1.8 RELATIVISTIC MECHANICS

1.340 From the formula for length contraction

$$\left(l_0 - l_0\sqrt{1-\frac{v^2}{c^2}}\right) = \eta\, l_0$$

So, $$1-\frac{v^2}{c^2} = (1-\eta)^2 \quad \text{or} \quad v = c\sqrt{\eta\,(2-\eta)}$$

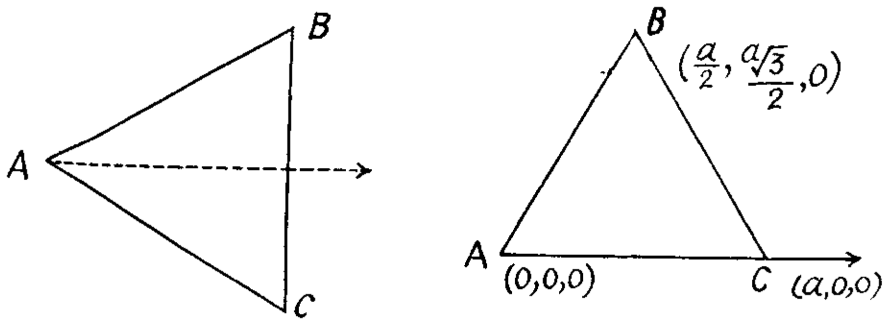
1.341 (a) In the frame in which the triangle is at rest the space coordinates of the vertices are (000), $\left(a\frac{\sqrt{3}}{2}, +\frac{a}{2}, 0\right)\left(a\frac{\sqrt{3}}{2}, -\frac{a}{2}, 0\right)$, all measured at the same time t. In the moving frame the corresponding coordinates at time t' are

$A : (vt', 0, 0), B : \left(\frac{a}{2}\sqrt{3}\sqrt{1-\beta^2} + vt', \frac{a}{2}, 0\right)$ and $C : \left(\frac{a}{2}\sqrt{3}\sqrt{1-\beta^2} + vt', -\frac{a}{2}, 0\right)$

The perimeter P is then

$$P = a + 2a\left(\frac{3}{4}(1-\beta^2) + \frac{1}{4}\right)^{1/2} = a\left(1+\sqrt{4-3\beta^2}\right)$$

(b) The coordinates in the first frame are shown at time t. The coordinates in the moving frame are,



$A : (vt', 0,0), B : \left(\frac{a}{2}\sqrt{1-\beta^2} + vt',\ a\frac{\sqrt{3}}{2}, 0\right),\ C : \left(a\sqrt{1-\beta^2} + vt', 0, 0\right)$

The perimeter P is then

$$P = a\sqrt{1-\beta^2} + \frac{a}{2}\left[1-\beta^2+3\right]^{1/2} \times 2 = a\left(\sqrt{1-\beta^2} + \sqrt{4-\beta^2}\right) \text{ here } \beta = \frac{V}{c}$$

1.342 In the rest frame, the coordinates of the ends of the rod in terms of proper length l_0

$$A : (0,0,0)\ B : (l_0\cos\theta_0, l_0\sin\theta_0, 0)$$

at time t. In the laboratory frame the coordinates at time t' are

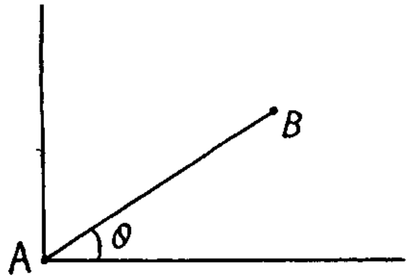
$$A : (vt', 0, 0), B : \left(l_0\cos\theta_0\sqrt{1-\beta^2} + vt', l_0\sin\theta_0, 0\right)$$

Therefore we can write,

$$l\cos\theta_0 = l_0\cos\theta_0\sqrt{1-\beta^2} \text{ and } l\sin\theta = l_0\sin\theta_0$$

Hence $l_0^2 = (l^2)\left(\dfrac{\cos^2\theta + (1-\beta^2)\sin^2\theta}{1-\beta^2}\right)$

or, $= \sqrt{\dfrac{1-\beta^2\sin^2\theta}{1-\beta^2}}$



1.343 In the frame K in which the cone is at rest the coordinates of A are (0,0,0) and of B are $(h, h\tan\theta, 0)$. In the frame K', which is moving with velocity v along the axis of the cone, the coordinates of A and B at time t' are

$A : (-vt', 0,0), B : \left(h\sqrt{1-\beta^2} - vt', h\tan\theta, 0\right)$

Thus the taper angle in the frame K' is

$$\tan\theta' = \frac{\tan\theta}{\sqrt{1-\beta^2}}\left(= \frac{y'_B - y'_A}{x'_B - x'_A}\right)$$

and the lateral surface area is,

$$S = \pi h'^2 \sec\theta' \tan\theta'$$

$$= \pi h^2(1-\beta^2)\frac{\tan\theta}{\sqrt{1-\beta^2}}\sqrt{1+\frac{\tan^2\theta}{1-\beta^2}} = S_0\sqrt{1-\beta^2\cos^2\theta}$$

Here $S_0 = \pi h^2 \sec\theta\tan\theta$ is the lateral surface area in the rest frame and

$$h' = h\sqrt{1-\beta^2}\,,\ \beta = v/c.$$

1.344 Because of time dilation, a moving clock reads less time. We write,

$$t - \Delta t = t\sqrt{1-\beta^2}\,,\ \beta = \frac{v}{c}$$

Thus, $$1 - \frac{2\Delta t}{t} + \left(\frac{\Delta t}{t}\right)^2 = 1-\beta^2$$

or, $$v = c\sqrt{\frac{\Delta t}{t}\left(2 - \frac{\Delta t}{t}\right)}$$

1.345 In the frame K the length l of the rod is related to the time of flight Δt by

$$l = v\,\Delta t$$

In the reference frame fixed to the rod (frame K')the proper length l_0 of the rod is given by

$$l_0 = v\,\Delta t'$$

But $$l_0 = \frac{l}{\sqrt{1-\beta^2}} = \frac{v\,\Delta t}{\sqrt{1-\beta^2}}\,,\ \beta = \frac{v}{c}$$

