Differential Equations

Introduction to Differential Equations

- An equation involving derivative (derivatives) of the dependent variable with respect to the independent variable (variables) is known as differential equation.

- A differential equation containing derivatives of the dependent variable with respect to only one independent variable is called ordinary differential equation. For example, $2\left(\frac{dy}{dx}\right)^2 + 5y = 0, \frac{dy}{dx} = \frac{x^2 + y^2}{1 + xy}$ are ordinary differential equations.

- Here, the derivatives are also denoted as $\frac{dy}{dx} = y', \frac{d^2y}{dx^2} = y'', \frac{d^3y}{dx^3} = y'''$.

- The order of a differential equation is defined as the order of the highest order derivative of the dependent variable with respect to the independent variable occurring in the given differential equation.
  For example, order of differential
  equations dydx+tan x=0, d2ydx2+ex=2dydx+tan x=0, d2ydx2+ex=2, and
  xd3ydx3+(dydx)2= 0xd3ydx3+dydx2= 0 are 1, 2, and 3 respectively.

- The degree of a differential equation is defined only when the differential equation is a polynomial equation in derivatives (i.e. $y', y'', y''', \ldots$). The degree of such differential equation is defined as the highest power of the highest order derivative.
  For example, degree of $e^x \frac{dy}{dx} + \tan x = 0, \left(\frac{d^2y}{dx^2}\right)^3 + x\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^4 + e^x,$ and
  xd3ydx3+(dydx)2= 0xd3ydx3+dydx2= 0 are 1, 3, and 1 respectively.

- Degree of $x\frac{d^2y}{dx^2} + \cos\left(\frac{dy}{dx}\right)^2 = 0$ is not defined because this is not a polynomial equation in derivative.

Solved Examples

Example: Find the degree and order of the following differential equations.

(i) $x\frac{d^2y}{dx^2} + \log x = 0$

(ii) $xdx + ydy = 0$

(iii) $\frac{dy}{dx}+\log\left(\frac{dy}{dx}\right)=\frac{1}{x}$

(iv) $\frac{d^3y}{dx^3}+\sqrt{1-\left(\frac{dy}{dx}\right)^2}$

Solution:

(i) $x\frac{d^2y}{dx^2}+\log x=0$

Here, the highest order derivative is $\frac{d^2y}{dx^2}$. Therefore, order is 2.

This is a polynomial equation in y'' and the highest power raised to y'' is 1. Therefore, the degree is 1.

(ii) $xdx + ydy = 0$

$$\Rightarrow \frac{dy}{dx}+\frac{x}{y}=0$$

This is a polynomial equation in y'. Therefore, its order is 1 and its degree is 1.

(iii) $\frac{dy}{dx}+\log\left(\frac{dy}{dx}\right)=\frac{1}{x}$ is not a polynomial equation. Therefore, its degree is not defined, but its order is 1.

(iv) $\frac{d^3y}{dx^3}+\sqrt{1-\left(\frac{dy}{dx}\right)^2}$

$$\Rightarrow \left(\frac{d^3y}{dx^3}\right)^2+\left(\frac{dy}{dx}\right)^2-1=0$$

Here, the highest order derivative is $\frac{d^3y}{dx^3}$ and its order is 3 and degree is 2.

General and Particular Solutions of a Differential Equation

- A function $y = \Phi(x)$ is said to be the solution of a differential equation if it satisfies the given differential equation.

- If the solution of a differential equation contains arbitrary constants, then the solution is called general solution of the differential equation.

- If the solution of a differential equation does not contain any arbitrary constants, then the solution is called particular solution of the differential equation.

Solved Examples

Example 1: Verify that $y = \tan(2x + a)$, $(a \in R)$ is the solution of the differential equation $\frac{dy}{dx} - 2y^2 - 2 = 0$.

