3 Mean Value Theorems

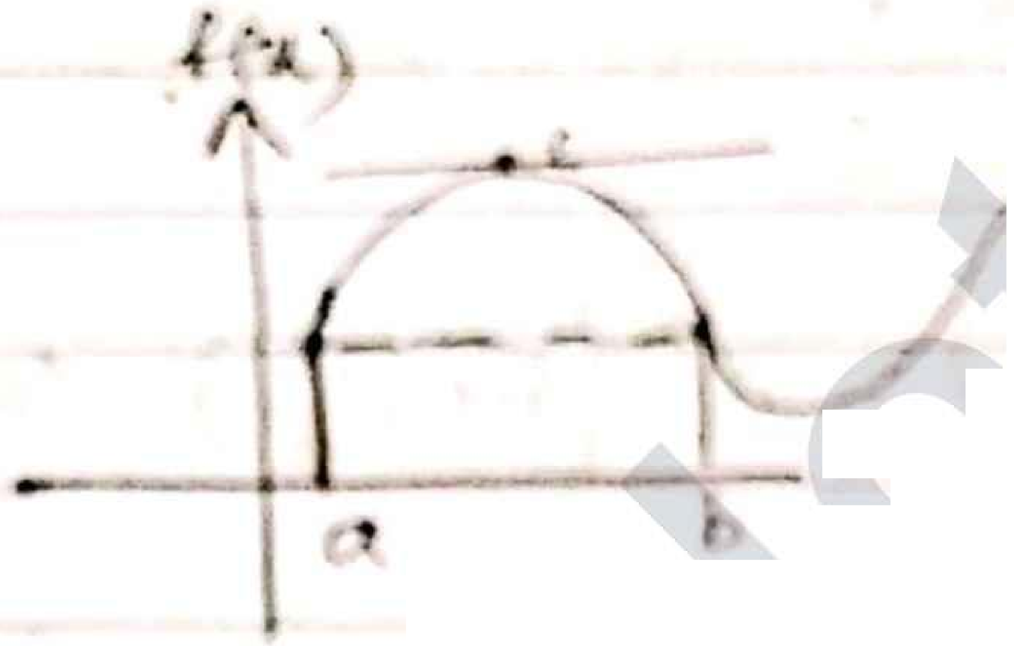
① Roles theorem :

let $f(x)$ be defined in $[a,b]$ such that it satisfies 3 conditions–

(i) $f(x)$ is continuous $[a,b]$
(ii) $f(x)$ is differentiable (a,b)
(iii) $f(a) = f(b)$

then $\exists$ atleast one point $c \in (a,b)$ such that

$f'(c) = 0$.



② Lagranges Mean Value theorem :–

let $f(x)$ be defined in $[a,b]$ such that it satisfies 2 conditions.

(i) $f(x)$ is continuous in $[a,b]$
(ii) $f(x)$ is differentiable in (a,b)

then $\exists$ atleast one point $c \in (a,b)$ such that.

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

③ Cauchy's Mean Value theorem :

Let $f(x)$ and $g(x)$ be defined in $[a,b]$ such that -

(i) $f(x)$ and $g(x)$ are continuous in $[a,b]$
(ii) $f(x)$ and $g(x)$ are differentiable in (a,b).
(iii) $g'(x) \neq 0 \quad \forall\, x \in (a,b)$

then. $\exists$ atleast one point $c \in (a,b)$ such that .

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

easy

Q1) The mean value c for the function - $e^x[\sin x - \cos x]$ in the interval $\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$ is ——

(a) 0 (b) $\frac{3\pi}{4}$ (c) $\frac{2\pi}{3}$ (d) π

$\Rightarrow$

if this two value equal then we rroles theorem.

$f(x) = e^x[\sin x - \cos x]$

$f\left(\frac{\pi}{4}\right) = e^x\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right)$
$= 0$

$f\left(\frac{5\pi}{4}\right) = e^x\left[\sin\left(\frac{5\pi}{4}\right) - \cos\left(\frac{5\pi}{4}\right)\right]$

$= e^{5\pi/4}(0) = 0 \longrightarrow$ use vertical calculator

$(0,0) \rightarrow$ now we can use rroles theorem —
$f(a) = f(b)$

$\exists\, c \in \left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$

$f'(c) = e^c[\sin c - \cos c]$

$f'(c) = e^c(2\sin c) = 0$

$= 2e^c \sin c = 0$

$\therefore \sin c = 0$ $(0, \pm\pi, \pm 2\pi, \cdots)$

$c = \pi \in \left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$

π

Q2 find mean value c for the function -

$f(x) = x(x+3) \cdot e^{-x/2}$, on $[-3,0]$

$\Rightarrow$

- $f(-3) = -3(0) = 0$
- $f(0) = 0$

$\boxed{f(a) = f(b)}$ → rolles theorem can apply (mean value)