Thus, $$v\,\Delta t' = \frac{v\,\Delta t}{\sqrt{1-\beta^2}}$$

So $$1-\beta^2 = \left(\frac{\Delta t}{\Delta t'}\right)^2 \quad \text{or} \quad v = c\sqrt{1-\left(\frac{\Delta t}{\Delta t'}\right)^2}$$

and $$l_0 = c\sqrt{(\Delta t')^2-(\Delta t)^2} = c\,\Delta t'\sqrt{1-\left(\frac{\Delta t}{\Delta t'}\right)^2}$$

1.346 The distance travelled in the laboratory frame of reference is $v\Delta t$ where v is the velocity of the particle. But by time dilation

$$\Delta t = \frac{\Delta t_0}{\sqrt{1-v^2/c^2}} \quad \text{So} \quad v = c\sqrt{1-(\Delta t_0/\Delta t)^2}$$

Thus the distance traversed is

$$c\Delta t\sqrt{1-(\Delta t_0/\Delta t)^2}$$

1.347 (a) If τ_0 is the proper life time of the muon the life time in the moving frame is

$$\frac{\tau_0}{\sqrt{1-v^2/c^2}} \quad \text{and hence} \quad l = \frac{v\,\tau_0}{\sqrt{1-v^2/c^2}}$$

Thus $$\tau_0 = \frac{l}{v}\sqrt{1-v^2/c^2}$$

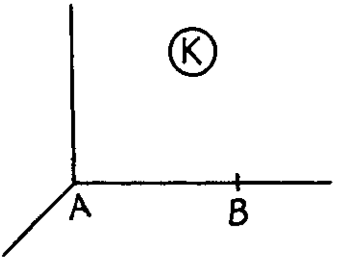
(The words "from the muon's stand point" are not part of any standard terminology)

1.348 In the frame K in which the particles are at rest, their positions are A and B whose coordinates may be taken as,

$$A:(0,0,0),\ B=(l_0,0,0)$$

In the frame K' with respect to which K is moving with a velocity v the coordinates of A and B at time t' in the moving frame are

$$A = (vt',0,0)\ B = \left(l_0\sqrt{1-\beta^2}+vt',0,0\right),\ \beta = \frac{v}{c}$$



Suppose B hits a stationary target in K' after time t'_B while A hits it after time $t_B+\Delta t$. Then,

$$l_0\sqrt{1-\beta^2}+vt'_B = v(t'_B+\Delta t)$$

So, $$l_0\,\frac{v\,\Delta t}{\sqrt{1-v^2/c^2}}$$

1.349 In the reference frame fixed to the ruler the rod is moving with a velocity v and suffers Lorentz contraction. If l_0 is the proper length of the rod, its measured length will be

$$\Delta x_1 = l_0\sqrt{1-\beta^2},\ \beta = \frac{v}{c}$$

In the reference frame fixed to the rod the ruler suffers Lorentz contraction and we must have

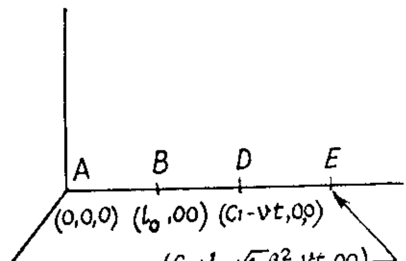
$$\Delta x_2 \sqrt{1-\beta^2} = l_0 \text{ thus } l_0 = \sqrt{\Delta x_1 \Delta x_2}$$

and
$$1-\beta^2 = \frac{\Delta x_1}{\Delta x_2} \text{ or } v = c\sqrt{1-\frac{\Delta x_1}{\Delta x_2}}$$

1.350 The coordinates of the ends of the rods in the frame fixed to the left rod are shown. The points B and D coincides when

$$l_0 = c_1 - vt_0 \text{ or } t_0 = \frac{c_1 - l_0}{v}$$

The points A and E coincide when



$$0 = c_1 + l_0\sqrt{1-\beta^2} - vt_1,\ t_1 = \frac{c_1 + l_0\sqrt{1-\beta^2}}{v}$$

Thus $\Delta t = t_1 - t_0 = \frac{l_0}{v}\left(1 + \sqrt{1-\beta^2}\right)$

or $\left(\frac{v\,\Delta t}{l_0} - 1\right)^2 = 1 - \beta^2 = 1 - \frac{v^2}{c^2}$

From this
$$v = \frac{2c^2\Delta t/l_0}{1 + c^2\Delta t^2/l_0^2} = \frac{2l_0/\Delta t}{1 + (l_0/c\,\Delta t)^2}$$

1.351 In K_0, the rest frame of the particles, the events corresponding to the decay of the particles are,

$$A : (0,0,0,0) \text{ and } (0, l_0, 0, 0) = B$$

In the reference frame K, the corresponding coordintes are by Lorentz transformation

$$A : (0,0,0,0), B : \left(\frac{vl_0}{c^2\sqrt{1-\beta^2}}, \frac{l_0}{\sqrt{1-\beta^2}}, 0, 0\right)$$

Now
$$l_0\sqrt{1-\beta^2} = l$$

by Lorentz Fitzgerald contraction formula. Thus the time lag of the decay time of B is

$$\Delta t = \frac{vl_0}{c^2\sqrt{1-\beta^2}} = \frac{v\,l}{c^2(1-\beta^2)} = \frac{vl}{c^2 - v^2}$$

