CHAPTER 14

Basic Operations and Factorization

Polynomials

An algebraic expression having a variable x, and its non-negative integral powers with real numbers as coefficients, is called a polynomial in x.

The highest power of x in this expression is called the degree of polynomial.

e.g., 1. $4x^3 - 9x^2 + 7x + 9$ is a polynomial in x of degree 3.

2. $\left(3x^2 + 9x - 1 + \frac{7}{x}\right)$ is not a polynomial. As it contains a term, namely $\frac{7}{x}$, having negative integral powers of x.

3. $(5x^2 - 7x^{7/2} + x - 1)$ is not a polynomial as the term $-7x^{7/2}$ contains rational power of x.

Various Types of Polynomial

1. **Linear Polynomial** A polynomial of degree 1, is called a linear polynomial.

   e.g., $9x + 7, x - 9, x + 2$ etc., are all linear polynomial.

2. **Quadratic Polynomial** A polynomial of degree 2 is called a quadratic polynomial.

   The general form of quadratic polynomial is $ax^2 + bx + c$, where a, b and c are real and $a \neq 0$.

   e.g., $x^2 - 7x + 12, 5y^2 - 7, z^2 - 4$ etc., are all quadratic polynomials.

3. **Cubic Polynomial** A polynomial of degree 3 is called a cubic polynomial.

   The general form of a cubic polynomial is $ax^3 + bx^2 + cx + d$, where a, b, c and d are real and $a \neq 0$.

   e.g., $3x^3 + 2x^2 + 7x - 1, 8y^3 + 5y + 2, x^3 - 8$ etc., are all cubic polynomials.

4. **Biquadratic Polynomial** A polynomial of degree 4 is called a biquadratic polynomials.

   The general form of a biquadratic polynomial is $ax^4 + bx^3 + cx^2 + dx + e$, where a, b, c, d and e are real numbers and $a \neq 0$.

   e.g., $(6x^4 + 3x^3 + 2x^2 + x + 2)$, $x^4 - 8$, all are biquadratic polynomials.

Factors

A polynomial $g(x)$, is called a factor of polynomial $p(x)$, if $g(x)$ divides $p(x)$ exactly.

e.g., As $(x + 5)$ satisfies the quadratic polynomial $x^2 + 6x + 5$ so it is a factor of polynomial.

Factorization

To express of polynomial as the product of polynomials of degree less than that of the given polynomial is called as factorization.

e.g., 1. $x^2 - 49 = x^2 - 7^2 = (x - 7)(x + 7)$

2. $x^2 - 7x + 12 = x^2 - 3x - 4x + 12$
$= x(x - 3) - 4(x - 3) = (x - 4)(x - 3)$

Required Formulae for Factorization

1. $(a + b)^2 = a^2 + b^2 + 2ab$ and $(a - b)^2 = a^2 + b^2 - 2ab$
2. $(a + b)^2 - (a - b)^2 = 4ab$ and $(a + b)^2 + (a - b)^2 = 2(a^2 + b^2)$
3. $(a + b)^3 = a^3 + b^3 + 3ab(a + b)$
4. $(a - b)^3 = a^3 - b^3 - 3ab(a - b)$
5. $(a^2 - b^2) = (a + b)(a - b)$
6. $(a^3 + b^3) = (a + b)(a^2 + b^2 - ab)$
7. $(a^3 - b^3) = (a - b)(a^2 + b^2 + ab)$

8. $(a+b+c)^2 = [a^2 + b^2 + c^2 + 2(ab + bc + ac)]$
9. $[a^3 + b^3 + c^3 - 3abc] = (a+b+c)(a^2 + b^2 + c^2 - ab - ac - bc)$
10. If $a+b+c=0 \Rightarrow a^3 + b^3 + c^3 = 3abc$
11. $a^4 + a^2b^2 + b^4 = (a^2 + ab + b^2)(a^2 - ab + b^2)$

Methods of Factorization

1. Factorization by taking Out the Common Factor

Method If each term of an expression has a common factor, take out the common factors.

Example 1. Factorize value of $16x^2 + 12xy$ is

(a) $4(4x+3y)$ (b) $4x+3y$
(c) $4x(4x+3y)$ (d) $4x-3y$

Sol. (c) $16x^2 + 12xy = 4\cdot 4x^2 + 3\cdot(4xy) = 4x(4x+3y)$

Example 2. Factorize value of $x^3 + 5x + 2x^2 + 10$ is

(a) $(x+5)(x+2)$ (b) $(x^2+5)(x+2)$
(c) $(x+5)(x^2+2)$ (d) None of these

Sol. (b) $x^3 + 2x^2 + 5x + 10 = x^2(x+2) + 5(x+2) = (x+2)(x^2+5)$

Example 3. Factorize value of $4(3a-2b)^2 - 5(3a-2b)$ is

(a) $(3a-2b)(12a-8b-5)$ (b) $(2a-3b)(12a-8b)$
(c) $(2a-3b)(8a-12b-5)$ (d) None of these

Sol. (a) $4(3a-2b)^2 - 5(3a-2b)$

$= (3a-2b)[4(3a-2b)-5] = (3a-2b)[12a-8b-5]$

2. Factorization by Grouping

Method Sometime in a given polynomial it is not possible to take out a common factor directly. However, on rearranging the terms of the polynomial and grouping such that all the terms have a common factor.

Example 4. Factorize value of $xy + yz + xa + za$ is

(a) $(x+z)(y+a)$ (b) $(x-z)(y+a)$
(c) $(x-z)(y-a)$ (d) $(x-z)(y^2+a^2)$

Sol. (a) $xy + yz + xa + za$

$= y(x+z) + a(x+z) = (x+z)(y+a)$

Example 5. Factorize value of $x^2 + y - xy - x$ is

(a) $(x-y)(x+1)$ (b) $(x-y)(x^2-1)$
(c) $(x+y)(x^2+1)$ (d) $(x-y)(x-1)$

Sol. (d) $x^2 + y - xy - x$

$= x^2 - x + y - xy = x(x-1) + y(1-x)$

$= -x(1-x) + y(1-x) = (1-x)(y-x)$

or $= x^2 - xy + y - x = x(x-y) + 1(y-x)$

$= x(x-y) - 1(x-y) = (x-y)(x-1)$

Example 6. Factorize value of $x^2 + \frac{4}{x^2} + 4 - 2x - \frac{4}{x}$ is

(a) $\left(x - \frac{2}{x}\right)\left(x + \frac{2}{x} + 2\right)$ (b) $\left(x - \frac{2}{x}\right)\left(x + \frac{2}{x} - 2\right)$
(c) $\left(x + \frac{2}{x}\right)\left(x + \frac{2}{x} - 2\right)$ (d) None of these

Sol. (c) $\left(x^2 + \frac{4}{x^2} + 4 - 2x - \frac{4}{x}\right) = \left(x^2 + \frac{4}{x^2} + 4\right) - 2\left(x + \frac{2}{x}\right)$

$= \left(x + \frac{2}{x}\right)^2 - 2\left(x + \frac{2}{x}\right) = \left(x + \frac{2}{x}\right)\left(x + \frac{2}{x} - 2\right)$

3. Factorizing a Perfect Square Trinomial

Method Here, the polynomial is given as the perfect square trinomial, which is the square of a binomial.

Formulae $(a+b)^2 = a^2 + b^2 + 2ab$

$(a-b)^2 = a^2 + b^2 - 2ab$

Example 7. Factorize value of $16x^2 - 8x + 1$ is

(a) $(2x-1)^2$ (b) $(x-4)^2$ (c) $(x-2)^2$ (d) $(4x-1)^2$

Sol. (d) $16x^2 - 8x + 1 = (4x)^2 - 2(4x) + 1^2 = (4x-1)^2$

Example 8. Factorize value of $1 - 10x + 25x^2$ is

(a) $(5x-1)^2$ (b) $(x-5)^2$ (c) $(2x-1)^2$ (d) $(2x-5)^2$

Sol. (a) $1 - 10x + 25x^2 = 25x^2 - 10x + 1 = (5x)^2 - 2(5x) + 1^2 = (5x-1)^2$

Example 9. Factorize value of $x^2 - 2\sqrt{5}x + 5$ is

(a) $(x\sqrt{5} - 1)^2$ (b) $(x-5)^2$
(c) $(x+\sqrt{5})(x-\sqrt{5})$ (d) $(x-\sqrt{5})^2$

Sol. (d) $x^2 - 2\sqrt{5}x + 5 = x^2 - 2\sqrt{5}x + (\sqrt{5})^2 = (x-\sqrt{5})^2$

Example 10. Factorize value of $x^2 + y^2 + 2(xy + yz + zx)$ is

(a) $(x+y)(x+y+z)$ (b) $(x+y)(x+y+2z)$
(c) $(x-y)(x+y)$ (d) $(x^2+y^2)(x-y)$

Sol. (b) $x^2 + y^2 + 2(xy + yz + zx) = x^2 + y^2 + 2xy + 2yz + 2zx$

$= (x^2 + y^2 + 2xy) + 2z(y+x)$

$= (x+y)^2 + 2z(y+x) = (x+y)([x+y+2z])$

Example 11. Factorize value of $x(x-2)(x-4) + 4x - 8$ is

(a) $(x-2)^2$ (b) $(x+2)^3$ (c) $(x-2)^3$ (d) $(x+2)^2$

Sol. (c) $x(x-2)(x-4) + 4x - 8 = x(x-2)(x-4) + 4(x-2)$

$= (x-2)([x(x-4)+4]) = (x-2)[x^2 - 4x + 4]$

$= (x-2)(x-2)^2 = (x-2)^3$

4. Factorizing the Difference of Two Squares

Method Here, the polynomial is given as the difference of two square or we will evaluate it to the condition, then apply formula $a^2 - b^2 = (a-b)(a+b)$.

