Solution of Recurrence Relations →

Normally, $a_n = f(a_{n-1}, a_{n-2}, \dots)$ — (A)

A function "$a_n = f(n)$" denoted by — (1)

is called a solution if the funct^n $f(n)$ satisfies (A).

3 Methods →

1) **Substitution method** →

In this method, the given recurrence relation is repeatedly used for $n = 1, 2, 3, \dots$ and then an appropriate algebraic simplification is used to get the reqd. sol^n.

Q.1. The solution of the rec. relation

$a_n = n \cdot a_{n-1}$ where $a_0 = 1$ is ?

→ put $n = 1$, $a_1 = 1 \cdot a_0 = 1 = 1!$

$a_2 = 2 \cdot a_1 = 2 = 2!$

$a_3 = 3 \cdot 2 = 6 = 3!$

$a_4 = 4 \cdot 6 = 24 = 4!$

$$\boxed{a_n = n!}$$

$n\cdot(n-1)(n-2)$

$a_n = 1 + n\cdot 3$ $\quad 2 + (3^1 + 3^2 + \cdots 3^{n-1})$ $\quad a_2 = 2 + 3^{n-1} = 5$

$a_0 = 1$ $\quad a_3 = 5 + 9 = 14$

$a_1 = 4$ $\quad a_2 = 7$ $\quad a_3 = 10$ $\quad a_4 = 14 + 27 = 41$

Q.2. The soln of the rec. relatn $a_n = a_{n-1} + 3^{n-1}$ where $a_1 = 2$

$\rightarrow$ $a_1 = 2$ $\qquad a_2 = 2 + 3^1$

$a_3 = a_2 + 3^2 = 2 + 3^1 + 3^2$

$a_4 = a_3 + 3^3 = 2 + 3^1 + 3^2 + 3^3.$

$\therefore a_n = 2 + 3 + 3^2 + 3^3 + \cdots 3^{n-1}$

$= 1 + 1 + 3 + 3^2 + 3^3 + \cdots + 3^{n-1}$

$= 1 + \dfrac{3^n - 1}{3 - 1} = \dfrac{3^n + 1}{2}$

$$\therefore \boxed{a_n = \frac{3^n + 1}{2}}$$

Q.3. The soln of the rec. relatn

$a_n = a_{n-1} + (2n+1)$ where $a_0 = 1$ is ?

$\rightarrow$ $a_1 = a_0 + 3 = 4 = 2^2$

$a_2 = a_1 + 5 = 9 = 3^2$

$a_3 = a_2 + 7 = 9 + 7 = 16 = 4^2.$

$a_n = (n+1)^2.$

$a_n = (n+1)^2$ = sum of first $(n+1)$ odd nos.

Q.4. $a_n = a_{n-1} + n$ where $a_0 = 1$

$\rightarrow a_1 = a_0 + 1 = 1 + 1 = 2$

$a_2 = a_1 + 1 = 2 + 1 = 3 = a_0 + 1 + 2$

$a_3 = a_2 + 1 = 3 + 1 = 4 = a_0 + 1 + 2 + 3$

$a_4 = a_3 + 1 = 4 + 1 = 5 = a_0 + 1 + 2 + 3 + 4.$

$$a_n = a_0 + \frac{n \cdot (n+1)}{2} = 1 + \frac{n(n+1)}{2}$$

$$\boxed{a_n = \frac{n^2 + n + 2}{2}}$$

Q.5. $a_n = a_{n-1} + n^2$. where $a_0 = 3$

$\rightarrow a_1 = a_0 + 1$

$a_2 = a_1 + 4 = a_0 + 1 + 4$

$a_3 = a_2 + 9 = a_0 + 1^2 + 2^2 + 3^2$

$\therefore a_n = a_0 + (1^2 + 2^3 + 3^2 + \cdots + n^2)$

$$= a_0 + \frac{n \cdot (n+1)(2n+1)}{6}$$

$= 3 +$

$\circledast$ $1^3 + 2^3 + 3^3 + \cdots + n^3$ $a_0 + \{1 - (1/n+1)\}$

$= \dfrac{n^2 (n+1)^2}{4}$

Q-6. The soln of the rec. relatn

$$a_n = a_{n-1} + \frac{1}{n(n+1)} \quad \text{where } a_0 = 1 \text{ is ?}$$

$$\Rightarrow \quad a_n = a_{n-1} + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$n = 1 \Rightarrow a_1 = a_0 + (1 - 1/2)$