Solution:

$$y = \tan(2x+a)$$
$$\frac{dy}{dx} = 2\sec^2(2x+a)$$
$$\therefore \frac{dy}{dx} - 2y^2 - 2 = 2\sec^2(2x+a) - 2\tan^2(2x+a) - 2$$
$$\frac{dy}{dx} - 2y^2 - 2 = 2\sec^2(2x+a) - 2\left[1+\tan^2(2x+a)\right]$$
$$\frac{dy}{dx} - 2y^2 - 2 = 2\sec^2(2x+a) - 2\sec^2(2x+a)$$
$$\frac{dy}{dx} - 2y^2 - 2 = 0$$

Thus, $y = \tan(2x + a)$ is a solution of the differential equation $\frac{dy}{dx} - 2y^2 - 2 = 0$

Example 2:

Verify that $y = 8e^{-2x} + 3x$ is a solution of the differential equation $e^{2x}\left(\frac{d^2y}{dx^2}\right) = 32$

Solution:

$$y = 8e^{-2x} + 3x$$
$$\frac{dy}{dx} = -16e^{-2x} + 3$$
$$\frac{d^2y}{dx^2} = 32e^{-2x}$$
$$\therefore e^{2x}\left(\frac{d^2y}{dx^2}\right) = 32$$

Thus, $y = 8e^{-2x} + 3x$ is a solution of the differential equation $e^{2x}\left(\frac{d^2y}{dx^2}\right) = 32$.

Example 3:

Verify that $\sin\frac{y}{x} = \log x$ is a solution of the differential equation $x\cos\left(\frac{y}{x}\right)\frac{dy}{dx} = y\cos\left(\frac{y}{x}\right) + x$.

Solution:

$$\sin\frac{y}{x} = \log x$$

Differentiating both sides with respect to *x*, we obtain

$$\cos\frac{y}{x}\cdot\frac{d}{dx}\left(\frac{y}{x}\right) = \frac{1}{x}$$
$$\Rightarrow \cos\left(\frac{y}{x}\right)\cdot\left[\frac{1}{x}\frac{dy}{dx} - y\frac{1}{x^2}\right] = \frac{1}{x}$$
$$\Rightarrow \cos\left(\frac{y}{x}\right)\left[x\frac{dy}{dx} - y\right] = x$$
$$\Rightarrow x\cos\left(\frac{y}{x}\right)\frac{dy}{dx} = y\cos\left(\frac{y}{x}\right) + x$$

Hence verified.

Formation of Differential Equation

- The order of a differential equation representing a family of curves is the same as the number of arbitrary constants present in the equation corresponding to the family of curves.
- The procedure of forming a differential equation whose general solution is given as follows:

- Let the given equation of a family of curves contain 'n' arbitrary constants $a_1, a_2 \dots a_n$. i.e., general solution of the equation is given as

  $f(x, y_1, y_2, \dots y_n, a_1, a_2, \dots a_n) = 0$, where $y_n = \frac{d^n y}{dx^n}$ ... (1)

- Step 1 − Differentiate the equation of family of curves (1) 'n' times to get 'n' more equations.

- Step 2 − Eliminate 'n' constants using $(n + 1)$ equations.

- The required differential equation is obtained in the form $f(x, y_1, y_2, \dots y_n) = 0$

- For example: Form the differential equation representing the family of curves $(y - a)^2 = 4b(x - 3)$

  Solution:

  $(y - a)^2 = 4b(x - 3)$ ... (1)

  On differentiating (1) with respect to 'x', we obtain

  $2(y - a)y_1 = 4b$

  $\Rightarrow (y - a)y_1 = 2b$ ... (2)

  On differentiating (2) with respect to 'x', we obtain

  $$(y-a)y_2 + y_1 y_1 = 0 \dots (3)$$
  $$\Rightarrow (y-a) = -\frac{y_1^2}{y_2} \dots (4)$$

  (2) and (4) $\Rightarrow$ $-\frac{y_1^3}{2y_2} = b \dots (5)$

  On substituting (4) and (5) in (1), we obtain

  $$\left(-\frac{y_1^2}{y_2}\right)^2 = 4\left(-\frac{y_1^3}{2y_2}\right)(x-3)$$
  $$\Rightarrow \frac{y_1^4}{y_2^2} + 4\frac{y_1^3}{2y_2}(x-3) = 0$$
  $$\Rightarrow y_1 + 2(x-3)y_2 = 0$$
  $$\text{or } 2(x-3)\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$$

This is the required differential equation.

