Chapter 5

Continuity and Differentiability

Exercise 5.8

Q. 1 Verify Rolle's theorem for the function f (x) = $x^2 + 2x – 8$, x ∈ [– 4, 2].

Answer:

The given function is f (x) = $x^2 + 2x – 8$ and x ∈ [-4, 2].

By Rolle's Theorem, for a function f: [a, b] → R, if

(a) f is continuous on [a, b]

(b) f is differentiable on (a, b)

(c) f(a) = f(b)

Then there exists some c in (a, b) such that f '(c) = 0.

As f(x) = $x^2 + 2x – 8$ is a polynomial function,

(a) f(x) is continuous in [-4, 2]

(b) f'(x) = 2x + 2

So, f(x) is differentiable in (-4, 2).

(c) f(a) = f (-4) = $(-4)^2 + 2(-4) – 8$ = 16 – 8 – 8 =16 – 16 = 0

f(b) = f (2) = $(2)^2 + 2(2) – 8$ = 4 + 4 – 8 = 8 – 8 = 0

Hence, f(a) = f(b).

∴ There is a point c ∈ (-4, 2) where f'(c) = 0.

f(x) = $x^2 + 2x – 8$

f'(x) = 2x + 2

$f'(c) = 0$

$\Rightarrow f'(c) = 2c + 2 = 0$

$\Rightarrow 2c = -2$

$\Rightarrow c = -2/2$

$\Rightarrow c = -1$ where $c = -1 \in (-4, 2)$

Hence, Rolle's Theorem is verified.

Q. 2 Examine if Rolle's theorem is applicable to any of the following functions. Can you say something about the converse of Rolle's theorem from these examples?

(i) $f(x) = [x]$ for $x \in [5, 9]$

(ii) $f(x) = [x]$ for $x \in [-2, 2]$

(iii) $f(x) = x^2 - 1$ for $x \in [1, 2]$

Answer:

By Rolle's Theorem, for a function $f: [a, b] \rightarrow R$, if

(a) f is continuous on [a, b]

(b) f is differentiable on (a, b)

(c) $f(a) = f(b)$

Then there exists some c in (a, b) such that $f'(c) = 0$.

If a function does not satisfy any of the above conditions, then Rolle's Theorem is not applicable.

(i) $f(x) = [x]$ for $x \in [5, 9]$

As the given function is a greatest integer function,

(a) f(x) is not continuous in [5, 9]

(b) Let y be an integer such that $y \in (5, 9)$

Left hand limit of f(x) at x = y: $\lim_{h\to 0^-} \frac{f(y+h)-f(y)}{h} = \lim_{h\to 0^-} \frac{[y+h]-[y]}{h} = \lim_{h\to 0^-} \frac{y-1-y}{h} = \lim_{h\to 0^-} \frac{-1}{h} = \infty$

Right hand limit of f(x) at x = y:

$$\lim_{h\to 0^+} \frac{f(y+h)-f(y)}{h} = \lim_{h\to 0^+} \frac{[y+h]-[y]}{h} = \lim_{h\to 0^+} \frac{y-y}{h} = \lim_{h\to 0^+} \frac{0}{h} = 0$$

Since, left and right hand limits of f(x) at x = y is not equal, f(x) is not differentiable at x=y.

So, f(x) is not differentiable in [5, 9]

(c) f(a)= f (5) = [5] = 5

f(b) = f (9) = [9] = 9

f(a) ≠ f(b)

Here, f(x) does not satisfy the conditions of Rolle's Theorem.

Rolle's Theorem is not applicable for f(x) = [x] for x ∈ [5, 9].

(ii) f (x) = [x] for x ∈ [– 2, 2]

As the given function is a greatest integer function,

(a) f(x) is not continuous in [-2, 2]

(b) Let y be an integer such that y ∈ (-2, 2)

Left hand limit of f(x) at x = y:

$$\lim_{h\to 0^-} \frac{f(y+h)-f(y)}{h} = \lim_{h\to 0^-} \frac{[y+h]-[y]}{h} = \lim_{h\to 0^-} \frac{y-1-y}{h} = \lim_{h\to 0^-} \frac{-1}{h} = \infty$$

Right hand limit of f(x) at x = y:

$$\lim_{h\to 0^+} \frac{f(y+h)-f(y)}{h} = \lim_{h\to 0^+} \frac{[y+h]-[y]}{h} = \lim_{h\to 0^+} \frac{y-y}{h} = \lim_{h\to 0^+} \frac{0}{h} = 0$$

Since, left and right hand limits of f(x) at x = y is not equal, f(x) is not differentiable at x = y.

