8 Second Order Linear Partial Differential Equations

oduction

Ve are about to study a simple type of partial differential equations (PDEs): the second order linear PDEs. ʀecall that a partial differential equation is any differential equation that contains two or more independent ·ariables. Therefore the derivative(s) in the equation are partial derivatives. We will examine the simplest case ıf equations with 2 independent variables. A few second order linear PDEs in 2 variables are:

$$a^2 u_{xx} = u_t \qquad \text{(one-dimensional heat conduction equation)}$$

$$a^2 u_{xx} = u_{tt} \qquad \text{(one-dimensional wave equation)}$$

$$u_{xx} + u_{yy} = 0 \qquad \text{(two-dimensional heat conduction equation)}$$

Classification of Second Order Linear PDEs

;onsider the general form of a second order linear partial differential equation in 2 variables with constant oefficients:

$$au_{xx} + bu_{xy} + cu_{yy} + du_x + eu_y + f_u = g(x, y)$$

or the equation to be of second order, a, b, and c cannot all be zero. Definen its discriminant to be $b^2 - 4ac$. he properties and behaviour of its solution are largely dependent of its type, as classified below.

If $b^2 - 4ac > 0$, then the equation is called **hyperbolic**. The wave equation is one such example.

If $b^2 - 4ac = 0$, then the equation is called **parabolic**. The heat conduction equation is one such example.

If $b^2 - 4ac < 0$, then the equation is called **elliptic**. The Laplace equation is one such example.

xample:

Consider the one-dimensional damped wave equation $9u_{xx} = u_{tt} + 6u_t$.

olution:

It can be rewritten as: $9u_{xx} - u_{tt} - 6u_t = 0$. It has coefficients $a = 9$, $b = 0$, and $c = -1$. Its discriminant is $9 > 0$. Therefore, the equation is hyperbolic.

Jndamped One-Dimensional Wave Equation: Vibrations of an Elastic String

onsider a piece of thin flexible string of length L, of negligible weight. Suppose the two ends of the string are 'mly secured ("clamped") at some supports so they will not move. Assume the set-up has no damping. Then, ıe vertical displacement of the string, $0 < x < L$, and at any time $t > 0$, is given by the displacement function (x, t). It satisfies the homogeneous one-dimensional undamped wave equation:

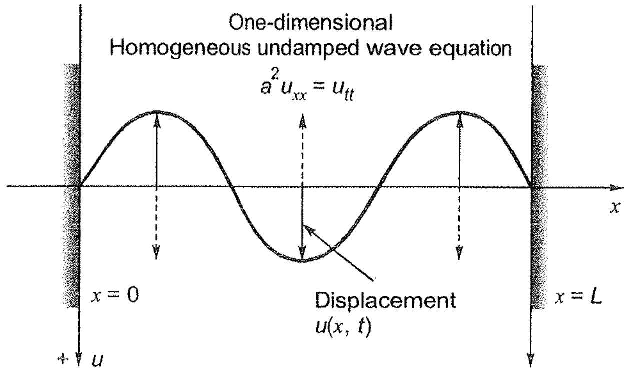
$$a^2 u_{xx} = u_{tt}$$

'here the constant coefficient a^2 is given by the formula $a^2 = T/\rho$, such that a = horizontal propagation speed ılso known as phase velocity) of the wave motion, T = force of tension exerted on the string, ρ = Mass density nass per unit length). It is subjected to the homogeneous boundary conditions.

$$u(0, t) = 0, \text{ and } u(L, t) = 0, \; t > 0$$

ıe two boundary conditions reflect that the two ends of the string are clamped in fixed positions. Therefore, ·ey are held motionless at all time.

The equation comes with 2 initial conditions, due to the fact that it contains the second partial derivative term u_{tt}. The two initial conditions are the $u_t(x, 0)$, both are arbitrary functions of x alone. (Note that the string is vibrates, vertically, in place. The resulting wave form, or the wave-like "shape" of the string, is what moves horizontally.)



Hence, what we have is the following initial-boundary value problem:

(wave equation)	$a^2 u_{xx} = u_{tt}$,	$0 < x < L,\ t > 0$
(Boundary conditions)	$u(0, t) = 0$, and	$u(L, t) = 0$,
(Initial conditions)	$u(x, 0) = f(x)$, and	$u_t(x, 0) = g(x)$

We first let $u(x, t) = X(x)\,T(t)$ and separate the wave equation into two ordinary differential equations. Substituting $u_{xx} = X''T$ and $u_{tt} = XT''$ into the wave equation, it becomes

$$a^2 X''T = XT''$$

Dividing both sides by $a^2 XT$:

$$\frac{X''}{X} = \frac{T''}{a^2T}$$

As for the heat conduction equation, it is customary to consider the constant a^2 as a function of t and group it with the rest of t-terms. Insert the constant of separation and break apart the equation:

$$\frac{X''}{X} = \frac{T''}{a^2T} = -\lambda$$

$$\frac{X''}{X} = -\lambda \quad \Rightarrow \quad X'' = -\lambda X \quad \Rightarrow \quad X'' + \lambda X = 0$$

$$\frac{T''}{a^2T} = -\lambda \quad \Rightarrow \quad T'' = -a^2\lambda T \Rightarrow T'' + a^2\lambda T = 0$$

