THE STRAIGHT LINE

The general form of the equation of a straight line is $ax + by + c = 0$ (the first degree equation in x and y) where a, b and c are real constants not all equal to zero simultaneously.

The line parallel to axes:

(i) The equation of a straight line $||^l$ to x-axis and at a distance b from it is given by $y = b$.

(ii) The equation of the straight line $||^l$ to y-axis and at a distance a from it is given by $x = a$.

(iii) The equation of the straight line $||^l$ to x-axis and y-axis respectively and passing through the pt. (a, b) are $y = b$, $x = a$.

(iv) Equation of x-axis is $y = 0$ and equation of y-axis is $x = 0$.

Slope intercept form: The equation of a straight line which cuts off an intercept c on y-axis and

makes an angle θ with the positive direction of x-axis is given by $y = mx + c$, where $m = \tan\theta$.
If θ is acute, then $\tan\theta$ *i.e.*, m is positive.
If θ is obtuse, then $\tan\theta$ *i.e.*, m is negative.
The number $m = \tan\theta$ is called the slope or gradient of line obviously $y = mx$ is the equation of the line through the origin.

Note: *(i)* If a line is parallel to x-axis $\theta = 0$ and its slope $m = \tan\theta = 0$

(ii) If a line parallel to y-axis it is perpendicular to x-axis so that $\theta = 90°$ and its slope $m = \tan 90° = \infty$

Condition of parallelism and perpendicularity of lines

Case I: Slopes of parallel lines are equal.
i.e., $m_1 = m_2$

Case II: The product of the slope of two perpendicular lines is –1.
i.e., $m_1 m_2 = -1$

Slope point form: The equation of a straight line passing through the point A (x_1, y_1) and having slope m is given by

$$y - y_1 = m(x - x_1)$$

This equation can also be written as

$$\frac{x-x_1}{\cos\theta} = \frac{y-y_1}{\sin\theta} = r \text{ (say)} \qquad [\because\ m = \tan\theta]$$

This form of the line is called parametric form or symmetrical form of the line.

The coordinates of any point P on this line are taken as $(x_1 + r\cos\theta, y_1 + r\sin\theta)$

$\therefore$ Distance of a point P from A (x_1, y_1)

$$= \sqrt{\left[(x_1 + r\cos\theta - x_1)^2 + (y_1 + r\sin\theta - y_1)^2\right]} = r$$

Thus, if the equation of a line through point A (x_1, y_1) is written in the above form, in which the coefficients of x and y are equal to 1 and the sum of squares of the terms in the denominators is equal to 1, then the coordinates of a point P on this line at a distance r from A (x_1, y_1) are

$$(x_1 + r_1\cos\theta, y_1 + r\sin\theta)$$

Two Point Form: The equation of a straight line passing through two points A (x_1, y_1) and B (x_2, y_2) is given by

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1).$$

Here, slope $(m) = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{\text{difference of ordinates}}{\text{difference of abscissae}}$

Intercept form: The equation of a straight line making intercepts a and b on the axes of x and y respectively is given by

$$\frac{x}{a}+\frac{y}{b}=1$$

Normal (or perpendicular) form: The equation of a straight line on which the perpendicular from origin is of length p and makes an angle α with the positive direction of x-axis is given by

$$x \cos \alpha + y \sin \alpha = p$$

Remember:

(i) If $\alpha = 0°$, the equation becomes
$x \cos 0° + y \sin 0° = p$ or, $x = p$
which is a line parallel to y-axis

(ii) If $\alpha = \pi/2$, the equation becomes
$x \cos \pi/2 + y \sin \pi/2 = p$
or, $y = p$
Which is a line parallel to x-axis.

(iii) If $\alpha = 0°$, $p = 0$, the equation becomes
$x \cos 0° + y \sin 0° = 0$
or, $x = 0$
which is the equation of y-axis

(iv) If $\alpha = \pi/2$, $p = 0$, the equation becomes
$x \cos \pi/2 + y \sin \pi/2 = 0$

or, $y = 0$

which is the equation of x-axis.

General Equation of a straight line: The equation of the first degree in x and y represent a straight line *i.e.*,

$$Ax + By + C = 0$$

where A, B, C $\in$ R and A, B are not zero simultaneously.

Reduction of the General Equation of the straight line to standard forms:

(*a*) Reduction of general equation of a straight line to the slope intercept form is

$$y = -\frac{A}{B}x - \frac{C}{B}$$

$$\text{Here, slope} = -\frac{A}{B} = -\frac{\text{coeff. of } x}{\text{coeff. of } y}$$

(*b*) Reduction of general equation of a straight line to the intercept form is

$$\frac{x}{\frac{-C}{A}} + \frac{y}{\frac{-C}{B}} = 1$$

(*c*) Reduction of the general equation of a straight line to the normal (or perpendicular) form $p^2 (A^2 + B^2) = C^2$

$$\therefore \qquad p = \pm \frac{C}{\sqrt{A^2 + B^2}}$$

But p is always positive.

Remember:

To reduce the general equation of a line to the normal form:

(i) Transpose the constant term to R.H.S. and make it positive. If the constant term is negative after transposition multiply both sides by (–1) to make it positive.

