CHAPTER VIII.

SURDS AND IMAGINARY QUANTITIES.

85. In the *Elementary Algebra*, Art. 272, it is proved that the denominator of any expression of the form $\frac{a}{\sqrt{b}+\sqrt{c}}$ can be rationalised by multiplying the numerator and the denominator by $\sqrt{b}-\sqrt{c}$, the surd *conjugate* to the denominator.

Similarly, in the case of a fraction of the form $\frac{a}{\sqrt{b}+\sqrt{c}+\sqrt{d}}$, where the denominator involves three quadratic surds, we may by two operations render that denominator rational.

For, first multiply both numerator and denominator by $\sqrt{b}+\sqrt{c}-\sqrt{d}$; the denominator becomes $(\sqrt{b}+\sqrt{c})^2-(\sqrt{d})^2$ or $b+c-d+2\sqrt{bc}$. Then multiply both numerator and denominator by $(b+c-d)-2\sqrt{bc}$; the denominator becomes $(b+c-d)^2-4bc$, which is a rational quantity.

Example. Simplify $\dfrac{12}{3+\sqrt{5}-2\sqrt{2}}$.

The expression
$$=\frac{12\,(3+\sqrt{5}+2\sqrt{2})}{(3+\sqrt{5})^2-(2\sqrt{2})^2}$$
$$=\frac{12\,(3+\sqrt{5}+2\sqrt{2})}{6+6\sqrt{5}}$$
$$=\frac{2\,(3+\sqrt{5}+2\sqrt{2})\,(\sqrt{5}-1)}{(\sqrt{5}+1)\,(\sqrt{5}-1)}$$
$$=\frac{2+2\sqrt{5}+2\sqrt{10}-2\sqrt{2}}{2}$$
$$=1+\sqrt{5}+\sqrt{10}-\sqrt{2}.$$

86. *To find the factor which will rationalise any given binomial surd.*

Case I. Suppose the given surd is $\sqrt[p]{a} - \sqrt[q]{b}$.

Let $\sqrt[p]{a} = x$, $\sqrt[q]{b} = y$, and let n be the L.C.M. of p and q; then x^n and y^n are both rational.

Now $x^n - y^n$ is divisible by $x - y$ for all values of n, and

$$x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \ldots\ldots + y^{n-1}).$$

Thus the rationalising factor is

$$x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \ldots\ldots + y^{n-1};$$

and the rational product is $x^n - y^n$.

Case II. Suppose the given surd is $\sqrt[p]{a} + \sqrt[q]{b}$.

Let x, y, n have the same meanings as before; then

(1) If n is even, $x^n - y^n$ is divisible by $x + y$, and

$$x^n - y^n = (x + y)(x^{n-1} - x^{n-2}y + \ldots\ldots + xy^{n-2} - y^{n-1}).$$

Thus the rationalising factor is

$$x^{n-1} - x^{n-2}y + \ldots\ldots + xy^{n-2} - y^{n-1};$$

and the rational product is $x^n - y^n$.

(2) If n is odd, $x^n + y^n$ is divisible by $x + y$, and

$$x^n + y^n = (x + y)(x^{n-1} - x^{n-2}y + \ldots\ldots - xy^{n-2} + y^{n-1}).$$

Thus the rationalising factor is

$$x^{n-1} - x^{n-2}y + \ldots\ldots - xy^{n-2} + y^{n-1};$$

and the rational product is $x^n + y^n$.

Example 1. Find the factor which will rationalise $\sqrt{3} + \sqrt[3]{5}$.

Let $x = 3^{\frac{1}{2}}$, $y = 5^{\frac{1}{3}}$; then x^6 and y^6 are both rational, and

$$x^6 - y^6 = (x + y)(x^5 - x^4y + x^3y^2 - x^2y^3 + xy^4 - y^5);$$

thus, substituting for x and y, the required factor is

$$3^{\frac{5}{2}} - 3^{\frac{4}{2}} \cdot 5^{\frac{1}{3}} + 3^{\frac{3}{2}} \cdot 5^{\frac{2}{3}} - 3^{\frac{2}{2}} \cdot 5^{\frac{3}{3}} + 3^{\frac{1}{2}} \cdot 5^{\frac{4}{3}} - 5^{\frac{5}{3}},$$

or $$3^{\frac{5}{2}} - 9 \cdot 5^{\frac{1}{3}} + 3^{\frac{3}{2}} \cdot 5^{\frac{2}{3}} - 15 + 3^{\frac{1}{2}} \cdot 5^{\frac{4}{3}} - 5^{\frac{5}{3}};$$

and the rational product is $3^{\frac{6}{2}} - 5^{\frac{6}{3}} = 3^3 - 5^2 = 2$.

Example 2. Express $\left(5^{\frac{1}{2}}+9^{\frac{1}{8}}\right) \div \left(5^{\frac{1}{2}}-9^{\frac{1}{8}}\right)$
as an equivalent fraction with a rational denominator.