The boundary conditions also separate:

$$u(0, t) = 0 \Rightarrow X(0)\,T(t) = 0 \Rightarrow X(0) = 0 \quad \text{or} \quad T(t) = 0$$

$$u(L, t) = 0 \Rightarrow X(L)\,T(t) = 0 \Rightarrow X(L) = 0 \quad \text{or} \quad T(t) = 0$$

As usual, in order to obtain nontrivial solutions, we need to choose $X(0) = 0$ and $X(L) = 0$ as the new boundary conditions. The result, after separation of variables, is the following simultaneous system of ordinary differential equations, with a set of boundary conditions:

$$X'' + \lambda X = 0, \quad X(0) = 0 \quad \text{and} \quad X(L) = 0,$$

$$T'' + a^2 \lambda T = 0$$

The next step is to solve the eigen value problem:

$$X'' + \lambda X = 0, \quad X(0) = 0 \quad \text{and} \quad X(L) = 0,$$

The solutions are given by taking λ negative

Eigen values: $\lambda = \dfrac{n^2 \pi^2}{L^2}, \qquad n = 1, 2, 3, \ldots$

Eigen functions: $$X_n = \sin\frac{n\pi x}{L}, \qquad n = 1, 2, 3, \ldots$$

Next, substitute the eigen values found above into the second equation to find $T(t)$. After putting eigen values λ into it, the equation of T becomes

$$T'' + a^2\frac{n^2\pi^2}{L^2}T = 0$$

It is a second order homogeneous linear equation with constant coefficients. It's characteristic have a pair of purely imaginary complex conjugate roots:

$$r = \pm\frac{an\pi}{L}i$$

Thus, the solutions are simple harmonic:

$$T_n(t) = A_n\cos\frac{an\pi t}{L} + B_n\sin\frac{an\pi t}{L}, \qquad n = 1, 2, 3, \ldots$$

Multiplying each pair of X_n and T_n together and sum them up, we find the general solution of the one-dimensional wave equation, with both ends fixed, to be

$$u(x, t) = \sum_{n=1}^{\infty}\left(A_n\cos\frac{an\pi t}{L} + B_n\sin\frac{an\pi t}{L}\right)\sin\frac{n\pi x}{L}$$

There are two sets of (infinitely many) arbitrary coefficients. We can solve for them using the two initial conditions. Set $t = 0$ and apply the first initial condition, the initial (vertical) displacement of the string $u(x, 0) = f(x)$, we have

$$u(x, 0) = \sum_{n=1}^{\infty}(A_n\cos(0) + B_n\sin(0))\sin\frac{n\pi x}{L}$$

$$= \sum_{n=1}^{\infty}A_n\sin\frac{n\pi x}{L} = f(x)$$

Therefore, we see that the initial displacement $f(x)$ needs to be a Fourier sine series. Since $f(x)$ can be an arbitrary function, this usually means that we need to expand it into its odd periodic extension (of period $2L$). the coefficients A_n are then found by the relation $A_n = b_n$, where b_n are the corresponding Fourier sine coefficients of $f(x)$. That is

$$A_n = b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx$$

Notice that the entire sequence of the coefficients A_n are determined exactly by the initial displacement. They are completely independent of the other sequence B_n, which are determined solely by the second initial condition, the initial (vertical) velocity of the string. To find B_n, we differentiate $u(x, t)$ with respect to t apply the initial velocity, $u_t(x, 0) = g(x)$.

$$u_t(x, t) = \sum_{n=1}^{\infty}\left(-A_n\frac{an\pi}{L}\sin\frac{an\pi t}{L} + B_n\frac{an\pi}{L}\cos\frac{an\pi t}{L}\right)\sin\frac{n\pi x}{L}$$

Set $t = 0$ and equate it with $g(x)$:

$$u_t(x, 0) = \sum_{n=1}^{\infty}B_n\frac{an\pi}{L}\sin\frac{n\pi x}{L} = g(x)$$

We see that $g(x)$ needs also be a Fourier sine series. Expand it into its odd periodic extension (period $2L$), if necessary. Once $g(x)$ is written into a sine series, the previous equation becomes

$$u_t(x, 0) = \sum_{n=1}^{\infty} B_n \frac{an\pi}{L} \sin\frac{n\pi x}{L} = g(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

Compare the coefficients of the like sine terms, we see

$$B_n \frac{an\pi}{L} = b_n = \frac{2}{L}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx$$

Therefore,
$$B_n = \frac{L}{an\pi} b_n = \frac{2}{an\pi}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx$$

As we have seen, half of the particular solution is determined by the initial displacement, the other half by the initial velocity. The two halves are determined independent of each other. Hence, if the initial displacement $f(x) = 0$, then all $A_n = 0$ and $u(x, t)$ contains no sine-terms of t. If the initial velocity $g(x) = 0$, then all $B_n = 0$ and $u(x, t)$ contains no cosine-terms of t.

Let us take another look and summarize the result for these 2 easy special cases, when either $f(x)$ or $g(x)$ is zero.

Special case I: Non-zero initial displacement, zero initial velocity: $f(x) \neq 0$, $g(x) = 0$.
Since $g(x) = 0$, then $B_n = 0$ for all n.

