* Dimensional Analysis *

(photocopy)

Q.10
Pg 35

$\cancel{\nu_m = 5\times10^{-4}}$ $\cancel{\nu_p = 10^{-6}}$

prototype Model

$\nu_p = 5\times10^{-5}$ $\nu_m = 10^{-6}$

$$\frac{Re_m}{Re_p} = Re_r = 1 = Fe_r = 1$$

$$Re_r = Fe_r$$

$$\frac{V_r L_r}{\nu_r} = \frac{V_r}{L_r}$$

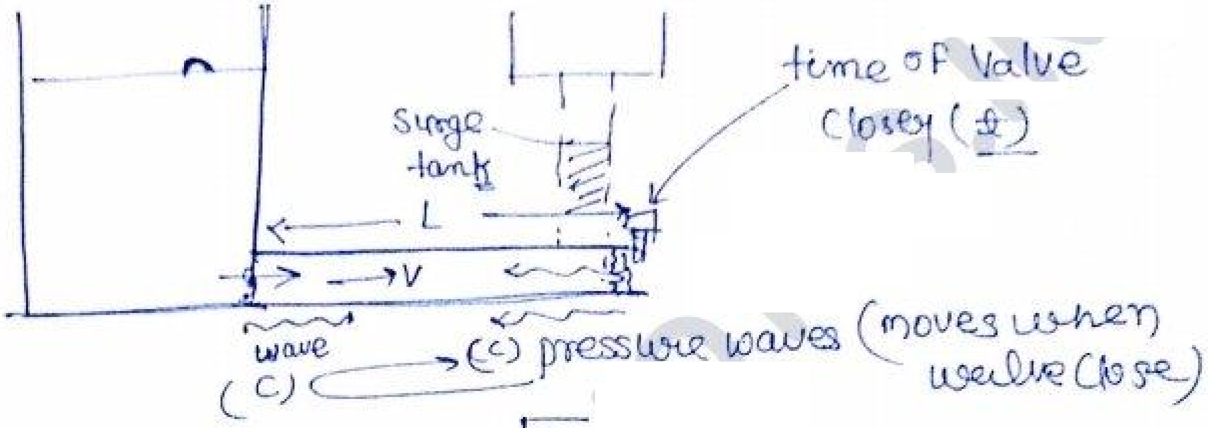
$$L_r = (\nu_r)^{2/3}$$

$$\frac{L_m}{L_p} = (\nu_r)^{2/3} = \cancel{\left(\frac{5\times10^{-6}}{10^{-6}}\right)^{2/3}}$$

$$\frac{L_m}{L_p} = \left(\frac{10^{-6}}{5\times10^{-5}}\right)^{2/3} =$$

$$\frac{L_p}{L_m} = 63$$

Water hammer:-



$$C = \sqrt{\frac{K}{\rho}}$$

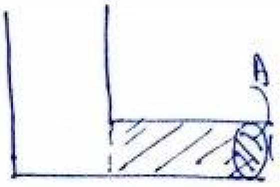
time travel by waves $t' = \frac{2L}{C}$

* Surge tank to Store water at the time of Sudden Closer.

Pressure Rise

Valve (momentum Analysis) Gradual Closer

$t' < t$



$V_i = V$

$V_f = 0$

$$\boxed{P = \frac{\rho V L}{t}}$$

Valve Sudden closer (energy analysis)

$\underline{t < t'}$

energy

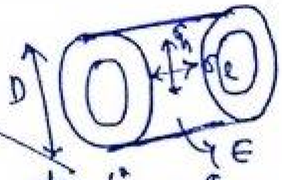
Rigid pipe

(K.E.) = (Strain energy)

$$\frac{1}{2}(\rho \text{vol})V^2 = \frac{1}{2}\frac{\sigma^2}{E} = \frac{1}{2}\frac{P^2}{K} \times \text{Vol}.$$

$$P^2 = \rho V^2 \frac{K}{\rho} \times \rho$$

$$\boxed{P = \rho V\sqrt{\frac{K}{\rho}} = \rho V C}$$



elastic pipe

elastic pepe

$$(K.E.)_{fluid} = (S.E.)_{f} + (S.E.)_{pipe}$$

$$P = \rho V \sqrt{\frac{K/\rho}{1 + \frac{DK}{E t_p}}}$$

t_p - thickness
K - bulk modulus
E - elastic modulus
$E \to \infty$ Riged

$$C' = \sqrt{\frac{K/\rho}{1 + \frac{DK}{E t_p}}}$$

$$P = \rho V C'$$

C' = water hammer pressure weave

Q

Dimensional Analysis

Dimensional analysis is a method to reduce number of experimental variables which affect a given physical phenomenon