B decays later (B is the forward particle in the direction of motion)

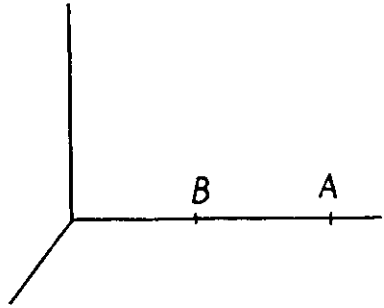
1.352 (a) In the reference frame K with respect to which the rod is moving with velocity v, the coordinates of A and B are

$$A : t, x_A + v(t - t_A), 0, 0$$

$$B : t, x_B + v(t - t_B), 0, 0$$

Thus $l = x_A - x_B - v(t_A - t_B) = l_0\sqrt{1-\beta^2}$

So $$l_0 = \frac{x_A - x_B - v(t_A - t_B)}{\sqrt{1 - v^2/c^2}}$$



(b) $\pm l_0 - v(t_A - t_B) = l = l_0\sqrt{1 - v^2/c^2}$

(since $x_A - x_B$ can be either $+l_0$ or $-l_0$.)

Thus $v(t_A - t_B) = \left(\pm 1 - \sqrt{1 - v^2/c_2}\right) l_o$

i.e. $$t_A - t_B = \frac{l_0}{v}\left(1 - \sqrt{1 - \frac{v^2}{c^2}}\right)$$

or $$t_B - t_A = \frac{l_0}{v}\left(1 + \sqrt{1 - v^2/c^2}\right)$$

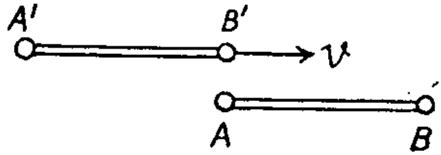
1.353 At the instant the picture is taken the coordintes of A, B, A', B' in the rest frame of $A\,B$ are

$A : (0, 0, 0, 0)$

$B : (0, l_0, 0, 0)$

$B' : (0, 0, 0, 0)$

$A' : (0_1 - l_0\sqrt{1 - v^2/c^2}, 0, 0)$



In this frame the coordinates of B' at other times are $B' : (t, vt, 0, 0)$. So B' is opposite to B at time $t(B) = \frac{l_0}{v}$. In the frame in which B', A' is at rest the time corresponding this is by Lorentz tranformation.

$$t^0(B') = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}\left(\frac{l_0}{v} - \frac{vl_0}{c^2}\right) = \frac{l_0}{v}\sqrt{1 - v^2/c^2}$$

Similarly in the rest frame of A, B, te coordinates of A at other times are

$$A' : \left(t, -l_0\sqrt{1 - \frac{v^2}{c^2}} + vt, 0, 0\right)$$

A' is opposite to A at time $t(A) = \frac{l_0}{v}\sqrt{1 - \frac{v^2}{c^2}}$

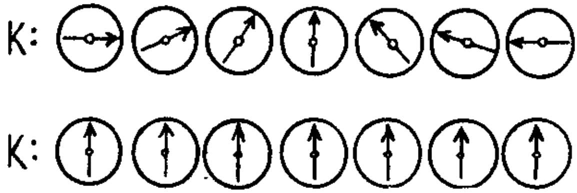
The corresponding time in the frame in which A', B' are at rest is

$$t(A') = \gamma\, t(A) = \frac{l_0}{v}$$

1.354 By Lorentz transformation $t' = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}\left(t - \frac{vx}{c^2}\right)$

So at time $t = 0,\ t' = \frac{vx}{c^2}\frac{1}{\sqrt{1 - v^2/c^2}}$

If $x > 0\ t' < 0$, if $x < 0$, $t' > 0$ and we get the diagram given below "in terms of the K–clock".



The situation in terms of the K' clock is reversed.

1.355 Suppose $x(t)$ is the locus of points in the frame K at which the readings of the clocks of both reference system are permanently identical, then by Lorentz transformation

$$t' = \frac{1}{\sqrt{1 - V^2/c^2}}\left(t - \frac{V x(t)}{c^2}\right) = t$$

So differentiating $x(t) = \frac{c^2}{V}\left(1 - \sqrt{1 - \frac{V^2}{c^2}}\right) = \frac{c}{\beta}\left(1 - \sqrt{1 - \beta^2}\right),\ \beta = \frac{V}{c}$

Let $\beta = \tan h\theta,\ 0 \le \theta < \infty$, Then

$$x(t) = \frac{c}{\tan h\theta}\left(1 - \sqrt{1 - \tan h^2\theta}\right) = c\frac{\cos h\theta}{\sin h\theta}\left(1 - \frac{1}{\cos h\theta}\right)$$

$$= c\frac{\cos h\theta - 1}{\sin h\theta} = c\sqrt{\frac{\cos h\theta - 1}{\cos h\theta + 1}} = c\tan h\frac{\theta}{2} \le v$$

(tan $h\,\theta$ is a monotonically increasing function of θ)

1.356 We can take the coordinates of the two events to be

$$A : (0, 0, 0, 0)\ \ B : (\Delta t, a, 0, 0)$$

For B to be the effect and A to be cause we must have $\Delta t > \frac{|a|}{c}$.

