CHAPTER 11

Structural Dynamics

Definition: Structural dynamics is a type of structural analysis which covers the behaviour of structures subjected to dynamic loading. Such dynamic loads include wind, waves, traffic, earthquakes and machine vibrations. Structural dynamics includes analysing amplitude, frequency and time period of motion response of structures subjected to dynamic load i.e., load which vary with respect to time.

A. Undamped Free Vibration of Single Degree of Freedom Systems

- Analytical solutions are obtained for simple problems.
- For problems involving complex material properties, loading and boundary conditions; assumptions are used to obtain approximate solutions.

11.1 Degree of Freedom

- The number of independent coordinates necessary to specify the configuration of a system at any time is known as the number of degree of freedom.

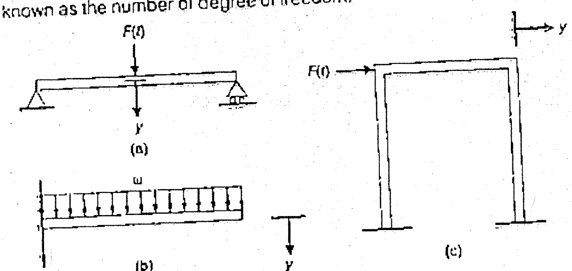


Fig. 11.1 Examples of structures modeled as one-degree-of-freedom systems

- These one-degree-of-freedom system may be described by mathematical model (as shown in figure below), which has the following elements.
  (i) m representing the mass and the inertial character of the structure.
  (ii) k representing the elastic restoring force and potential energy capacity of the structure.

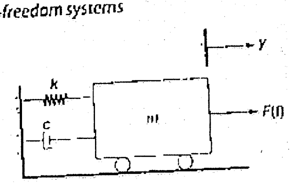


Fig. 11.2 Mathematical model for on-degree-of-freedom systems

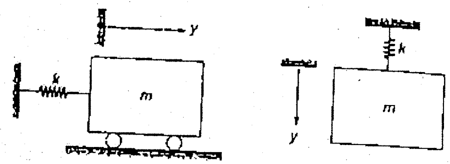
(iii) damping element C represents the frictional characteristics and energy losses of the structure.
(iv) an excitation force $F(t)$ represents the external force acting on the structural system.

11.2 Undamped System

- In the analysis of single-degree-of-freedom undamped system, effect of frictional forces or damping is neglected. Also, system is considered to be free from any external actions or forces during its motion or vibration.
- Under these conditions, motion of system is governed by the initial conditions that is given displacement and velocity at time $t = 0$.
- This undamped one-degree-of-freedom system is also referred as the simple undamped oscillator.
- Simple examples of it are shown in figure 11.3.
- In these models the mass m is restrained by the spring k and goes through rectilinear motion along one coordinate axis.
- The relation between force and displacement for a linear spring is

$$F_s = ky$$

(where, y is a small displacement in y direction in spring)



(a) (b)

Fig. 11.3 Alternate representations of mathematical models for one-degree-of-freedom systems

11.3 Springs in Parallel or in Series

- For n springs in parallel, the equivalent spring constant is

$$k_e = \sum_{i=1}^{n} k_i$$

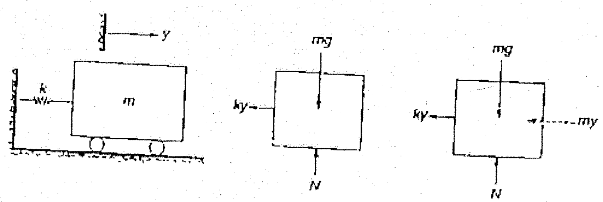
(where, displacement of all springs are same)

- For n springs in series, the equivalent spring constant is

$$\frac{1}{k_e} = \sum_{i=1}^{n} \frac{1}{k_i}$$

(where forces in all the springs are same)

11.4 Free Body Diagram



(a) (b) (c)

Fig. 11.4 FBD (a) Single degree-of-freedom system (b) Showing only external forces (c) Showing external and inertial forces.

- Forces acting in vertical direction do not enter into the equation of motion for the y-direction.
- The application of Newton's second law of motion in 'y' direction gives

$$F = m\ddot{y} \Rightarrow m\ddot{y} + ky = 0$$

where, y is displacement and $\ddot{y}$ is acceleration in y-direction (double overdots represents the second derivative with respect to time)

11.5 D'Alembert's Principle

- The equation $m\ddot{y} + ky = 0$ is also obtained by D'Alembert's principle which states that a system may be set in a dynamic equilibrium state by adding a fictitious force known as the inertial force to the external forces.
- Figure (c) shown above shows free body diagram with the inertial forces $m\ddot{y}$. This force in mass multiplied by acceleration and always directed oppositely to the corresponding coordinate.