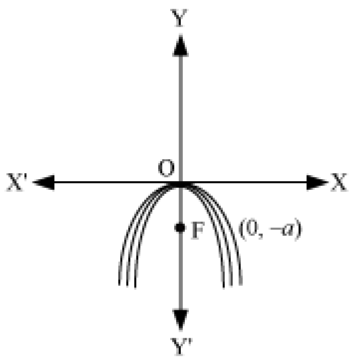
Solved Examples

Example 1

Form the differential equation of the family of all parabolas having vertex at the origin and axis along the negative direction of the *y*-axis.

Solution:

Let Φ be the given family of parabolas. Let (0, −*a*) be the focus of a member of the family, where *a* is an arbitrary constant.



∴ The equation of Φ is given by $x^2 = -4ay$ … (1)

On differentiating both sides with respect to ‘*x*’, we obtain

$$2x = -4a\frac{dy}{dx} \;\ldots\; (2)$$

From (1), $-4a = \frac{x^2}{y}$

On substituting in (2), we obtain

$$2x = \frac{x^2}{y}\frac{dy}{dx}$$

$$\Rightarrow 2xy = x^2\frac{dy}{dx}$$

$$\Rightarrow x^2\frac{dy}{dx} - 2xy = 0$$

This is the required differential equation of the given family of parabolas.

Example 2

Form a differential equation whose family of curve is $y = ax^3 + bx$.

Solution:

$$y = ax^3 + bx \ \dots \ (1)$$
$$\Rightarrow y_1 = 3ax^2 + b \ \dots \ (2)$$
$$\Rightarrow y_2 = 6ax \ \dots \ (3)$$
$$(3) \Rightarrow a = \frac{y_2}{6x} \ \dots \ (4)$$

(2) and (4) ⇒ $y_1 = 3\left(\frac{y_2}{6x}\right)x^2 + b$

$$\Rightarrow y_1 = \frac{y_2 x}{2} + b$$
$$\Rightarrow b = y_1 - \frac{1}{2}xy_2 \ \dots \ (5)$$

On substituting (4) and (5) in (1), we obtain

$$y = \frac{y_2}{6x}x^3 + \left(y_1 - \frac{1}{2}xy_2\right)x$$
$$\Rightarrow y = \frac{1}{6}y_2x^2 + y_1x - \frac{x^2}{2}y_2$$
$$\Rightarrow y = -\frac{1}{3}x^2y_2 + xy_1$$
$$\Rightarrow -\frac{1}{3}x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0$$
$$\Rightarrow x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 3y = 0$$

Method of Solving Differential Equations with Variables Separable

- The first order, first degree differential equation of the form $\frac{dy}{dx} = F(x, y)$, where $F(x, y)$ can be expressed as $f(x)\,g(y)$ [where $f(x)$ is a function of x and $g(y)$ is a function of y], is said to be of variable separable type. That is, differential equation is of the form

$$\frac{dy}{dx} = f(x)g(y)$$

- The variable separable equation, i.e. $\frac{dy}{dx} = f(x)g(y)$, can be solved as:
- If $g(y) \neq 0$, then separating the variables and rewriting the equation as

$$\frac{dy}{g(y)} = f(x)dx \qquad \text{... (1)}$$

- Then, integrating expression (1) to obtain

$$\int \frac{1}{g(y)} dy = \int f(x)dx \qquad \text{... (2)}$$

- Equation (2) gives the solution of the given differential equation of the form, G (y) = H (x) + C, where G (y) and H(x) are anti-derivatives of $\frac{1}{g(y)}$ and $f(x)$ respectively and C is an arbitrary constant.
- For example, consider the differential equation

$$y\frac{dy}{dx} - x(1+y^2) = 0$$

$$\frac{dy}{dx} = \frac{x(1+y^2)}{y}$$

$$\Rightarrow \frac{ydy}{1+y^2} = xdx$$

Integrating both sides, we obtain

$$\frac{1}{2}\int\frac{2y}{1+y^2}dx = \int xdx$$

$$\Rightarrow \frac{1}{2}\log\left(1+y^2\right) = \frac{x^2}{2} + C_1$$

$$\Rightarrow \log\left(1+y^2\right) = x^2 + C_2 \qquad \left(C_2 = 2C_1\right)$$

$$\Rightarrow 1+y^2 = e^{x^2+C_2}$$

$$\Rightarrow 1+y^2 = ke^{x^2} \qquad \left(k = e^{C_2}\right)$$

This is the required solution.