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx, \qquad n = 1, 2, 3, \ldots$$

Therefore,
$$u(x, t) = \sum_{n=1}^{\infty} A_n \cos\frac{an\pi t}{L} \sin\frac{n\pi x}{L}$$

ILLUSTRATIVE EXAMPLES

Example:

Solve the one-dimensional wave problem.

$$9u_{xx} = u_{tt}, \qquad 0 < x < 5, \quad t > 0,$$
$$u(0, t) = 0, \quad \text{and} \quad u(5, t) = 0,$$
$$u(x, t) = 4\sin(\pi x) - \sin(2\pi x) - 3\sin(5\pi x),$$
$$u_t(x, 0) = 0.$$

Solution:

First note that $a^2 = 9$ (so, $a = 3$), and $L = 5$

The general solution is, therefore,

$$u(x, t) = \sum_{n=1}^{\infty}\left(A_n \cos\frac{3n\pi t}{5} + B_n \sin\frac{3n\pi t}{5}\right)\sin\frac{n\pi x}{5}$$

Since $g(x) = 0$, it must be that all $B_n = 0$. We just need to find A_n. We also see that $u(x, 0) = f(x)$ is already in the form of a Fourier sine series. Therefore, we just need to extract the corresponding Fourier sine coefficients:

$$A_5 = b_5 = 4,$$
$$A_{10} = b_{10} = -1,$$
$$A_{25} = b_{25} = -3,$$
$$A_n = b_n = 0, \text{ for all other } n,\ n \neq 5, 10, \text{ or } 25.$$

Hence, the particular solution is

$$u(x, t) = 4\cos(3\pi t)\sin(\pi x) - \cos(6\pi t)\sin(2\pi x) - 3\cos(15\pi t)\sin(5\pi x)$$

Example:

Solve the one-dimensional wave problem.

$$9u_{xx} = u_{tt}, \qquad 0 < x < 5, \quad t > 0,$$
$$u(0, t) = 0, \quad \text{and} \qquad u(5, t) = 0,$$
$$u(x, 0) = 0$$
$$u_t(x, 0) = 4.$$

Solution:

As in the previous example, $a^2 = 9$ (so, $a = 3$), and $L = 5$

Therefore, the general solution remains

$$u(x, t) = \sum_{n=1}^{\infty}\left(A_n \cos\frac{3n\pi t}{5} + B_n \sin\frac{3n\pi t}{5}\right)\sin\frac{n\pi x}{5}$$

Now, $f(x) = 0$, consequently all $A_n = 0$. We just need to find B_n. The initial velocity $g(x) = 4$ is a constant function. It is not an odd periodic function. Therefore, we need to expand it into its odd periodic extension (period $T = 10$), then equate it with $u_t(x, 0)$. In short:

$$B_n = \frac{2}{an\pi}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx = \frac{2}{3n\pi}\int_0^5 4\sin\frac{n\pi x}{5}\,dx$$

$$= \begin{cases} \dfrac{80}{3n^2\pi^2}, & n = \text{odd} \\ 0, & n = \text{even} \end{cases}$$

Therefore,

$$u(x, t) = \sum_{n=1}^{\infty}\frac{80}{3(2n-1)^2\pi^2}\sin\frac{3(2n-1)\pi t}{5}\sin\frac{(2n-1)\pi x}{5}$$

8.2.1 Summary of Wave Equation: Vibrating String Problems

The vertical displacement of a vibrating string of length L, securely clamped at both ends, of negligible weight and without damping, is described by the homogeneous undamped wave equation initial-boundary value problem:

$$a^2 u_{xx} = u_{tt}, \qquad 0 < x < L, \quad t > 0,$$
$$u(0, t) = 0, \quad \text{and} \qquad u(L, t) = 0,$$
$$u(x, 0) = f(x), \quad \text{and} \qquad u_t(x, 0) = g(x)$$

The general solution is

$$u(x, t) = \sum_{n=1}^{\infty}\left(A_n \cos\frac{an\pi t}{L} + B_n \sin\frac{an\pi t}{L}\right)\sin\frac{n\pi x}{L}$$

The particular solution can be found by the formulas:

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx, \quad \text{and}$$

$$B_n = \frac{2}{an\pi}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx$$

The solution waveform has a constant (Horizontal) propagation speed, in both directions of the x-axis, of a. The vibrating motion has a (vertical) velocity given by $u_t(x, t)$ at any location $0 < x < L$ along the string.

Exercise:

1. Solve the vibrating string problem of the given initial conditions.

$$4u_{xx} = u_{tt}, \quad 0 < x < \pi, \quad t > 0,$$
$$u(0, t) = 0, \quad u(\pi, t) = 0,$$

(a) $u(x, 0) = 12\sin(2x) - 16\sin(5x) + 24\sin(6x)$; $u_t(x, 0) = 0$.