So, f(x) is not differentiable in (-2, 2)

(c) f(a)= f (-2) = [-2] = -2

f(b) = f (2) = [2] = 2

f(a) ≠ f(b)

Here, f(x) does not satisfy the conditions of Rolle's Theorem.

Rolle's Theorem is not applicable for f(x) = [x] for x ∈ [-2, 2].

(iii) f (x) = $x^2 - 1$ for x ∈ [1, 2]

As the given function is a polynomial function,

(a) f(x) is continuous in [1, 2]

(b) f'(x) = 2x

So, f(x) is differentiable in [1, 2]

(c) f(a) = f (1) = $1^2 - 1 = 1 - 1 = 0$

f(b) = f (2) = $2^2 - 1 = 4 - 1 = 3$

f(a) ≠ f(b)

Here, f(x) does not satisfy a condition of Rolle's Theorem.

Rolle's Theorem is not applicable for f(x) = $x^2 - 1$ for x ∈ [1, 2].

Q. 3

If f: [– 5, 5] → R is a differentiable function and if f′(x) does not vanish anywhere, then prove that f (– 5) ≠ f (5).

Answer:

Given: f: [-5, 5] → R is a differentiable function.

Mean Value Theorem states that for a function f: [a, b] → R, if

(a)f is continuous on [a, b]

(b)f is differentiable on (a, b)

Then there exists some c ∈ (a, b) such that

We know that a differentiable function is a continuous function.

So,

(a) f is continuous on [-5, 5]

(b) f is differentiable on (-5, 5)

∴ By Mean Value Theorem, there exists c ∈ (-5, 5) such that

⇒ $f'(c) = \frac{f(5)-f(-5)}{5-(-5)}$

⇒ $10 f'(c) = f(5) – f(-5)$

It is given that f'(x) does not vanish anywhere.

∴ $f'(c) \neq 0$

$10 f'(c) \neq 0$

$f(5) – f(-5) \neq 0$

$f(5) \neq f(-5)$

Hence proved.

By Mean Value Theorem, it is proved that $f(5) \neq f(-5)$.

Q. 4 Verify Mean Value Theorem, if $f(x) = x^2 – 4x – 3$ in the interval [a, b], where a = 1 and b =4.

Answer:

Given: $f(x) = x^2 – 4x – 3$ in the interval [1, 4]

Mean Value Theorem states that for a function f: [a, b] → R, if

(a)f is continuous on [a, b]

(b)f is differentiable on (a, b)

Then there exists some c ∈ (a, b) such that f ‘ (c) = $\frac{f(b)-f(a)}{b-a}$

As f(x) is a polynomial function,

(a) f(x) is continuous in [1, 4]

(b) f'(x) = 2x – 4

So, f(x) is differentiable in (1, 4).

$\therefore \frac{f(b)-f(a)}{b-a} = \frac{f(4)-f(1)}{4-1}$

f (4) = 4^2 – 4(4) – 3 = 16 – 16 – 3 = -3

f (1) = 1^2 – 4(1) – 3 = 1 – 4 – 3 = -6

$= \frac{f(4)-f(1)}{4-1} = \frac{-3-(-6)}{4-1} = \frac{3}{3} = 1$

∴ There is a point c ∈ (1, 4) such that f'(c) = 1

⇒ f'(c) = 1

⇒ 2c – 4 = 1

⇒ 2c = 1+4 =5

⇒ c = 5/2 where c ∈ (1,4)

The Mean Value Theorem is verified for the given f(x).

Q. 5 Verify Mean Value Theorem, if f (x) = $x^3 - 5x^2 - 3x$ in the interval [a, b], where a = 1 and b = 3. Find all c ∈ (1, 3) for which f′(c) = 0.

Answer:

Given: f(x) = $x^3 - 5x^2 - 3x$ in the interval [1, 3]

Mean Value Theorem states that for a function f: [a, b] → R, if

(a)f is continuous on [a, b]

(b)f is differentiable on (a, b)

Then there exists some c ∈ (a, b) such that f ' (c) = $\frac{f(b)-f(a)}{b-a}$

As f(x) is a polynomial function,

(a) f(x) is continuous in [1, 3]

(b) f'(x) = 3x2 - 10x - 3

So, f(x) is differentiable in (1, 3).