$n = 2 \Rightarrow a_2 = a_1 + (1/2 - 1/3)$

$= a_0 + (1 - 1/2) + (1/2 - 1/3)$

$= a_0 + (1 - 1/3)$

$n = 3 \Rightarrow a_3 = a_2 + (1/3 - 1/4).$

$= a_0 + (1 - 1/3) + (1/3 - 1/4)$

$= a_0 + (1 - 1/4).$

$$\therefore a_n = a_0 + \left\{1 - \left(\frac{1}{n+1}\right)\right\}$$

$$= 1 + 1 - \left\{\frac{1}{n+1}\right\}$$

$$\boxed{a_n = \left(2 - \frac{1}{n+1}\right)}$$

Shift operator (E) →

$E(a_n) = a_{n+1}$

$E^2(a_n) = a_{n+2}$

$E^3(a_n) = a_{n+3}$

∴ For any +ve integer k,

$\boxed{E^k(a_n) = a_{n+k}.}$ (for $k \geq 1$)

Method of characteristic roots →

Consider the linear recurrence relation

lambda — $\lambda_0 a_n + \lambda_1 . a_{n-1} + \dots + \lambda_k . a_{n-k} = f(n).$ — (1)

Replacing 'n' with 'n+k', we have,

$\Rightarrow \lambda_0 . a_{n+k} + \lambda_1 . a_{n+k-1} + \dots + \lambda_k . a_n = f(n+k)$

$\Rightarrow \lambda_0 E^k a_n + \lambda_1 . E^{k-1}(a_n) + \dots + \lambda_k . a_n = f(n).$

(K is const.)

$\Rightarrow (\lambda_0 . E^k + \lambda_1 . E^{k-1} + \dots + \lambda_k .) a_n = f(n).$

$\Rightarrow \phi(E) \, a_n = f(n)$ — (2)

The characteristic eq^n is

$$\boxed{\phi(t) = 0}$$

The roots of this eq^n are called "characteristic roots".

Let $t = t_1, t_2, \dots, t_k$ be the characteristic roots.

Complementary Function → (C.F.)

This is solution of eq^n ① when $f(n) = 0$, i.e. the solution of homogeneous part of eq^n ①.

Rules for finding complementary $func^n$ (C.F.) ?

characteristic roots.	Complementary $Func^n$.
1) Roots are real and distinct.	$C_1 \cdot t_1^n + C_2 \cdot t_2^n + \dots + C_k \cdot t_k^n$.
2) Roots are real and 2 roots are equal. say, $t_1, t_1, t_3, \dots, t_k$.	$(C_1 + C_2 \cdot n) t_1^n + C_3 \cdot t_3^n + \dots + C_k \cdot t_k^n$.
3) Roots are real and 3 roots are equal. say, $t_1, t_1, t_1, t_4, \dots, t_k$.	$(C_1 + C_2 \cdot n + C_3 \cdot n^2) t_1^n + C_4 \cdot t_4^n + \dots + C_k \cdot t_k^n$.

4) If all the roots are Same! $(c_1 + c_2 \cdot n + c_3 \cdot n^2 + \cdots + c_k \cdot n^{k-1})$

5) A pair of roots are complex say $(\alpha \pm i\beta)$ $r^n \{ c_1 \cos(n\theta) + c_2 \sin(n\theta) \}$

where $r = \sqrt{\alpha^2 + \beta^2}$

and $\theta = \tan^{-1}\left(\frac{\beta}{\alpha}\right)$

Particular solution → (P.S.) →

From eq^n ②,

$$\text{Particular soln (P.S.)} = \frac{1}{\phi(E)} \{ f(n) \}.$$

complete of eq^n ①
The soln is

$$a_n = \underset{(C.F.)}{\text{complementary func}^n} + \underset{(P.S.)}{\text{particular soln}}$$

Rule to find Particular Solution (P.S.) →

when R.H.S. of eq^n ② i.e. $f(n)$

$f(n) = b^n$ where b is constant.

then particular soln is $\boxed{\frac{b^n}{\phi(E)}}$

$$\Rightarrow \boxed{\frac{b^n}{\phi(b)}} \quad (\because \phi(b) \neq 0)$$

$a_n = 2 \cdot a_{n-1} - 1$
$a_1 = 2 \cdot a_0 - 1$
$a_2 =$

t_1^n

Case of failure i.e. when $\phi(b) = 0$,

$$\frac{b^n}{(E-b)^k} = C(n,k) \cdot b^{n-k}$$
$(\because k = 1, 2, 3, \dots)$

Q.1. The solution of the recurrence relation $a_n = 2 \cdot a_{n-1} - 1$ where $a_1 = 2$ ~~$a_0 = 1$~~ is what ?