eg: $F = \underbrace{f(L, V, \rho, \mu)}_{\text{independent variables}}$

F → Drag force

Dependent Variables

or $\Delta P = f(D, \rho, \mu, V)$

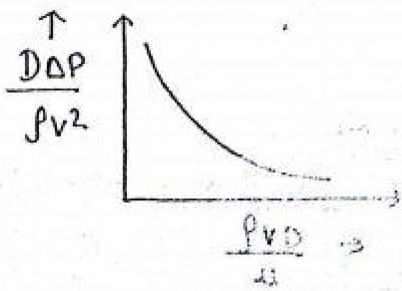
ΔP → pressure drop per unit length in pipe flow.

To perform the experiments in a meaningful and systematic manner, It would be necessary to change one of the variable, such as velocity while holding all other constant and measure the corresponding pressure drop.

In this analysis while finding the functional relation between ΔP & ρ making constant μ is not feasible & process become complex by increasing the number of variables.

To eleminate this difficulty we can collect these variable into two non dimensional variables

$$\frac{D\,\Delta P_1}{\rho V^2} = f\left(\frac{\rho V D}{\mu}\right)$$



There are two methods of grouping of variables of physical phenominon.

(i) Rayleigh method

(ii) Buckingham Pi theorem

Dimensionless Numbers:

(1) Reynold's number: $Re = \dfrac{F_i}{F_v}$ ← Inertia / ← viscous

$F_i = ma$

$= \rho . L^3 \left(\frac{V}{t}\right)$ — L = Characterotic length

$= \rho L^2 \left(\frac{L}{t}\right) . V$

$F_i = \rho V^2 L^2$

$F_v = \tau A$

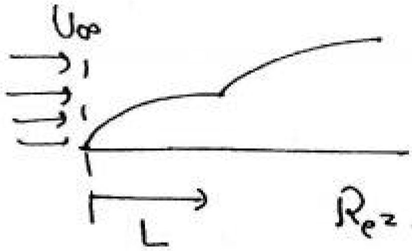
$F_v = \dfrac{\mu V}{h} . A$

$F_v = \dfrac{\mu . V . L^2}{L} = \mu V L$

$Re = \dfrac{\rho V^{\cancel{2}} L^{\cancel{2}}}{\mu \cancel{V} \cancel{L}}$

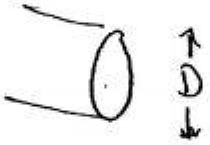
$$\boxed{Re = \frac{\rho V L}{\mu}}$$

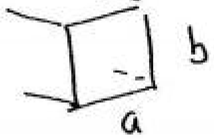
L For Plate (External flow)



$Re = \dfrac{\rho U_\infty L}{\mu}$

Duct / Pipe flow (Internal flow)





$L = D_h = \dfrac{4A}{P}$

$D_h = \dfrac{4 (\pi/4) D^2}{\pi D} = D$

$Re = \dfrac{\rho V D_h}{\mu}$

$L = D_h = \dfrac{4A}{P}$

$L = \dfrac{4ab}{2(a+b)}$

$D_h = \dfrac{2ab}{(a+b)}$

$Re = \dfrac{\rho V D_h}{\mu}$

Application: pipe flow (viscous), Flow inside boundary layer, Flow over submerged body.

(2) Euler's No:

$$Eu = \sqrt{\frac{F_i}{F_p}} = \sqrt{\frac{\rho \cancel{L^2} v^2}{P.\cancel{L^2}}} = \sqrt{\frac{\rho . v^2}{P}}$$

$F_p = PA = PL^2$

$$Eu = \sqrt{\left(\frac{\rho}{P}\right)} . V$$

Application: Pressurised pipe flow flow in penstoke.

(3) Weber No:

$$We = \sqrt{\frac{F_i}{F_s}} = \sqrt{\frac{\rho V^2 L^2}{\sigma \times L}} = \sqrt{\frac{\rho v^2 L}{\sigma}}$$

Application:
Flow of blood in arteries & veins, Capilary action, Capillary movement of water in soil, Atomization of fluid.