In the moving frame the coordinates of A and B become

$$A : (0, 0, 0, 0), B : \left[\gamma\left(\Delta t - \frac{aV}{c^2}\right),\ \gamma(a - V\Delta t), 0, 0\right] \text{ where } \gamma = \frac{1}{\sqrt{1 - \left(\frac{V^2}{c^2}\right)}}$$

Since

$$(\Delta t')^2 - \frac{a'^2}{c^2} = \gamma^2\left[\left(\Delta t - \frac{aV}{c^2}\right)^2 - \frac{1}{c^2}(a - V\Delta t)^2\right] = (\Delta t)^2 - \frac{a^2}{c^2} > 0$$

we must have $\Delta t' > \frac{|a'|}{c}$

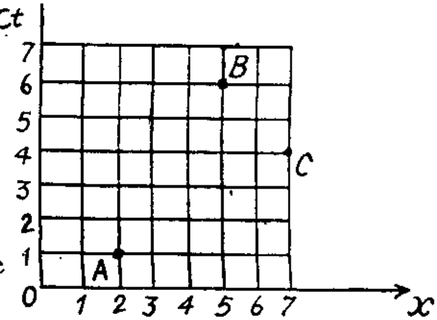
1.357 (a) The four-dimensional interval between A and B (assuming $\Delta y = \Delta z = 0$) is :

$$5^2 - 3^2 = 16 \text{ units}$$

Therefore the time interval between these two events in the reference frame in which the events occurred at the same place is

$$c\,(t'_B - t'_A) = \sqrt{16} = 4\text{ m}$$

or $$t'_B - t'_A = \frac{4}{c} = \frac{4}{3} \times 10^{-8}\text{ s}$$



(b) The four dimensional interval between A and C is (assuming $\Delta y = \Delta z = 0$)

$$3^2 - 5^2 = -16$$

So the distance between the two events in the frame in which they are simultaneous is 4 units = 4m.

1.358 By the velocity addition formula

$$v_x' = \frac{v_x - V}{1 - \frac{V v_x}{c^2}}, \quad v'_y = \frac{v_y \sqrt{1 - V^2/c^2}}{1 - \frac{v_x V}{c^2}}$$

and $$v' = \sqrt{v'^2_x + v'^2_y} = \frac{\sqrt{(v_x - V)^2 + v_y^2 (1 - V^2/c^2)}}{1 - \frac{v_x V}{c^2}}$$

1.359 (a) By definition the velocity of apporach is

$$v_{\text{approach}} = \frac{dx_1}{dt} - \frac{dx_2}{dt} = v_1 - (-v_2) = v_1 + v_2$$

in the reference frame K.

(b) The relative velocity is obtained by the transformation law

$$v_r = \frac{v_1 - (-v_2)}{1 - \frac{v_1(-v_2)}{c^2}} = \frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}}$$

1.360 The velocity of one of the rods in the reference frame fixed to the other rod is

$$V = \frac{v + v}{1 + \frac{v^2}{c^2}} = \frac{2v}{1 + \beta^2}$$

The length of the moving rod in this frame is

$$l = l_0 \sqrt{1 - \frac{4 v^2/c^2}{(1 + \beta^2)^2}} = l_0 \frac{1 - \beta^2}{1 + \beta^2}$$

1.361 The approach velocity is defined by

$$\vec{V}_{\text{approach}} = \frac{d\vec{r}_1}{dt} - \frac{d\vec{r}_2}{dt} = \vec{V}_1 - \vec{V}_2$$

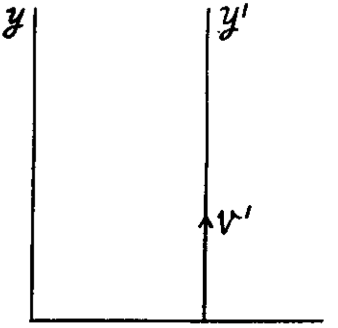
in the laboratory frame. So $V_{\text{approach}} = \sqrt{v_1^2 + v_2^2}$

On the other hand, the relative velocity can be obtained by using the velocity addition formula and has the components

$$\left[-v_1,\ v_2\sqrt{1-\left(\frac{v_1^2}{c^2}\right)}\right] \text{ so } V_r = \sqrt{v_1^2+v_2^2-\frac{v_1 v_2^2}{c^2}}$$

1.362 The components of the velocity of the unstable particle in the frame K are

$$\left(V, v'\sqrt{1-\frac{V^2}{c^2}}, 0\right)$$



so the velocity relative to K is

$$\sqrt{V^2+v'^2-\frac{v'^2 V^2}{c^2}}$$

The life time in this frame dilates to

$$\Delta t_0 \Big/ \sqrt{1-\frac{V^2}{c^2}-\frac{v'^2}{c^2}+\frac{v'^2 V^2}{c^4}}$$

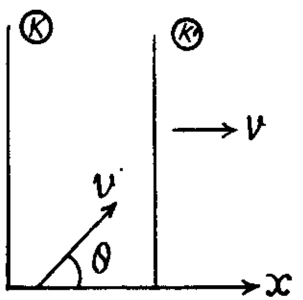
and the distance traversed is

$$\Delta t_0 \frac{\sqrt{V^2+v'^2-(v'^2 V^2)/c^2}}{\sqrt{1-V^2/c^2}\,\sqrt{1-v'^2/c^2}}$$

1.363 In the frame K' the components of the velocity of the particle are

$$v'_x = \frac{v\cos\theta - V}{1-\dfrac{vV\cos\theta}{c^2}}$$

$$v'_y = \frac{v\sin\theta\sqrt{1-V^2/c^2}}{1-\dfrac{vV}{c^2}\cos\theta}$$



Hence, $\tan\theta' = \dfrac{v'_y}{v'_x} = \dfrac{v\sin\theta}{v\cos\theta - V}\sqrt{(1-V^2)/c^2}$