→ Replacing 'n' with 'n+1',

$\Rightarrow a_{n+1} - 2 \cdot a_n = -1$

$\Rightarrow E(a_n) - 2a_n = -1$

$\Rightarrow (E-2) \cdot a_n = -1$ ——— ①

The characteristic eqn is

$t - 2 = 0$ $\boxed{\therefore t = 2}$

For $t = 2$, the complementary funcn is

$\Rightarrow \boxed{c_1 \cdot 2^n}$ e C.F.

From eqn ①, P.S. is

$$\left(\frac{-1}{E-2}\right) = -\left(\frac{1^n}{E-2}\right)$$

$$\Rightarrow -\left(\frac{1^n}{1-2}\right) \qquad \left(\because \frac{b^n}{\phi(E)} = \frac{b^n}{\phi(b)}\right)$$

$\Rightarrow \boxed{1} \in$ p.s.

$\therefore$ the solution is

$a_n = cf + ps$

$a_n = c_1 \cdot 2^n + 1$ — ②.

for $n = 1$

$\therefore 2 = 2 \cdot c_1 + 1$

$\therefore c_1 = 1/2$

Substitute c_1 in eq.^{n} ②, we have

$a_n = 1/2 \cdot 2^n + 1$

$a_n = 2^{n-1} + 1$

Q.2. The solution of the recurrence relation

$T(2^k) = 3T(2^{k-1}) + 1$ where $T(1) = 1$.

is what?

$\rightarrow$ Let $T(2^k) = a_k$.

$\therefore \Rightarrow a_k = 3 \cdot a_{k-1} + 1$

Replace 'k' with 'k+1'

$\Rightarrow a_{k+1} = 3 \cdot a_k + 1$

$\Rightarrow \therefore E(a_k) - 3a_k = 1$

$\Rightarrow \therefore (E-3) \cdot a_k = 1$ —— (1)

Now, the characteristic eq^n is

$(t-3) = 0 \quad \therefore t = 3.$

$\therefore$ The complementary function is

$\Rightarrow C_1 \cdot 3^K$

from (1), particular soln is

$$\frac{1}{E-3} \Rightarrow \frac{1^K}{(E-3)} \Rightarrow \frac{1^K}{1-3}.$$

replace E by 1.

$\therefore$ p.s. $\Rightarrow -\left(\frac{1^K}{2}\right) = -1/2.$

$\therefore$ The solution is

$a_k = T(2^K) = C_1 \cdot 3^K - 1/2$ —— (2).

$a_0 = 1 = C_1 - 1/2,$

$\therefore C_1 = 3/2.$

$$\therefore \boxed{T(2^k) = \left(\frac{3^{K+1} - 1}{2}\right)}$$

Q.3. The solution of the recurrence relation

$a_n - 2 \cdot a_{n-1} = 2^n$ where $a_0 = 1$.

is what ?

→ put $n = n+1$

$\therefore a_{n+1} - 2a_n = 2^{n+1}$

$\Rightarrow E(a_n) - 2a_n = 2^{n+1}$

$\Rightarrow \therefore (E-2) \cdot a_n = 2(2^n)$ — ①

characteristic eqn is

$(t-2) = 0 \quad \therefore t = 2.$

∴ the complementary funcn is

$c_1 \cdot 2^n$

From ①, P.S. is

$$\frac{2(2^n)}{E-2} = 2 \cdot C(n,1) \, 2^{n-1} = n \cdot 2^n$$

cant replace E by 2 here

$$\left(\frac{b^n}{(E-b)^k} = C(n,k) \cdot b^{n-k} \right)$$

$a_n = n+1$ $\quad$ $c_3 = 6$ $\quad$ $a_n = 8 \cdot 3 \quad 5$ $\quad$ $a_n = 2a_{n+1} - a_{n-2}$ $\quad$ $a_2 = 2a_q = a_0$ $\quad$ $a_{-1} = 3$

$\therefore$ The solution is

$a_n = CF + PS$

$= C_1 2^n + D \cdot 2^n$ — (2).