(4) Froude Number:

$$Fr = \sqrt{\frac{F_i}{F_g}} = \sqrt{\frac{\rho v^2 L^2}{mg}} = \sqrt{\frac{\rho v^2 \cancel{L^2}}{\rho L^3 g}}$$

$$\boxed{Fr = \frac{V}{\sqrt{Lg}}}$$

Application: Open channel flow, notches & wiers, Spillways, Surface waves over sea.

(5) Mach No:

$$Ma = \sqrt{\frac{F_i}{F_e}} \leftarrow \text{elastic forces}$$

when fluid is compressed there is a rise in pressure & rise of pressure give rise of force is known as elastic force.

$$\Delta P \propto K \leftarrow \text{bulk modulus } (N/m^2)$$

$$F_e = K.A$$

$$F_e = K.L^2$$

$$Ma = \sqrt{\frac{F_i}{F_e}} = \sqrt{\frac{\rho v^2}{K}} = \frac{V}{\sqrt{K/\rho}} = \frac{V}{C}$$

<u>Cauchy Number:</u>

$$\text{Cauchy No} = \frac{F_i}{F_e} = \frac{\rho v^2 \cancel{L^2}}{K \cancel{L^2}} = \frac{\rho v^2}{K}$$

<u>Application</u>: Aerodynamic testing, water hammer

Similitude & Modeling

A physical <u>model</u> is a representation of a physical system that may be used to predict the behaviour of the system

The physical system for which predictions are to made is called <u>prototype</u>.

<u>Usually models are Smaller than the</u> protype (but Some time large eg: nozzles, Carburetors)

models handled more easily in labaratory and is less expensive to construct and operate than a large prototype.

The fundamental requirement for the physical similarity between two is that the ~~physics~~ physics of the two must be same. eg: Pressurised pipe flow (Duct flow) will never be made similar to open channel flow.

In pipe flow, Pressure & viscous forces are dominant.

In open channel flow gravity forces are dominant.

Note:

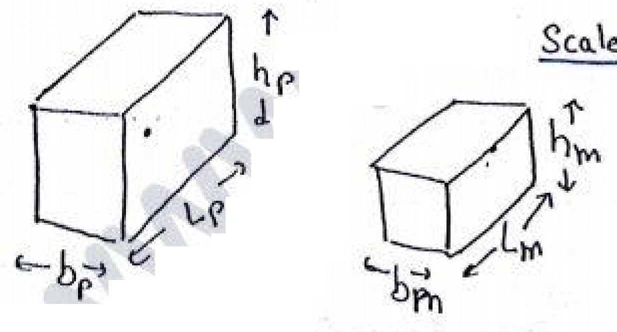
For the similarity physical quantities between two must be similar.

If Physical quantities are geometrical dimensions, the similarity is called Geometric similarity.

If quantities are related to motion then Kinematic similari

If " " " " forces then Dynamic Similiarity.

(1) Geometric Similarity: Model & prototype are said to be in geometrical similar if the ratio of dimensions in model & prototype are same.



Scale ratio

$$L_r = \frac{L_m}{L_p} = \frac{h_m}{h_p} = \frac{b_m}{b_p}$$

Area ratio (A_r) $\quad A_m = l_m \times b_m \quad A_p = l_p \times b_p$

$$A_r = \frac{A_m}{A_p} = \frac{l_m \times b_m}{l_r \times b_r} = l_r \cdot l_r$$

$$\therefore A_r = l_r^2$$

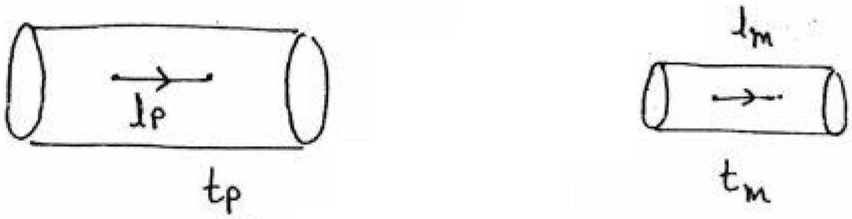
Volume ratio (V_r) :

$$V_r = \frac{l_m \times b_m \times h_m}{l_p \times b_p \times h_p}$$

$$V_r = l_r^3$$

(2) **Kinematic Similarity**: The model & prototype are said to be in Kinematic similarity if the ratio of velocity & acc^n at corresponding point in model & prototype is same.

