Trigonometric Functions

- If in a circle of radius r, an arc of length l subtends an angle of θ radians, then $l = r\theta$.
- Radian measure $= \frac{\pi}{180} \times$ Degree measure
- Degree measure $= \frac{180}{\pi} \times$ Radian measure
- A degree is divided into 60 minutes and a minute is divided into 60 seconds. One sixtieth of a degree is called a minute, written as 1', and one sixtieth of a minute is called a second, written as1".
  Thus, $1^\circ = 60'$ and $1' = 60''$
- **Signs of trigonometric functions in different quadrants:**

Trigonometric function	Quadrant I	Quadrant II	Quadrant III	Quadrant IV
$\sin x$	+ ve (Increases from 0 to 1)	+ ve (Decreases from 1 to 0)	–ve (Decreases from 0 to –1)	–ve (Increases from –1 to 0)
$\cos x$	+ ve (Decreases from 1 to 0)	–ve (Decreases from 0 to –1)	–ve (Increases from –1 to 0)	+ ve (Increases from 0 to 1)
$\tan x$	+ ve (Increases from 0 to ∞)	–ve (Increases from –∞ to 0)	+ ve (Increases from 0 to ∞)	–ve (Increases from –∞ to 0)
$\cot x$	+ ve (Decreases from ∞ to 0)	–ve(Decreases from 0 to –∞)	+ ve (Decreases from ∞ to 0)	–ve (Decreases from 0 to –∞)
$\sec x$	+ ve (Increases from 1 to ∞)	–ve (Increases from –∞ to –1)	–ve (Decreases from –1 to –∞)	+ ve (Decreases from ∞ to 1)
$\operatorname{cosec} x$	+ ve (Decreases from ∞ to 1)	+ ve (Increases from 1 to ∞)	–ve (Increases from –∞ to –1)	–ve (Decreases from –1 to –∞)

Example 1:

If $\sin\theta = -\frac{1}{\sqrt{3}}$, where $\pi < \theta < \frac{3\pi}{2}$, then find the value of $3\tan\theta - \sqrt{3}\sec\theta$.

Solution:

Since θ lies in the third quadrant, therefor $\tan\theta$ is positive and $\cos\theta$ (or $\sec\theta$) is negative.

$$\cos^2\theta + \sin^2\theta = 1$$

$$\Rightarrow \cos\theta = \pm\sqrt{1-\sin^2\theta}$$

$$\Rightarrow \cos\theta = \pm\sqrt{1-\left(-\frac{1}{\sqrt{3}}\right)^2} = \pm\sqrt{1-\frac{1}{3}} = \pm\sqrt{\frac{2}{3}}$$

$$\therefore \cos\theta = -\sqrt{\frac{2}{3}}$$

$$\Rightarrow \sec\theta = -\sqrt{\frac{3}{2}}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{-\frac{1}{\sqrt{3}}}{-\sqrt{\frac{2}{3}}} = \frac{1}{\sqrt{2}}$$

$$\therefore 3\tan\theta - \sqrt{3}\sec\theta = 3\times\frac{1}{\sqrt{2}} - \sqrt{3}\times\left(-\sqrt{\frac{3}{2}}\right) = \frac{3}{\sqrt{2}} + \frac{3}{\sqrt{2}} = 3\sqrt{2}$$

Example 2: Find the value of cos 390° cos 510° + sin 390° cos (–660°).
Solution:

$$\cos 390° = \cos(2 \times 180° + 30°) = \cos 30° = \frac{\sqrt{3}}{2}$$

$$\cos 510° = \cos(3 \times 180° - 30°) = -\cos 30° = -\frac{\sqrt{3}}{2}$$

$$\sin 390° = \sin(2 \times 180° + 30°) = \sin 30° = \frac{1}{2}$$

$$\cos(-660°) = \cos 660° = \cos(4 \times 180° - 60°) = \cos 60° = \frac{1}{2}$$

$$\therefore \cos 390° \cos 510° + \sin 390° \cos(-660°)$$

$$= \frac{\sqrt{3}}{2} \times \left(-\frac{\sqrt{3}}{2}\right) + \left(\frac{1}{2}\right) \times \left(\frac{1}{2}\right)$$

$$= -\frac{3}{4} + \frac{1}{4}$$

$$= -\frac{2}{4}$$

$$= -\frac{1}{2}$$

- **Domain and Range of trigonometric functions:**

Trigonometric function	Domain	Range
$\sin x$	**R**	[-1, 1]
$\cos x$	**R**	[-1, 1]
$\tan x$	**R** - $\left\{x : x = \frac{(2n+1)\pi}{2}, n \in Z\right\}$	**R**
$\cot x$	**R** - $\{x : x = n\pi, n \in \mathbf{Z}\}$	**R**
$\sec x$	**R** - $\left\{x : x = \frac{(2n+1)\pi}{2}, n \in Z\right\}$	**R** - [-1, 1]
$\text{cosec}\, x$	**R** - $\{x : x = n\pi, n \in \mathbf{Z}\}$	**R** - [-1, 1]

- **Trigonometric identities and formulas:**
  - $\text{cosec}\, x = \frac{1}{\sin x}$
  - $\sec x = \frac{1}{\cos x}$
  - $\tan x = \frac{\sin x}{\cos x}$
  - $\cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x}$
  - $\cos^2 x + \sin^2 x = 1$
  - $1 + \tan^2 x = \sec^2 x$