∴ $\frac{f(b)-f(a)}{b-a} = \frac{f(3)-f(1)}{3-1}$

f (3) = $3^3 - 5(3)^2 - 3(3)$ = 27 – 45 - 9 = -27

f (1) = $1^3 - 5(1)^2 - 3(1)$ = 1 – 5 - 3 = -7

⇒ $\frac{f(3)-f(1)}{3-1} = \frac{-27-(-7)}{3-1} = \frac{-20}{0} = -10$

∴ There is a point c ∈ (1, 4) such that f'(c) = -10

⇒ f'(c) = -10

⇒ $3c^2 - 10c - 3 = -10$

⇒ $3c^2 - 10c + 7 = 0$

⇒ $3c^2 - 3c - 7c + 7 = 0$

⇒ 3c (c – 1) – 7(c – 1) = 0

⇒ (c – 1) (3c – 7) = 0

⇒ c = 1, 7/3 where c = 7/3 ∈ (1, 3)

The Mean Value Theorem is verified for the given f(x) and c = 7/3 ∈ (1, 3) is the only point for which f'(c) = 0.

Q. 6 Examine the applicability of Mean Value Theorem for all three functions given in the above exercise 2.

Answer:

Mean Value Theorem states that for a function f: [a, b] → R, if

(a) f is continuous on [a, b]

(b) f is differentiable on (a, b)

Then there exists some c ∈ (a, b) such that f' (c) = $\frac{f(b)-f(a)}{b-a}$

If a function does not satisfy any of the above conditions, then Mean Value Theorem is not applicable.

(i) f (x) = [x] for x ∈ [5, 9]

As the given function is a greatest integer function,

(a)f(x) is not continuous in [5, 9]

(b) Let y be an integer such that y ∈ (5, 9)

Left hand limit of f(x) at x = y:

$$\lim_{h\to 0^-}\frac{f(y+h)-f(y)}{h} = \lim_{h\to 0^-}\frac{[y+h]-[y]}{h} = \lim_{h\to 0^-}\frac{y-1-y}{h} = \lim_{h\to 0^-}\frac{-1}{h} = \infty$$

Right hand limit of f(x) at x = y:

$$\lim_{h\to 0^+}\frac{f(y+h)-f(y)}{h} = \lim_{h\to 0^+}\frac{[y+h]-[y]}{h} = \lim_{h\to 0^+}\frac{y-y}{h} = \lim_{h\to 0^+}\frac{0}{h} = 0$$

Since, left and right hand limits of f(x) at x=y is not equal, f(x) is not differentiable at x=y.

So, f(x) is not differentiable in [5, 9].

Here, f(x) does not satisfy the conditions of Mean Value Theorem.

Mean Value Theorem is not applicable for f(x) = [x] for x ∈ [5, 9].

(ii) f (x) = [x] for x ∈ [– 2, 2]

As the given function is a greatest integer function,

(a)f(x) is not continuous in [-2, 2]

(b) Let y be an integer such that y ∈ (-2, 2)

Left hand limit of f(x) at x = y:

$$\lim_{h\to 0^-} \frac{f(y+h)-f(y)}{h} = \lim_{h\to 0^-} \frac{[y+h]-[y]}{h} = \lim_{h\to 0^-} \frac{y-1-y}{h} = \lim_{h\to 0^-} \frac{-1}{h} = \infty$$

Right hand limit of f(x) at x = y:

$$\lim_{h\to 0^+} \frac{f(y+h)-f(y)}{h} = \lim_{h\to 0^+} \frac{[y+h]-[y]}{h} = \lim_{h\to 0^+} \frac{y-y}{h} = \lim_{h\to 0^+} \frac{0}{h} = 0$$

Since, left and right hand limits of f(x) at x=y is not equal, f(x) is not differentiable at x=y.

So, f(x) is not differentiable in (-2, 2)

Here, f(x) does not satisfy the conditions of Mean Value Theorem.

Mean Value Theorem is not applicable for f(x) = [x] for x ∈ [-2, 2].

(iii) f (x) = x2 – 1 for x ∈ [1, 2]

As the given function is a polynomial function,

(a) f(x) is continuous in [1, 2]

(b) f'(x) = 2x

So, f(x) is differentiable in [1, 2].

Here, f(x) satisfies the conditions of Mean Value Theorem.

So, Mean Value Theorem is applicable for f(x).

$$\therefore \frac{f(b)-f(a)}{b-a} = \frac{f(2)-f(1)}{2-1}$$

f (2) = 2^2 – 1 = 4 -1 = 3

f (1) = 1^2 – 1 = 1 – 1 = 0

$$\Rightarrow \frac{f(2)-f(1)}{2-1} = \frac{3-0}{2-1} = \frac{3}{1} = 3$$

∴ There is a point c ∈ (1, 2) such that f'(c) = 3

$\Rightarrow f'(c) = 3$

$\Rightarrow 2c = 3$

$\Rightarrow c = 3/2$ where $c \in (1, 2)$

Mean Value Theorem is applicable for $f(x) = x^2 - 1$ for $x \in [1, 2]$.