Example 12. Factorize value of $16x^2 - 25y^2$ is

(a) $(4x-5y)^2$ (b) $(4x+5y)^2$
(c) $(4x-5y)(4x+5y)$ (d) None of these

Sol. (c) $16x^2 - 25y^2 = (4x)^2 - (5y)^2 = (4x-5y)(4x+5y)$

Example 13. Factorize value of $2a^5 - 32a$ is

(a) $(a^2+4)(a-2)(a+2)$ (b) $2a(a^2+4)(a-2)(a+2)$
(c) $2a(a^2+4)$ (d) None of these

Sol. (b) $2a^5 - 32a = 2a(a^4 - 16)$

$$= 2a[(a^2)^2 - (4)^2] = 2a(a^2+4)(a^2-4)$$
$$= 2a(a^2+4)(a^2-2^2) = 2a(a^2+4)(a-2)(a+2)$$

Example 14. Factorize value of $x^4 + x^2y^2 + y^4$ is

(a) $(x^2+y^2-xy)(x^2+y^2+xy)$
(b) $(x+y)^2(x-y)^2$
(c) $(x+y-xy)(x+y+xy)$
(d) None of the above

Sol. (a) $x^4 + x^2y^2 + y^4 = x^4 + 2x^2y^2 + y^4 - x^2y^2$

$$= (x^2+y^2)^2 - (xy)^2 = (x^2+y^2+xy)(x^2+y^2-xy)$$

Example 15. Factorize value of $x^4 + x^2 + 1$ is

(a) $(x^2+1+x)(x^2-x+1)$ (b) $(x^2+1+x)(x^2-x-1)$
(c) $(x^2-1+x)(x^2+x+1)$ (d) None of the above

Sol. (a) $x^4 + x^2 + 1 = x^4 + 2x^2 + 1 - x^2$

$$= (x^2+1)^2 - x^2 = (x^2+1+x)(x^2-x+1)$$

5. Factorizing a Trinomial (by splitting the middle term)

Method Here, the general form of trinomial is $ax^2 + bx + c$

Find two number such that their sum is 'b' and product is 'c'.

Then, if numbers are p and q.

Then, $ax^2 + bx + c = ax^2 + (p+q)x + pq$

$$= ax^2 + px + qx + pq$$

Now, factorize by grouping.

Example 16. Factorize value of $x^2 + 9x + 14$ is

(a) $(x+2)(x+7)$ (b) $(x-2)(x-7)$
(c) $(x+2)(x-7)$ (d) $(x-2)(x+7)$

Sol. (a) $x^2 + 9x + 14 = x^2 + (7+2)x + 14$

$$= x^2 + 7x + 2x + 14 = x(x+7) + 2(x+7)$$
$$= (x+7)(x+2)$$

Example 17. Factorize value of $4\sqrt{3}x^2 + 5x - 2\sqrt{3}$ is

(a) $(2x+\sqrt{3})(4x-\sqrt{3})$ (b) $(2x-\sqrt{3})(4x+\sqrt{3})$
(c) $(\sqrt{3}x+2)(4x-\sqrt{3})$ (d) None of these

Sol. (c) $4\sqrt{3}x^2 + 5x - 2\sqrt{3} = 4\sqrt{3}x^2 + 8x - 3x - 2\sqrt{3}$

$$[\because 4\sqrt{3} \times (-2\sqrt{3}) = -24 = 8 \times (-3)]$$
$$= 4x(\sqrt{3}x+2) - \sqrt{3}(\sqrt{3}x+2) = (\sqrt{3}x+2)(4x-\sqrt{3})$$

Example 17. Factorize value of $5(2a+b)^2 + 6(2a+b) - 8$ is

(a) $(2a+b+2)$ (b) $(2a+b+2)[5(2a+b)-4]$
(c) $[5(2a+b)-4]$ (d) None of these

Sol. (b) $5(2a+b)^2 + 6(2a+b) - 8$

Put $(2a+b) = x = 5x^2 + 6x - 8$

$$= 5x^2 + 10x - 4x - 8 = 5x(x+2) - 4(x+2)$$
$$= (x+2)(5x-4)$$

Hence, $5(2a+b)^2 + 6(2a+b) - 8 = (2a+b+2)[5(2a+b)-4]$

6. Factorization of Sum and Difference of Cubes

Method The polynomial will be given in the form of $a^3 + b^3$ or $a^3 - b^3$ or evaluate, then to the form and apply formulae.

1. $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$
2. $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

Example 19. Factorize value of $8a^3 - 343b^3$ is

(a) $(2a-7b)(4a^2+14ab+49b^2)$
(b) $(2a+7b)(4a^2+14ab+49b^2)$
(c) $(2a+7b)$
(d) $(2a-7b)$

Sol. (a) $8a^3 - 343b^3 = (2a)^3 - (7b)^3$

$$= (2a-7b)[(2a)^2 + (2a)(7b) + (7b)^2]$$
$$= (2a-7b)(4a^2 + 14ab + 49b^2)$$

Example 20. Factorize value of $(2x+3y)^3 - (2x-3y)^3$ is

(a) $18(4x^2+3y^2)$ (b) $18y(3x^2+4y^2)$
(c) $xy(4x^2+3y^2)$ (d) $18y(4x^2+3y^2)$

Sol. (d) $(2x+3y)^3 - (2x-3y)^3$

Put $2x+3y = a$

$2x - 3y = b = a^3 - b^3 = (a-b)(a^2+ab+b^2)$

$$= (2x+3y-2x+3y)$$
$$\times [(2x+3y)^2 + (2x+3y)(2x-3y) + (2x-3y)^2]$$
$$= 6y[4x^2 + 9y^2 + 12xy + 4x^2 - 9y^2 + 4x^2 + 9y^2 - 12xy]$$
$$= 6y[12x^2 + 9y^2] = 18y[4x^2 + 3y^2]$$

Hence, $(2x+3y)^3 - (2x-3y)^3 = 18y[4x^2 + 3y^2]$

Example 21. Factorize value of $a^3 - \dfrac{1}{a^3} - 2a + \dfrac{2}{a}$ is

(a) $\left(a+\dfrac{1}{a}\right)\left(a^2+\dfrac{1}{a^2}\right) - 2\left(a-\dfrac{1}{a}\right)$
(b) $\left(a-\dfrac{1}{a}\right)\left(a^2+\dfrac{1}{a^2}+1\right) + 2\left(a-\dfrac{1}{a}\right)$
(c) $\left(a+\dfrac{1}{a}\right)\left(a-\dfrac{1}{a}\right)$
(d) None of the above

Sol. (d) $a^3-\frac{1}{a^3}-2a+\frac{2}{a}=a^3-\left(\frac{1}{a}\right)^3-2\left(a-\frac{1}{a}\right)$

$$=\left(a-\frac{1}{a}\right)\left(a^2+\frac{1}{a^2}+1\right)-2\left(a-\frac{1}{a}\right)$$

$$=\left(a-\frac{1}{a}\right)\left[a^2+\frac{1}{a^2}+1-2\right]=\left(a-\frac{1}{a}\right)\left(a^2+\frac{1}{a^2}-1\right)$$

7. Factorization of $a^3+b^3+c^3-3abc$

Method Here, it is easy to use.