$a_0 = 1$ given. $\therefore 1 = C_1 \cdot 1 + D \cdot 1$

$1 = C_1 + 0$ $\quad \therefore C_1 = 1$ $\quad \boxed{C_1 = 1}$

$\therefore a_n = (1-n) \cdot 2^n + n \cdot 2^n$

$= (1-n+n) \cdot 2^n$

$= \boxed{2^n}$.

$\therefore a_n = 2^n(n+1)$.

Q.4. The solution of the rec. relat^n

$a_n - 2a_{n-1} + a_{n-2} = 0$ where

$a_0 = 1$ and $a_1 = 2$ is what?

$\rightarrow$ $a_{n+2} - 2a_{n+1} + a_n = 0$.

$\Rightarrow (E^2 - 2 \cdot E + 1) a_n = 0$.

The char. eq^n is

$t^2 - 2t + 1 = 0$ $\quad t = 1, 1$.

The solution is

$$a_n = (c_1 + c_2 n) \cdot t_1^{\,n} \quad -(1).$$

$$= (c_1 + c_2 n) \cdot 1^n.$$

put $n = 0$ in eqn (1),

$$a_0 = 1 = c_1.$$

$n = 1 \quad a_1 = 2 = 1 + c_2$

$$c_2 = 1.$$

$$\therefore \boxed{a_n = (1+n).}$$

Q.5. The soln of the rec. relatn

$$a_n - 7a_{n-1} + 12 \cdot a_{n-2} = 0.$$

$$\Rightarrow a_{n+2} - 7a_{n+1} + 12a_n = 0.$$

$$\Rightarrow (E^2 - 7E + 12)a_n = 0$$

The char. eqn is

$$t^2 - 7t + 12 = 0.$$

$$\therefore t = 3, 4.$$

The soln is $a_n = c_1 3^n + c_2 4^n$

$a_n = 7a_{n-1} - 12a_{n-2}$ $\quad a_0 = 2$, $a_1 = 5$

$a_2 = 35 - 24 = 11$

$a_3 = 179 - 60 = 17.$

$a_4 =$

$n = 0. \Rightarrow 2 = C_1 + C_2$

$n = 1 \Rightarrow 5 = 3C_1 + 4C_2.$

$C_1 = 3 \quad C_2 = -1.$

$$\therefore \boxed{a_n = 3^{n+1} - 4^n}$$

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Q.6. The solution of the recurrence relation

$a_n - a_{n-1} + a_{n-2} = 0$ where $a_0 = a_1 = 1$ is ?

→ The characteristic eq^n (directly).

$t^2 - t + 1 = 0.$

$$\therefore t = \left.\frac{1 \pm \sqrt{3}\, i}{2}\right\} \text{complex} = 1/2 \pm \frac{\sqrt{3}}{2} i$$

$$r = \sqrt{\alpha^2 + \beta^2} = 1$$

$$\theta = \tan^{-1}\left(\frac{\beta}{\alpha}\right) = \pi/3.$$

$$\therefore a_n = c_1 \cdot \cos\left(\frac{n\pi}{3}\right) + c_2 \cdot \sin\left(\frac{n\pi}{3}\right) \quad \text{--- (1)}$$

$n = 0 \Rightarrow 1 = C_1$

$n = 1 \Rightarrow 1 = C_1/2 + C_2 \cdot \sqrt{3}/2$

$$c_2 = \frac{1}{\sqrt{3}}$$

$$\therefore a_n = \cos(n\pi/3) + \frac{1}{\sqrt{3}} \cdot \sin(n\pi/3).$$

Q.7. The solution of the rec. relation

$a_n - 3 \cdot a_{n-1} + 2 \cdot a_{n-2} = 2^n$ is ?

$\rightarrow$ put $n = n+2$.

$\therefore a_{n+2} - 3a_{n+1} + 2a_n = 2^{n+2}$

$(E^2 - 3E + 2)a_n = 4(2^n)$ — ①

The characteristic eqn is

$t^2 - 3t + 2 = 0.$

$\therefore t = 2, t = 1.$ (Roots $\rightarrow$ reals and distinct).