Solved Examples

Example 1: Find the general solution of the differential equation $2ye^x\frac{dy}{dx} = \cos x$.

Solution:

$$2ye^x\frac{dy}{dx} = \cos x$$

$$\Rightarrow 2ydy = e^{-x}\cos xdx$$

Integrating both sides, we obtain

$$\int 2ydy = \int e^{-x}\cos xdx \qquad \dots(1)$$

$$\int 2ydy = 2\int ydy = 2-\frac{y^2}{2} = y^2 \qquad \dots(2)$$

$$\text{Let } I = \int e^{-x}\cos x\,dx$$

$$I = \cos x\int e^{-x}dx - \int\left[\frac{d}{dx}\cos x\int e^{-x}dx\right]dx$$

$$I = -\cos x\,e^{-x} - \int \sin x\,e^{-x}dx$$

$$I = -\cos x\,e^{-x} - \left[\sin x\int e^{-x}dx - \int\left[\frac{d}{dx}\sin x\int e^{-x}dx\right]dx\right]$$

$$I = -\cos xe^{-x} - \left[-\sin x\,e^{-x} + \int \cos x\,e^{-x}dx\right]$$

$$\Rightarrow 2I = -\cos x\,e^{-x} + \sin x\,e^{-x}$$

$$\Rightarrow I = \frac{1}{2}e^{-x}\left(\sin x - \cos x\right) \qquad \dots(3)$$

From equations (2) and (3), substituting in equation (1), we obtain

$$y^2 = \frac{1}{2}e^{-x}(\sin x - \cos x) + C$$
$$\Rightarrow 2y^2 e^x = \sin x - \cos x + Ce^x$$

Example 2: Find the particular solution of the differential equation $(2y+1)dy - (4-2x)(y-2)^2\,dx = 0$, if $x = 4$ at $y = 3$.

Solution:

$$(2y+1)dy - (4-2x)(y-2)^2\,dx = 0$$

$$\Rightarrow (2y+1)dy = (4-2x)(y-2)^2\,dx$$
$$\Rightarrow \frac{2y+1}{(y-2)^2}dy = (4-2x)dx$$

Integrating both sides, we obtain

$$\int \frac{2y+1}{(y-2)^2}dy = \int (4-2x)dx$$
$$\Rightarrow \int \left[\frac{2}{y-2} + \frac{5}{(y-2)^2}\right]dy = \int (4-2x)dx$$
$$\Rightarrow 2\log|y-2| - 5\left(\frac{1}{y-2}\right) = 4x - x^2 + C$$

It is given that at $x = 4$, $y = 3$

$$\therefore 2\log|3-2| - 5 = 4\times 4 - 4^2 + C$$
$$\Rightarrow C = -5$$
$$\therefore 2\log|y-2| - \frac{5}{(y-2)} = 4x - x^2 - 5$$

This is the required particular solution of the given differential equation.

Example 3: Find the equation of a curve passing through (1, 2), if the slope of normal to the curve at any point (x, y) is $-\frac{3y^2}{32x}$.

Solution:

The slope of normal to the curve at point (x, y) is given as $-\dfrac{3y^2}{32x}$.

$\therefore$ Slope of tangent at $(x, y) = \dfrac{-1}{\left(-\dfrac{3y^2}{32x}\right)} = \dfrac{32x}{3y^2}$

$$\therefore \frac{dy}{dx} = \frac{32x}{3y^2}$$
$$\Rightarrow 3y^2 dy = 32x dx$$
$$\int 3y^2 dy = \int 32x dx$$
$$\Rightarrow \frac{3y^3}{3} = \frac{32x^2}{2} + C$$
$$\therefore y^3 = 16x^2 + C$$

At $x = 1$ and $y = 2$, we obtain $C = -8$.