1.364 In K' the coordinates of A and B are

$$A:(t', 0, -v't', 0);\ B:(t', l, -v't', 0)$$

After performing Lorentz transformation to the frame K *we get*

$$A: t = \gamma t' \quad B: t = \gamma\left(t' + \frac{Vl}{c^2}\right)$$

$$x = \gamma V t' \quad x = \gamma(l + Vt')$$

$$y = v't' \quad y = -v't'$$

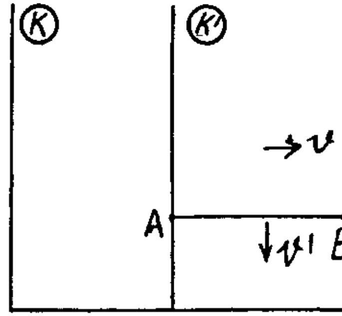
$$z = 0 \quad z = 0$$

By translating $t' \to t' - \dfrac{Vl}{c^2}$, we can write

the coordinates of B as $B: t = \gamma t'$

$$x = \gamma l\left(1 - \frac{V^2}{c^2}\right) + V t' \gamma = l\sqrt{1 - \frac{v^2}{c^2}} + V t' \gamma$$

$$y = -v'\left(t' - \frac{Vl}{c^2}\right), \; z = 0$$



Thus $$\Delta x = l\sqrt{1 - \left(\frac{V^2}{c^2}\right)}, \; \Delta y = \frac{v' V l}{c^2}$$

Hence $$\tan\theta' - \frac{v' V}{c^2\sqrt{1 - \frac{v' V}{c^2}}}$$

1.365 t $\quad$ $t + dt$ .... (K)

$\vec{v}$ $\quad$ $\vec{v} + \vec{w}\,dt$

In K the velocities at time t and $t + dt$ are respectively v and $v + wdt$ along x – axis which is parallel to the vector $\vec{V}$. In the frame K' moving with velocity $\vec{V}$ with respect to K, the velocities are respectively,

$$\frac{v - V}{1 - \frac{vV}{c^2}} \text{ and } \frac{v + w\,dt - V}{1 - (v + w\,dt)\frac{V}{c^2}}$$

The latter velocity is written as

$$\frac{v - V}{1 - v\frac{V}{c^2}} + \frac{w\,dt}{1 - v\frac{V}{c^2}} + \frac{v - V}{\left(1 - \frac{vV}{c^2}\right)}\frac{wV}{c^2}dt = \frac{v - V}{1 - v\frac{V}{c^2}} + \frac{w\,dt\left(1 - \frac{V^2}{c^2}\right)}{\left(1 - \frac{vV}{c^2}\right)^2}$$

Also by Lorentz transformation

$$dt' = \frac{dt - V\,dx/c^2}{\sqrt{1 - V^2/c^2}} = dt\,\frac{1 - vV/c^2}{\sqrt{1 - V^2/c^2}}$$

Thus the acceleration in the K' frame is

$$w' = \frac{dv'}{dt'} = \frac{w}{\left(1 - \frac{vV}{c^2}\right)^3}\left(1 - \frac{V^2}{c^2}\right)^{3/2}$$

(b) In the K frame the velocities of the particle at the time t and $t + dt$ are repectively

$$(0, v, 0) \text{ and } (0, v + wdt, 0)$$

where $\vec{V}$ is along x–axis. In the K' frame the velocities are

$$\left(-V, v\sqrt{1 - V^2/c^2}, 0\right)$$

and $$\left(-V, (v + wdt)\sqrt{1 - V^2/c^2}, 0\right) \text{ respectively}$$

Thus the acceleration

$$w' = \frac{w\,dt\sqrt{(1 - V^2/c^2)}}{dt'} = w\left(1 - \frac{V^2}{c^2}\right) \text{ along the } y\text{-axis.}$$

We have used $dt' = \dfrac{dt}{\sqrt{1 - V^2/c^2}}$

1.366 In the instantaneous rest frame $v = V$ and

$$w' = \frac{w}{\left(1 - \frac{V^2}{c^2}\right)^{3/2}} \text{ (from 1.365a)}$$

So,

$$= \frac{dv}{\left(1 - \frac{V^2}{c^2}\right)^{3/2}} = w'\,dt$$

w' is constant by assumption. Thus integration gives

$$v = \frac{w'\,t}{\sqrt{1 + \left(\frac{w'\,t}{c}\right)^2}}$$

Integrating once again $x = \dfrac{c^2}{w'}\left(\sqrt{1 + \left(\dfrac{w'\,t}{c}\right)^2} - 1\right)$

1.367 The boost time τ_0 in the reference frame fixed to the rocket is related to the time τ elapsed on the earth by

$$\tau_0 = \int_0^\tau \sqrt{1 - \frac{v^2}{c^2}}\,dt = \int_0^\tau \left[1 - \frac{\left(\frac{w'\,t}{c}\right)^2}{1 + \left(\frac{w'\,t}{c}\right)^2}\right]^{1/2} dt$$

$$= \int_0^\tau \frac{dt}{\sqrt{1 + \left(\frac{w'\,t}{c}\right)^2}} = \frac{c}{w'}\int_0^{(w'\tau)/c} \frac{d\xi}{\sqrt{1 + \xi^2}} = \frac{c}{w'}\ln\left[\frac{w'\,\tau}{c} + \sqrt{1 + \left(\frac{w'\,\tau}{c}\right)^2}\right]$$

1.368 $m = \dfrac{m_0}{\sqrt{1 - \beta^2}}$

For

$$\beta \to 1,\ \frac{m}{m_0} = \frac{1}{\sqrt{2(1 - \beta)}} = \frac{1}{\sqrt{2\eta}}$$