For Kinematic similarity geometric similarity is must.



$$V_p = \frac{l_p}{t_p} \qquad V_m = \frac{l_m}{t_m}$$

$$V_r = \frac{V_m}{V_p} = \frac{l_m / t_m}{l_p / t_p} = \frac{(l_m/l_p)}{(t_m/t_p)} = \frac{l_r}{t_r}$$

$$V_r = \frac{l_r}{t_r} \text{ or } \frac{V_m}{V_p}$$

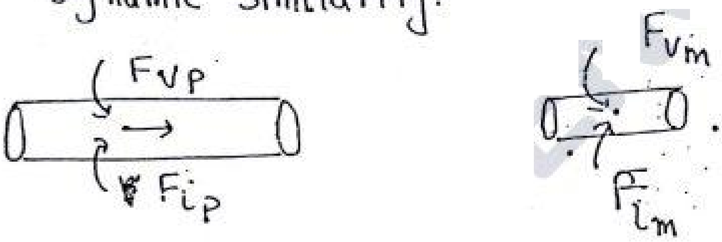
$$a_r = \frac{V_m/t_m}{V_p/t_p} = \frac{a_m}{a_p}$$

$$a_r = \frac{V_m/t_m}{V_p/t_p} = \frac{V_m}{V_p} \times \frac{1}{(t_m/t_p)} = \frac{l_r}{t_r \cdot t_r} = \frac{l_r}{t_r^2}$$

(3) Dynamic Similarity:

When the ratio of forces at corresponding point in model & prototype are equal.

Different dimensionless numbers are used for Dynamic Similarity.



$$\frac{F_{i_p}}{F_{i_m}} = \frac{F_{v_p}}{F_{v_m}}$$

$$\frac{F_{v_m}}{F_{i_m}} = \frac{F_{v_p}}{F_{i_p}}$$

$$\boxed{Re_m = Re_p}$$

So dynamic similarity is obtained by different model laws

When viscous forces are dominant

Re model law $\quad Re_m = Re_p$

Eu model law : when Pressure is dominant

$$Eu_m = Eu_p$$

Fr model law: $\quad Fr_m = Fr_p \leftarrow$ Gravity forces

We " " : $\quad We_m = We_p \leftarrow$ Surface tension

Ma " " : For elastic forces $\quad Ma_m = M_p$

Q: A Fluid flow phenomenon is studied in a model which is to be constructed by using Reynold's model law. Find the expression for model to prototype for velocity & discharge?

Ans: $\Rightarrow Re_m = Re_p$

$$\left(\frac{VL}{\nu}\right)_m = \left(\frac{VL}{\nu}\right)_p$$

$$\frac{V_m}{V_p} = \frac{L_p}{L_m} \times \frac{\nu_m}{\nu_p}$$

$$\boxed{V_r = \frac{1}{L_r} \times \nu_r}$$

$$Q_r = \frac{Q_m}{Q_p} = \frac{A_m V_m}{A_p V_p} = L_r^2 V_r$$

$$Q_r = L_r^{\cancel{2}} \cdot \frac{1}{\cancel{L_r}} \cdot \nu_r$$

$$\boxed{Q_r = L_r \nu_r}$$

$F_{r_m} = F_{r_p}$

$\frac{F_{r_m}}{F_{r_p}} = 1$

$F_{r_r} = 1$

$\frac{V_r}{\sqrt{L_r}} = 1 \Rightarrow V_r = L_r^{1/2}$

$Q = A_r V_r = L_r^2 V_r = (L_r)^{5/2}$

Q: For the froude model law find the ratio of velocity & discharge?

Ans: $F_{r_m} = F_{r_p}$

$$\frac{V_m}{\sqrt{L_m g}} = \frac{V_p}{\sqrt{L_p g}}$$

$$V_r = \frac{V_m}{V_p} = \sqrt{\frac{L_m}{L_p}} = L_r^{1/2}$$

$$\boxed{V_r = L_r^{1/2}}$$

$$Q_r = \frac{Q_m}{Q_p} = \frac{A_m V_m}{A_p V_p}$$

$$Q_r = A_r V_r = L_r^2 L_r^{1/2}$$

$$\boxed{Q_r = L_r^{5/2}}$$

Q: To obtain the expression for scale ratio of model & prototype which has to satisfy both froude model law and Reynold's model law and express your answer in terms of kinematic viscosity?