$\exists\, c \in (-3, 0)$

$f(c) = c(c+3)\, e^{-c/2}$

$f(c) = (c^2+3c)\, e^{-c/2}$

$f'(c) = (2c+3)\, e^{-c/2} + (c^2+3c)\, e^{-c/2}(-1/2)$

$= e^{-c/2}(2c+3) - \frac{1}{2} e^{-c/2} (c^2+3c)$

$= \frac{1}{2} e^{-c/2} (4c+6-c^2-3c)$

$= \frac{1}{2} \cdot e^{-c/2} (c - c^2 + 6) = 0$ → only this part can be '0'

$\therefore c - c^2 + 6 = 0$

$\Rightarrow c^2 - c - 6 = 0$

$\Rightarrow c^2 - 3c + 2c - 6 = 0$

$\Rightarrow c(c-3) + 2(c-3) = 0$

$c = 3, -2$

$c = -2 \in [-3, 0]$

✓ → mean value of c.

Q3) The mean value c for the function

$f(x) = \sqrt{x^2 - 4}$ in $[2, 4]$.

a) $\sqrt{6}$ b) $2\sqrt{2}$ c) $8/3$ d) none.

(i) continuous,
(ii) Differentiable.

$\Rightarrow$

$f(2) = \sqrt{2^2 - 4} = 0$

$f(4) = \sqrt{16 - 4} = \sqrt{12}$

$f(2) \neq f(4)$

Using Lagrange theorem –

$f'(c) = \frac{f(b) - f(a)}{b - a}$ — (1)

$\Rightarrow \frac{c}{\sqrt{c^2 - 4}} = \frac{f(b) - f(a)}{b - a}$

$\Rightarrow \frac{c}{\sqrt{c^2-4}} = \frac{\sqrt{12} - 0}{4-2}$

$\Rightarrow 2c = \sqrt{12} \cdot \sqrt{c^2 - 4}$

$\Rightarrow 4c^2 = 12 \cdot (c^2 - 4)$

$\Rightarrow 4c^2 = 12c^2 - 48$

$\Rightarrow 8c^2 = 48$

$c^2 = \sqrt{\frac{48}{8}} = \sqrt{6}$

$f(x) = \sqrt{x^2 - 4}$

$f'(x) = (x^2 - 4)^{1/2}$

$f'(x) = \frac{1}{2}(x^2-4)^{1/2 - 1} \cdot 2x$

$= \frac{x}{\sqrt{x^2 - 4}}$

very easy.

Q4 Find mean value c for the function –

$$f(x) = lx^2 + mx + n, \quad (l \neq 0) \text{ in } [a,b].$$

→ since, $f(x)$ is quadratic equ it is continuous on $[a,b]$ and differentiable on (a,b).

∴ $$f'(c) = \frac{f(b) - f(a)}{b - a} \quad \text{—— Lagrange theorem.}$$

$$\Rightarrow l \cdot 2c + m = \frac{lb^2 + mb + \cancel{n} - (la^2 + ma + \cancel{n})}{b - a}$$

$$\Rightarrow 2lc + m = \frac{l(b^2 - a^2) + m(b - a)}{(b - a)}$$

$$= \frac{l(b+a)(b-a) + m(b-a)}{(b-a)}$$

$$\Rightarrow 2lc + \cancel{m} = l(b+a) + \cancel{m}$$

$$\Rightarrow c = \frac{\cancel{l}(b+a)}{2\cancel{l}} = \left(\frac{a+b}{l}\right) \in (a,b).$$

Q5) The mean value of c for the functions $f(x) = \sin x$ and $g(x) = \cos x$ in $[-\frac{\pi}{2}, 0]$ is-

$\Rightarrow$ ~~g(x) =~~

Cauchy's mean value theorem-

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)} \quad --- (1)$$

~~Ex~~ $\cos x$

$f(-\frac{\pi}{2}) = -1 \qquad f(0) = 0$

$g(-\frac{\pi}{2}) = 0 \qquad g(0) = 1$

$$\frac{\cos c}{-\sin c} = \frac{0 - (-1)}{1 - 0}$$

$\Rightarrow$ ~~cot~~ $-\frac{\cos c}{\sin c} = \frac{1}{1} = 1$

$\tan c = -1$

$\therefore c = -\frac{\pi}{4} \in (-\frac{\pi}{2}, 0)$.