To rationalise the denominator, which is equal to $5^{\frac{1}{2}}-3^{\frac{1}{4}}$, put $5^{\frac{1}{2}}=x$, $3^{\frac{1}{4}}=y$; then since $x^4-y^4=(x-y)(x^3+x^2y+xy^2+y^3)$

the required factor is $5^{\frac{3}{2}}+5^{\frac{2}{2}}\,.\,3^{\frac{1}{4}}+5^{\frac{1}{2}}\,.\,3^{\frac{2}{4}}+3^{\frac{3}{4}}$;

and the rational denominator is $5^{\frac{4}{2}}-3^{\frac{4}{4}}=5^2-3=22$.

$$\therefore \text{ the expression } = \frac{\left(5^{\frac{1}{2}}+3^{\frac{1}{4}}\right)\left(5^{\frac{3}{2}}+5^{\frac{2}{2}}\,.\,3^{\frac{1}{4}}+5^{\frac{1}{2}}\,.\,3^{\frac{2}{4}}+3^{\frac{3}{4}}\right)}{22}$$

$$= \frac{5^{\frac{4}{2}}+2\,.\,5^{\frac{3}{2}}\,.\,3^{\frac{1}{4}}+2\,.\,5^{\frac{2}{2}}\,.\,3^{\frac{2}{4}}+2\,.\,5^{\frac{1}{2}}\,.\,3^{\frac{3}{4}}+3^{\frac{4}{4}}}{22}$$

$$= \frac{14+5^{\frac{3}{2}}\,.\,3^{\frac{1}{4}}+5\,.\,3^{\frac{1}{2}}+5^{\frac{1}{2}}\,.\,3^{\frac{3}{4}}}{11}$$

87. We have shewn in the *Elementary Algebra*, Art. 277, how to find the square root of a binomial quadratic surd. We may sometimes extract the square root of an expression containing more than two quadratic surds, such as $a+\sqrt{b}+\sqrt{c}+\sqrt{d}$.

Assume $\sqrt{a+\sqrt{b}+\sqrt{c}+\sqrt{d}}=\sqrt{x}+\sqrt{y}+\sqrt{z}$;

$$\therefore a+\sqrt{b}+\sqrt{c}+\sqrt{d}=x+y+z+2\sqrt{xy}+2\sqrt{xz}+2\sqrt{yz}.$$

If then $2\sqrt{xy}=\sqrt{b}$, $2\sqrt{xz}=\sqrt{c}$, $2\sqrt{yz}=\sqrt{d}$,

and if, at the same time, the values of x, y, z thus found satisfy $x+y+z=a$, we shall have obtained the required root.

Example. Find the square root of $21-4\sqrt{5}+8\sqrt{3}-4\sqrt{15}$.

Assume $\sqrt{21-4\sqrt{5}+8\sqrt{3}-4\sqrt{15}}=\sqrt{x}+\sqrt{y}-\sqrt{z}$;

$$\therefore 21-4\sqrt{5}+8\sqrt{3}-4\sqrt{15}=x+y+z+2\sqrt{xy}-2\sqrt{xz}-2\sqrt{yz}.$$

Put $2\sqrt{xy}=8\sqrt{3}$, $2\sqrt{xz}=4\sqrt{15}$, $2\sqrt{yz}=4\sqrt{5}$;

by multiplication, $xyz=240$; that is $\sqrt{xyz}=4\sqrt{15}$;
whence it follows that $\sqrt{x}=2\sqrt{3}$, $\sqrt{y}=2$, $\sqrt{z}=\sqrt{5}$.

And since these values satisfy the equation $x+y+z=21$, the required root is $2\sqrt{3}+2-\sqrt{5}$.

88. *If* $\sqrt[3]{a + \sqrt{b}} = x + \sqrt{y}$, *then will* $\sqrt[3]{a - \sqrt{b}} = x - \sqrt{y}$.

For, by cubing, we obtain

$$a + \sqrt{b} = x^3 + 3x^2\sqrt{y} + 3xy + y\sqrt{y}.$$

Equating rational and irrational parts, we have

$$a = x^3 + 3xy, \quad \sqrt{b} = 3x^2\sqrt{y} + y\sqrt{y};$$

$$\therefore \quad a - \sqrt{b} = x^3 - 3x^2\sqrt{y} + 3xy - y\sqrt{y};$$

that is,

$$\sqrt[3]{a - \sqrt{b}} = x - \sqrt{y}.$$

Similarly, by the help of the Binomial Theorem, Chap. XIII., it may be proved that if

$$\sqrt[n]{a + \sqrt{b}} = x + \sqrt{y}, \text{ then } \sqrt[n]{a - \sqrt{b}} = x - \sqrt{y},$$

where n is any positive integer.

89. By the following method the cube root of an expression of the form $a \pm \sqrt{b}$ may sometimes be found.

Suppose $\sqrt[3]{a + \sqrt{b}} = x + \sqrt{y}$;

then $\sqrt[3]{a - \sqrt{b}} = x - \sqrt{y}$.

$$\therefore \quad \sqrt[3]{a^2 - b} = x^2 - y \quad \text{.........................(1).}$$

Again, as in the last article,

$$a = x^3 + 3xy. \quad \text{.............................(2).}$$

The values of x and y have to be determined from (1) and (2).