Ans: $Re_m = Re_p$

$$\frac{V_m L_m}{\nu_m} = \frac{V_p L_p}{\nu_p} \qquad V_r = \frac{V_m}{V_p} = \frac{1}{L_r} \times \nu_r \quad \text{—(i)}$$

$Fr_m = Fr_p$

$$\frac{V_m}{\sqrt{L_m g}} = \frac{V_p}{\sqrt{L_p g}} \qquad V_r = \sqrt{\frac{L_m}{L_p}} = L_r^{1/2}$$

$$L_r = V_r^2 \quad \text{—(ii)}$$

By Re model law

$$V_r = \frac{1}{V_r^2} \cdot \nu_r$$

$$V_r^3 = \nu_r$$

$$\boxed{V_r = \nu_r^{1/3}}$$

$$V_r = \frac{1}{L_r} \cdot \nu_r$$

$$L_r^{1/2} = \frac{1}{L_r} \nu_r$$

$$L_r^{3/2} = \nu_r$$

$$\boxed{L_r = \nu_r^{2/3}}$$

Q: The force required to sail a $\frac{1}{30}$ scale model of a boat in a lake at a speed of 2m/s is 0.5 N. Assuming that the viscous resistance due to water and air is negligible in compair with the wave resistance. Calculate the corresponding speed of the prototype for dynamic similarity condition. What would be the force required to propel the prototype at that velocity in the same lake?

Ans: $L_r = \frac{1}{30}$ $\quad V_m = 2$ m/s $\quad F_m = 0.5$

because of water wave gravity forces are dominant

$Fr_m = Fr_p$

$$\frac{V_m}{\sqrt{L_m g}} = \frac{V_p}{\sqrt{L_p g}} \qquad V_r = \sqrt{\frac{L_m}{L_p}} \qquad \frac{V_m}{V_p} = L_r^{1/2}$$

$$V_p = \frac{V_m}{\sqrt{L_r}} = \frac{2}{\sqrt{(1/30)}} = 10.95 \text{ m/s}$$

$F_i = \rho v^2 L^2$

Force ratio $F_r = \frac{F_{im}}{F_{ip}} = \frac{\rho V_m^2 L_m^2}{\rho V_p^2 L_p^2}$

$$F_r = \frac{V_m^2 L_r^2}{V_p^2} = \frac{2^2}{(10.95)^2} \times \left(\frac{1}{30}\right)^2 = 3.706 \times 10^{-5}$$

$$F_r = \frac{F_{Vm}}{F_{Vp}}$$

$$F_{Vp} = \frac{F_{Vm}}{F_r} = \frac{0.5}{3.706 \times 10^{-5}} = 13500\,N$$

Q: Oil of density 917 Kg/m³ and viscosity 0.29 Ns/m² flows in a pipe of 15 cm dia at a velocity of 2 m/s what would be the velocity of flow of water in a 1 cm dia pipe to make to flows dynamicaly Similar.

If $\rho_w = 998\ Kg/m^3$

$\mu_w = 1.31 \times 10^{-3}\ Ns/m^2$

Sol: $Re_{oil} = Re_{water}$

$$\left(\frac{\rho V D}{\mu}\right)_{oil} = \left(\frac{\rho V D}{\mu}\right)_w$$

$$\left(\frac{917 \times 2 \times 0.15}{0.29}\right) = \left(\frac{998 \times V \times 0.01}{1.31 \times 10^{-3}}\right)$$

$$V = 0.1245\ m/s$$

Rayleigh's Method: It is based on fundamental principal of dimensional homogeneity of physical variables.

This method gives relationship for dimensionless parameters, but variables should not be more (about 3 to 4) Other wise Buckingham π method will be used.

Que: For a Laminar flow in a pipe for pressure drop ΔP is a function of pipe length (L), dia (D), velocity (V), viscosity (μ). By using Rayleigh method develope an expression for ΔP.