(b) $u(x, 0) = 0$; $u_{tt}(x, 0) = 6$

(c) $u(x, 0) = 0$; $u_t(x, 0) = 12\sin(2x) - 16\sin(5x) + 24\sin(6x)$

2. Solve the vibrating string problem.

$$100\, u_{xx} = u_{tt}, \quad 0 < x < 2, \quad t > 0,$$
$$u(0, t) = 0, \quad u(2, t) = 0,$$
$$u(x, 0) = 32\sin(\pi x) + e^2 \sin(3\pi x) + 25\sin(6\pi x),$$
$$u_t(x, 0) = 6\sin(2\pi x) - 16\sin(5\pi x/2)$$

3. Solve the vibrating string problem.

$$25\, u_{xx} = u_{tt}, \quad 0 < x < 1, \quad t > 0,$$
$$u(0, t) = 0 \text{ and } u(2, t) = 0,$$
$$u(x, 0) = x - x^2,$$
$$u_t(x, 0) = \pi$$

4. Verify that the D'Alembert solution, $u(x, t) = [F(x - at) + F(x + at)]/2$, where $F(x)$ is an odd periodic function of period $2L$ such that $F(x) = f(x)$ on the interval $0 < x < L$, indeed satisfies the initial-boundary value problem by checking that it satisfies the wave equation, boundary conditions, and initial conditions.

$$a^2 u_{xx} = u_{tt}, \qquad 0 < x < L, \quad t > 0,$$
$$u(0, t) = 0, \qquad u(L, t) = 0,$$
$$u(x, 0) = f(x), \qquad u_t(x, 0) = 0.$$

5. Use the method of separation of variables to solve the following wave equation problem where the string is rigid, but not fixed in place, at both ends (i.e., it is inflexible at the end points such that the slope of displacement curve is always zero at both ends, but the two ends of the string are allowed to freely slide in the vertical direction).

$$a^2 u_{xx} = u_{tt}, \qquad 0 < x < L, \quad t > 0,$$
$$u_x(0, t) = 0, \qquad u_x(L, t) = 0,$$
$$u(x, 0) = f(x), \qquad u_t(x, 0) = g(x).$$

6. What is the steady-state displacement of the string in #5? What is $\lim_{t \to \infty} u(x, t)$? Are they the same?

Answers:

1. (a) $u(x, t) = 12\cos(4t)\sin(2x) - 16\cos(10t)\sin(5x) + 24\cos(12t)\sin(6x)$.

(c) $u(x, t) = 3\sin(4t)\sin(2x) - 1.6\sin(10t)\sin(5x) + 24\sin(12t)\sin(6x)$.

5. (a) The general solution is $u(x, t) = A_0 + B_0 t + \sum_{n=1}^{\infty}\left(A_n \cos\frac{an\pi t}{L} + B_n \sin\frac{an\pi t}{L}\right)\cos\frac{n\pi x}{L}$

The particular solution can be found by the formulas:

$$A_0 = \frac{1}{L}\int_0^L f(x)\,dx, \quad A_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx, \quad B_0 = \frac{1}{L}\int_0^L g(x)\,dx, \text{ and } B_n = \frac{2}{an\pi}\int_0^L g(x)\cos\frac{n\pi x}{L}\,dx$$

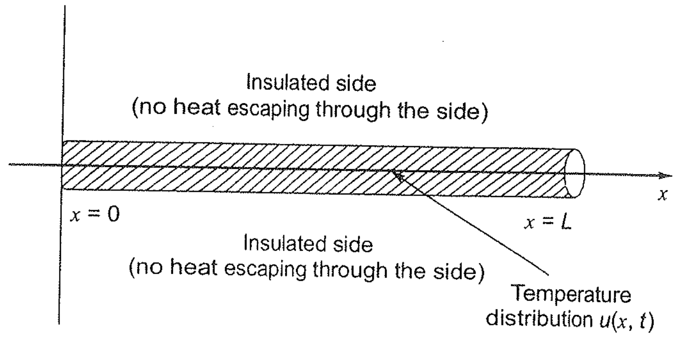
6. The steady-state displacement is the constant term of the solution, A_0. The limit does not exist unless $u(x, t) = C$ is a constant function, which happens when $f(x) = C$ and $g(x) = 0$, in which case the limit is C. They are not the same otherwise.

8.3 The One-Dimensional heat Conduction Equation

Consider a thin bar of length L, of uniform cross-section and constructed of homogeneous material. Suppose that the side of the bar is perfectly insulated so no heat transfer could occur through it (heat could possibly still move into or out of the bar through the two ends of the bar). Thus, the movement of heat inside the bar could occur only in the x-direction. then, the amount of heat content at any place inside the bar, $0 < x < L$, and at any time $t > 0$, is given by the temperature distribution function $u(x, t)$. It satisfies the homogeneous one-dimensional heat conduction equation:

$$a^2 u_{xx} = u_t$$

Where the constant coefficient α^2 is the thermo diffusivity of the bar, given by $a^2 = k/\rho s$. (k = thermal conductivity, ρ = density, s = specific heat, of the material of the bar.)



Further, let us assume that both ends of the bar are kept constantly at 0 degree temperature.

(Heat conduction equation) $a^2 u_{xx} = u_t$, $0 < x < L$, $t > 0$,

(Boundary conditions) $u(0, t) = 0$, and $u(L, t) = 0$,

(Initial condition) $u(x, 0) = f(x)$

8.3.1 Conduction Problem

The general solution of the initial-boundary value problem given by the one-dimensional heat conduction modeling a bar that has both of its ends at 0 degree. The general solution is

$$u(x, t) = \sum_{n=1}^{\infty} C_n e^{-a^2 n^2 \pi^2 t / L^2} \sin\frac{n\pi x}{L}$$

Setting $t = 0$ and applying the initial condition $u(x, 0) = f(x)$, we get

$$u(x, 0) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{L} = f(x)$$

We know that the above equation says that the initial condition needs to be an odd periodic function of period $2L$. Since the initial condition could be an arbitrary function, it usually means that we would need to "force the issue" and expand it into an odd periodic function of period $2L$. That is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