In (1) suppose that $\sqrt[3]{a^2 - b} = c$; then by substituting for y in (2) we obtain

$$a = x^3 + 3x(x^2 - c);$$

that is,

$$4x^3 - 3cx = a.$$

If from this equation the value of x can be determined by trial, the value of y is obtained from $y = x^2 - c$.

NOTE. We do not here assume $\sqrt{x} + \sqrt{y}$ for the cube root, as in the extraction of the square root; for with this assumption, on cubing we should have

$$a + \sqrt{b} = x\sqrt{x} + 3x\sqrt{y} + 3y\sqrt{x} + y\sqrt{y},$$

and since every term on the right hand side is irrational we cannot equate rational and irrational parts.

Example. Find the cube root of $72-32\sqrt{5}$.

Assume $$\sqrt[3]{72-32\sqrt{5}}=x-\sqrt{y};$$

then $$\sqrt[3]{72+32\sqrt{5}}=x+\sqrt{y}.$$

By multiplication, $$\sqrt[3]{5184-1024\times 5}=x^2-y;$$

that is, $$4=x^2-y \quad\text{.........................}\quad (1).$$

Again $$72-32\sqrt{5}=x^3-3x^2\sqrt{y}+3xy-y\sqrt{y};$$

whence $$72=x^3+3xy \quad\text{.........................}\quad (2).$$

From (1) and (2), $$72=x^3+3x(x^2-4);$$

that is, $$x^3-3x=18.$$

By trial, we find that $x=3$; hence $y=5$, and the cube root is $3-\sqrt{5}$.

90. When the binomial whose cube root we are seeking consists of *two* quadratic surds, we proceed as follows.

Example. Find the cube root of $9\sqrt{3}+11\sqrt{2}$.

$$\sqrt[3]{9\sqrt{3}+11\sqrt{2}}=\sqrt[3]{3\sqrt{3}\left(3+\frac{11}{3}\sqrt{\frac{2}{3}}\right)}$$
$$=\sqrt{3}\sqrt[3]{3+\frac{11}{3}\sqrt{\frac{2}{3}}}.$$

By proceeding as in the last article, we find that

$$\sqrt[3]{3+\frac{11}{3}\sqrt{\frac{2}{3}}}=1+\sqrt{\frac{2}{3}};$$

$\therefore$ the required cube root $$=\sqrt{3}\left(1+\sqrt{\frac{2}{3}}\right)$$
$$=\sqrt{3}+\sqrt{2}.$$

91. We add a few harder examples in surds.

Example 1. Express with rational denominator $\dfrac{4}{\sqrt[3]{9}-\sqrt[3]{3}+1}$.

The expression $$=\frac{4}{3^{\frac{2}{3}}-3^{\frac{1}{3}}+1}$$
$$=\frac{4\left(3^{\frac{1}{3}}+1\right)}{\left(3^{\frac{1}{3}}+1\right)\left(3^{\frac{2}{3}}-3^{\frac{1}{3}}+1\right)}$$
$$=\frac{4\left(3^{\frac{1}{3}}+1\right)}{3+1}=3^{\frac{1}{3}}+1.$$

Example 2. Find the square root of

$$\frac{3}{2}(x-1)+\sqrt{2x^2-7x-4}.$$

The expression $=\frac{1}{2}\{3x-3+2\sqrt{(2x+1)(x-4)}\}$

$$=\frac{1}{2}\{(2x+1)+(x-4)+2\sqrt{(2x+1)(x-4)}\};$$

hence, by inspection, the square root is

$$\frac{1}{\sqrt{2}}(\sqrt{2x+1}+\sqrt{x-4}).$$

Example 3. Given $\sqrt{5}=2{\cdot}23607$, find the value of

$$\frac{\sqrt{3-\sqrt{5}}}{\sqrt{2}+\sqrt{7-3\sqrt{5}}}.$$

Multiplying numerator and denominator by $\sqrt{2}$,

the expression $=\dfrac{\sqrt{6-2\sqrt{5}}}{2+\sqrt{14-6\sqrt{5}}}$

$$=\frac{\sqrt{5}-1}{2+3-\sqrt{5}}$$

$$=\frac{1}{\sqrt{5}}=\frac{\sqrt{5}}{5}={\cdot}44721.$$

EXAMPLES. VIII. a.

Express as equivalent fractions with rational denominator:

1. $\dfrac{1}{1+\sqrt{2}-\sqrt{3}}$.

2. $\dfrac{\sqrt{2}}{\sqrt{2}+\sqrt{3}-\sqrt{5}}$.

3. $\dfrac{1}{\sqrt{a}+\sqrt{b}+\sqrt{a+b}}$.

4. $\dfrac{2\sqrt{a+1}}{\sqrt{a-1}-\sqrt{2a}+\sqrt{a+1}}$.