$\therefore$ complementary funcn is

$c_1 \cdot 1^n + c_2 \cdot 2^n$

$\Rightarrow c_1 + c_2 \cdot 2^n$

from eqn ①

The particular soln is

$$\frac{4 \cdot (2^n)}{E^2 - 3E + 2} \Rightarrow \frac{4 \cdot (2^n)}{(E-2)(E-1)}$$

$$\Rightarrow 4 \cdot \frac{2^n}{(E-2)(E-1)}$$

$$\Rightarrow \frac{4}{(E-2)} \cdot \frac{(2^n)}{(E-1)} \Rightarrow \frac{4}{(E-2)} \cdot \frac{2^n}{(2-1)}$$

$$\Rightarrow \frac{4}{E-2} \cdot 2^n \Rightarrow \frac{4 \cdot 2^n}{E-2} \Rightarrow \frac{4 \cdot 2^n}{(E-2)}$$

$$\Rightarrow 4 \cdot {}^nC_1 \cdot 2^{n-1} \quad \left(\because \frac{b^n}{(E-b)^k} = C(n,k) \cdot b^{n-k} \right)$$

$$= 2n \cdot 2^n$$

The solution is

$$\boxed{a_n = C_1 + C_2 \cdot 2^n + 2n \cdot 2^n} \quad \text{(General sol}^n\text{)}.$$

Q.8. The sol^n of

$a_n - 6a_{n-1} + 9a_{n-2} = 8^n$ is ?

* (no initial cond^{n}s given).

→

$$a_{n+2} - 6a_{n+1} + 9a_n = 3^{n+2}$$

$$\therefore (E^2 - 6E + 9) \cdot a_n = 3^{n+2}$$

$$\Rightarrow (E^2 - 6E + 9) \cdot a_n = 3^n (9) \quad — (1)$$

The characteristic eq^n is

$$t^2 - 6t + 9 = 0 \qquad t = 3$$

∴ Complementary Function is

$(C_1 + C_2 n) . 3^n$

Particular soln is

$$\frac{9(3^n)}{E^2 - 6E + 9}$$

$$\Rightarrow \frac{9(3^n)}{(E-3)(E-3)} \Rightarrow \frac{9 \cdot (3^n)}{(E-3)^2}$$

Here, $\frac{b^n}{(E-b)^k} = C(n,k) \, b^{n-k}$.

$k = 2$ here

$$\therefore \Rightarrow 9 \cdot \frac{3^n}{(E-3)^2} \Rightarrow 9 \cdot {}^nC_2 \cdot 3^{n-2}$$

$$\Rightarrow \frac{n(n-1)}{2} \cdot 3^n.$$

The soln is

$$\boxed{a_n = 3^n \left(C_1 + C_2 n + \frac{n(n-1)}{2} \right).}$$

(**) Recurrence relations reduceable to linear form by substitution →

Q.9: The soln of the rec relatn

$a_n^2 - 2 \cdot a_{n-1}^2 = 1$ where $a_0 = 2$ is what?

→ Let $a_n^2 = x_n$, then

$x_n - 2x_{n-1} = 1$

(now, same as prev.)

put $n = n+1$

$\therefore x_{n+1} - 2x_n = 1$

$\therefore E(x_n) - 2x_n = 1$

$\therefore (E-2) \cdot x_n = 1$ — ①

$\therefore$ Complementary funcn is

$c_1 \cdot 2^n$.

Particular soln is

$\frac{1}{E-2} \Rightarrow \frac{1^n}{E-2} \Rightarrow \frac{1^n}{1-2}$

$\Rightarrow -1$

$\therefore$ The solution is

$x_n = a_n^2 = c_1 \cdot 2^n - 1$

Given → $a_0 = 2$.

$\therefore \; 4 = c_1 \cdot 1 - 1$

$\therefore \; c_1 = 5$

$$\therefore \; \boxed{a_n^2 = 5 \cdot 2^n - 1}$$

$$\Rightarrow \; \boxed{a_n = \sqrt{5 \cdot 2^n - 1}}$$

Q.10. The solution of the recurrence relation

$\sqrt{a_n} - \sqrt{a_{n-1}} - 2 \cdot \sqrt{a_{n-2}} = 0$ ← because of this, complementary funcn

where $a_0 = a_1 = 1$ is what? is the soln.