(i) $a^3+b^3+c^3-3abc$

$$=(a+b+c)(a^2+b^2+c^2-ab-bc-ac)$$

(ii) If $a+b+c=0$, then

$$a^3+b^3+c^3=3abc$$

Example 22. Factorize value of

$x^3+27y^3+8z^3-18xyz$ is

(a) $(x+3y+2z)(x^2+9y^2+4z^2+3xy+6yz+2xz)$

(b) $(x+3y+2z)(x^2+9y^2+4z^2-3xy-6yz-2xz)$

(c) $(x-3y-2z)(x^2+9y^2-4z^2-3xy+6yz+2xz)$

(d) None of the above

Sol. (b) $x^3+27y^3+8z^3-18xyz=x^3+(3y)^3+(2z)^3-3(x)(3y)(2z)$

$$=(x+3y+2z)[x^2+9y^2+4z^2-x(3y)-3y(2z)-x(2z)]$$

$$=(x+3y+2z)([x^2+9y^2+4z^2-3xy-6yz-2xz])$$

Example 23. Factorize value of

$(2x-3y)^3+(3y-5z)^3+(5z-2x)^3$ is

(a) $3(2x-3y)(3y-5z)(5z-2x)$

(b) $(2x-3y)(3y-5z)(5z-2x)$

(c) $3(2x-3y)(3y+5z)(5z-2x)$

(d) None of the above

Sol. (a) Here, $2x-3y=a, 3y-5z=b, 5z-2x=c$

So, $a+b+c=0$

Here, $(2x-3y)^3+(3y-5z)^3+(5z-2x)^3$

$$=a^3+b^3+c^3=3abc=3(2x-3y)(3y-5z)(5z-2x)$$

Hence,

$$(2x-3y)^3+(3y-5z)^3+(5z-2x)^3$$
$$=3(2x-3y)(3y-5z)(5z-2x)$$

8. Factorization by Factor Theorem/ Remainder Theorem/ Synthetic Division Method

Remainder Theorem Let $p(x)$ be a polynomial in 'x' of degree not less than one and 'α' be a real number. If $p(x)$ is divided by $(x-\alpha)$, then remainder is $f(\alpha)$. Remainder can be evaluated by substituting $x-\alpha=0$, *i.e.*, $x=\alpha$ in $p(x)$.

Example 24. The remainder when

$12x^3-13x^2-5x+9$ is divided by $(3x+2)$ is

(a) 1 (b) 2 (c) 3 (d) 0

Sol. (c) Let $p(x)=12x^3-13x^2-5x+9$

and $q(x)=3x+2=3\left(x+\frac{2}{3}\right)=3\left[x-\left(\frac{-2}{3}\right)\right]$

When $p(x)$ is divided by $(3x+2)$ the remainder is $p\left(\frac{-2}{3}\right)$.

Now, $p\left(\frac{-2}{3}\right)=12\left(\frac{-2}{3}\right)^3-13\left(\frac{-2}{3}\right)^2-5\left(\frac{-2}{3}\right)+9$

$$=12\times\left(\frac{-8}{27}\right)-13\times\frac{4}{9}+\frac{10}{3}+9=3$$

$\therefore$ The required remainder $=3$

Factor Theorem Let $p(x)$ be a polynomial in x of degree not less than 1 and α be a real number.

If $p(\alpha)=0$, then $(x-\alpha)$ is a factor of $p(x)$.

If $(x-a)$ is a factor of $p(x)$, then $p(a)=0$.

Example 25. Check, if $x-3$ is a factor of $x^3+x^2-17x+15$, then the value of remainder is

(a) 0 (b) 1 (c) 2 (d) 4

Sol. (a) Here, $p(x)=x^3+x^2-17x+15$ and $(x-3)$ will be a factor of $p(x)$, if $p(3)=0$.

So, $p(3)=(3)^3+(3)^2-17\times3+15$

$$=27+9-51+15=51-51=0$$

$\therefore x-3$ is a factor of $p(x)$.

Synthetic Division Method (Shortcut Method)

Horners Method

> This method is to find the quotient and the remainder when a polynomial is divided by a binomial.

Rule for Synthetic Division

Rule 1 First we make the given polynomial $f(x)$ complete by supplying the missing term with zero coefficients.

Rule 2 Write the successive coefficients $a_0, a_1, a_2, \ldots, a_n$ of the polynomial $f(x)$.

Rule 3 If we are to divide the polynomial by $x-h$, then write 'h' is the left corner.

Rule 4 In third row write b_0 below $a_0 (a_0=b_0)$ and first term is obtained by multiplying b_0 (or a_0) by h and product $hb_0=a_1$.

Rule 5 Add hb_0 to a_0, we get b_1, which hb_1 to a_2, we get b_2 which is third term in third row.

Rule 6 Repeat this till you get last term which is remainder R

If $R=0$, then 'h' is the root of the polynomial $f(x)=0$ and the equation can be depressed by one dimension.

Example 26. The remainder, when $f(x)$ is divide by $(x-5)$ is

where

$$f(x)=x^5-4x^4+7x^3-11x-13$$

(a) 289 (b) 432 (c) 1432 (d) 1289

Sol. (c) Here, $f(x)=x^5-4x^4+7x^3-11x-13$

Rule 1 Now, $a_0=1, a_2=-4, a_2=7, a_3=0, a_4=-11, a_5=-13$

Rule 2

	5	1	−4	7	0	−11	−13
Rule 3			5	5	60	300	1445
		$b_0=1$	$b_1=1$	$b_2=12$	$b_3=60$	$b_4=289$	$R=1432$

The quotient $Q=b_0x^4+b_1x^3+b_2x^2+b_3x+b_4$

So, $Q=x^4+x^3+12x^2+60x+289$

and remainder $R=1432$

Example 27. Factorize value of $2x^4+3x^3-2x^2+3x+6$, when divided by $(x+2)$ is

(a) $(x+2)(2x^3-x^2-3)$

(b) $(x-2)(2x^3-x^2+3)$

(c) $(x+2)(2x^3-x^2+3)$

(d) None of the above

Sol. (c) So, here $f(x)=2x^4+3x^3-2x^2+3x+6$ and $x+2$ is divisor

−2	2	3	−2	3	6
		−4	2	0	−6
	2	−1	0	3	0

$\therefore$ $Q=2x^3-x^2+3$

So, $f(x)=(x+2)(2x^3-x^2+3)$

Exercise

1. The factors of $5x^2-20xy$ are
   (a) $5x(x-4y)$ (b) $10x(x-2y)$
   (c) $5(x^2-2y)$ (d) None of these

2. The factors of $5x(y+z)-7y(y+z)$ are
   (a) $(5x-7y)(y-z)$ (b) $(5x-7y)(y+z)$
   (c) $(5x+7y)(y+z)$ (d) $(5x+7y)(y-z)$

3. If $\left(x+\frac{1}{x}\right)=6$, then $\left(x^2+\frac{1}{x^2}\right)$ is equal to
   (a) 32 (b) 38
   (c) 34 (d) 44

4. If $\left(x-\frac{1}{x}\right)=\frac{1}{2}$, then the value of $\left(6x^2+\frac{6}{x^2}\right)$ is
   (a) $\frac{25}{4}$ (b) $\frac{9}{2}$ (c) $\frac{9}{4}$ (d) $\frac{27}{2}$

5. What are the factors of $x^2+4y^2+4y-4xy-2x-8$? **(CDS 2010 II)**
   (a) $(x-2y-4)$ and $(x-2y+2)$
   (b) $(x-y+2)$ and $(x-4y+4)$
   (c) $(x-y+2)$ and $(x-4y-4)$
   (d) $(x+2y-4)$ and $(x+2y+2)$

6. If $(x-3)$ is a factor of $(x^2+4px-11p)$, then what is the value of p? **(CDS 2011 II)**
   (a) −9 (b) −3 (c) −1 (d) 1

7. If $x+\frac{1}{x}=\sqrt{5}$, then the value of $x^3+\frac{1}{x^3}$ is
   (a) $8\sqrt{5}$ (b) $2\sqrt{5}$
   (c) $5\sqrt{5}$ (d) $7\sqrt{5}$

8. The factors $8-4x-2x^3+x^4$ are
   (a) $(2-x)(4-x^3)$ (b) $(2+x)(4-x^3)$
   (c) $(2+x)(3-x^3)$ (d) $(2-x)(x^3-4)$

9. If $a+b=5$ and $ab=6$, then the value of a^3+b^3 is
   (a) 25 (b) 30 (c) 35 (d) 45

10. If $x=5, y=3$ and $z=2$, then the value of $x^2+y^2+z^2-2xy+2yz-2zx$ is
    (a) 125 (b) 0
    (c) −25 (d) 10

11. Which one of the following is the factor of $x^4+xy^3+xz^3+x^3y+y^4+yz^3$? **(CDS 2007 I)**
    (a) $x+y+z$ (b) $x^2+y^2+z^2$
    (c) $x^3+y^3+z^3$ (d) x^2+y^2

12. The factors of $x^2-2\sqrt{3}x+3$ are
    (a) $(x+\sqrt{3})^2$ (b) $(x-\sqrt{3})^2$
    (c) $(x+\sqrt{3})(x-\sqrt{3})$ (d) $(x+2)(x+\sqrt{3})$