Therefore, the particular solution is found by setting all the coefficients $C_n = b_n$, where b_n's are the Fourier sine coefficients of (or the odd periodic extension of) the initial condition $f(x)$:

$$C_n = b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

ILLUSTRATIVE EXAMPLES

Example:

Solve the heat conduction problem.

$$8\,u_{xx} = u_{tt}, \quad 0 < x < 5, \quad t > 0,$$
$$u(0, t) = 0 \text{ and } u(5, t) = 0,$$

$$u(x, 0) = 2\sin(\pi x) - 4\sin(2\pi x) - \sin(5\pi x)$$

Solution:

Since the standard form of the heat conduction equation $a^2 u_{xx} = u_t$, we see that $a^2 = 8$; and we also note that $L = 5$. Therefore, the general solution is

$$u(x, t) = \sum_{n=1}^{\infty} C_n e^{-a^2 n^2 \pi^2 t / L^2} \sin\frac{n\pi x}{L}$$

$$= \sum_{n=1}^{\infty} C_n e^{-8n^2 \pi^2 t / 25} \sin\frac{n\pi x}{5}$$

The initial condition, $f(x)$, is already an odd periodic function (notice that it is a Fourier sine series) of the correct period $T = 2L = 10$.

Therefore, no additional calculation is needed, and all we need to do is to extract the correct Fourier sine coefficients from $f(x)$. To wit

$$C_5 = b_5 = 2,$$
$$C_{10} = b_{10} = -4,$$
$$C_{25} = b_{25} = 1,$$
$$C_n = b_n = 0, \text{ for all other } n,\ n \neq 5, 10, \text{ or } 25.$$

Hence,

$$u(x, t) = 2e^{-8(5^2)\pi^2 t/25} \sin(\pi x) - 4e^{-8(10)^2 \pi^2 t/25} \sin(2\pi x) + e^{-8(25^2)\pi^2 t/25} \sin(5\pi x)$$

Example:

What will the particular solution be if the initial condition is $u(x, 0) = x$ instead? That is, solve the following heat conduction problem:

$$8\, u_{xx} = u_t, \quad 0 < x < 5, \quad t > 0,$$
$$u(0, t) = 0 \text{ and } u(5, t) = 0,$$
$$u(x, 0) = x$$

Solution:

The general solution is still

$$u(x, t) = \sum_{n=1}^{\infty} C_n e^{-8n^2 \pi^2 t / 25} \sin\frac{n\pi x}{5}$$

The initial condition is an odd function, but it is not a periodic function. Therefore, it needs to be expanded into its odd periodic extension of period 10($T = 2L$). Its coefficients are, for $n = 1, 2, 3, \ldots$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx = \frac{2}{5}\int_0^5 x\sin\frac{n\pi x}{5}\,dx$$

$$= \frac{2}{5}\left(\frac{-5x}{n\pi}\cos\frac{n\pi x}{5}\Big|_0^5 - \frac{-5}{n\pi}\int_0^5 \cos\frac{n\pi x}{5}\,dx\right)$$

$$= \frac{2}{5}\left(\frac{-5x}{n\pi}\cos\frac{n\pi x}{5} + \frac{25}{n^2\pi^2}\sin\frac{n\pi x}{5}\Big|_0^5\right)$$

$$= \frac{2}{5}\left[\left(\frac{-25}{n\pi}\cos(n\pi) - 0\right) - (0 - 0)\right] = \frac{-10}{n\pi}\cos(n\pi)$$

$$= \begin{cases} \dfrac{10}{n\pi}, & n = \text{odd} \\ \dfrac{-10}{n\pi}, & n = \text{even} \end{cases} = \frac{(-1)^{n+1}10}{n\pi}$$

The resulting sine series is (representing the function $f(x) = x$, $-5 < x < 5$, $f(x + 10) = f(x)$):

$$f(x) = \frac{10}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{5}$$

The particular solution can then be found by setting each coefficient, C_n, to be the corresponding Fourier sine coefficient of the series above, $C_n = b_n = \dfrac{(-1)^{n+1}(10)}{n\pi}$. Therefore, the particular solution is

$$u(x, t) = \frac{10}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}e^{-8n^2\pi^2 t/25}\sin\frac{n\pi x}{5}$$

The Steady-State Solution

The steady-state solution, $v(x)$, of a heat conduction problem is the part of the temperature distribution function that is independent of time t. It represents the equilibrium temperature distribution. To find it, we note the fact that it is a function of x alone, yet it has to satisfy the heat conduction equation. Since $v_{xx} = v''$ and $v_t = 0$, substituting them into the heat conduction equation, we get,

$$a^2 v_{xx} = 0$$

Divide both sides by a^2 and integrate twice with respect to x, we find that $v(x)$ must be in the form of a degree 1 polynomial:

$$v(x) = Ax + B$$

Then, rewrite the boundary conditions in terms of v: $u(0, t) = v(0) = T_1$, and $u(L, t) = v(L) = T_2$. Apply those 2 conditions to find that:

$$v(0) = T_1 = A(0) + B = B \quad \Rightarrow \quad B = T_1$$
$$v(L) = T_2 = AL + B = AL + T_1 \quad \Rightarrow \quad A = (T_2 - T_1)/L$$

Therefore, $$v(x) = \frac{T_2 - T_1}{L}x + T_1$$

Further examples of steady-state solutions of the heat conduction equation:

ILLUSTRATIVE EXAMPLES

Example:

Find $v(x)$, given each set of boundary conditions below:

1. $u(0, t) = 50$, $u_x(6, t) = 0$
2. $u(0, t) - 4u_x(0, t) = 0$, $u_x(10, t) = 25$

Solution:

1. We are looking for a function of the form $v(x) = Ax + B$ that satisfies the given boundary conditions. Its derivative is then $v'(x) = A$. The two boundary conditions can be rewritten to be $u(0, t) = v(0) = 50$, and $u_x(6, t) = v'(6) = 0$. Hence,

$$v(0) = 50 = A(0) + B = B \quad \Rightarrow \quad B = 50$$
$$v'(0) = 0 = A \quad \Rightarrow \quad A = 0$$

Therefore, $v(x) = 0x + 50 = 50$

2. The two boundary conditions can be rewritten be $v(0) - 4v'(0) = 0$, and $v'(10) = 25$.

Hence, $$v(0) - 4v'(0) = 0 = (A(0) + B) - 4A = -4A + B$$

$$4v'(10) = 25 = A \qquad \Rightarrow \quad A = 25$$

Substitute $A = 25$ into the first equation: $0 = -4A + B = -100 + B$

$$\Rightarrow \qquad B = 100$$

Therefore, $v(x) = 25x + 100$.

8.4 Laplace Equation for a Rectangular Region

Consider a rectangular of length a and width b. Suppose the top, bottom, and left sides border free-space; while beyond the right side there lies a source of heat/gravity/magnetic flux, whose strength is given by $f(y)$. The potential function at any point (x, y) within this rectangular region, $u(x, y)$, is then described by the boundary value problem:

(2-dim. Laplace equation) $\quad u_{xx} + u_{yy} = 0, \qquad 0 < x < a,\ 0 < y < b,$

(Boundary conditions) $\quad u(x, 0) = 0$, and $\quad u(x, b) = 0,$

$\quad u(0, y) = 0$, and $\quad u(a, b) = f(y).$

The separation of variables proceeds similarly. A slight difference here is that $Y(y)$ is used in the place of $T(t)$. Let $u(x, y) = X(x)\,Y(y)$ and substituting $u_{xx} = XY''$ into the wave equation, it becomes

$$XY'' + XY'' = 0,$$

$$X''Y = -XY''$$

Dividing both sides by XY:

$$\frac{X''}{X} = -\frac{Y''}{Y}$$

Now that the independent variables are separated to the two sides, we can insert the constant of separation. Unlike the previous instances, it is more convenient to denote the constant as positive λ instead.

$$\frac{X''}{X} = -\frac{Y''}{Y} = \lambda$$

$$\frac{X''}{X} = \lambda \qquad \Rightarrow \quad X'' = \lambda X \qquad \Rightarrow \quad X'' - \lambda X = 0$$

$$-\frac{Y''}{Y} = \lambda \qquad \Rightarrow \quad Y'' = \lambda Y \qquad \Rightarrow \quad Y'' + \lambda Y = 0$$

The boundary conditions also separate:

$$u(x, 0) = 0 \quad \Rightarrow \quad X(x)\,Y(0) = 0 \quad \Rightarrow \quad X(x) = 0 \text{ or } Y(0) = 0$$

$$u(x, b) = 0 \quad \Rightarrow \quad X(x)\,Y(b) = 0 \quad \Rightarrow \quad X(x) = 0 \text{ or } Y(b) = 0$$

$$u(0, y) = 0 \quad \Rightarrow \quad X(0)\,Y(y) = 0 \quad \Rightarrow \quad X(0) = 0 \text{ or } Y(y) = 0$$

$$u(a, y) = f(y) \quad \Rightarrow \quad X(a)\,Y(y) = f(y) \quad \Rightarrow \quad \text{[cannot be simplified further]}$$

$$X'' - \lambda X = 0, \qquad X(0) = 0,$$

$$Y'' - \lambda Y = 0, \qquad Y(0) = 0 \text{ and } Y(b) = 0$$

Plus the fourth boundary condition, $u(a, y) = f(y)$

The next step is to solve the eigen value problem. Notice that there is another slight difference. Namely that this time it is the equation of Y that gives rise to the two-point boundary value problem which we need to solve.

$$Y'' + \lambda Y = 0, \qquad Y(0) = 0, \quad Y(b) = 0$$

However, except for the fact that the variables is y and the function is Y, rather than x and X, respectively, we have already seen this problem before (more than once, as a matter of fact ; here the constant $L = b$). The eigen values of this problem are

$$\lambda = \sigma^2 = \frac{n^2\pi^2}{b^2}, \qquad n = 1, 2, 3, \ldots$$

Their corresponding eigen function are

$$Y_n = \sin\frac{n\pi y}{b}, \qquad n = 1, 2, 3, \ldots$$

Once we have found the eigen values, substitute l into the equation of x. We have the equation, together with one boundary condition:

$$X'' - \frac{n^2\pi^2}{b^2}X = 0, \qquad X(0) = 0.$$

Its characteristic equation, $r^2 - \frac{n^2\pi^2}{b^2} = 0$, has real roots $r = \pm\frac{n\pi}{b}$.