1.369 We define the density ρ in the frame K in such a way that $\rho\,dx\,dy\,dz$ is the rest mass dm_0 of the element. That is $\rho\,dx\,dy\,dz = \rho_0\,dx_0\,dy_0\,dz_0$, where ρ_0 is the proper density dx_0, dy_0, dz_0 are the dimensions of the element in the rest frame K_0. Now

$$dy = dy_0,\ dz = dz_0,\ dx = dx_0\sqrt{1 - \frac{v^2}{c^2}}$$

if the frame K is moving with velocity, v relative to the frame K_0. Thus

$$\rho = \frac{\rho_0}{\sqrt{1-\frac{v^2}{c^2}}}$$

Defining η by $\rho = \rho_0(1+\eta)$

We get $\quad 1+\eta = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ or, $\frac{v^2}{c^2} = 1 - \frac{1}{(1+\eta)^2} = \frac{\eta(2+\eta)}{(1+\eta)^2}$

or
$$v = c\sqrt{\frac{\eta(2+\eta)}{(1+\eta)^2}} = \frac{c\sqrt{\eta(2+\eta)}}{1+\eta}$$

1.370 We have

$$\frac{m_0 v}{\sqrt{1-\frac{v^2}{c^2}}} = p \quad \text{or,} \quad \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}} = \sqrt{m_0^2 + \frac{p^2}{c^2}}$$

or
$$1 - \frac{v^2}{c^2} = \frac{m_0^2 c^2}{m_0^2 c^2 + p^2} = 1 - \frac{p^2}{p^2 + m_0^2 c^2}$$

or
$$v = \frac{c_p}{\sqrt{p^2 + m_0^2 c^2}} = \frac{c}{\sqrt{1+\left(\frac{m_0 c}{p}\right)^2}}$$

So $\quad \frac{c-v}{c} = \left[1 - \left(1 + \left(\frac{m_0 c}{p}\right)^2\right)^{-1/2}\right] \times 100\,\% = \frac{1}{2}\left(\frac{m_0 c}{p}\right)^2 \times 100\,\%$

1.371 By definition of η,

$$\frac{m_0 v}{\sqrt{1-\frac{v^2}{c^2}}} = \eta\, m_0 v \quad \text{or} \quad 1 - \frac{v^2}{c^2} = \frac{1}{\eta^2}$$

or
$$v = c\sqrt{1-\frac{1}{\eta^2}} = \frac{c}{\eta}\sqrt{\eta^2 - 1}$$

1.372 The work done is equal to change in kinetic energy which is different in the two cases Classically i.e. in nonrelativistic mechanics, the change in kinetic energy is

$$\frac{1}{2} m_0 c^2 \left((0{\cdot}8)^2 - (0{\cdot}6)^2\right) = \frac{1}{2} m_0 c^2\, 0{\cdot}28 = 0{\cdot}14\, m_0 c^2$$

Relativistically it is,

$$\frac{m_0 c^2}{\sqrt{1-(0{\cdot}8)^2}} - \frac{m_0 c^2}{\sqrt{1-(0{\cdot}6)^2}} = \frac{m_0 c^2}{0{\cdot}6} - \frac{m_0 c^2}{0{\cdot}8} = m_0 c^2\,(1{\cdot}666 - 1{\cdot}250)$$

$$= 0{\cdot}416\, m_0 c^2 = 0{\cdot}42\, m_0 c^2$$

1.373 $$\frac{m_0 c^2}{\sqrt{1-\frac{v^2}{c^2}}} = 2\, m_0 c^2$$

or $$\sqrt{1-\frac{v^2}{c^2}} = \frac{1}{2} \quad \text{or} \quad 1-\frac{v^2}{c^2} = \frac{1}{4}$$

or $$\frac{v}{c} = \frac{\sqrt{3}}{2} \cdot \text{i.e. } v = c\,\frac{\sqrt{3}}{2}$$

1.374 Relativistically

$$\frac{T}{m_0 c^2} = \left(\frac{1}{\sqrt{1-\beta^2}} - 1\right) = \frac{1}{2}\beta^2 + \frac{3}{8}\beta^4$$

So $$\beta_{rel}^2 = \frac{2T}{m_0 c^2} - \frac{3}{4}\left(\beta_{rel}^2\right)^2 = \frac{2T}{m_0 c^2} - \frac{3}{4}\left(\frac{2T}{m_0 c^2}\right)^2$$

Thus $$\beta_{rel} = \left[\frac{2T}{m_0 c^2} - 3\,\frac{T^2}{m_0^2 c^4}\right]^{1/2} = \sqrt{\frac{2T}{m_0 c^2}}\left(1 - \frac{3}{4}\,\frac{T}{m_0 c^2}\right)$$

But Classically, $\beta_{cl} = \sqrt{\frac{2T}{m_0 c^2}}$ so $\frac{\beta_{rel} - \beta_{cl}}{\beta_{cl}} = \frac{3}{4}\,\frac{T}{m_0 c^2} = \varepsilon$

Hence if $$\frac{T}{m_0 c^2} < \frac{4}{3}\,\varepsilon$$

the velocity β is given by the classical formula with an error less than ε.

1.375 From the formula

$$E = \frac{m_0 c^2}{\sqrt{1-\frac{v^2}{c^2}}}, \quad p = \frac{m_0 v}{\sqrt{1 - v^2/c^2}}$$

we find $$E^2 = c^2 p^2 + m_0^2 c^4 \quad \text{or} \quad (m_0 c^2 + T)^2 = c^2 p^2 + m_0^2 c^4$$

or $$T\,(2\, m_0 c^2 + T) = c^2 p^2 \quad \text{i.e.} \quad p = \frac{1}{c}\sqrt{T\,(2 m_0 c^2 + T)}$$

1.376 Let the total force exerted by the beam on the target surface be F and the power liberated there be P. Then, using the result of the previous problem we see

$$F = Np = \frac{N}{c}\sqrt{T\,(T + 2 m_0 c^2)} = \frac{I}{ec}\sqrt{T\,(T + 2 m_0 c^2)}$$

since $I = Ne$, N being the number of particles striking the target per second. Also,

$$P = N\left(\frac{m_0 c^2}{\sqrt{1 - v^2/c^2}} - m_0 c^2\right) = \frac{I}{e}\,T$$

These will be, respectively, equal to the pressure and power developed per unit area of the target if I is current density.