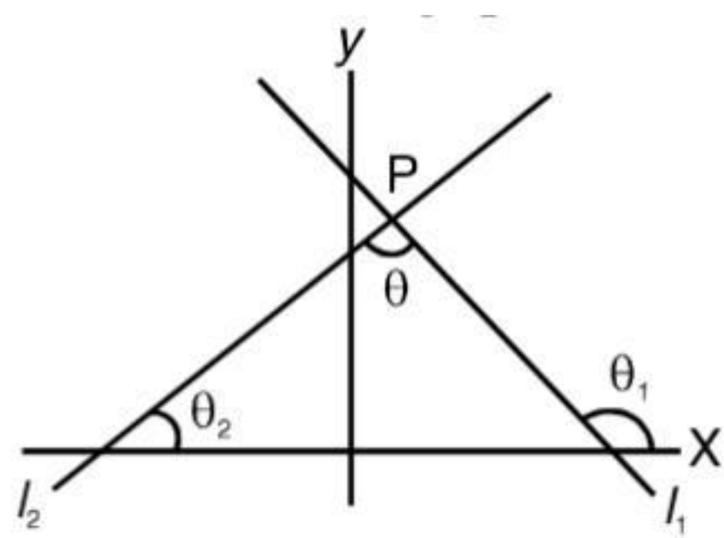
(ii) Dividing throughout by

$$\sqrt{A^2 + B^2} = \sqrt{(\text{coeff. of } x)^2 + (\text{coeff. of } y)^2}$$

Angle between two straight lines: The angle between the lines

$$y = m_1 x + c_1 \text{ and } y = m_2 x + c_2 \text{ is}$$

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}$$



Note: If $\tan\theta$ is +ve, θ is acute.

If $\tan\theta$ is –ve, θ is obtuse.

$\therefore$ The acute angle between the two lines is

given by $\tan\theta = \left|\frac{m_1 - m_2}{1 - m_1 m_2}\right|$

(a) **Condition of parallelism:**

If lines are parallel; $\theta = 0°$ or $180°$

$\therefore \quad \tan\theta = 0$ or $\tan 180°$

$\therefore \quad m_1 = m_2$

(b) **Condition of perpendicularity:**

If lines are perpendicular, $\theta = 90°$

$\therefore \quad \tan\theta = \tan 90° = \infty$

$\therefore \quad m_1 m_2 = -1$

Equation of line parallel to $Ax + By + C = 0$:

The equation of the straight line which is parallel to the line $Ax + By + C = 0$ is given by $Ax + By + \lambda = 0$. Thus, to write the equation of any line parallel to a given line, do not change the coefficients of x and y and change to constant term only.

Equation of line perpendicular to $Ax + By + C = 0$:

The equation of a straight line which is perpendicular to the line $Ax + By + C = 0$ is given by $Bx - Ay + \lambda = 0$. Thus, to write the equation of any line perpendicular to given line interchange the coefficient of x and y in the given equation and change the sign of one of those coefficient and also change the constant term.

Point of intersection of two given lines: Let the two given lines be $a_1x + b_1 y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$. Solving these equations, the point of intersection of these two lines is given by

$$\left(\frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}\right)$$

Concurrent lines: Three or more straight lines are said to be concurrent lines if they meet in a point.

Method to prove the three lines to be concurrent:

(i) Find the point of intersection of any two lines and if this point satisfy the equation of the third line, then the three lines are concurrent, and this point is the point of intersection of the three lines.

(ii) The three lines $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$ and $a_3x + b_3y + c_3 = 0$ are concurrent if

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$$

(*iii*) Let P = 0, Q = 0 and R = 0 be three given lines. Then these lines are concurrent if we can find three constants l, m, n not all zero (*i.e.*, at least one of these is not equal to zero) such that $lP + mQ + nR = 0$ takes the form $0.x + 0.y + 0 = 0$

Any line through the point of intersection of two lines: Let P = 0 and Q = 0 be the two given lines. Then the equation of any line passing through the point of intersection of these two lines is $P + \lambda Q = 0$, where λ is any real number.

The value of λ will be obtained by the condition given in the question.

The length of perpendicular from a given point (x_1, y_1) to a line $ax + by + c = 0$ is

$$\frac{ax_1 + by_1 + c}{\sqrt{(a^2 + b^2)}}$$

Distance between parallel lines:

Let the two parallel lines be

$ax + by + c_1 = 0$ and $ax + by + c_2 = 0$.

Now, to find the distance between these two parallel lines, we proceed as follows:

(A) First Method: Find a point on any of the line. For this put $x = 0$ and find y, put $y = 0$

and find x. Then the distance between the lines is equal to the length of perpendicular from this point on one line to the other line.

(B) IInd Method: Let P (h,k) be a point on one line, say $ax + by + c_1 = 0$,

$\therefore \quad ah + bk = -c_1$

Then, distance between lines

= Length of perpendicular from P (h, k) to the other line $ax + by + c_2 = 0$

$$= \left|\frac{ah + bk + c_2}{\sqrt{(a^2 + b^2)}}\right| = \left|\frac{c_2 - c_1}{\sqrt{(a^2 + b^2)}}\right|$$

(C) IIIrd Method: Find the distance of each line from the origin. If these distances are p_1 and p_2, then $p_1 = \dfrac{c_1}{\sqrt{(a^2 + b^2)}}$ and $p_2 = \dfrac{c_2}{\sqrt{(a^2 + b^2)}}$

and retain their signs.

Then, the perpendicular distance between

$$\text{lines} = |p_2 - p_1| = \left|\frac{c_2 - c_1}{\sqrt{a^2 + b^2}}\right|$$

Equation of the bisectors of the angles between two lines:

Let the two given lines be

$a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$

Then, the equation of the bisectors of the angles between two lines are

$$\frac{a_1x + b_1y + c_1}{\sqrt{(a_1^2 + b_1^2)}} = \pm \frac{a_2x + b_2y + c_2}{\sqrt{(a_2^2 + b_2^2)}}$$

Remember that, every point of the bisectors of angles is equidistant from the given lines.