Hence, the general solution for the equation of x is

$$X = C_1 e^{\frac{n\pi}{b}x} + C_2 e^{\frac{-n\pi}{b}x}$$

The single boundary condition gives

$$X(0) = 0 = C_1 + C_2 \quad \Rightarrow \quad C_2 = C_1$$

Therefore, for $n = 1, 2, 3, \ldots$

$$X_n = C_n\left(e^{\frac{n\pi}{b}x} - e^{\frac{-n\pi}{b}x}\right)$$

Because of the identity for the hyperbolic sine function

$$\sinh\theta = \frac{e^{\theta} - e^{-\theta}}{2},$$

the previous expression is often rewritten in terms of hyperbolic sine:

$$X_n = K_n \sinh\frac{n\pi x}{b}, \quad n = 1, 2, 3, \ldots$$

The coefficients satisfy the relation: $K_n = 2C_n$.

Combining the solutions of the two equations, we get the set of solutions that satisfies the two-dimensional Laplace equation, given the specified boundary conditions:

$$u_n(x, y) = X_n(x)Y_n(y) = K_n \sin\frac{n\pi x}{b}\sin\frac{n\pi y}{b}, \quad n = 1, 2, 3, \ldots$$

$$u(x, y) = \sum_{n=1}^{\infty} K_n \sinh\frac{n\pi x}{b}\sin\frac{n\pi y}{b}$$

This solution, of course, is specific to the set of boundary conditions

$$u(x, 0) = 0, \text{ and} \qquad u(x, b) = 0,$$
$$u(0, y) = 0, \text{ and} \qquad u(a, y) = f(y)$$

To find the particular solution, we will use the fourth boundary condition, namely, $u(a, y) = f(y)$.

$$u(a, y) = \sum_{n=1}^{\infty} K_n \sinh\frac{an\pi}{b}\sin\frac{n\pi y}{b} = f(y)$$

We have seen this story before. There is nothing really new here. the summation above is a sine series whose Fourier sine coefficients are $b_n = K_n \sin(an\pi/b)$. Therefore, the above relation says that the last boundary condition, $f(y)$, must either be an odd periodic function (period = $2b$), or it needs to be expanded into one. Once we have $f(y)$ as a Fourier sine series, the coefficients K_n of the particular solution can then be computed:

$$K_n \sinh\frac{an\pi}{b} = b_n = \frac{2}{b}\int_0^b f(y)\sin\frac{n\pi y}{b}dy$$

Therefore,

$$K_n = \frac{b_n}{\sinh\frac{an\pi}{b}} = \frac{2}{b\sinh\frac{an\pi}{b}}\int_0^b f(y)\sin\frac{n\pi y}{b}dy$$

Previous GATE and ESE Questions

Q.1 The solution of the partial differential equation $\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$ is of the form

(a) $C\cos(kt)\left|C_1 e^{(\sqrt{k/\alpha})} + C_2 e^{-(\sqrt{k/\alpha})x}\right|$

(b) $Ce^{kt}\left|C_1 e^{(\sqrt{k/\alpha})x} + C_2 e^{-(\sqrt{k/\alpha})x}\right|$

(c) $Ce^{kt}\left|C_1 \cos(\sqrt{k/\alpha})x + C_2 \sin(-\sqrt{k/\alpha})x\right|$

(d) $C\sin(kt)\left|C_1 \cos(\sqrt{k/\alpha})x + C_2 \sin-(\sqrt{k/\alpha})x\right|$

[CE, GATE-2016 : 1 Mark]

Q.2 The type of partial differential equation

$$\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} + 3\frac{\partial^2 P}{\partial x \partial y} + 2\frac{\partial P}{\partial x} - \frac{\partial P}{\partial y} = 0 \text{ is}$$

(a) elliptic (b) parabolic
(c) hyperbolic (d) none of these

[CE, GATE-2016 : 1 Mark]

Q.3 Consider the following partial differential equation $u(x, y)$ with the constant $c > 1$:

$$\frac{\partial u}{\partial y} + c\frac{\partial u}{\partial x} = 0$$

Solution of this equation is
(a) $u(x, y) = f(x + cy)$ (b) $u(x, y) = f(x - cy)$
(c) $u(x, y) = f(cx + y)$ (d) $u(x, y) = f(cx - y)$

[ME, GATE-2017 : 1 Mark]

Q.4 Consider a function $f(x, y, z)$ given by
$f(x, y, z) = (x^2 + y^2 - 2z^2)(y^2 + z^2)$
The partial derivative of this function with respect to x at the point, $x = 2$, $y = 1$ and $z = 3$ is ________.

[EE, GATE-2017 : 1 Mark]

Q.5 Consider the following partial differential equation:

$$3\frac{\partial^2 \phi}{\partial x^2} + B\frac{\partial^2 \phi}{\partial x \partial y} + 3\frac{\partial^2 \phi}{\partial y^2} + 4\phi = 0$$

For this equation to be classified as parabolic, the value of B^2 must be ________.