→ Let $x_n^2 = a_n$.

$\therefore \; x_n - x_{n-1} - 2 \cdot x_{n-2} = 0$

$\therefore$ put $n = n+2$

$\therefore \; x_{n+2} - x_{n+1} - 2 \cdot x_n = 0$.

$\therefore \; (E^2 - E - 2) \cdot x_n = 0$.

The characteristic eqn is

$t^2 - t - 2 = 0$.

$\therefore \; t^2 - 2t + t - 2 = 0$.

$\therefore \; t(t-2) + 1(t-2) = 0$.

$\therefore \quad t = 2, -1.$

$\therefore$ The complementary func^n is

$c_1 \cdot 2^n + c_2 (-1)^n$

$\therefore$ The sol^n is

$x_n = c_1 \cdot 2^n + c_2 \cdot (-1)^n$

put $n = 0 \Rightarrow 1 = c_1 + c_2$

put $n = 1 \Rightarrow 1 = 2c_1 - c_2$

$\therefore c_1 = 2/3, \quad c_2 = 1/3.$

$\therefore x_n = \sqrt{a_n} = 2/3 \cdot 2^n + 1/3 \cdot (-1)^n.$

$$\boxed{\therefore a_n = \left(\frac{2^{n+1} + (-1)^n}{3}\right)^2}$$

Q.H.

Divide-And-Conquer Relations :→

A rec relat^n of the form

$$\boxed{T(n) = c \cdot T\left(\frac{n}{d}\right) + f(n)}$$

is called "Divide-And-Conquer Relation".

where 'c' and 'd' are constants.

This rec. relat^n can be reduced to linear form by substituting

$n = 2^k$.

Q.11. The sol^n of the rec. relat^n

$T(n) = 2 \cdot T(n/2) + (n-1)$ where $T(1) = 0$ is ?

→ put $n = 2^k$.

So. $T(2^k) = 2 \cdot T(2^{k-1}) + (2^k - 1)$

Let $T(2^k) = a_k$.

$\therefore a_k - 2a_{k-1} = 2^k - 1$

put $k = k+1$

$\therefore a_{k+1} - 2a_k = 2^{k+1} - 1$

$\therefore (E-2) \cdot a_k = 2^{k+1} - 1$

$\therefore (E-2) \cdot a_k = 2 \cdot (2^k) - 1$ — ①

$\therefore$ The characteristic eq^n is

$t - 2 = 0 \quad \therefore t = 2.$

$\therefore$ The complementary func^n is

$c_1 \cdot 2^k$.

The particular sol^n is

$$\frac{2 \cdot 2^K}{E-2} - \frac{1}{E-2}$$

$\Rightarrow$ ~~nc_2~~ $\cdot b$

$$\Rightarrow 2 \cdot K \cdot 2^{K-1} - \left(\frac{1^K}{1-2} \right) \Rightarrow K \cdot 2^K + 1$$

The sol^n is

$$a_K = c_1 \cdot 2^K + K \cdot 2^K + 1 \qquad — (2)$$

put $K = 0$, $\therefore a_0 = 0 = c_1 + 1$

$\therefore c_1 = -1$

$\therefore T(2^K) = 2^K(K-1) + 1$.

$\because$ replace 2^K with n,

$$\boxed{T(n) = n \cdot (\log_2 n - 1) + 1}$$

Q.12. The solution of the rec. relat^n

[illegible]

$T(n) = 7T(n/3) + 2n$ where, $T(1) = 5/2$ is ?

→ put $n = 3^k$.

$\therefore T(3^k) = 7 \cdot T(3^{k-1}) + 2 \cdot 3^k$.

$\therefore$ put $T(3^k) = a_k$.

$\therefore a_k - 7 \cdot a_{k-1} = 2 \cdot 3^k$.

put $k = k+1$

$\therefore a_{k+1} - 7 \cdot a_k = 2 \cdot 3^{k+1}$

$\therefore (E - 7) \cdot a_k = 2 \cdot 3^{k+1}$

$\therefore$ Characteristic eqn.

$t = 7$.

$\therefore$ complementary funcn is

$c_1 \cdot 7^k$.