1.377 In the frame fixed to the sphere :- The momentum transferred to the eastically scatterred particle is

$$\frac{2\,mv}{\sqrt{1-\frac{v^2}{c^2}}}$$

The density of the moving element is, from 1.369, $n\,\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$

and the momentum transferred per unit time per unit area is

$$p = \text{the pressure} = \frac{2\,mv}{\sqrt{1-\frac{v^2}{c^2}}}\,n\,\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\cdot v = \frac{2\,mn\,v^2}{1-\frac{v^2}{c^2}}$$

In the frame fixed to the gas :- When the sphere hits a stationary particle, the latter recoils with a velocity

$$= \frac{v+v}{1+\frac{v^2}{c^2}} = \frac{2\,v}{1+\frac{v^2}{c^2}}$$

$$\text{The momentum transferred is } \frac{\frac{m\cdot 2v}{1+v^2/c^2}}{\sqrt{1-\frac{4v^2/c^2}{(1-v^2/c^2)^2}}} = \frac{2\,mv}{1-\frac{v^2}{c^2}}$$

$$\text{and the pressure is } \frac{2\,mv}{1-\frac{v^2}{c^2}}\cdot n\cdot v = \frac{2\,mnv^2}{1-\frac{v^2}{c^2}}$$

1.378 The equation of motion is

$$\frac{d}{dt}\left(\frac{m_0\,v}{\sqrt{1-\frac{v^2}{c^2}}}\right) = F$$

$$\text{Integrating} = \frac{v/c}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{\beta}{\sqrt{1-\beta^2}} = \frac{F_t}{m_0\,c}, \text{ using } v = 0 \text{ for } t = 0$$

$$\frac{\beta^2}{1-\beta^2} = \left(\frac{Ft}{m_0\,c}\right)^2 \text{ or, } \beta^2 = \frac{(Ft)^2}{(Ft)^2+(m_0\,c)^2} \text{ or, } v = \frac{Fct}{\sqrt{(m_0\,c)^2+(Ft)^2}}$$

$$\text{or}\quad x = \int\frac{Fct\;dt}{\sqrt{F^2\,t^2+m_0^2\,c^2}} = \frac{c}{F}\int\frac{\xi\,d\xi}{\sqrt{\xi^2+(m_0\,c)^2}} = \frac{c}{F}\sqrt{F^2\,t^2+m_0^2\,c^2} + \text{constant}$$

$$\text{or using } x = 0 \text{ at } t = 0, \text{ we get, } x = \sqrt{c^2\,t^2+\left(\frac{m_0\,c^2}{F}\right)^2} - \frac{m_0\,c^2}{F}$$

1.379 $x = \sqrt{a^2 + c^2 t^2}$, so $\dot{x} = v = \frac{c^2 t}{a^2 + c^2 t^2}$

or, $$\frac{v}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{c^2 t}{a}. \text{ Thus } \frac{d}{dt}\left(\frac{m_0 v}{\sqrt{1 - \frac{v^2}{c^2}}}\right) = \frac{m_0 c^2}{a} = F$$

1.380 $$\vec{F} = \frac{d}{dt}\left(\frac{m_0 \vec{v}}{\sqrt{1 - \frac{v^2}{c^2}}}\right) = m_0 \frac{\dot{\vec{v}}}{\sqrt{1 - \frac{v^2}{c^2}}} + m_0 \frac{\vec{v}}{c^2} \vec{v} \cdot \dot{\vec{v}} \frac{1}{\left(1 - \frac{v^2}{c^2}\right)^{3/2}}$$

Thus $$\vec{F}_\perp = m_0 \frac{\vec{w}}{\sqrt{1 - \beta^2}}, \ \vec{w} = \dot{\vec{v}}, \ \vec{w} \perp \vec{v}$$

$$\vec{F}_\parallel = m_0 \frac{\vec{w}}{(1 - \beta^2)^{3/2}}, \ \vec{w} = \dot{\vec{v}}, \ \vec{w} \parallel \vec{v}$$

1.381 By definition,

$$E = m_0 \frac{c^2}{\sqrt{1 - \frac{v_x^2}{c^2}}} = \frac{m_0 c^3 \, dt}{ds}, \quad p_x = m_0 \frac{v_x}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{c \, m_0 \, dx}{ds}$$

where $ds^2 = c^2 dt^2 - dx^2$ is the invariant interval ($dy = dz = 0$)

Thus, $$p'_x = c m_0 \frac{dx'}{ds} = c m_0 \gamma \frac{(dx - V dt)}{ds} = \frac{p_x - VE/c^2}{\sqrt{1 - V^2/c^2}}$$

$$E' = m_0 c^3 \frac{dt'}{ds} - c^3 m_0 \gamma \frac{\left(dt - \frac{V d_x}{c^2}\right)}{ds} = \frac{E - V p_x}{\sqrt{1 - \frac{V^2}{c^2}}},$$

1.382 For a photon moving in the x direction

$$\varepsilon = c p_x, \ p_y = p_z = 0,$$

In the moving frame, $$\varepsilon' = \frac{1}{\sqrt{1 - \beta^2}}\left(\varepsilon - V \frac{\varepsilon}{c}\right) = \varepsilon \sqrt{\frac{1 - V/c}{1 + V/c}}$$

Note that $$\varepsilon' = \frac{\varepsilon}{2} \text{ if, } \frac{1}{4} = \frac{1 - \beta}{1 + \beta} \text{ or } \beta = \frac{3}{5}, \ V = \frac{3c}{5}.$$