5. $\dfrac{\sqrt{10}+\sqrt{5}-\sqrt{3}}{\sqrt{3}+\sqrt{10}-\sqrt{5}}$.

6. $\dfrac{(\sqrt{3}+\sqrt{5})(\sqrt{5}+\sqrt{2})}{\sqrt{2}+\sqrt{3}+\sqrt{5}}$.

Find a factor which will rationalise:

7. $\sqrt[3]{3}-\sqrt{2}$.

8. $\sqrt[6]{5}+\sqrt[3]{2}$.

9. $a^{\frac{1}{6}}+b^{\frac{1}{4}}$.

10. $\sqrt[3]{3}-1$.

11. $2+\sqrt[4]{7}$.

12. $\sqrt[3]{5}-\sqrt[4]{3}$.

Express with rational denominator:

13. $\dfrac{\sqrt[3]{3}-1}{\sqrt[3]{3}+1}$. 14. $\dfrac{\sqrt[6]{9}-\sqrt[6]{8}}{\sqrt[6]{9}+\sqrt[6]{8}}$. 15. $\dfrac{\sqrt{2}\,.\sqrt[3]{3}}{\sqrt[3]{3}+\sqrt{2}}$.

16. $\dfrac{\sqrt[3]{3}}{\sqrt{3}+\sqrt[6]{9}}$. 17. $\dfrac{\sqrt{8}+\sqrt[3]{4}}{\sqrt{8}-\sqrt[3]{4}}$. 18. $\dfrac{\sqrt[6]{27}}{3-\sqrt[6]{9}}$.

Find the square root of

19. $16-2\sqrt{20}-2\sqrt{28}+2\sqrt{35}$. 20. $24+4\sqrt{15}-4\sqrt{21}-2\sqrt{35}$.

21. $6+\sqrt{12}-\sqrt{24}-\sqrt{8}$. 22. $5-\sqrt{10}-\sqrt{15}+\sqrt{6}$.

23. $a+3b+4+4\sqrt{a}-4\sqrt{3b}-2\sqrt{3ab}$.

24. $21+3\sqrt{8}-6\sqrt{3}-6\sqrt{7}-\sqrt{24}-\sqrt{56}+2\sqrt{21}$.

Find the cube root of

25. $10+6\sqrt{3}$. 26. $38+17\sqrt{5}$. 27. $99-70\sqrt{2}$.

28. $38\sqrt{14}-100\sqrt{2}$. 29. $54\sqrt{3}+41\sqrt{5}$. 30. $135\sqrt{3}-87\sqrt{6}$.

Find the square root of

31. $a+x+\sqrt{2ax+x^2}$. 32. $2a-\sqrt{3a^2-2ab-b^2}$.

33. $1+a^2+(1+a^2+a^4)^{\frac{1}{2}}$. 34. $1+(1-a^2)^{-\frac{1}{2}}$.

35. If $a=\dfrac{1}{2-\sqrt{3}}$, $b=\dfrac{1}{2+\sqrt{3}}$, find the value of $7a^2+11ab-7b^2$.

36. If $x=\dfrac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}$, $y=\dfrac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}$, find the value of $3x^2-5xy+3y^2$.

Find the value of

37. $\dfrac{\sqrt{26-15\sqrt{3}}}{5\sqrt{2}-\sqrt{38+5\sqrt{3}}}$. 38. $\sqrt{\dfrac{6+2\sqrt{3}}{33-19\sqrt{3}}}$.

39. $(28-10\sqrt{3})^{\frac{1}{2}}-(7+4\sqrt{3})^{-\frac{1}{2}}$. 40. $(26+15\sqrt{3})^{\frac{2}{3}}-(26+15\sqrt{3})^{-\frac{2}{3}}$.

41. Given $\sqrt{5}=2{\cdot}23607$, find the value of

$$\frac{10\sqrt{2}}{\sqrt{18}-\sqrt{3+\sqrt{5}}}-\frac{\sqrt{10}+\sqrt{18}}{\sqrt{8}+\sqrt{3-\sqrt{5}}}.$$

42. Divide $x^3+1+3x\sqrt[3]{2}$ by $x-1+\sqrt[3]{2}$.

43. Find the cube root of $9ab^2+(b^2+24a^2)\sqrt{b^2-3a^2}$.

44. Evaluate $\dfrac{\sqrt{x^2-1}}{x-\sqrt{x^2-1}}$, when $2x=\sqrt{a}+\dfrac{1}{\sqrt{a}}$,

Imaginary Quantities.

92. Although from the rule of signs it is evident that a negative quantity cannot have a real square root, yet imaginary quantities represented by symbols of the form $\sqrt{-a}$, $\sqrt{-1}$ are of frequent occurrence in mathematical investigations, and their use leads to valuable results. We therefore proceed to explain in what sense such roots are to be regarded.