Thus, the required curve is $y^3 = 16x^2 - 8$.

Homogenous Differential Equations

- A function $f(x, y)$ is said to be a homogeneous function of degree n, if $f(\lambda x, \lambda y) = \lambda^n f(x, y)$ for any non-zero λ.

  For example, $f(x, y) = x^2 - 3xy$ is a homogenous function because $f(\lambda x, \lambda y) = \lambda^2 (x^2 - 3xy) = \lambda^2 f(x, y)$

  $g(x, y) = \log x + \log y$ is not a homogenous function because $g(\lambda x, \lambda y) = \log \lambda x + \log \lambda y \neq \lambda^n g(x, y)$ for any $n \in \mathbf{N}$.

- If $f(x, y)$ is a homogenous function of degree n, then $f(x, y)$ can be expressed as $f(x, y) = x^n g\left(\dfrac{y}{x}\right)$ or $y^n h\left(\dfrac{x}{y}\right)$.

- A differential equation of the form $\dfrac{dy}{dx} = f(x, y)$ is said to be homogenous, if $f(x, y)$ is a homogenous function of degree zero.

For example, $\frac{dy}{dx}=\frac{y^2-x^2}{2x^2+y^2}$

$$\left[\frac{dy}{dx}=f(x,y),\ f(x,y)=\frac{y^2-x^2}{2x^2+y^2};\ f(\lambda x,\lambda y)=\frac{\lambda^2(y^2-x^2)}{\lambda^2(2x^2+y^2)}=\lambda^0 f(x,y)\right]$$

- A homogenous differential equation of type $\frac{dy}{dx}=f(x,y)=g\left(\frac{y}{x}\right)$ can be solved by substituting *y* = *vx* in the given equation and then differentiating it with respect to *x*. That is, *y* = *vx* . . . (1)

  Differentiating it with respect to *x*, we obtain $\frac{dy}{dx}=v+x\frac{dv}{dx}$

  Now, substituting the value of $\frac{dy}{dx}$ in given equation (1), we obtain $v+x\frac{dv}{dx}=g(v)\Rightarrow x\frac{dv}{dx}=g(v)-v$

  Separate the variables and integrate i.e., $\int\frac{dv}{g(v)-v}=\int\frac{dx}{x}+C$

  This gives the general solution of the given differential equation.

- The homogenous differential equations of the form $\frac{dx}{dy}=f(x,y)$ can be similarly solved by substituting *x* = *vy.*

  For example, consider the differential equation, $\frac{dx}{dy}=\frac{x-y}{x+y}$

  Let *x* = *vy*

  $\therefore\frac{dx}{dy}=v+y\frac{dv}{dy}$

  Substituting the value of *x* in the given equation, we obtain

$$v + y\frac{dv}{dy} = \frac{v-1}{v+1}$$

$$\Rightarrow y\frac{dv}{dy} = \frac{v-1}{v+1} - v$$

$$\Rightarrow y\frac{dv}{dy} = \frac{v-1-v-v^2}{1+v} = -\frac{\left(1+v^2\right)}{1+v}$$

$$\Rightarrow \left(\frac{1+v}{1+v^2}\right)dv = -\frac{dy}{y}$$

$$\Rightarrow \left(\frac{1}{1+v^2}\right)dv + \frac{1}{2}\left(\frac{2v}{1+v^2}\right)dv = -\frac{dy}{y}$$

Integrating both sides, we obtain

$$\Rightarrow \left(\frac{1}{1+v^2}\right)dv + \frac{1}{2}\left(\frac{2v}{1+v^2}\right)dv = -\frac{dy}{y}$$

$$\tan^{-1} v + \frac{1}{2}\log\left(1+v^2\right) = -\log y + \mathrm{C}$$

$$\tan^{-1}\left(\frac{x}{y}\right) + \frac{1}{2}\log\left(1+\frac{x^2}{y^2}\right) = -\log y + \mathrm{C}$$

$$\tan^{-1}\left(\frac{x}{y}\right) + \frac{1}{2}\log\left(y^2+x^2\right) - \log y = -\log y + \mathrm{C}$$

$$\tan^{-1}\left(\frac{x}{y}\right) + \frac{1}{2}\log\left(y^2+x^2\right) = \mathrm{C}$$