Particular soln is

$$\frac{2 \cdot 3^{k+1}}{E - 7} = \frac{6 \cdot 3^k}{(E-7)}$$

$$= \frac{6 \cdot 3^k}{-4} = -\frac{3 \cdot 3^k}{2}$$

$$= -\frac{3^{k+1}}{2}$$

Master's theorem →

$$T(n) = a \cdot T\left(\frac{n}{b}\right) + f(n)$$

where $f(n) = \theta(n^k)$.

1) if $a < b^k$ then $T(n) = \theta(n^k)$

2) if $a = b^k$ then $T(n) = \theta(n^k \cdot \log n)$

3) if $a > b^k$ then $T(n) = \theta\left(n^{\log_b a}\right)$

Q.13. The solution of

$T(n) = 2T(n/2) + (n-1)$ is ?

a) $\theta(n)$ ✓b) $\theta(n \cdot \log n)$ c) $\theta(n^2)$ d) $\theta(\log n)$.

→ $a = 2$, $b = 2$, $k = 1$

$f(n) = n - 1$ it is of order 'n^k' where $k = 1$

we have,

$a = b^k$.

By Master's theorem, the solⁿ is $n^k \cdot \log n$

$\therefore \boxed{\theta(n \cdot \log n)}$

$$\boxed{n^{\log_3 7} = 7^{\log_3 n}}$$

Q.14. The soln of the rec. relatn

$T(n) = 7 \cdot T(n/3) + 2n$ is ?

$a = 7$, $b = 3$, and $k = 1$.

we have $a > b^k$

$\therefore$ By master's theorem,

$$\boxed{T(n) = \Theta\left(\cancel{n \log}\left(n^{(\log_3 7)}\right)\right).}$$

Q.15. The soln of the rec relatn

$T(n) = 6 \cdot T(n/3) + 3 \cdot n^4$ is ?

$\rightarrow$ $a = 6$, $b = 3$, $k = 4$.

$a < b^k$.

$\therefore$ $\boxed{T(n) = \Theta(n^4).}$

Method of undetermined coefficients $\rightarrow$

Consider the linear rec. relatn

$\phi(E) \cdot a_n = f(n)$

The characteristic eqn is

$\phi(t) = 0$.

* multiplicity → no. of times 1 appears as char. root

So, we get characteristic roots

$t =$

The complementary funcn. → same.

*Rules to find particular soln →

$f(n)$	$\phi(t)$	Particular soln.
) n^k $k=(1,2,3,\ldots)$ (or) any poly. funcn of degree 'k'	$\phi(1) \neq 0$, i.e. 1 is not a characteristic root.	$A_0 \cdot n^k + A_1 \cdot n^{k-1} + \cdots + A_k$ where, $A_0, A_1, \ldots, A_k$ are undetermined coefficients.
). same condn as above.	$\phi(1) = 0$ i.e. 1 is a characteristic root with multiplicity 'm'.	$(A_0 \cdot n^k + A_1 \cdot n^{k-1} + \cdots + A_k) \cdot n^m$
) $b^n \cdot n^k$ $(k = 1, 2, 3, \ldots)$	$\phi(b) \neq 0$ i.e. 'b' is not a characteristic root.	$b^n (A_0 \cdot n^k + A_1 \cdot n^{k-1} + \cdots + A_k)$
) same condn as above	$\phi(b) = 0$ i.e. 'b' is a char. root with multiplicity 'm'.	$b^n \cdot n^m (A_0 \cdot n^k + A_1 \cdot n^{k-1} + \cdots + A_k)$

Q. 16. The solution of

$a_n - 2 \cdot a_{n-1} = (n+2)$ is ?

→ The characteristic eqn is

$(t-2) = 0 \quad \therefore \quad t = 2.$

The complementary funcn is

$c_1 \cdot 2^n .$

Let particular soln is

$a_n = $

$p.s. = (c \cdot n + d) - ①$, where c and d are undetermined coefficients.

Substituting in given rec. relatn, we have,

$(cn+d) - 2\{c \cdot (n-1) + d\} = n+2 .$

equating the coeff. of 'n' ⇒

$-c = 1 \Rightarrow \underline{c = -1}.$

equating constants ⇒

$2c - d = 2.$

$\therefore d = 2c - 2 = \underline{-4} = d.$

$\therefore$ particular solⁿ is $\underline{-n-4}$.