13. The factors of $(a^4b^4-16c^4)$ are
    (a) $4(a^2b^2+c^2)(ab-2c)(ab+2c)$
    (b) $(a^2b^2-4c^2)(ab+2c)^2$
    (c) $(a^2b^2+4c^2)(ab+2c)(ab-2c)$
    (d) $(a^2b^2-4c^2)^2(ab+2c)(ab+4c)$

14. If $\left(x+\frac{1}{x}\right)=\sqrt{3}$, then the value of $\left(x^3+\frac{1}{x^3}\right)$ will be
    (a) $3\sqrt{3}$ (b) $3(\sqrt{3}-1)$
    (c) 0 (d) $3(\sqrt{3}+1)$

15. If $pqr=1$, what is the value of the expression $\frac{1}{1+p+q^{-1}}+\frac{1}{1+q+r^{-1}}+\frac{1}{1+r+p^{-1}}$? **(CDS 2007 I)**
    (a) 1 (b) −1
    (c) 0 (d) $\frac{1}{3}$

16. The factors of $(x^8 - y^8)$ are
(a) $(x^4+y^4)(x^2+y^2)(x+y)(x-y)$
(b) $(x^2+y^2)^2(x+y)(x-y)$
(c) $(x^4+y^4)(x^2+y^2)^2(x+y)(x-y)$
(d) $(x^2+y^2)(x-y)^2$

17. The factors of $(a^2 - b^2 - 4ac + 4c^2)$ are
(a) $(a+2c+b)(a-2c-b)$ (b) $(a-2c+b)(a-2c-b)$
(c) $(a-2b+c)(a+b+2c)$ (d) $(a-2b)(a+2b+2c)$

18. Factors of $[(2x-3)^2 - 8x + 12]$ are
(a) $(3x-2)(7x-2)$ (b) $(2x-3)(2x-7)$
(c) $(4x-3)(3x-4)$ (d) $(9x-3)(4x-2)$

19. If $x^5 - 9x^2 + 12x - 14$ is divisible by $(x-3)$, what is the remainder? **(CDS 2011 I)**
(a) 0 (b) 1 (c) 56 (d) 184

20. The factors of $\left(a^2 + a + \frac{1}{4}\right)$ are
(a) $\left(a+\frac{1}{2}\right)^2$ (b) $\left(a+\frac{1}{2}\right)(a+2)$
(c) $\left(a+\frac{1}{2}\right)\left(a-\frac{1}{2}\right)$ (d) $\left(a-\frac{1}{2}\right)^2$

21. What are the factors of $(a^3 - 2\sqrt{2}b^3)$?
(a) $(a-\sqrt{2}b)(a^2-\sqrt{2}ab+2b^2)$
(b) $(a-\sqrt{2}b)(a^2-\sqrt{2}ab-2b^2)$
(c) $(a-\sqrt{2}b)(a^2+\sqrt{2}ab+2b^2)$
(d) $(a+\sqrt{2}b)(a^2-\sqrt{2}ab-2b^2)$

22. If $x = 2+\sqrt{3}$, then what is $(x^2 - x^{-2})$ equal to? **(CDS 2009 II)**
(a) 12 (b) 13 (c) 14 (d) 15

23. The factors of $[(a-b) - a^3 + b^3]$ are
(a) $(a+b)(a^2+b^2+ab+1)$
(b) $(a-b)(a^2+b^2-ab)$
(c) $(a-b)[1-a^2-b^2-ab]$
(d) $(a-b)(a^2+b^2+ab)$

24. The factors of $\left(8x^3 - \frac{1}{27y^3}\right)$ are
(a) $\left(2x-\frac{1}{3y}\right)\left(4x^2-\frac{2x}{3y}+\frac{1}{9y^2}\right)$
(b) $\left(2x+\frac{1}{3y}\right)\left(4x^2-\frac{2x}{3y}-\frac{1}{9y^2}\right)$
(c) $\left(2x-\frac{1}{3y}\right)\left(4x^2+\frac{2x}{3y}+\frac{1}{9y^2}\right)$
(d) $\left(2x+\frac{1}{3y}\right)\left(4x^2+\frac{2x}{3y}+\frac{1}{9y^2}\right)$

25. If $a+b+c=0$, then what is the value of $\frac{a^2}{bc}+\frac{b^2}{ca}+\frac{c^2}{ab}$? **(CDS 2010 II)**
(a) –3 (b) 0 (c) 1 (d) 3

26. The factors of $\left(a^3 - \frac{8}{a^3} - 4a + \frac{8}{a}\right)$ are
(a) $\left(a+\frac{2}{a}\right)\left(a^2+\frac{4}{a^2}-2\right)$ (b) $\left(a+\frac{2}{a}\right)\left(a^2-\frac{4}{a^2}+2\right)$
(c) $\left(a-\frac{2}{a}\right)\left(a^2-\frac{4}{a^2}+2\right)$ (d) $\left(a-\frac{2}{a}\right)\left(a^2+\frac{4}{a^2}-2\right)$

27. The factors of $(9x^2 + 12xy)$ are
(a) $3x(3x+4y)$ (b) $4x(3x+3y)$
(c) $(3x+2y)(3x+6y)$ (d) None of these

28. The factors of $(x^2 - 1 - 2a - a^2)$ are
(a) $(x-a-1)(x+a+1)$
(b) $(x-a+1)(x+a+1)$
(c) $(x+a+1)(x+a-1)$
(d) $(x+a-1)(x-a+1)$

29. If the remainder of the polynomial $a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$ when divided by $(x-1)$ is 1, then which one of the following is correct? **(CDS 2009 I)**
(a) $a_0 + a_2 + \ldots = a_1 + a_3 + \ldots$
(b) $a_0 + a_2 + \ldots = 1 + a_1 + a_3 + \ldots$
(c) $1 + a_0 + a_2 + \ldots = -(a_1 + a_3 + \ldots)$
(d) $1 - a_0 - a_2 - \ldots = a_1 + a_3 + \ldots$

30. What are the factors of $[(5a-7b)^3 + (9c-5a)^3 + (7b-9c)^3]$?
(a) $3(5a-7b)(7b-9c)(9c-5a)$
(b) $3(5a+7b)(7b+9c)(9c+5a)$
(c) $\frac{1}{3}(5a-7b)(7b-9c)(9c-5a)$
(d) None of the above

31. What is the remainder when $(4x^3 - 3x^2 + 2x - 1)$ is divided by $(x+2)$?
(a) 49 (b) 48 (c) –49 (d) –48

32. What is the remainder when $12x^3 - 13x^2 - 5x + 7$ is divided by $(3x+2)$, is
(a) 1 (b) –1 (c) 2 (d) –2

33. If $a+b+c=6$, $a^2+b^2+c^2=26$, then what is the value of $ab+bc+ca$? **(CDS 2011 II)**
(a) 0 (b) 2 (c) 4 (d) 5

34. The factors of $(m^2 + 17mn - 84n^2)$ are
(a) $(m-21n)(m-4n)$ (b) $(m+21n)(m-4n)$
(c) $(m+n)(14m-2n)$ (d) None of these

35. What are the factors of $\left(\frac{1}{3}x^2 - 2x - 9\right)$?
(a) $\frac{1}{3}(x-9)(x+3)$ (b) $\frac{1}{3}(x-9)(x-3)$
(c) $\frac{1}{3}(x+9)(x+3)$ (d) $\frac{1}{3}(x+9)(x-3)$

36. What are the factors of $(6\sqrt{3}x^2 - 47x + 5\sqrt{3})$?
(a) $(3\sqrt{3}x+5\sqrt{3})(2x-5\sqrt{3})$
(b) $(3\sqrt{3}x-5\sqrt{3})(x\sqrt{3}x+1)$
(c) $(2x+5\sqrt{3})(3\sqrt{3}x+1)$
(d) $(2x-5\sqrt{3})(3\sqrt{3}x-1)$

37. Which one of the following statements is correct? **(CDS 2009 II)**
(a) Remainder theorem is a special case of factor theorem
(b) Factor theorem is a special case of remainder theorem
(c) Factor theorem and remainder theorem are two independent results
(d) None of the above

38. The polynomial $f(x) = ax^3 + 9x^2 + 4x - 8$, when divided by $(x + 3)$ leaves the remainder (-20). Then, the value of a is
(a) 1 (b) 2 (c) 3 (d) 5

39. The factors of $(8a^3 + 125b^3 - 64c^3 + 120abc)$ are
(a) $(2a + 5b - 4c)(2a + 5b + 4c)$
(b) $(2a - 5b - 4c)(2a + 5b + 4c)$
(c) $(2a + 5b - 4c)(4a^2 + 25b^2 + 16c^2 - 10ab + 20bc + 8ac)$
(d) $(2a + 5b + 4c)(4a^2 + 25b^2 + 16c^2 - 10ab + 20bc + 8ac)$

