MIND MAP : LEARNING MADE SIMPLE

CHAPTER - 5

Continuity and Differentiability

Derivatives of functions in parametric form

Let $x = f(t)$, $y = g(t)$ be two functions of parameter 't'.

Then, $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$ or $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \left(\frac{dx}{dt} \neq 0 \right)$

Jhus, $\frac{dy}{dx} = \frac{g'(t)}{f'(t)} \left[\text{provided } f'(t) \neq 0 \right]$

For *eg* : if $x = a\cos\theta$, $y = a\sin\theta$ then $\frac{dx}{d\theta} = -a\sin\theta$ and $\frac{dy}{d\theta} = a\cos\theta$, and so $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = -\frac{a\cos\theta}{a\sin\theta} = -\cot\theta$.

Continuous Function

Suppose f is a real function on a subset of the real numbers and let 'c' be a point in the domain of f. Then f is continuous at c if $\lim_{x \to c} f(x) = f(c)$

A real function f is said to be continuous if it is continuous at every point in the domain of f. For *eg*: The function $f(x) = \frac{1}{x}$, $x \neq 0$ is continuous

Let 'C' be any non-zero real number, then $\lim_{x \to c} f(x) \lim_{x \to c} \frac{1}{x} = \frac{1}{c}$. For $c = 0, f(c) = \frac{1}{c}$ So $\lim_{x \to c} f(x) = f(c)$ and hence f is continuous at every point in the domain of f.

Algebra of continuous functions

Suppose f and g are two real functions continuous at a real number c, then, $f+g$, $f-g$, $f.g$ and $\frac{f}{g}$ are continuous at $x = c \left(g(c) \neq 0 \right)$.

Differentiability

Suppose f is a real function and c is a point in its domain. The derivative of f at c is $f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$

Every differentiable function is continuous, but the converse is not true.

Second order derivative

Let $y = f(x)$ then $\frac{dy}{dx} = f'(x)$, if $f'(x)$ is differentiable, then $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx} f'(x)$ *i.e.*, $\frac{d^2y}{dx^2} = f''(x)$ is the second order derivative of y *w.r.t.x*.

For *eg* : if $y = 3x^2 + 2$, then $y' = 6x$ and $y'' = 6$.

Rolle's theorem

if $f : [a, b] \to R$ is continuous on $[a, b]$ and differentiable on (a, b). Such that $f(a) = f(b)$, then $\exists$ some c in (a, b) s.t. $f'(c) = 0$.

Chain Rule

if $f = vou$, $t = u(x)$ and if both $\frac{dt}{dx}, \frac{dv}{dt}$ exists, then $\frac{df}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx}$.

Mean Value Theorem

if $f : [a, b] \to R$ continuous on $[a, b]$ and differentiable on (a, b).

Then $\exists$ some c in (a, b) such that $f'(c) = \frac{f(b) - f(a)}{b - a}$

e.g. Let $f(x) = x^2$ defined in the interval $[2, 4]$. Since $f(x) = x^2$ is continuous in $[2, 4]$ and differentiable in $(2, 4)$ as $f'(x) = 2x$ defined in $(2, 4)$. So,

$$f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{16 - 4}{4 - 2} = 6, c \in (2, 4).$$

Some Standard derivatives

(i) $\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1 - x^2}}$

(ii) $\frac{d}{dx}\left(\cos^{-1} x\right) = -\frac{1}{\sqrt{1 - x^2}}$

(iii) $\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1 + x^2}$

(iv) $\frac{d}{dx}\left(\cot^{-1} x\right) = -\frac{1}{1 + x^2}$

(v) $\frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{x\sqrt{1 - x^2}}$

(vi) $\frac{d}{dx}\left(\operatorname{cosec}^{-1} x\right) = -\frac{1}{x\sqrt{1 - x^2}}$

(vii) $\frac{d}{dx}\left(e^x\right) = e^x$

(viii) $\frac{d}{dx}(\log x) = \frac{1}{x}$

Logarithmic differentiation

Let $y = f(x) = \left[u(x)\right]^{v(x)}$

$\log y = v(x) \log\left[u(x)\right]$

$\frac{1}{y} = v(x) \frac{1}{u(x)} u'(x) + v'(x) \log\left[u(x)\right]$

$\frac{dy}{dx} = y\left[\frac{v(x)}{u(x)}\right] u'(x) + v'(x) \log\left[u(x)\right]$

For *e.g.* : Let $y = a^x$ Then $\log y = x \log a$

$\frac{1}{y} \cdot \frac{dy}{dx} = \log a$

$\frac{dy}{dx} = y \log a = a^x \log a$.