- $1+\cot^2 x = \mathrm{cosec}^2 x$
- $\cos(2n\pi + x) = \cos x, n \in Z$
- $\sin(2n\pi + x) = \sin x, n \in Z$
- $\sin(-x) = -\sin x$
- $\cos(-x) = \cos x$
- $\cos(x+y) = \cos x \cos y - \sin x \sin y$
- $\cos(x-y) = \cos x \cos y + \sin x \sin y$
- $\cos\left(\frac{\pi}{2} - x\right) = \sin x$
- $\sin\left(\frac{\pi}{2} - x\right) = \cos x$
- $\sin(x+y) = \sin x \cos y + \cos x \sin y$
- $\sin(x-y) = \sin x \cos y - \cos x \sin y$
- $\cos\left(\frac{\pi}{2} + x\right) = -\sin x$
- $\sin\left(\frac{\pi}{2} + x\right) = \cos x$
- $\cos(\pi - x) = -\cos x$
- $\sin(\pi - x) = \sin x$
- $\cos(\pi + x) = -\cos x$
- $\sin(\pi + x) = -\sin x$
- $\cos(2\pi - x) = \cos x$
- $\sin(2\pi - x) = -\sin x$
- If none of the angles x, y and $(x \pm y)$ is an odd multiple of $\frac{\pi}{2}$, then

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}, \text{ and } \tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

- If none of the angles x, y and $(x \pm y)$ is a multiple of π, then $\cot(x+y) = \frac{\cot x \cot y - 1}{\cot y + \cos x}$, and $\cot(x-y) = \frac{\cot x \cot y + 1}{\cot y - \cot x}$

- $\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x = \frac{1-\tan^2 x}{1+\tan^2 x}$
- In particular, $\cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2} = 2\cos^2\frac{x}{2} - 1 = 1 - 2\sin^2\frac{x}{2} = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$
- $\sin 2x = 2\sin x \cos x = \frac{2\tan x}{1+\tan^2 x}$

- In particular, $\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$
- $\tan 2x = \frac{2\tan x}{1-\tan^2 x}$

- In particular,

- **General solutions of some trigonometric equations:**
  - $\sin x = 0 \Rightarrow x = n\pi,$, where $n \in \mathbf{Z}$
  - $\cos x = 0 \Rightarrow x = (2n+1)\frac{\pi}{2}$,where $n \in \mathbf{Z}$
  - $\sin x = \sin y \Rightarrow x = n\pi + (-1)^n y$, where $n \in \mathbf{Z}$
  - $\cos x = \cos y \Rightarrow x = 2n\pi \pm y$, where $n \in \mathbf{Z}$
  - $\tan x = \tan y \Rightarrow x = n\pi + y$, where $n \in \mathbf{Z}$

Example 1: Solve $\cot x \cos^2 x = 2 \cot x$
Solution:

$\cot x \cos^2 x = 2\cot x$

$\Rightarrow \cot x \cos^2 x - 2\cot x = 0$

$\Rightarrow \cot x (\cos^2 x - 2) = 0$

$\Rightarrow \cot x = 0$ or $\cos^2 x = 2$

$\Rightarrow \frac{\cos x}{\sin x} = 0$ or $\cos x = \pm\sqrt{2}$

$\Rightarrow \cos x = 0$ or $\cos x = \pm\sqrt{2}$

Now, $\cos x = 0 \Rightarrow x = (2n+1)\frac{\pi}{2}$, where $n \in \mathbf{Z}$

and $\cos x = \pm\sqrt{2}$

But this is not possible as $-1 \le \cos x \le 1$

Thus, the solution of the given trigonometric equation is $x = (2n+1)\frac{\pi}{2}$ where $n \in Z$.

Example 2: Solve $\sin 2x + \sin 4x + \sin 6x = 0$.
Solution:

$$\sin 4x + (\sin 2x + \sin 6x) = 0$$

$$\Rightarrow \sin 4x + 2\sin\left(\frac{2x+6x}{2}\right)\cos\left(\frac{2x-6x}{2}\right) = 0$$

$$\Rightarrow \sin 4x + 2\sin 4x \cos 2x = 0$$

$$\Rightarrow \sin 4x(1 + 2\cos 2x) = 0$$

$$\Rightarrow \sin 4x = 0 \text{ or } 1 + 2\cos 2x = 0$$

$$\Rightarrow \sin 4x = 0 \text{ or } \cos 2x = -\frac{1}{2}$$

$$\sin 4x = 0$$

$$\Rightarrow 4x = n\pi, n \in Z$$

$$\Rightarrow x = \frac{n\pi}{4}, n \in Z$$

$$\cos 2x = -\frac{1}{2}$$

$$\Rightarrow \cos 2x = \cos\frac{2\pi}{3}$$

$$\Rightarrow 2x = 2m\pi \pm \frac{2\pi}{3}, m \in Z$$

$$\Rightarrow x = m\pi \pm \frac{\pi}{3}, m \in Z$$

Thus, $x = \frac{n\pi}{4}$ or $x = m\pi \pm \frac{\pi}{3}$, where m, $n \in \mathbf{Z}$