40. What is the value of $x(y - z)(y + z) + y(z - x)(z + x) + z(x - y)(x + y)$ equal to? **(CDS 2009 II)**
(a) $(x + y)(y + z)(z + x)$ (b) $(x - y)(x - z)(z - y)$
(c) $(x + y)(z - y)(x - z)$ (d) $(y - x)(z - y)(x - z)$

41. What is the value of the polynomial $r(x)$, so that $f(x) = g(x)q(x) + r(x)$ and $\deg r(x) < \deg g(x)$, where $f(x) = x^2 + 1$ and $g(x) = x + 1$? **(CDS 2008 II)**
(a) 1 (b) -1 (c) 2 (d) -2

42. Which one of the following is one of the factors of $x^2(y - z) + y^2(z - x) - z(xy - yz - zx)$? **(CDS 2007 II)**
(a) $x - y$ (b) $x + y - z$ (c) $x - y - z$ (d) $x + y + z$

43. What must be added to $\frac{1}{x}$ to make it equal to x?
(a) $\frac{x^2 + 1}{x}$ (b) $\frac{x^2 - 1}{x}$ (c) $\frac{x + 1}{x - 1}$ (d) $\frac{x - 1}{x + 1}$

44. The factors of $(2x^3 + 19x^2 + 38x + 21)$ are
(a) $(2x + 3)(x + 5)(x + 1)$ (b) $(4x + 9)(2x + 1)(x + 1)$
(c) $(x + 1)(x + 4)(2x + 1)$ (d) $(x + 1)(x + 7)(2x + 3)$

45. If one of the two factors of an expression which is the difference of two cubes is $(x^4 + x^2y + y^2)$, then what is the other factor? **(CDS 2007 II)**
(a) $x + y$ (b) $x - y$ (c) $x^2 + y$ (d) $x^2 - y$

46. If $a + b + c = 10$ and $ab + bc + ac = 31$, then the value of $a^2 + b^2 + c^2$ is
(a) 48 (b) 38 (c) 28 (d) 18

47. If $a = 5, b = 3$ and $c = 2$, then the value of $a^2 + b^2 + c^2 - 2ab + 2bc - 2ac$ is
(a) 0 (b) 100 (c) 225 (d) 289

48. Let S be a set of all even integers. If the operations
I. addition II. subtraction
III. multiplication IV. division
are applied to any pair of numbers from S, then for which operations is the resulting number in S? **(CDS 2008 I)**
(a) I, II, III and IV (b) I, II and III only
(c) I and III only (d) II and IV only

49. The factors of $(x^4 + x^2 + 25)$ are
(a) $(x^2 + 3x + 5)(x - 2)$ (b) $(x^2 + 5 + 3x)(x^2 + 5 - 3x)$
(c) $(x^2 + x + 5)(x^2 - x + 5)$ (d) $(x^2 + 2x + 3)(x - 2)$

50. The factors of $(8a^3 + b^3 - 6ab + 1)$ are
(a) $(2a - 1 + b)(4a^2 + 1 - 4a - b^2 - 2ab)$
(b) $(2a - b + 1)(4a^2 + b^2 - 4ab + 1 - 2ab)$
(c) $(2a + b - 1)(4a^2 + b^2 + 1 - 3ab - 2a)$
(d) $(2a + b + 1)(4a^2 + b^2 + 1 - 2ab - b - 2a)$

51. The factors of $6(2a + 3b)^2 - 8(2a + 3b)$ are
(a) $2(2a + 3b)(6a + 9b - 4)$ (b) $2(2a + 3b)(6a + 9b + 4)$
(c) $2a(2 + 3b)(6a + 9b - 4)$ (d) $3(2a + 3b)(6a + 9b - 4)$

52. If $(ab - b + 1 = 0)$ and $(bc - c + 1 = 0)$, then what is the value of $(a - ac)$? **(CDS 2008 I)**
(a) -1 (b) 0 (c) 1 (d) 2

53. The factors of $x^2 + \frac{1}{4x^2} + 1 - 2x - \frac{1}{x}$ are
(a) $\left(x + \frac{1}{2x}\right)\left(x - \frac{1}{2x} + 2\right)$ (b) $\left(x + \frac{1}{2x}\right)\left(x + \frac{1}{2x} - 2\right)$
(c) $\left(x - \frac{1}{2x}\right)\left(x - \frac{1}{2x} - 2\right)$ (d) $\left(x + \frac{1}{2x}\right)\left(x - \frac{1}{2x} - 2\right)$

54. $(x^2 + 3x + 5)(x^2 - 3x + 5)$ can be expressed as a difference of squares which are
(a) $(x^2 + 5)^2 - (3x)^2$ (b) $(x^2 + 5)^2 + (3x)^2$
(c) $(x^2 - 5)^2 - (3x)^2$ (d) $(x^2 + 5)^2 + (6x)^2$

55. The factors of $(a^2 - b^2)(c^2 - d^2) - 4abcd$ are
(a) $(ac + bd + bc + ad)(ac - bd - bc - ad)$
(b) $(ac - bd - bc + ad)(ac + bd + dc + ab)$
(c) $(ac - bd + bc + ad)(ac - bd - bc - ad)$
(d) $(ac - bd - bc - ad)(ac + bd + dc + ab)$

56. Suppose, $p * q = 2p + 2q - pq$, where p and q are natural numbers. If $8 * x = 4$, then what is the value of x? **(CDS 2008 II)**
(a) 1 (b) 2 (c) 3 (d) 4

57. The value of $a^3 - 2\sqrt{2}\, b^3$ is
(a) $(a + \sqrt{2}\, b)(a^2 - \sqrt{2}\, ab + 2b^2)$
(b) $(a - \sqrt{2}\, b)(a^2 + \sqrt{2}\, ab + 2b^2)$
(c) $(a - \sqrt{2}\, b)(a^2 - \sqrt{2}\, ab + 2b^2)$
(d) $(a - \sqrt{2}\, b)(a^2 + 2ab + b^2)$

58. The remainder when $4a^3 - 12a^2 + 14a - 3$ is divided by $(2a - 1)$, is
(a) $\frac{5}{2}$ (b) $\frac{3}{2}$ (c) 0 (d) $\frac{-1}{2}$

59. If $(x/y) = (z/w)$, then what is the value of $(xy + zw)^2$? **(CDS 2009 II)**
(a) $(x^2 + z^2)(y^2 + w^2)$ (b) $x^2y^2 + z^2w^2$
(c) $x^2w^2 + y^2z^2$ (d) $(x^2 + w^2)(y^2 + z^2)$

60. The factors of $4x^3 + 23x^2 - 41x - 42$ are
(a) $(x + 7)(x - 7)(3x + 4)$ (b) $(x - 2)(x + 7)(4x + 3)$
(c) $(x + 7)(x - 2)(3x + 4)$ (d) $(x + 7)(4x + 3)(x + 2)$

61. If (x^3+ax^2+bx+6) has $(x-2)$ as a factor and leaves a remainder 3 when divided by $(x-3)$, then the values of 'a' and 'b' are
(a) $a=-3, b=-1$ (b) $a=3, b=1$
(c) $a=2, b=1$ (d) $a=-3, b=1$

62. The value of
$(a^{1/8}+a^{-1/8})(a^{1/8}-a^{-1/8})(a^{1/4}+a^{-1/4})(a^{1/2}+a^{-1/2})$ is
(a) $(a+a^{-1})$ (b) $(a-a^{-1})$ (c) (a^2+a^{-2}) (d) (a^2-a^{-2})

63. What must be subtracted from $(x^3+4x^2-6x-14)$ to obtain a polynomial exactly divisible by (x^2-x-2)?
(a) $x+4$ (b) $2x+4$ (c) $x-4$ (d) $x+8$

64. What must be added to $(x^4+2x^3-2x^2-2x-1)$ to obtain a polynomial which is exactly divisible by (x^2+2x-3)?
(a) $4x-2$ (b) $4x+2$ (c) $2x+4$ (d) $2x-4$

65. If m and n are two integers such that $m=n^2-n$, then (m^2-2m) is always divisible by **(CDS 2009 II)**
(a) 9 (b) 16 (c) 24 (d) 48

66. What must be added to $(x^3-3x^2+4x-15)$ to obtain a polynomial, which is exactly divisible by $(x-3)$?
(a) 3 (b) -8 (c) 14 (d) -4

67. The polynomial $f(x)=x^4-2x^3+3x^2-ax+b$, when divided by $(x-1)$ and $(x+1)$ leaves the remainders 5 and 19, respectively. The values of a and b are
(a) $a=4, b=3$ (b) $a=3, b=4$
(c) $a=5, b=8$ (d) $a=9, b=7$

