\sin t\end{aligned}$$

$$\begin{aligned}y'(t)&=\frac{d}{dt}(t\sin t)\\&=\sin t+t\cos t\end{aligned}$$

The partial derivatives of f with respect to *x* and *y* is:

$$f_x = \frac{d}{dx}\left((x(t))^2 + (y(t))^2\right) \quad f_y = \frac{d}{dy}\left((x(t))^2 + (y(t))^2\right)$$
$$= 2x(t) \qquad\qquad = 2y(t)$$
$$= 2t\cos t \qquad\qquad = 2t\sin t$$

Substitute all these values in $\mathbf{v}(t)$.

$$\mathbf{v}(t) = \left\langle x'(t), y'(t), \left[f_x x'(t) + f_y y'(t)\right]\right\rangle$$
$$= \left\langle \cos t - t\sin t, \sin t + t\cos t, \left[2t\cos t(\cos t - t\sin t) + 2t\sin t(\sin t + t\cos t)\right]\right\rangle$$
$$= \left\langle \cos t - t\sin t, \sin t + t\cos t, \begin{bmatrix} 2t\cos^2 t - 2t^2\cos t\sin t \\ +2t\sin^2 t + 2t^2\sin t\cos t \end{bmatrix}\right\rangle$$
$$= \left\langle \cos t - t\sin t, \sin t + t\cos t, \left[2t\cos^2 t + 2t\sin^2 t\right]\right\rangle$$
$$= \left\langle \cos t - t\sin t, \sin t + t\cos t, \left[2t\left(\cos^2 t + \sin^2 t\right)\right]\right\rangle$$
$$= \boxed{\left\langle \cos t - t\sin t, \sin t + t\cos t, 2t\right\rangle} \quad \text{Since}: \sin^2 t + \cos^2 t = 1$$

From part (a), the kinetic energy is

$$K=\frac{1}{2}m\left(\left(1+f_x^{\,2}\right)\left[x'(t)\right]^2+2f_xf_yx'(t)y'(t)+\left(1+f_y^{\,2}\right)\left[y'(t)\right]^2\right)$$

$$=\frac{1}{2}m\begin{pmatrix}\left(1+(2t\cos t)^2\right)\left[\cos t-t\sin t\right]^2\\ +2(2t\cos t)(2t\sin t)(\cos t-t\sin t)(\sin t+t\cos t)\\ +\left(1+(2t\sin t)^2\right)\left[\sin t+t\cos t\right]^2\end{pmatrix}$$

$$=\frac{1}{2}m\begin{pmatrix}\left(1+2t^2\cos^2 t\right)\left[\cos^2 t-2\cdot\cos t\cdot t\sin t+t^2\sin^2 t\right]\\ +8t^2\cos t\sin t\left(\cos t\sin t-t\sin^2 t+t\cos^2 t-t^2\cos t\sin t\right)\\ +\left(1+4t^2\sin^2 t\right)\left[\sin^2 t+2\cdot\sin t\cdot t\cos t+t^2\cos^2 t\right]\end{pmatrix}$$

$$=\frac{1}{2}m\begin{pmatrix}\begin{bmatrix}\cos^2 t+2t^2\cos^4 t-2t\cos t\sin t-4t^3\cos^3 t\sin t\\ +t^2\sin^2 t+2t^4\sin^2 t\cos^2 t\end{bmatrix}\\ +\begin{pmatrix}8t^2\cos^2 t\sin^2 t-8t^3\cos t\sin^3 t\\ +8t^3\cos^3 t\sin t-8t^4\cos^2 t\sin^2 t\end{pmatrix}\\ +\begin{bmatrix}\sin^2 t+4t^2\sin^4 t+2t\sin t\cos t+8t^3\sin^3 t\cos t\\ +t^2\cos^2 t+4t^4\cos^2 t\sin^2 t\end{bmatrix}\end{pmatrix}$$

$$=\frac{1}{2}m\begin{pmatrix}1+2t^2\cos^4 t+t^2+8t^2\cos^2 t\sin^2 t+4t^3\cos^3 t\sin t\\ -2t^4\cos^2 t\sin^2 t+4t^2\sin^4 t\end{pmatrix}$$

$$=\boxed{\frac{1}{2}m\left(1+2t^2\cos^4 t+t^2+8t^2\cos^2 t\sin^2 t+4t^3\cos^3 t\sin t-2t^4\cos^2 t\sin^2 t+4t^2\sin^4 t\right)}$$

From part (b), the acceleration vector is

$$\mathbf{a}(t)=\left\langle x''(t),y''(t),\begin{bmatrix}\frac{d}{dt}(f_x)\cdot\left[x'(t)\right]^2+f_xx''(t)\\ +\frac{d}{dt}(f_y)\cdot\left[y'(t)\right]^2+f_yy''(t)\end{bmatrix}\right\rangle$$

For this, find $x''(t), y''(t), \frac{d}{dt}(f_x)$ and $\frac{d}{dt}(f_y)$.

$$\begin{aligned} x''(t) &= \frac{d}{dt}(x(t)) \\ &= \frac{d}{dt}(\cos t - t\sin t) \\ &= \sin t - \sin t - t\cos t \\ &= -t\cos t \end{aligned}$$

$$\begin{aligned} y''(t) &= \frac{d}{dt}(y(t)) \\ &= \frac{d}{dt}(\sin t + t\cos t) \\ &= \cos t + \cos t - t\sin t \\ &= 2\cos t - t\sin t \end{aligned}$$

$$\begin{aligned} \frac{d}{dt}(f_x) &= \frac{d}{dt}(2t\cos t) \\ &= 2(\cos t - t\sin t) \end{aligned}$$

$$\begin{aligned} \frac{d}{dt}(f_y) &= \frac{d}{dt}(2t\sin t) \\ &= 2(\sin t + t\cos t) \end{aligned}$$

Substitute all these values in $\mathbf{a}(t)$.

$$\mathbf{a}(t) = \left\langle \begin{array}{l} -t\cos t, 2\cos t - t\sin t, \\ \left[\begin{array}{l} 2(\cos t - t\sin t)\cdot[\cos t - t\sin t]^2 + (2t\cos t)(-t\cos t) \\ +2(\sin t + t\cos t)\cdot[\sin t + t\cos t]^2 + (2t\sin t)(2\cos t - t\sin t) \end{array} \right] \end{array} \right\rangle$$

$$= \left\langle \begin{array}{l} -t\cos t, 2\cos t - t\sin t, \\ \left[\begin{array}{l} 2[\cos t - t\sin t]^3 - 2t^2\cos^2 t \\ +2[\sin t + t\cos t]^3 + 4t\sin t\cos t - 2t^2\sin^2 t \end{array} \right] \end{array} \right\rangle$$

$$= \left\langle \begin{array}{l} -t\cos t, 2\cos t - t\sin t, \\ \left[\begin{array}{l} 2[\cos t - t\sin t]^3 - 2t^2\left(\cos^2 2t + \sin^2 2t\right) \\ +2[\sin t + t\cos t]^3 + 4t\sin t\cos t \end{array} \right] \end{array} \right\rangle$$

$$= \boxed{\left\langle \begin{array}{l} -t\cos t, 2\cos t - t\sin t, \\ \left[2[\cos t - t\sin t]^3 - 2t^2 + 2[\sin t + t\cos t]^3 + 4t\sin t\cos t \right] \end{array} \right\rangle}$$

Since: $\cos^2 t + \sin^2 t = 1$