When the quantity under the radical sign is negative, we can no longer consider the symbol $\sqrt{}$ as indicating a possible arithmetical operation; but just as $\sqrt{a}$ may be defined as a symbol which obeys the relation $\sqrt{a} \times \sqrt{a} = a$, so we shall define $\sqrt{-a}$ to be such that $\sqrt{-a} \times \sqrt{-a} = -a$, and we shall accept the meaning to which this assumption leads us.

It will be found that this definition will enable us to bring imaginary quantities under the dominion of ordinary algebraical rules, and that through their use results may be obtained which can be relied on with as much certainty as others which depend solely on the use of real quantities.

93. By definition, $\sqrt{-1} \times \sqrt{-1} = -1.$

$$\therefore \quad \sqrt{a} \, . \sqrt{-1} \times \sqrt{a} \, . \sqrt{-1} = a(-1);$$

that is, $(\sqrt{a} \, . \sqrt{-1})^2 = -a.$

Thus the product $\sqrt{a} \, . \sqrt{-1}$ may be regarded as equivalent to the imaginary quantity $\sqrt{-a}$.

94. It will generally be found convenient to indicate the imaginary character of an expression by the presence of the symbol $\sqrt{-1}$; thus

$$\sqrt{-4} = \sqrt{4 \times (-1)} = 2\sqrt{-1}.$$

$$\sqrt{-7a^2} = \sqrt{7a^2 \times (-1)} = a\sqrt{7}\sqrt{-1}.$$

95. We shall always consider that, in the absence of any statement to the contrary, of the signs which may be prefixed before a radical the positive sign is to be taken. But in the use of imaginary quantities there is one point of importance which deserves notice.

Since $$(-a) \times (-b) = ab,$$
by taking the square root, we have
$$\sqrt{-a} \times \sqrt{-b} = \pm \sqrt{ab}.$$

Thus in forming the product of $\sqrt{-a}$ and $\sqrt{-b}$ it would appear that either of the signs $+$ or $-$ might be placed before $\sqrt{ab}$. This is not the case, for
$$\begin{aligned}\sqrt{-a} \times \sqrt{-b} &= \sqrt{a} \,.\, \sqrt{-1} \times \sqrt{b} \,.\, \sqrt{-1}\\ &= \sqrt{ab}\,(\sqrt{-1})^2\\ &= -\sqrt{ab}.\end{aligned}$$

96. It is usual to apply the term 'imaginary' to all expressions which are not wholly real. Thus $a + b\sqrt{-1}$ may be taken as the general type of all imaginary expressions. *Here* a *and* b *are real quantities, but not necessarily rational.*

97. In dealing with imaginary quantities we apply the laws of combination which have been proved in the case of other surd quantities.

Example 1. $a + b\sqrt{-1} \pm (c + d\sqrt{-1}) = a \pm c + (b \pm d)\sqrt{-1}$.

Example 2. The product of $a + b\sqrt{-1}$ and $c + d\sqrt{-1}$
$$\begin{aligned}&= (a + b\sqrt{-1})\,(c + d\sqrt{-1})\\ &= ac - bd + (bc + ad)\sqrt{-1}.\end{aligned}$$

98. *If* $a + b\sqrt{-1} = 0$, *then* $a = 0$, *and* $b = 0$.

For, if $$a + b\sqrt{-1} = 0,$$
then $$b\sqrt{-1} = -a;$$
$$\therefore\ -b^2 = a^2;$$
$$\therefore\ a^2 + b^2 = 0.$$

Now a^2 and b^2 are both positive, therefore their sum cannot be zero unless each of them is separately zero; that is, $a = 0$, and $b = 0$.

99. *If* $a + b\sqrt{-1} = c + d\sqrt{-1}$, *then* $a = c$, *and* $b = d$.

For, by transposition, $a - c + (b - d)\sqrt{-1} = 0$;
therefore, by the last article, $a - c = 0$, and $b - d = 0$;
that is $$a = c, \text{ and } b = d.$$

Thus in order that two imaginary expressions may be equal it is necessary and sufficient that the real parts should be equal, and the imaginary parts should be equal.

100. Definition. When two imaginary expressions differ only in the sign of the imaginary part they are said to be **conjugate.**

Thus $a-b\sqrt{-1}$ is conjugate to $a+b\sqrt{-1}$.

Similarly $\sqrt{2}+3\sqrt{-1}$ is conjugate to $\sqrt{2}-3\sqrt{-1}$.

101. *The sum and the product of two conjugate imaginary expressions are both real.*

For $\qquad a+b\sqrt{-1}+a-b\sqrt{-1}=2a.$

Again

$$(a+b\sqrt{-1})(a-b\sqrt{-1})=a^2-(-b^2)$$
$$=a^2+b^2.$$

102. Definition. The positive value of the square root of a^2+b^2 is called the **modulus** of each of the conjugate expressions

$$a+b\sqrt{-1} \text{ and } a-b\sqrt{-1}.$$

103. *The modulus of the product of two imaginary expressions is equal to the product of their moduli.*

Let the two expressions be denoted by $a+b\sqrt{-1}$ and $c+d\sqrt{-1}$.