68. What is the remainder when $(x^{11}+1)$ is divided by $(x+1)$? **(CDS 2011 II)**
(a) 0 (b) 2 (c) 11 (d) 12

69. If 'a' is an integer such that $a+\frac{1}{a}=\frac{17}{4}$, then the value of $\left(a-\frac{1}{a}\right)$ is
(a) 4 (b) $\frac{13}{4}$ (c) $\frac{17}{4}$ (d) $\frac{15}{4}$

70. If $x+y+z=0$, then what is the value of $\frac{xyz}{(x+y)(y+z)(z+x)}$?
$[x\neq -y, y\neq -z, z\neq -x]$ **(CDS 2011 II)**
(a) -1 (b) 1
(c) $xy+yz+zx$ (d) None of these

Answers

1. (a)	2. (b)	3. (c)	4. (d)	5. (a)	6. (a)	7. (b)	8. (a)	9. (c)	10. (b)
11. (c)	12. (b)	13. (c)	14. (c)	15. (a)	16. (a)	17. (b)	18. (b)	19. (d)	20. (a)
21. (c)	22. (c)	23. (c)	24. (c)	25. (d)	26. (d)	27. (a)	28. (a)	29. (d)	30. (a)
31. (c)	32. (a)	33. (d)	34. (b)	35. (a)	36. (d)	37. (b)	38. (c)	39. (c)	40. (b)
41. (c)	42. (c)	43. (b)	44. (d)	45. (d)	46. (b)	47. (a)	48. (b)	49. (b)	50. (d)
51. (a)	52. (c)	53. (b)	54. (a)	55. (c)	56. (b)	57. (b)	58. (b)	59. (a)	60. (b)
61. (a)	62. (b)	63. (c)	64. (a)	65. (c)	66. (a)	67. (c)	68. (a)	69. (d)	70. (a)

Hints and Solutions

1. $5x^2-20xy=5x(x-4y)$

2. $5x(y+z)-7y(y+z)=(5x-7y)(y+z)$

3. $\left(x+\frac{1}{x}\right)=6$

On squaring, we get
$$x^2+\frac{1}{x^2}+2=36 \Rightarrow x^2+\frac{1}{x^2}=34$$

4. $x-\frac{1}{x}=\frac{1}{2}$

On squaring, we get
$$x^2+\frac{1}{x^2}-2=\frac{1}{4}$$
$$x^2+\frac{1}{x^2}=\frac{1}{4}+2=\frac{9}{4} \Rightarrow 6\left(x^2+\frac{1}{x^2}\right)=6\times\frac{9}{4}=\frac{27}{2}$$

5. $x^2+4y^2+4y-4xy-2x-8=(x-2y)^2-2(x-2y)-8$
$$=(x-2y)^2-4(x-2y)+2(x-2y)-8$$
$$=(x-2y-4)(x-2y+2)$$

6. Let $f(x)=x^2+4px-11p$

Since, $(x-3)$ is a factor of $f(x)$.
$\therefore$ $f(3)=0$
$\Rightarrow$ $(3)^2+4p(3)-11p=0 \Rightarrow p=-9$

7. $$\left(x+\frac{1}{x}\right)^3=x^3+\frac{1}{x^3}+3\left(x+\frac{1}{x}\right)$$
$$(\sqrt{5})^3=x^3+\frac{1}{x^3}+3(\sqrt{5}) \qquad \left(\because x+\frac{1}{x}=\sqrt{5}\right)$$
$\Rightarrow$ $$x^3+\frac{1}{x^3}=5\sqrt{5}-3\sqrt{5}=2\sqrt{5}$$

8. $8-4x-2x^3+x^4=4(2-x)-x^3(2-x)=(2-x)(4-x^3)$

9. $a^3+b^3=(a+b)^3-3ab(a+b)$
$$=(5)^3-3(6)(5) \qquad (\therefore a+b=5, ab=6)$$
$$=125-90=35$$

10. $x^2+y^2+z^2-2xy+2yz-2zx$ $\quad (\because x=5, y=3, z=2)$

$=(x-y-z)^2=(5-3-2)^2=0$

11. $x^4+xy^3+xz^3+x^3y+y^4+yz^3=x(x^3+y^3+z^3)+y(x^3+y^3+z^3)$

$=(x+y)(x^3+y^3+z^3)$

Clearly, $(x^3+y^3+z^3)$ is a factor of

$x^4+xy^3+xz^3+x^3y+y^4+yz^3$

12. $x^2-2\sqrt{3}\,x+3=x^2-\sqrt{3}\,x-\sqrt{3}\,x+3$

$=x($[redacted]$\sqrt{3})-\sqrt{3}(x-\sqrt{3})=(x-\sqrt{3})(x-\sqrt{3})=(x-\sqrt{3})^2$

13. $(a^4b^4-16c^4)$

$=[(a^2b^2)^2-(4c^2)^2]$ $\quad [\because a^2-b^2=(a-b)(a+b)]$

$=[a^2b^2+4c^2][a^2b^2-4c^2]=[a^2b^2+4c^2][(ab)^2-(2c)^2]$

$=[a^2b^2+4c^2][(ab+2c)(ab-2c)]$

14. $x^3+\frac{1}{x^3}=\left(x+\frac{1}{x}\right)^3-3x\cdot\frac{1}{x}\left(x+\frac{1}{x}\right)$

$=(\sqrt{3})^3-3(\sqrt{3})$ $\quad \left(\because x+\frac{1}{x}=\sqrt{3}\right)$

$=3\sqrt{3}-3\sqrt{3}=0$

15. $\frac{1}{1+p+q^{-1}}+\frac{1}{1+q+r^{-1}}+\frac{1}{1+r+p^{-1}}$

$=\frac{1}{1+p+\frac{1}{q}}+\frac{1}{1+q+\frac{1}{r}}+\frac{1}{1+r+\frac{1}{p}}$

$=\frac{q}{1+pq+q}+\frac{r}{r+rq+1}+\frac{p}{p+rp+1}$

$=\frac{q}{1+pq+q}+\frac{r}{\frac{1}{pq}+\frac{1}{p}+1}+\frac{p}{p+\frac{1}{q}+1}$ $\quad (\because pqr=1)$

$=\frac{q}{1+pq+q}+\frac{rpq}{1+q+pq}+\frac{pq}{pq+1+q}$

$=\frac{q+rpq+pq}{1+pq+q}$ $\quad (\because pqr=1)$

$=\frac{q+1+pq}{1+pq+q}=1$

16. $x^8-y^8=[(x^4)^2-(y^4)^2]=[x^4+y^4][x^4-y^4]$

$=[x^4+y^4][(x^2)^2-(y^2)^2]=(x^4+y^4)(x^2+y^2)(x^2-y^2)$

$=(x^4+y^4)(x^2+y^2)(x+y)(x-y)$

17. $(a^2-b^2-4ac+4c^2)=(a^2-4ac+4c^2)-b^2$

$=(a^2-2ac-2ac+4c^2)-(b^2)$

$=(a-2c)^2-b^2=(a-2c-b)(a-2c+b)$

18. $(2x-3)^2-8x+12=(2x-3)^2-4(2x-3)$

$=(2x-3)[(2x-3)-4]=(2x-3)(2x-7)$

19. On putting $x=3$ in $x^5-9x^2+12x-14$, we get

Remainder $=(3)^5-9(3)^2+12\times3-14$

$=243-81+36-14=184$

20. $\left(a^2+a+\frac{1}{4}\right)=a^2+\frac{1}{2}a+\frac{1}{2}a+\frac{1}{4}$

$=a\left(a+\frac{1}{2}\right)+\frac{1}{2}\left(a+\frac{1}{2}\right)=\left(a+\frac{1}{2}\right)\left(a+\frac{1}{2}\right)=\left(a+\frac{1}{2}\right)^2$

21. $a^3-2\sqrt{2}\,b^3=a^3-(\sqrt{2}\,b)^3=(a-\sqrt{2}b)(a^2+\sqrt{2}\,ab+2b^2)$

$[\because (a^3-b^3)=(a-b)(a^2+ab+b^2)]$

22. $\therefore x^2+\frac{1}{x^2}=\left(x+\frac{1}{x}\right)^2-2$

$=\left(2+\sqrt{3}+\frac{1}{2+\sqrt{3}}\right)^2-2$ $\quad (\because x=2+\sqrt{3})$

