10. Plane

- **Equation of a plane in normal form:**
  - **Vector form:** Equation of a plane which is at a distance of *d* from the origin, and the unit vector $\hat{n}$ normal to the plane through the origin is $\vec{r}.\hat{n} = d$, where $\vec{r}$ is the position vector of a point in the plane
  - **Cartesian form:** Equation of a plane which is at a distance *d* from the origin and the d.c.'s of the normal to the plane as l, m, n is $lx + my + nz = d$

- **Equation of a plane perpendicular to a given vector and passing through a given point:**
  - **Vector form:** Equation of a plane through a point whose position vector is $\vec{a}$ and perpendicular to the vector $\vec{N}$ is $(\vec{r} - \vec{a}).\vec{N} = 0$, where $\vec{r}$ is the position vector of a point in the plane
  - **Cartesian form:** Equation of plane passing through the point (x_1, y_1, z_1) and perpendicular to a given line whose d.r.'s are A, B, C is $A(x - x_1) + B(y - y_1) + C(z - z_1) = 0$

- **Equation of a plane passing through three non-collinear points:**
  - **Cartesian form:** Equation of a plane passing through three non-collinear points (x_1, y_1, z_1), (x_2, y_2, z_2), and (x_3, y_3, z_3) is $\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$
  - **Vector form:** Equation of a plane that contains three non-collinear points having position vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ is $(\vec{r} - \vec{a}).[(\vec{b} - \vec{a}) \times (\vec{c} - \vec{a})] = 0$, where $\vec{r}$ is the position vector of a point in the plane

- **Planes passing through the intersection of two planes:**
  - **Vector form:** Equation of the plane passing through intersection of two planes $\vec{r}.\vec{n_1} = d_1$ and $\vec{r}.\vec{n_2} = d_2$ is given by, $\vec{r}.(\vec{n_1} + \lambda\vec{n_2}) = d_1 + \lambda d_2$, where λ is a non-zero constant
  - **Cartesian form:** Equation of a plane passing through the intersection of two planes $A_1x + B_1y + C_1z + D_1 = 0$ and $A_2x + B_2y + C_2z + D_2 = 0$, is given by, $(A_1x + B_1y + C_1z + D_1) + \lambda(A_2x + B_2y + C_2z + D_2) = 0$, where λ is a non-zero constant

- **Angle between two planes:** The angle between two planes is defined as the angle between their normals.
  - **Vector form:** If θ is the angle between the two planes $\vec{r}.\vec{n_1} = d_1$ and $\vec{r}.\vec{n_2} = d_2$, then $\cos\theta = \left|\frac{\vec{n_1}.\vec{n_2}}{|\vec{n_1}||\vec{n_2}|}\right|$

Note that if two planes are perpendicular to each other, then $\vec{n_1} \cdot \vec{n_2} = 0$; and if they are parallel to each other, then $\vec{n_1}$ is parallel to $\vec{n_2}$.

- **Cartesian form:** If θ is the angle between the two planes $A_1x + B_1y + C_1z + D_1 = 0$ and $A_2x + B_2y + C_2z + D_2 = 0$, then $\cos\theta = \left|\dfrac{A_1A_2 + B_1B_2 + C_1C_2}{\sqrt{A_1^2 + B_1^2 + C_1^2}\sqrt{A_2^2 + B_2^2 + C_2^2}}\right|$

  Note that if two planes are perpendicular to each other, then $A_1A_2 + B_1B_2 + C_1C_2 = 0$; and if they are parallel to each other, then $\dfrac{A_1}{A_2} = \dfrac{B_1}{B_2} = \dfrac{C_1}{C_2}$

- **Angle between a line and a plane:** The angle Φ between a line $\vec{r} = \vec{a} + \lambda\vec{b}$ and the plane $\vec{r} \cdot \vec{n} = d$ is the complement of the angle between the line and the normal to the plane and is given by $\Phi = \sin^{-1}\left|\dfrac{\vec{b} \cdot \vec{n}}{\left|\vec{b}\right|\left|\vec{n}\right|}\right|$.

- **Co-planarity of two lines**
  - **Vector form:** Two lines $\vec{r} = \vec{a_1} + \lambda\vec{b_1}$ and $\vec{r} = \vec{a_2} + \mu\vec{b_2}$ are co-planar, if $\left(\vec{a_2} - \vec{a_1}\right) \cdot \left(\vec{b_1} \times \vec{b_2}\right) = 0$
  - **Cartesian form:** Two lines $\dfrac{x - x_1}{a_1} = \dfrac{y - y_1}{b_1} = \dfrac{z - z_1}{c_1}$ and $\dfrac{x - x_2}{a_2} = \dfrac{y - y_2}{b_2} = \dfrac{z - z_2}{c_2}$ and $\dfrac{x - x_2}{a_2} = \dfrac{y - y_2}{b_2} = \dfrac{z - z_2}{c_2}$ are co-planar, if $\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$

- **Distance of a point from a plane:**
  - **Vector form:** The distance of a point, whose position vector is $\vec{a}$, from the plane $\vec{r} \cdot \hat{n} = d$ is $\left|d - \vec{a} \cdot \hat{n}\right|$.

    Note:

    - If the equation of the plane is in the form of $\vec{r} \cdot \vec{N} = d$, where $\vec{N}$ is the normal to the plane, then the perpendicular distance is $\dfrac{\left|\vec{a} \cdot \vec{N} - d\right|}{\left|\vec{N}\right|}$.

- Length of the perpendicular from origin to the plane $\vec{r} \cdot \vec{N} = d$ is $\frac{|d|}{|\vec{N}|}$.

- **Cartesian form:** The distance from a point (x_1, y_1, z_1) to the plane $Ax + By + Cz + D = 0$ is $\left|\frac{Ax_1+By_1+Cz_1+D}{\sqrt{A^2+B^2+C^2}}\right|$.