Solved examples

Example 1:

Solve: $y\frac{dx}{dy} = y\cos\frac{x}{y} + x$

Solution:

$$y\frac{dx}{dy} = y\cos\frac{x}{y} + x$$

$$\Rightarrow \frac{dx}{dy} = \cos\left(\frac{x}{y}\right) + \frac{x}{y} \qquad \ldots(1)$$

Let $x = vy$

$$\Rightarrow \frac{dx}{dy} = y\frac{dv}{dy} + v$$

Thus, equation (1) reduces to

$$y\frac{dv}{dy} + v = \cos v + v$$

$$\Rightarrow \frac{dv}{\cos v} = \frac{dy}{y}$$

$$\Rightarrow \int \sec v dv = \int \frac{dy}{y}$$

$$\Rightarrow \log\left|\sec v + \tan v\right| = \log y + \log C$$

$$\Rightarrow \sec v + \tan v = yC$$

$$\therefore \sec\frac{x}{y} + \tan\frac{x}{y} = yC$$

Example 2:

Solve: $x^2\frac{dy}{dx} = x^2 + y^2 - 2xy$

Solution:

$$x^2\frac{dy}{dx} = x^2 + y^2 - 2xy$$

$$\Rightarrow \frac{dy}{dx} = 1 + \frac{y^2}{x^2} - 2\frac{y}{x} = \left(1 - \frac{y}{x}\right)^2 \qquad \ldots(1)$$

Let $\frac{y}{x} = v \Rightarrow y = xv$

So, $\frac{dy}{dx} = x\frac{dv}{dx} + v$

Substituting in equation (1), we obtain

$$x\frac{dv}{dx}+v=(1-v)^2$$
$$\Rightarrow \frac{xdv}{dx}=1+v^2-3v$$
$$\frac{dv}{v^2-3v+1}=\frac{dx}{x}$$

Integrating both sides, we obtain

$$\int\frac{dv}{\left(v-\frac{3}{2}\right)^2-\left(\frac{\sqrt{5}}{2}\right)^2}=\int\frac{dx}{x}$$
$$\Rightarrow \frac{1}{2.\left(\frac{\sqrt{5}}{2}\right)}\cdot\log\left|\frac{\left(v-\frac{3}{2}\right)-\left(\frac{\sqrt{5}}{2}\right)}{v-\frac{3}{2}+\frac{\sqrt{5}}{2}}\right|=\log x+\log C$$
$$\Rightarrow \frac{1}{\sqrt{5}}\log\left|\frac{2v-3-\sqrt{5}}{2v-3+\sqrt{5}}\right|=\log Cx$$
$$\Rightarrow \frac{1}{\sqrt{5}}\log\left|\frac{2y-\left(3+\sqrt{5}\right)x}{2y-\left(3-\sqrt{5}\right)x}\right|=\log Cx$$

Example 3:

Solve: $x\frac{dy}{dx}=\sqrt{x^2-y^2}+y$

Solution:

$$x\frac{dy}{dx}=\sqrt{x^2-y^2}+y$$
$$\Rightarrow \frac{dy}{dx}=\sqrt{1-\frac{y^2}{x^2}}+\frac{y}{x} \qquad \dots(1)$$
$$\text{Put } \frac{y}{x}=v\Rightarrow y=xv$$
$$\Rightarrow \frac{dy}{dx}=x\frac{dv}{dx}+v$$