$=\left(2+\sqrt{3}+\frac{2-\sqrt{3}}{1}\right)^2-2=16-2=14$

23. $[a-b-a^3+b^3]=[(a-b)-(a^3-b^3)]$

$=[(a-b)-(a-b)(a^2+ab+b^2)]$

$=(a-b)[1-(a^2+ab+b^2)]=(a-b)(1-a^2-b^2-ab)$

24. $8x^3-\frac{1}{27y^3}=(2x)^3-\left(\frac{1}{3y}\right)^3=\left(2x-\frac{1}{3y}\right)\left((2x)^2+2x\cdot\frac{1}{3y}+\left(\frac{1}{3y}\right)^2\right)$

$=\left(2x-\frac{1}{3y}\right)\left(4x^2+\frac{2x}{3y}+\frac{1}{9y^2}\right)$

25. Given, $a+b+c=0$

$\Rightarrow$ $a^3+b^3+c^3=3abc$

$\Rightarrow$ $\frac{a^3}{abc}+\frac{b^3}{abc}+\frac{c^3}{abc}=3\Rightarrow\frac{a^2}{bc}+\frac{b^2}{ac}+\frac{c^2}{ab}=3$

26. $a^3-\frac{8}{a^3}-4a+\frac{8}{a}=a^3-\frac{8}{a^3}-4\left(a-\frac{2}{a}\right)$

$=\left[(a)^3-\left(\frac{2}{a}\right)^3\right]-4\left(a-\frac{2}{a}\right)$

$=\left(a-\frac{2}{a}\right)\left(a^2+2+\frac{4}{a^2}\right)-4\left(a-\frac{2}{a}\right)$

$=\left(a-\frac{2}{a}\right)\left(a^2+\frac{4}{a^2}-4+2\right)=\left(a-\frac{2}{a}\right)\left(a^2+\frac{4}{a^2}-2\right)$

27. $9x^2+12xy=3x(3x+4y)$

28. $x^2-1-2a-a^2=x^2-(1+2a+a^2)$

$=x^2-(a+1)^2=(x-a-1)(x+a+1)$

29. Let $f(x)=a_0+a_1x+a_2x^2+\ldots+a_nx^n$

$\therefore$ $f(1)=a_0+a_1+a_2+\ldots+a_n$

$\Rightarrow$ $1=a_0+a_1+a_2+\ldots+a_n$

$\Rightarrow$ $1-a_0-a_2-\ldots=a_1+a_3+\ldots$

30. As $(5a-7b)+(9c-5a)+(7b-9c)=0$

$\because$ $a^3+b^3+c^3=3abc$

If $a+b+c=0$

So, $(5a-7b)^3+(9c-5a)^3+(7b-9c)^3$

$=3(5a-7b)(7b-9c)(9c-5a)$

31. Let $f(x)=4x^3-3x^2+2x-1$

Put $x=-2$

$f(-2)=4(-2)^3-3(-2)^2+2(-2)-1$

$=-32-12-4-1=-49$

32. Here, $f(x)=12x^3-13x^2-5x+7$

Put $3x+2=0 \Rightarrow x=\frac{-2}{3}$

$$f\left(-\frac{2}{3}\right)=12\left(\frac{-2}{3}\right)^3-13\left(\frac{-2}{3}\right)^2-5\left(\frac{-2}{3}\right)+7$$

$$=\frac{-96}{27}-\frac{52}{9}+\frac{10}{3}+7=\frac{-32}{9}-\frac{52}{9}+\frac{10}{3}+7$$

$$=\frac{-32-52+30+63}{9}=\frac{-84+93}{9}=\frac{9}{9}=1$$

33. $\because$ $(a+b+c)^2=a^2+b^2+c^2+2(ab+bc+ca)$

$\therefore$ $(6)^2=26+2(ab+bc+ca)$

$\Rightarrow$ $2(ab+bc+ca)=10 \Rightarrow ab+bc+ca=5$

34. $m^2+17mn-84n^2=m^2+21mn-4mn-84n^2$

$=m(m+21n)-4n(m+21n)$

$=(m+21n)(m-4n)$

35. $\left(\frac{1}{3}x^2-2x-9\right)=\frac{(x^2-6x-27)}{3}=\frac{1}{3}[x^2-9x+3x-27]$

$=\frac{1}{3}[x(x-9)+3(x-9)]=\frac{1}{3}(x+3)(x-9)$

36. $6\sqrt{3}\,x^2-47x+5\sqrt{3}=6\sqrt{3}\,x^2-45x-2x+5\sqrt{3}$

$=3\sqrt{3}\,x(2x-5\sqrt{3})-1(2x-5\sqrt{3})$

$=(2x-5\sqrt{3})(3\sqrt{3}\,x-1)$

37. Factor theorem is a special case of remainder theorem.

38. Here, $f(x)=ax^3+9x^2+4x-8$

So, put $x=-3$

$f(-3)=a(-3)^3+9(-3)^2+4(-3)-8=-20$

$=-27a+81-12-8=-20$ (given)

$-27a=-81$

$a=\frac{81}{27}=3$

$a=3$

39. $8a^3+125b^3-64c^3+120abc=(2a)^3+(5b)^3-(4c)^3+120abc$

$=(2a)^3+(5b)^3+(-4c)^3-3(2a)(5b)(-4c)$

$=(2a+5b-4c)(4a^2+25b^2+16c^2-10ab+20bc+8ac)$

$[\because a^3+b^3+c^3-3abc=(a+b+c)(a^2+b^2+c^2-ab-bc-ac)]$

40. $x(y-z)(y+z)+y(z-x)(z+x)+z(x-y)(x+y)$

$=x(y^2-z^2)+y(z^2-x^2)+z(x^2-y^2)$

$=x(y^2-z^2)+yz^2-yx^2+zx^2-zy^2$

$=x(y-z)(y+z)+x^2(z-y)+yz(z-y)$

$=(y-z)(xy+xz-x^2-yz)$

$=(y-z)[y(x-z)+x(z-x)]$

$=(y-z)(z-x)(x-y)$

$=(x-y)(x-z)(z-y)$

41. Now,

$$\begin{array}{r|l} & x-1 \\ \hline x+1 & x^2+1 \\ & x^2+x \\ & -\;- \\ \hline & -x+1 \\ & -x-1 \\ & +\;+ \\ \hline & 2 \end{array}$$

Hence, $r(x)=2$

42. $x^2(y-z)+y^2(z-x)-z(xy-yz-zx)$

$=x^2y-x^2z+y^2z-y^2x-zxy+yz^2+z^2x$

$=xy(x-y-z)-z(x^2-y^2)+z^2(x+y)$

$=xy(x-y-z)-z(x+y)[x-y-z]$

$=(x-y-z)(xy-yz-zx)$

43. Let a be added, then

$a+\frac{1}{x}=x \Rightarrow a=x-\frac{1}{x}$

$a=\frac{x^2-1}{x}$

44. As $f(x)=2x^3+19x^2+38x+21$

As $x=-1$ satisfies the equation, so $(x+1)$ is a factor of $f(x)$.
Now, by synthetic division method

−1	2	19	38	21
		−2	−17	−21
	2	17	21	0

$\Rightarrow 2x^3+19x^2+38x+21=(x+1)(2x^2+17x+21)$

$=(x+1)[2x^2+14x+3x+21]$

$=(x+1)[2x(x+7)+3(x+7)]$

$=(x+1)(x+7)(2x+3)$

45. Now, $(x^2)^3-y^3=(x^2-y)(x^4+y^2+x^2y)$

Hence, other factor is (x^2-y).

46. $(a+b+c)^2=a^2+b^2+c^2+2ab+2bc+2ac$

$(10)^2=(a^2+b^2+c^2)+2(31)$

$a^2+b^2+c^2=100-62 \Rightarrow a^2+b^2+c^2=38$

47. As $a^2+b^2+c^2-2ab-2ac+2bc=(a-b-c)^2=(5-3-2)^2=0$

48. Let $S=\{\ldots,-6,-4,-2,0,2,4,6,\ldots\}$

I. Now, $2+(-2)=0\in S$, it is applied.

II. Now, $-2(2)=-4\in S$, it is applied.

III. Now, $-4x(4)=-16\in S$, it is applied.

IV. Now, $-4\div 4=-1\notin S$, it is not applied.

49. $x^4+x^2+25=(x^4+25)+x^2$

$=(x^4+25+10x^2)-10x^2+x^2=(x^4+25+10x^2)-9x^2$

$=(x^2+5)^2-(3x)^2$ $[\because a^2-b^2=(a+b)(a-b)]$

$=(x^2+5-3x)(x^2+5+3x)$

50. $8a^3+b^3-6ab+1=(2a)^3+b^3+1^3-3(2a)(b)(1)$

$=(2a+b+1)(4a^2+b^2+1-2ab-b-2a)$

$\because a^3+b^3+c^3-3abc=(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$