1.383 As before

$$E = m_0 c^3 \frac{dt}{ds}, \ p_x = m_0 c \frac{dx}{ds}.$$

Similarly $$p_y = m_0 c \frac{dy}{ds},\ p_z = m_0 c \frac{dz}{ds}$$

Then $$E^2 - c^2 p^2 = E^2 - c^2 \left(p_x^2 + p_y^2 + p_x^2\right)$$

$$= m_0^2 c^4 \frac{(c^2 dt^2 - dx^2 - dy^2 - dz^2)}{ds^2} = m_0^2 c^4 \text{ is invariant}$$

1.384 (b) & (a) In the CM frame, the total momentum is zero, Thus

$$\frac{V}{c} = \frac{cp_{1x}}{E_1 + E_2} = \frac{\sqrt{T(T + 2m_0 c^2)}}{T + 2m_0 c^2} = \sqrt{\frac{T}{T + 2m_0 c^2}}$$

where we have used the result of problem (1.375)

Then

$$\frac{1}{\sqrt{1 - V^2/c^2}} = \frac{1}{\sqrt{1 - \frac{T}{T + 2m_0 c^2}}} = \sqrt{\frac{T + 2m_0 c^2}{2m_0 c^2}}$$

Total energy in the CM frame is

$$\frac{2m_0 c^2}{\sqrt{1 - V^2/c^2}} = 2m_0 c^2 \sqrt{\frac{T + 2m_0 c^2}{2m_0 c^2}} = \sqrt{2\, m_0 c^2 (T + 2m_0 c^2)} = \tilde{T} + 2m_0 c^2$$

So $$\tilde{T} = 2m_0 c^2 \left(\sqrt{1 + \frac{T}{2m_0 c^2}} - 1\right)$$

Also $2\sqrt{c^2 \tilde{p}^2 + m_0^2 c^4} = \sqrt{2m_0 c^2 (T + 2m_0 c^2)}$, $4 c^2 \tilde{p}^2 = 2m_0 c^2 T$, or $\tilde{p} = \sqrt{\frac{1}{2} m_0 T}$

1.385 $M_0 c^2 = \sqrt{E^2 - c^2 p^2}$

$$\sqrt{(2m_0 c^2 + T)^2 - T(2m_0 c^2 + T)} = \sqrt{2m_0 c^2 (2m_0 c^2 + T)} = c\sqrt{2m_0 (2m_0 c^2 + T)}$$

Also $$cp = \sqrt{T(T + 2m_0 c^2)},\ v = \frac{c^2 p}{E} = c\sqrt{\frac{T}{T + 2m_0 c^2}}$$

1.386 Let T' = kinetic energy of a proton striking another stationary particle of the same rest mass. Then, combined kinetic energy in the CM frame

$$= 2m_0 c^2 \left(\sqrt{1 + \frac{T'}{2m_0 c^2}} - 1\right) = 2T,\ \left(\frac{T}{m_0 c^2} + 1\right)^2 = 1 + \frac{T'}{2m_0 c^2}$$

$$\frac{T'}{2m_0 c^2} = \frac{T(2m_0 c^2 + T)}{m_0^2 c^4},\ T' = \frac{2T(T + 2m_o c^2)}{m_0 c^2}$$

1.387 We have

$$E_1 + E_2 + E_3 = m_0 c^2, \ \vec{p_1} + \vec{p_2} + \vec{p_3} = 0$$

Hence $$(m_0 c^2 - E_1)^2 - c^2 \vec{p}_1^{\,2} = (E_2 + E_3)^2 - \left(\vec{p_2} + \vec{p_3}\right)^2 c^2$$

The L.H.S. $= (m_0^2 c^4 - E_1)^2 - c^2 \vec{p}_1^{\,2} = (m_0^2 + m_1^2) c^4 - 2 m_0 c^2 E_1$

The R.H.S. is an invariant. We can evaluate it in any frame. Choose the CM frame of the particles 2 and 3.

In this frame R.H.S. $= (E'_2 + E'_3)^2 = (m_2 + m_3)^2 c^4$

Thus $(m_0^2 + m_1^2) c^4 - 2 m_0 c^2 E_1 = (m_2 + m_3)^2 c^4$

or $2 m_0 c^2 E_1 \le \left\{ m_0^2 + m_1^2 - (m_2 + m_3)^2 \right\} c^4$, or $E_1 \le \dfrac{m_0^2 + m_1^2 - (m_2 + m_3)^2}{2 m_0} c^2$

1.388 The velocity of ejected gases is u realtive to the rocket. In an earth centred frame it is

$$\frac{v - u}{1 - \dfrac{vu}{c^2}}$$

in the direction of the rocket. The momentum conservation equation then reads

$$(m + dm)(v + dv) + \frac{v - u}{1 - \dfrac{uv}{c^2}}(-dm) = mv$$

or $$m dv - \left(\frac{v - u}{1 - \dfrac{uv}{c^2}} - v \right) dm = 0$$

Here $-dm$ is the mass of the ejected gases. so

$$m dv - \frac{-u + \dfrac{uv^2}{c^2}}{1 - \dfrac{uv}{c^2}} dm = 0, \text{ or } m dv + u \left(1 - \frac{v^2}{c^2}\right) dm = 0$$

(neglecting $1 - \dfrac{uv}{c^2}$ since u is non- relativistic.)

Integrating $\left(\beta = \dfrac{v}{c}\right)$, $\displaystyle\int \frac{d\beta}{1 - \beta^2} + \frac{u}{c} \int \frac{dm}{m} = 0$, $\ln \dfrac{1 + \beta}{1 - \beta} + \dfrac{u}{c} \ln m = \text{constant}$

The constant $= \dfrac{u}{c} \ln m_0$ since $\beta = 0$ initially.

Thus $$\frac{1 - \beta}{1 + \beta} = \left(\frac{m}{m_0}\right)^{u/c} \text{ or } \beta = \frac{1 - \left(\dfrac{m}{m_0}\right)^{u/c}}{1 + \left(\dfrac{m}{m_0}\right)^{u/c}}$$