Substituting in equation (1), we obtain

$$x\frac{dv}{dx}+v=\sqrt{1-v^2}+v$$

$$\Rightarrow \frac{dv}{\sqrt{1-v^2}}=\frac{dx}{x}$$

Integrating both sides, we obtain

$\sin^{-1}v = \log x + \log C$

$\therefore \sin^{-1}v = \log (Cx)$

Linear Differential Equation

- The differential equation of the form $\frac{dy}{dx}+Py=Q$, where P and Q are constants or functions of x only, is known as a first order linear differential equation. For example, $\frac{dy}{dx}+e^x y=\cos x, \frac{dy}{dx}+2y=\log x$
- Another form of first order linear differential equation is $\frac{dx}{dy}+P_1x=Q_1$, where P_1 and Q_1 are constants or functions of y only. For example, $\frac{dx}{dy}+3x=e^y$, $\frac{dy}{dx}+\frac{x}{y\log y}=y^2$
- The linear differential equations can be solved as:
- Firstly, reduce the given differential equation in the form $\frac{dy}{dx}+Py=Q$, where P and Q are constants or functions of x only.
- Then, find the integrating factor (I.F) given by, $e^{\int P\,dx}$.
- The solution of the given differential equation is given by, $y(\text{I.F})=\int Q(\text{I.F})\,dx+C$
- If the first order linear differential equation is in the form $\frac{dx}{dy}+P_1x=Q_1$, where P_1 and Q_1 are constants or functions of y, then I.F. is given by $e^{\int P_1\,dy}$ and the solution is $x(\text{I.F})=\int Q_1(\text{I.F})\,dy+C$.

- For example, consider $\frac{dy}{dx}+\frac{y}{x}=x$

  This differential equation is of the form $\frac{dy}{dx}+\mathrm{P}y=\mathrm{Q}$, where $\mathrm{P}=\frac{1}{x}$ and Q = x

  $\mathrm{I.F}=\mathrm{e}^{\int\frac{1}{x}dx}=\mathrm{e}^{\log x}=x$

  The solution is given by,

  $y.x=\int x.x\ dx+\mathrm{C}$

  $\Rightarrow xy=\int x^2\ dx+\mathrm{C}$

  $\Rightarrow xy=\frac{x^3}{3}+\mathrm{C}$

Solved Examples

Example 1:

Solve: $(1+y^2)\frac{dx}{dy}-x=\tan^{-1}y$

Solution:

The given differential equation can be written as

$$\frac{dx}{dy}-\left(\frac{1}{1+y^2}\right)x=\frac{\tan^{-1}y}{1+y^2}$$

I.F. = $\mathrm{e}^{-\int\frac{1}{1+y^2}dy}=\mathrm{e}^{-\tan^{-1}y}$

The solution is given by,

$$x\mathrm{e}^{-\tan^{-1}y}=\int\frac{\tan^{-1}y\cdot\mathrm{e}^{-\tan^{-1}y}}{1+y^2}dy+\mathrm{C} \qquad ...(1)$$

$$\text{Let I}=\int\frac{\tan^{-1}y\cdot\mathrm{e}^{-\tan^{-1}y}}{1+y^2}dy$$

Put $\tan^{-1}y=t\Rightarrow\frac{1}{1+y^2}dy=dt$

$$\therefore \text{I} = \int te^{-t} dt$$

$$\text{I} = t\int e^{-t} dt - \int \frac{d}{dt} t \cdot \int e^{-t} dt$$

I = $-te^{-t} - e^{-t}$

Substituting in equation (1), we obtain

$$xe^{-\tan^{-1} y} = e^{-\tan^{-1} y} - e^{-\tan^{-1} y} \tan^{-1} y + \text{C}$$

∴ x=1−tan−1y+Cetan−1yx=1-tan-1y+Cetan-1y

This is the required solution.

Example 2:

Solve: $\frac{dy}{dx} + y \tan x = \sin x$

Solution:

$$\frac{dy}{dx} + y \tan x = \sin x$$

Here, P = tan x and Q = sin x

∴ I.F = $e^{\int \tan x \, dx} = e^{-\log \cos x} = \sec x$

The solution is given by,

$$y \sec x = \int \sec x \ \sin x \ dx + \text{C}$$

$$y \sec x = \int \frac{\sin x}{\cos x} dx + \text{C}$$

$$\therefore y \sec x = \log \sec x + \text{C}$$