Ans:

$$\Delta P = f(L, D, V, \mu)$$

$$\Delta P = K L^a D^b V^c \mu^d$$

$$\Delta P = \frac{N}{m^2} = [M L^{-1} T^{-2}]$$

$$[M L^{-1} T^{-2}] = K [L^a] [L^b] [L T^{-1}]^c [M L^{-1} T^{-1}]^d$$

$d = 1$ $\qquad M = M^d$

$L^{-1} = L^{a+b+c-d}$ $\qquad -1 = a+b+c-d$

$T^{-2} = T^{-c-d}$ $\qquad -2 = -c-d$

$-c = -2+1 = -1$

$c = 1$

$-1 = a+b+1-1$

$b = -1-a$

$$\Delta P = K L^a D^{-1-a} V^1 \mu^1$$

$$\Delta P = \frac{K \mu V L^a}{D^{1+a}}$$

$$\boxed{\Delta P = K \frac{\mu V}{D} \left(\frac{L}{D}\right)^a}$$

Buckingham π - theorem : If there are n - number of total variables and m - number of fundamental quantities then the phenomenon can be grouped into $(n-m)\,\pi$ terms.

$$X = f(x_1, x_2, x_3 \ldots x_{n-1})$$

X: Dependent Variables

$\underbrace{x_1, x_2, x_3 \ldots x_{n-1}}_{\text{Independent Variables}}$

implicit function : $\phi(\underbrace{X, x_1, x_2, x_3 \ldots x_{n-1}}_{\text{total number of } n \text{ variables including dependent \& independent variables}}) = c$

Therefore the analytical phenomenon can be reduced to

$$F(\pi_1, \pi_2 \ldots \pi_{n-m}) = 0$$

Ques: The resistance force F of the ship is function of length L, velocity V, acceleration due to gravity g, and fluid properties like density ρ & viscosity write this in dimensional form & in Buckingham $-\pi$ theorem.

Ans:

$$F = f(L, V, g, \rho, \mu)$$

$$\phi(F, L, V, g, \rho, \mu) = 0$$

no of total variables $(n) = 6$

$F = MLT^{-2}$ $\quad V = LT^{-1}$ $\quad \rho = ML^{-3}$

$L = L$ $\quad g = LT^{-2}$ $\quad \mu = ML^{-1}T^{-1}$

number of repeating variables $m = 3$

i.e M, L, T

Repeating variables should not include dependent variable i.e Select repeating variables from independent variables.

$$F = f(\underbrace{L}, \underbrace{V, g}, \underbrace{\rho, \mu})$$

Geometric Similarity $[L]$

Kinematic Similarity $[LT^{-1}], [LT^{-2}]$

Dynamic Similarity $[ML^{-3}], [ML^{-1}T^{-1}]$

Repeating variables must include all fundamental dimensions

Note: If $n = m$ there will be zero π terms

So $n > m$

So n must be greater than m

$\boxed{n \geq m+1}$ for 1π term $n - m = 1\pi$

only one solution

π_1

Now $(n-m)\pi$ term

$(6-3)\pi$ terms 3π terms

$$\pi_1 = [L^{a_1} V^{b_1} \rho^{c_1}] F$$

$$\pi_2 = [L^{a_2} V^{b_2} \rho^{c_2}] g$$

$$\pi_2 = [L^{a_3} V^{b_3} \rho^{c_3}] \mu$$

For π_1 term:

$$\pi_1 [M^0 L^0 T^0] = [L^{a_1} (LT^{-1})^{b_1} (ML^{-3})^{c_1}] [MLT^{-2}]$$

$$\pi_1 [M^0 L^0 T^0] = [L^{a_1+b_1-3c_1} T^{-b_1} M^{c_1}] [MLT^{-2}]$$

$c_1 + 1 = 0$

$c_1 = -1$

$a_1 + b_1 - 3c_1 + 1 = 0$

$a_1 + b_1 + 3 + 1 = 0$

$a_1 + b_1 = -4$

$a_1 - 2 = -4$

$a_1 = -2$

$-b_1 - 2 = 0$

$b_1 = -2$

$$\pi_1 = [L^{-2} V^{-2} \rho^{-1}] F$$

$$\pi_1 = \frac{F}{\rho V^2 L^2}$$

For π_2 - term:

$$\pi_2 = [L^{a_2} V^{b_2} \rho^{c_2}] g$$

$$\pi_2 [M^0 L^0 T^0] = [L^{a_2+b_2-3c_2} T^{-b_2} M^{c_2}] [LT^{-2}]$$

by solving

$a_2 = 1$

$c_2 = 0$

$b_2 = -2$

$$\pi_2 = [L^1 V^{-2} \rho^0] g \qquad \pi_2 = \frac{g \cdot L}{V^2}$$

For π_3 - term:

$$\pi_3 = [L^{a_3} V^{b_3} \rho^{c_3}] \mu$$

$$\pi_3 [M^0 L^0 T^0] = [L^{a_3+b_3-3c_3} T^{-b_3} M^{c_3}] [ML^{-1}T^{-1}]$$

By solving

$a_3 = -1$, $b_3 = -1$, $c_3 = -1$

$$\pi_3 = \frac{\mu}{\rho V L}$$

So $\pi_1 = \phi(\pi_2, \pi_3)$

$$\frac{F}{\rho V^2 L^2} = \phi\left\{\frac{g \cdot L}{V^2}, \frac{\mu}{\rho V L}\right\}$$

Rules of Selection of Repeating Variables :

(i) Repeated variable must be selected from independent variables.

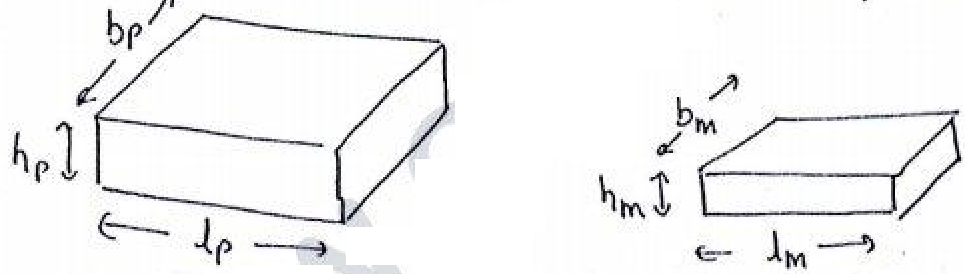
(ii) Number of repeated variables equal to number of fundamental quantities.

(iii) Repeated variable groups must contains all fundamental units.

(iv) Repeating variables must have their own dimensions. They do not have same dimensions and any two variables or more variables do not form dimensionless quantity.

Distorted Models :

This is used when the models are not geometrical similar to the prototype.



In this model different scale ratio's are taken for different ~~for~~ dimensions

$$(L_r)_H = \left(\frac{b_m}{b_p}\right) \qquad (L_r)_V = \left(\frac{h_m}{h_p}\right)$$

Horizontal Scale ratio   Vertical

This model is used for design of rivers & harbours because in rivers the vertical depth is small in comparision to horizontal length, So different scale ratio's are selected for different dimensions, So these models are known as distorted models.

$$A_r = \frac{A_m}{A_p} = \frac{b_m}{b_p} \times \frac{h_m}{h_p}$$

$$A_r = (L_r)_H (L_r)_V$$

$$Q_r = A_r V_r = (L_r)_H (L_r)_V V_r$$

<u>Que:</u> A river model is constructed to a horizontal scale of 1:1000 and a vertical scale of 1:100 A model discharge of $0.1\ m^3/s$ would corresponds to a discharge in the prototype of what magnitude?

<u>Ans:</u>

$$(F_r)_m = (F_r)_p$$

$$\frac{V_m}{\sqrt{L_m g}} = \frac{V_p}{\sqrt{L_p g}} \qquad V_r = \sqrt{\frac{L_m}{L_p}} = (L_{r_V})^{1/2}$$

$$Q_r = A_r V_r$$

$$= (L_r)_H (L_r)_V \cdot V_r$$

$$= (L_r)_H (L_r)_V \cdot (L_r)_V^{1/2}$$

$$Q_r = (L_r)_H (L_r)_V^{3/2} = \left(\frac{1}{1000}\right) \cdot \left(\frac{1}{100}\right)^{3/2}$$

$$\frac{Q_m}{Q_p} = \frac{1}{10^6} \qquad Q_p = Q_m \times 10^6 \qquad Q_p = 0.1 \times 10^6$$

$$\boxed{Q_p = 10^5}$$