Then their product $=ac-bd+(ad+bc)\sqrt{-1}$, which is an imaginary expression whose modulus

$$=\sqrt{(ac-bd)^2+(ad+bc)^2}$$
$$=\sqrt{a^2c^2+b^2d^2+a^2d^2+b^2c^2}$$
$$=\sqrt{(a^2+b^2)(c^2+d^2)}$$
$$=\sqrt{a^2+b^2}\times\sqrt{c^2+d^2};$$

which proves the proposition.

104. If the denominator of a fraction is of the form $a+b\sqrt{-1}$, it may be rationalised by multiplying the numerator and the denominator by the conjugate expression $a-b\sqrt{-1}$.

For instance

$$\frac{c+d\sqrt{-1}}{a+b\sqrt{-1}} = \frac{(c+d\sqrt{-1})(a-b\sqrt{-1})}{(a+b\sqrt{-1})(a-b\sqrt{-1})}$$

$$= \frac{ac+bd+(ad-bc)\sqrt{-1}}{a^2+b^2}$$

$$= \frac{ac+bd}{a^2+b^2} + \frac{ad-bc}{a^2+b^2}\sqrt{-1}.$$

Thus by reference to Art. 97, we see that *the sum, difference, product, and quotient of two imaginary expressions is in each case an imaginary expression of the same form.*

105. *To find the square root of* $a+b\sqrt{-1}$.

Assume $\sqrt{a+b\sqrt{-1}} = x+y\sqrt{-1}$,

where x and y are real quantities.

By squaring, $a+b\sqrt{-1} = x^2-y^2+2xy\sqrt{-1}$;

therefore, by equating real and imaginary parts,

$$x^2-y^2 = a \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1),$$

$$2xy = b \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2);$$

$$\therefore\ (x^2+y^2)^2 = (x^2-y^2)^2+(2xy)^2$$

$$= a^2+b^2;$$

$$\therefore\ x^2+y^2 = \sqrt{a^2+b^2} \ldots\ldots\ldots\ldots\ldots\ldots(3).$$

From (1) and (3), we obtain

$$x^2 = \frac{\sqrt{a^2+b^2}+a}{2},\ y^2 = \frac{\sqrt{a^2+b^2}-a}{2};$$

$$\therefore\ x = \pm\left\{\frac{\sqrt{a^2+b^2}+a}{2}\right\}^{\frac{1}{2}},\ y = \pm\left\{\frac{\sqrt{a^2+b^2}-a}{2}\right\}^{\frac{1}{2}}.$$

Thus the required root is obtained.

Since x and y are real quantities, x^2+y^2 is positive, and therefore in (3) the positive sign must be prefixed before the quantity $\sqrt{a^2+b^2}$.

Also from (2) we see that the product xy must have the same sign as b; hence x and y must have like signs if b is positive, and unlike signs if b is negative.

Example 1. Find the square root of $-7-24\sqrt{-1}$.

Assume $$\sqrt{-7-24\sqrt{-1}}=x+y\sqrt{-1};$$
then $$-7-24\sqrt{-1}=x^2-y^2+2xy\sqrt{-1};$$
$$\therefore\ x^2-y^2=-7 \qquad\qquad (1),$$
and $$2xy=-24.$$
$$\therefore\ (x^2+y^2)^2=(x^2-y^2)^2+(2xy)^2$$
$$=49+576$$
$$=625;$$
$$\therefore\ x^2+y^2=25 \qquad\qquad (2).$$

From (1) and (2), $x^2=9$ and $y^2=16$;
$$\therefore\ x=\pm 3,\ y=\pm 4.$$

Since the product xy is negative, we must take
$$x=3,\ y=-4;\ \text{ or } x=-3,\ y=4.$$

Thus the roots are $3-4\sqrt{-1}$ and $-3+4\sqrt{-1}$;

that is, $$\sqrt{-7-24\sqrt{-1}}=\pm(3-4\sqrt{-1}).$$

Example 2. To find the value of $\sqrt[4]{-64a^4}$.

$$\sqrt[4]{-64a^4}=\sqrt{\pm 8a^2\sqrt{-1}}$$
$$=2a\sqrt{2}\sqrt{\pm\sqrt{-1}}.$$

It remains to find the value of $\sqrt{\pm\sqrt{-1}}$.