51. $6(2a+3b)^2-8(2a+3b)$

$$=2(2a+3b)[3(2a+3b)-4]$$
$$=2(2a+3b)([6a+9b-4])$$

52. Given, $ab-b+1=0 \Rightarrow b(a-1)=-1$

$\Rightarrow \quad b=\dfrac{1}{1-a}$ ...(i)

Also, $bc-c+1=0$

$\Rightarrow \quad b=\dfrac{-1+c}{c}$ ...(ii)

From Eqs. (i) and (ii), we get

$$\frac{1}{1-a}=\frac{-1+c}{c}$$

$\Rightarrow \quad c=(1-a)(-1+c)$

$\Rightarrow \quad c=-1+c+a-ac \Rightarrow a-ac=1$

53. $x^2+\dfrac{1}{4x^2}+1-2x-\dfrac{1}{x}=\left(x^2+\dfrac{1}{4x^2}+1\right)-2\left(x+\dfrac{1}{2x}\right)$

$$=\left(x+\frac{1}{2x}\right)^2-2\left(x+\frac{1}{2x}\right)=\left(x+\frac{1}{2x}\right)\left[x+\frac{1}{2x}-2\right]$$

54. $(x^2+3x+5)(x^2-3x+5)=[(x^2+5)+3x][(x^2+5)-3x]$

$$=(x^2+5)^2-(3x)^2 \quad [\because (a-b)(a+b)=a^2-b^2]$$

55. $(a^2-b^2)(c^2-d^2)-4abcd$

$$=a^2c^2-a^2d^2-b^2c^2+b^2d^2-4abcd$$
$$=(a^2c^2+b^2d^2-2abcd)-(b^2c^2+a^2d^2+2abcd)$$
$$=(ac-bd)^2-(bc+ad)^2$$
$$=(ac-bd+bc+ad)(ac-bd-bc-ad)$$

56. Given, $p * q=2p+2q-pq$

$\because \quad 8*x=4$

$\Rightarrow \quad 2(8)+2(x)-8x=4$

$-6x=-12 \Rightarrow x=2$

57. $a^3-2\sqrt{2}b^3=a^3-(\sqrt{2}b)^3=(a-\sqrt{2}b)(a^2+\sqrt{2}ab+2b^2)$

58. Here, $f(a)=4a^3-12a^2+14a-3$ and $g(a)=2a-1$

Put $\quad g(a)=0 \Rightarrow a=\dfrac{1}{2}$

Put the value of a in $f(a)$, we get

$$f\left(\frac{1}{2}\right)=4\left(\frac{1}{2}\right)^3-12\times\left(\frac{1}{2}\right)^2+14\times\frac{1}{2}-3$$
$$=4\times\frac{1}{8}-12\times\frac{1}{4}+7-3=\frac{1}{2}-3+7-3=\frac{3}{2}$$

59. Given, $\dfrac{x}{y}=\dfrac{z}{w} \Rightarrow xw=yz$ ...(i)

Now, $(xy+zw)^2=x^2y^2+z^2w^2+2(xy\cdot zw)$

$=x^2y^2+z^2w^2+2(yz\cdot yz)$ [from Eq. (i)]

$=x^2y^2+y^2z^2+z^2w^2+y^2z^2$ [from Eq. (i)]

$=y^2(x^2+z^2)+z^2w^2+x^2w^2$ [from Eq. (i)]

$=y^2(x^2+z^2)+w^2(x^2+z^2)$

$=(x^2+z^2)(y^2+w^2)$

60. As for $x=2$, $4x^3+23x^2-41x-42$ eliminates, so $(x-2)$ is a factor of it.

By synthetic division method

2	4	23	−41	−42
		8	62	42
	4	31	21	0

$\therefore 4x^3+23x^2-41x-42=(x-2)(4x^2+31x+21)$

$$=(x-2)[4x^2+28x+3x+21]$$
$$=(x-2)[4x(x+7)+3(x+7)]$$
$$=(x-2)(x+7)(4x+3)$$

61. As $(x-2)$ is factor, so $2^3+a(2)^2+b(2)+6=0$

$4a+2b=-14 \Rightarrow 2a+b=-7$ ...(i)

and $(x-3)$ leaves remainder 3, so

$(3)^3+a(3)^2+b(3)+6=3$

$9a+3b=-30 \Rightarrow 3a+b=-10$ ...(ii)

On solving Eqs. (i) and (ii), we get $a=-3$ and $b=-1$

62. As $(a+b)(a-b)=a^2-b^2$

So, $[(a^{1/8}+a^{-1/8})(a^{1/8}-a^{-1/8})(a^{1/4}+a^{-1/4})(a^{1/2}+a^{-1/2})]$

$$=[(a^{1/4}-a^{-1/4})(a^{1/4}+a^{1/4})](a^{1/2}+a^{-1/2})$$
$$=[a^{1/2}-a^{-1/2}][a^{1/2}+a^{-1/2}]=(a-a^{-1})$$

63.

$$\begin{array}{r|l} & x+5 \\ x^2-x-2 & x^3+4x^2-6x-14 \\ & x^3-x^2-2x \\ & -\ \ +\ \ + \\ \hline & 5x^2-4x-14 \\ & 5x^2-5x-10 \\ & -\ \ +\ \ + \\ \hline & x-4 \end{array}$$

Hence, polynomial to be subtracted is $(x-4)$.

64. Let $\quad f(x)=x^2+2x-3=(x+3)(x-1)$

$g(x)=x^4+2x^3-2x^2-2x-1$

when $g(x)$ is divided by a quadratic polynomial $f(x)$, it will leave a remainder, which is a polynomial of degree less than 2.

Let $(ax+b)$ to be added to $g(x)$ to obtain $q(x)$ as a factor.

$\therefore \quad q(x)=g(x)+ax+b=(x^4+2x^3-2x^2-2x-1)+(ax+b)$

So, $q(x)$ will be exactly divisible by $(x+3)$ and $(x-1)$.

$\therefore \quad q(-3)=[(-3)^4+2(-3^3)-2(-3^2)-2(-3)-1$
$+a(-3)+b]=0$

$\Rightarrow \quad b-3a=-14$ ...(i)

$q(1)=0$

$[1^4+2(1)^3-2\times1^2-2\times1-1]+a+b=0$

$\Rightarrow \quad b+a=2$ ...(ii)

On solving Eqs. (i) and (ii), we get $a=4$ and $b=-2$

$\therefore$ The polynomial to be added $=(4x-2)$

65. Given, $\quad m=n^2-n$

$\therefore \quad m^2-2m=(n^2-n)^2-2(n^2-n)$

$$= n(n-1)(n^2-n-2)$$
$$= (n+1)n(n-1)(n-2)$$

It is a product of consecutive number.
Hence, it is divisible by 24 (4!).

66. Let $f(x) = x^3 - 3x^2 + 4x - 15$

Let m to be added, then

$\therefore \quad g(x) = f(x) + m$
$$g(x) = x^3 - 3x^2 + 4x - 15 + m$$
$$g(3) = 0$$
$$g(3) = 3^3 - 3\times 3^2 + 4(3) - 15 + m = 0$$
$$m - 3 = 0 \Rightarrow m = 3$$

67. $f(x) = x^4 - 2x^3 + 3x^2 - ax + b$

Put $x = 1$ as $f(1) = 5$, $f(-1) = 19$
$$f(1) = 1^4 - 2(1)^3 + 3(1)^2 - a + b$$
$$1 - 2 + 3 - a + b = 5 \Rightarrow -a + b = 3 \quad \ldots(\text{i})$$
$$f(-1) = (-1)^4 - 2(-1)^3 + 3(-1)^2 - a(-1) + b$$
$$= 1 + 2 + 3 + a + b = 19 \Rightarrow a + b = 13 \quad \ldots(\text{ii})$$

On solving Eqs. (i) and (ii), we get $2b = 16 \Rightarrow b = 8$
Put $b = 8$ in Eq. (i), we get
$$-a + 8 = 3$$
$$-a = 3 - 8 \Rightarrow -a = -5 \Rightarrow a = 5$$

68. Let $f(x) = x^{11} + 1$

Put $x = -1$, we get
$$f(-1) = (-1)^{11} + 1 = -1 + 1 = 0$$

69. $\because \quad \left(a - \frac{1}{a}\right) = \sqrt{\left(a + \frac{1}{a}\right)^2 - 4}$
$$= \sqrt{\left(\frac{17}{4}\right)^2 - 4} \quad \left[\because \text{Given } \left(a + \frac{1}{a}\right) = \frac{17}{4}\right]$$
$$= \sqrt{\frac{289 - 64}{16}} = \sqrt{\frac{225}{16}} = \frac{15}{4}$$

70. Given, $x + y + z = 0$

$\therefore \quad \frac{xyz}{(x+y)(y+z)(z+x)} = \frac{xyz}{(-z)(-x)(-y)} = \frac{xyz}{-xyz} = -1$