The solⁿ is

$$\boxed{a_n = c_1 \cdot 2^n - n - 4}$$

* 2nd order Nh. rec. relatⁿ

Q.17. The solution of the rec. relatⁿ

$$a_n - 2 \cdot a_{n-1} + a_{n-2} = (3n+5) \text{ is ?}$$

$\rightarrow$ The char. eqⁿ is.

$$t^2 - 2t + 1 = 0.$$

$\therefore t = 1, 1$

$\therefore$ complementory funcⁿ is

$$(c_1 + c_2 \cdot n) \cdot 1^n.$$

By rule 2, let particular solⁿ

$$P.S. = (cn+d) \cdot n^2 \quad (n^m \because m = 2)$$

$$= (cn^3 + dn^2)$$

substituting in the given rec. relaⁿ, we have,

$$(cn^3 + dn^2) - 2(c(n-1)^3 + d(n-1)^2) + c(n-2)^3 + d(n-2)^2$$

$$= 3n+5$$

put $n=1$, $c+d-c+d = 3(1)+5$

$2d = 8$.

$\therefore \boxed{d=4.}$

$n=2$, $8c+(-2c) = 6c+2d = 11$
$+4d-2d$.

$\therefore \boxed{c = 1/2.}$

$\therefore$ P.S. is.

$P.S. = 1/2\, n^3 + 4n^2$.

The soln is

$= (c_1 + c_2 . n) . + 4n^2 + \frac{n^3}{2}$.

Q.18. The soln of the rec. relatn

$a_n - 2.a_{n-1} = 3^n.(n+2)$
$\uparrow b$.

$\rightarrow$ The characteristic eqn is

$t-2=0 \quad \therefore t=2.$

Complementary funcn is

C.F. $c_1 . 2^n$.

By rule 3, let
Particular soln $p.s = 3^n (An+B)$ $(\because k=1)$

OR

$a_n = p.s. = 3^n (cn+d)$. — ①.

Substituting in the given rec. relatn

$$3^n(cn+d) - 2.3^{n-1}\{c\cdot(n-1)+d\}$$

$$= 3^n.(n+2)$$

Dividing the eqn by 3^n,

$$(cn+d) - 2/3\cdot\{c\cdot(n-1)+d\} = (n+2)$$

comparing coeff of n on both sides,

$$c - 2/3\,c = 1$$

$$\therefore \; 1/3\,c = 1 \qquad \boxed{\therefore c = 3}$$

equating the constants.

$$\frac{2c}{3} + d/3 = 2.$$

$$\therefore \boxed{d = 0.}$$

$$\therefore p.s = 3^n.3n = n.3^{(n+1)}$$

$\therefore$ The soln is

$$\boxed{a_n = c_1.2^n + n.3^{(n+1)}}$$

Q.19. The solⁿ of the recurrence relatⁿ $(a_n$ →

→) $a_n - 2 \cdot a_{n-1} = \underbrace{2^n \cdot n}_{b^n \cdot n^k}$ $(b = 2, k = 1)$.

→ characteristic eqⁿ is

$t - 2 = 0 \qquad \therefore t = 2$

Complementary funcⁿ is

$c_1 \cdot 2^n$.

By rule 4, let particular solⁿ

$P.S. = 2^n(cn + d) \cdot n^1$ (by rule 4)

$P.S. = 2^n(cn^2 + dn)$. — ①

substituting in the given rec relatⁿ,

$\underbrace{2^n \cdot (cn^2 + dn)}_{a_n} - \underbrace{2 \cdot \{2^{n-1}(c \cdot (n-1)^2 + d(n-1))\}}_{a_{n-1}}$

$= 2^n \cdot n$.

Div. the eqⁿ by 2^n,

$cn^2 + dn - \{c \cdot (n-1)^2 + d \cdot (n-1)\} = n$

put $n = 1$. → $c + d = 1$

put $n = 0$ → $-c + d = 0$.

$\therefore c = d = 1/2$.

$\therefore$ P.S. is

$$P.S. = \frac{2^n(n^2+n)}{2} = 2^{n-1}\cdot(n^2+n).$$

The solⁿ is

$$\boxed{a_n = C_1 \cdot 2^n + 2^{n-1}\cdot(n^2+n)}$$