Assume $$\sqrt{+\sqrt{-1}}=x+y\sqrt{-1};$$
then $$+\sqrt{-1}=x^2-y^2+2xy\sqrt{-1};$$
$$\therefore\ x^2-y^2=0 \text{ and } 2xy=1;$$
whence $$x=\frac{1}{\sqrt{2}},\ y=\frac{1}{\sqrt{2}};\ \text{ or } x=-\frac{1}{\sqrt{2}},\ y=-\frac{1}{\sqrt{2}};$$
$$\therefore\ \sqrt{+\sqrt{-1}}=\pm\frac{1}{\sqrt{2}}(1+\sqrt{-1}).$$

Similarly $$\sqrt{-\sqrt{-1}}=\pm\frac{1}{\sqrt{2}}(1-\sqrt{-1})$$
$$\therefore\ \sqrt{\pm\sqrt{-1}}=\pm\frac{1}{\sqrt{2}}(1\pm\sqrt{-1});$$
and finally $$\sqrt[4]{-64a^4}=\pm 2a(1\pm\sqrt{-1}).$$

106. The symbol $\sqrt{-1}$ is often represented by the letter i; but until the student has had a little practice in the use of imaginary quantities he will find it easier to retain the symbol $\sqrt{-1}$. It is useful to notice the successive powers of $\sqrt{-1}$ or i; thus

$$(\sqrt{-1})^1 = \sqrt{-1}, \qquad i = i;$$
$$(\sqrt{-1})^2 = -1, \qquad i^2 = -1;$$
$$(\sqrt{-1})^3 = -\sqrt{-1}, \qquad i^3 = -i;$$
$$(\sqrt{-1})^4 = 1, \qquad i^4 = 1;$$

and since each power is obtained by multiplying the one before it by $\sqrt{-1}$, or i, we see that the results must now recur.

107. We shall now investigate the properties of certain imaginary quantities which are of very frequent occurrence.

Suppose $\quad x = \sqrt[3]{1}$; then $x^3 = 1$, or $x^3 - 1 = 0$;

that is, $\quad (x-1)(x^2+x+1) = 0.$

$\therefore$ either $\quad x - 1 = 0$, or $x^2 + x + 1 = 0$;

whence $\quad x = 1$, or $\quad x = \dfrac{-1 \pm \sqrt{-3}}{2}$.

It may be shewn by actual involution that each of these values when cubed is equal to unity. Thus unity has three cube roots,

$$1, \quad \frac{-1+\sqrt{-3}}{2}, \quad \frac{-1-\sqrt{-3}}{2};$$

two of which are imaginary expressions.

Let us denote these by α and β; then since they are the roots of the equation

$$x^2 + x + 1 = 0,$$

their product is equal to unity;

that is, $\quad \alpha\beta = 1;$

$$\therefore \alpha^3\beta = \alpha^2;$$

that is, $\quad \beta = \alpha^2, \quad$ since $\alpha^3 = 1$.

Similarly we may shew that $\alpha = \beta^2$.

108. Since *each of the imaginary roots is the square of the other*, it is usual to denote the three cube roots of unity by $1, \omega, \omega^2$.

Also ω satisfies the equation $x^2 + x + 1 = 0$;

$$\therefore\ 1 + \omega + \omega^2 = 0;$$

that is, *the sum of the three cube roots of unity is zero.*

Again, $\omega \, . \, \omega^2 = \omega^3 = 1$;

therefore (1) *the product of the two imaginary roots is unity;*
(2) *every integral power of ω^3 is unity.*

109. It is useful to notice that the successive positive integral powers of ω are 1, ω, and ω^2; for, if n be a multiple of 3, it must be of the form $3m$; and $\omega^n = \omega^{3m} = 1$.

If n be not a multiple of 3, it must be of the form $3m + 1$ or $3m + 2$.

If $n = 3m + 1$, $\omega^n = \omega^{3m+1} = \omega^{3m} \, . \, \omega = \omega.$

If $n = 3m + 2$, $\omega^n = \omega^{3m+2} = \omega^{3m} \, . \, \omega^2 = \omega^2.$

110. We now see that every quantity has three cube roots, two of which are imaginary. For the cube roots of a^3 are those of $a^3 \times 1$, and therefore are a, $a\omega$, $a\omega^2$. Similarly the cube roots of 9 are $\sqrt[3]{9}$, $\omega\sqrt[3]{9}$, $\omega^2\sqrt[3]{9}$, where $\sqrt[3]{9}$ is the cube root found by the ordinary arithmetical rule. In future, unless otherwise stated, the symbol $\sqrt[3]{a}$ will always be taken to denote the arithmetical cube root of a.

Example 1. Reduce $\dfrac{(2+3\sqrt{-1})^2}{2+\sqrt{-1}}$ to the form $A + B\sqrt{-1}$.

The expression
$$= \frac{4-9+12\sqrt{-1}}{2+\sqrt{-1}}$$
$$= \frac{(-5+12\sqrt{-1})(2-\sqrt{-1})}{(2+\sqrt{-1})(2-\sqrt{-1})}$$
$$= \frac{-10+12+29\sqrt{-1}}{4+1}$$
$$= \frac{2}{5} + \frac{29}{5}\sqrt{-1};$$

which is of the required form.

Example 2. *Resolve* $x^3 + y^3$ *into three factors of the first degree.*

Since $x^3 + y^3 = (x+y)(x^2 - xy + y^2)$

$$\therefore x^3 + y^3 = (x+y)(x+\omega y)(x+\omega^2 y);$$

for $\omega + \omega^2 = -1$, and $\omega^3 = 1$.