[CE, GATE-2017 : 1 Mark]

Q.6 The solution of the following partial differential equation $\frac{\partial^2 u}{\partial x^2} = 9\frac{\partial^2 u}{\partial y^2}$ is:

(a) $\sin(3x - y)$ (b) $3x^2 + y^2$
(c) $\sin(3x - 3y)$ (d) $(3y^2 - x^2)$

[ESE Prelims-2017]

Q.7 What is the value of m for which $2x - x^2 + my^2$ is harmonic?
(a) 1 (b) –1
(c) 2 (d) –2

[EE, ESE-2017]

Q.8 The solution at $x = 1$, $t = 1$ of the partial differential equation $\frac{\partial^2 u}{\partial x^2} = 25\frac{\partial^2 u}{\partial t^2}$ subject to initial conditions of $u(0) = 3x$ and $\frac{\partial u}{\partial t}(0) = 3$ is

(a) 1 (b) 2
(c) 4 (d) 6

[CE, GATE-2018 : 2 Marks]

Q.9 Consider a function u which depends on position x and time t. The partial differential equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ is known as the

(a) Wave equation (b) Heat equation
(c) Laplace's equation (d) Elasticity equation

[ME, GATE-2018 : 1 Mark]

Q.10 Let $r = x^2 + y - z$ and $z^3 - xy + yz + y^3 = 1$. Assume that x and y are independent variables. At $(x, y, z) = (2, -1, 1)$, the value (correct to two decimal places) of $\frac{\partial r}{\partial x}$ is ______.

[EC, GATE-2018 : 2 Marks]

Q.11 The general integral of the partial differential equation $y^2p - xyq = x(z - 2y)$ is
(a) $\phi(x^2 + y^2, y^2 - yz) = 0$
(b) $\phi(x^2 - y^2, y^2 + yz) = 0$
(c) $\phi(xy, yz) = 0$
(d) $\phi(x + y, \ln x - z) = 0$

[EE, ESE-2018]

Answers Second Order Linear Partial Differential Equations

1. (b) 2. (c) 3. (b) 4. (40) 5. (36) 6. (a) 7. (a) 8. (d) 9. (b)
10. (4.50) 11. (a)

Explanations Second Order Linear Partial Differential Equations

1. (b)

The PDE $\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$ ...(i)

Solution of (i) is

$$u(x, t) = (A\cos px + B\sin px)Ce^{-p^2\alpha t}$$

Put $-p^2\alpha = k$

$$\Rightarrow \quad p = \sqrt{-\frac{k}{\alpha}} = \sqrt{\frac{k}{\alpha}}\, i$$

Putting value of p in eq. (i)

$$u(x, t) = \left(A\cos\sqrt{\frac{k}{\alpha}}x + b\sinh\sqrt{\frac{k}{\alpha}}x\right)Ce^{kt}$$

$$= Ce^{kt}\left[A\left\{\frac{e^{\sqrt{\frac{k}{\alpha}}x} + e^{-\sqrt{\frac{k}{\alpha}}x}}{2}\right\} + B\left\{\frac{e^{\sqrt{\frac{k}{\alpha}}x} - e^{-\sqrt{\frac{k}{\alpha}}x}}{2}\right\}\right]$$

$$= Ce^{kt}\left[e^{\sqrt{\frac{k}{\alpha}}x}\left\{\frac{A+B}{2}\right\} + e^{-\sqrt{\frac{k}{\alpha}}x}\left\{\frac{A-B}{2}\right\}\right]$$

$$= Ce^{kt}\left[c_1 e^{\sqrt{\frac{k}{\alpha}}x} + c_2 e^{-\sqrt{\frac{k}{\alpha}}x}\right]$$

2. (c)

Comparing the given equation with the general form of second order partial differential equation, we have $A = 1$, $B = 3$, $C = 1 \Rightarrow B^2 - 4AC = 5 > 0$

$\therefore$ PDE is Hyperbola.

3. (b)

$$u = f(x - cy)$$

$$\frac{\partial u}{\partial x} = f'(x - cy)(1)$$

$$\frac{\partial u}{\partial y} = f'(x - cy)(-c)$$

$$= -c\cdot f'(x - cy) = -c\cdot\frac{\partial u}{\partial x}$$

$$\therefore \quad \frac{\partial u}{\partial y} + c\frac{\partial u}{\partial x} = 0$$

4. Sol.

$$f(x, y, z) = (x^2 + y^2 - 2z^2)(y^2 + z^2)$$

$$\frac{\partial f}{\partial x} = (x^2 + y^2 - 2z^2)(0) + (y^2 + z^2)(2x)$$

$$= 0 + (y^2 + z^2)(2x)$$

$$\left.\frac{\partial f}{\partial x}\right|_{\substack{x=2\\y=1\\z=3}} = (1 + 9) \times 2 \times (2) = 40$$

5. Sol.

Given that the partial differential equation is parabolic.

$\therefore \quad B^2 - 4AC = 0$ Here $A = 3$

$\therefore \quad B^2 - 4(3)(3) = 0$ $C = 3$

$$B^2 - 36 = 0$$

$$B^2 = 36$$

6. (a)

$$u = \sin(3x - y)$$

$$u_x = 3\cos(3x - y)$$

$$u_{xx} = -9\sin(3x - y)$$

$$u_y = -\cos(3x - y)$$

$$u_{yy} = -[-\sin(3x - y) \times -1]$$

$$= -\sin(3x - y)$$

7. (a)

$f(x) = 2x - x^2 + my^2$ is harmonic

$$\therefore \quad f_{xx} + f_{yy} = 0$$

$$f_x = 2 - 2x$$

$$f_y = 2my$$

$$f_{xx} = -2$$

$$f_{yy} = 2m$$

$$f_{xx} + f_{yy} = 0$$

$$-2 + 2m = 0$$

$$2m = 2$$

$$m = 1$$

8. (d)

$$\frac{c^2\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$