Example 3. Shew that

$$(a+\omega b+\omega^2 c)(a+\omega^2 b+\omega c)=a^2+b^2+c^2-bc-ca-ab.$$

In the product of $a+\omega b+\omega^2 c$ and $a+\omega^2 b+\omega c$,

the coefficients of b^2 and c^2 are ω^3, or 1;

the coefficient of bc $=\omega^2+\omega^4=\omega^2+\omega=-1$;

the coefficients of ca and $ab=\omega^2+\omega=-1$;

$$\therefore\ (a+\omega b+\omega^2 c)(a+\omega^2 b+\omega c)=a^2+b^2+c^2-bc-ca-ab.$$

Example 4. Shew that

$$(1+\omega-\omega^2)^3-(1-\omega+\omega^2)^3=0.$$

Since $1+\omega+\omega^2=0$, we have

$$\begin{aligned}(1+\omega-\omega^2)^3-(1-\omega+\omega^2)^3&=(-2\omega^2)^3-(-2\omega)^3\\&=-8\omega^6+8\omega^3\\&=-8+8\\&=0.\end{aligned}$$

EXAMPLES. VIII. b.

1. Multiply $2\sqrt{-3}+3\sqrt{-2}$ by $4\sqrt{-3}-5\sqrt{-2}$.

2. Multiply $3\sqrt{-7}-5\sqrt{-2}$ by $3\sqrt{-7}+5\sqrt{-2}$.

3. Multiply $e^{\sqrt{-1}}+e^{-\sqrt{-1}}$ by $e^{\sqrt{-1}}-e^{-\sqrt{-1}}$.

4. Multiply $x-\dfrac{1+\sqrt{-3}}{2}$ by $x-\dfrac{1-\sqrt{-3}}{2}$.

Express with rational denominator:

5. $\dfrac{1}{3-\sqrt{-2}}$.

6. $\dfrac{3\sqrt{-2}+2\sqrt{-5}}{3\sqrt{-2}-2\sqrt{-5}}$.

7. $\dfrac{3+2\sqrt{-1}}{2-5\sqrt{-1}}+\dfrac{3-2\sqrt{-1}}{2+5\sqrt{-1}}$.

8. $\dfrac{a+x\sqrt{-1}}{a-x\sqrt{-1}}-\dfrac{a-x\sqrt{-1}}{a+x\sqrt{-1}}$.

9. $\dfrac{(x+\sqrt{-1})^2}{x-\sqrt{-1}}-\dfrac{(x-\sqrt{-1})^2}{x+\sqrt{-1}}$.

10. $\dfrac{(a+\sqrt{-1})^3-(a-\sqrt{-1})^3}{(a+\sqrt{-1})^2-(a-\sqrt{-1})^2}$.

11. Find the value of $(-\sqrt{-1})^{4n+3}$, when n is a positive integer.

12. Find the square of $\sqrt{9+40\sqrt{-1}}+\sqrt{9-40\sqrt{-1}}$.

Find the square root of

13. $-5+12\sqrt{-1}$. **14.** $-11-60\sqrt{-1}$. **15.** $-47+8\sqrt{-3}$.

16. $-8\sqrt{-1}$. **17.** $a^2-1+2a\sqrt{-1}$.

18. $4ab-2(a^2-b^2)\sqrt{-1}$.

Express in the form $A+iB$

19. $\dfrac{3+5i}{2-3i}$. **20.** $\dfrac{\sqrt{3}-i\sqrt{2}}{2\sqrt{3}-i\sqrt{2}}$. **21.** $\dfrac{1+i}{1-i}$.

22. $\dfrac{(1+i)^2}{3-i}$. **23.** $\dfrac{(a+ib)^2}{a-ib}-\dfrac{(a-ib)^2}{a+ib}$.

If 1, ω, ω^2 are the three cube roots of unity, prove

24. $(1+\omega^2)^4=\omega$. **25.** $(1-\omega+\omega^2)(1+\omega-\omega^2)=4$.

26. $(1-\omega)(1-\omega^2)(1-\omega^4)(1-\omega^5)=9$.

27. $(2+5\omega+2\omega^2)^6=(2+2\omega+5\omega^2)^6=729$.

28. $(1-\omega+\omega^2)(1-\omega^2+\omega^4)(1-\omega^4+\omega^8)\ldots$ to $2n$ factors $=2^{2n}$.

29. Prove that

$$x^3+y^3+z^3-3xyz=(x+y+z)(x+y\omega+z\omega^2)(x+y\omega^2+z\omega).$$

30. If $$x=a+b,\quad y=a\omega+b\omega^2,\quad z=a\omega^2+b\omega,$$
shew that

(1) $xyz=a^3+b^3$.

(2) $x^2+y^2+z^2=6ab$.

(3) $x^3+y^3+z^3=3(a^3+b^3)$.

31. If $ax+cy+bz=X$, $cx+by+az=Y$, $bx+ay+cz=Z$,
shew that $(a^2+b^2+c^2-bc-ca-ab)(x^2+y^2+z^2-yz-zx-xy)$
$$=X^2+Y^2+Z^2-YZ-XZ-XY.$$