11.6 Solution of Differential Equation of Motion

For differential equation, $m\ddot{y} + ky = 0$ ...1(i)

Taking $y = e^{bt}$ as general solution

Putting solution in differential equation

$$mb^2 . e^{bt} + ke^{bt} = 0 \quad ...1(ii)$$

From eq. 1(ii)

$$b = \pm\sqrt{\frac{-k}{m}} = \pm\sqrt{\frac{k}{m}}\, i$$ (where i is square root of −1)

So solution of differential equation will be

$$y = C_1 \cos\sqrt{\frac{k}{m}}\,t + C_2 \sin\sqrt{\frac{k}{m}}\,t$$

(where C_1 and C_2 are constants of integration)

Taking, $\omega = \sqrt{\frac{k}{m}}$ in equation above

[ω is called angular frequency of motion]

Displacement, $y = C_1 \cos \omega t + C_2 \sin \omega t$ ...1(iii)

Differentiating above equation

Velocity, $\dot{y} = -C_1\omega \sin \omega t + C_2\omega \cos \omega t$ ...(iv)

Acceleration, $\ddot{y} = -C_1\omega^2 \cos \omega t - C_2\omega^2 \sin \omega t$ ...1(v)

Using boundary conditions

i.e., when $t = 0$, $y = y_0$

Also when $t = 0$, velocity $\dot{y} = V_0$

Putting these values in 1(iii) and 1(iv)

$$C_1 = y_0 \quad ...1(vi)$$

$$C_2 = \frac{V_0}{\omega} \quad ...1(vii)$$

Using 1(vi) and 1(vii) in 1(iii)

Displacement,

$$y = y_0 \cos\omega t + \frac{V_0}{\omega}\sin\omega t \qquad ...1(viii)$$

$$\Rightarrow \quad y = \left\{\frac{y_0}{\sqrt{y_0^2+\left(\frac{V_0}{\omega}\right)^2}}\cos\omega t + \frac{V_0/\omega}{\sqrt{y^2+\left(\frac{V_0}{\omega}\right)^2}}\sin\omega t\right\}\sqrt{y_0^2+\left(\frac{V_0}{\omega}\right)^2}$$

Taking an angle 'θ' such that

$$\sin\theta = \frac{y_0}{\sqrt{y_0^2+\left(\frac{V_0}{\omega}\right)^2}}$$

and

$$\cos\theta = \frac{V_0/\omega}{\sqrt{y_0^2+\left(\frac{V_0}{\omega}\right)^2}}$$

Also

$$A = \sqrt{y_0^2+\left(\frac{V_0}{\omega}\right)^2}$$

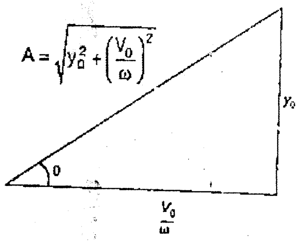


Fig. 11.5 Triangular Transformation of equation

Displacement,	$y = A\sin(\omega t + \theta)$	...1(ix)
Velocity,	$\dot{y} = A\omega\cos(\omega t + \theta)$	...1(x)
Acceleration,	$\ddot{y} = -A\omega^2\sin(\omega t + \theta)$	...1(xi)

From above equations

Maximum value of displacement, also called as amplitude = A

Maximum value of velocity, $V_{max} = A\omega$

Maximum value of acceleration, $a_{max} = A\omega^2$

11.7 Frequency, Period and Energy

- The motion described by equation of displacement is simple harmonic motion as it is represented by sine and cosine function having same angular frequency ω.
- As sine and cosine both are repetitive functions having same period equal to 2π. So period of motion can be easily found as

$$\omega T = 2\pi$$

Time period, $T = \frac{2\pi}{\omega}$

- Since natural frequency is inverse of time period i.e., natural frequency,

$$f = \frac{1}{T} = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

- For any instant in motion

Potential energy (energy stored in spring)

$$U(t) = \frac{1}{2}ky^2$$

$$\Rightarrow \quad U(t) = \frac{1}{2}k\left(A\sin(\omega t+\theta)\right)^2 = \frac{1}{2}kA^2\sin^2(\omega t+\theta)$$

Kinematic energy of motion, $k(t) = \frac{1}{2}mV^2$

$$k(t) = \frac{1}{2}m\left(A\omega\cos(\omega t+\theta)\right)^2$$

$$= \frac{1}{2}m\frac{k}{m}\cos^2(\omega t+\theta)A^2 = \frac{1}{2}kA^2\cos^2(\omega t+\theta)$$

Total energy, $E = U(t) + k(t) = \frac{1}{2}kA^2$

Note: Unit of natural frequency (f) is hertz or cycles per second. Where as unit of circular or angular frequency is radians per second (rad/second).

11.8 Graphical Representation of Motion

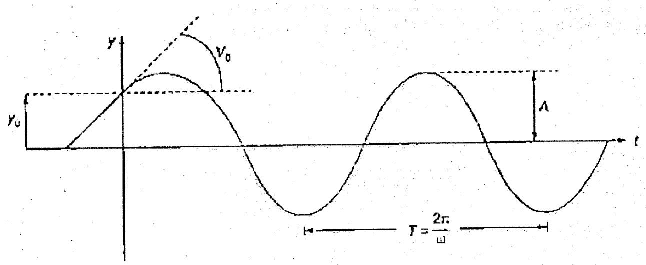


Fig. 11.6 Graphical representation of free vibration response

Summary

- General solution of single degree of freedom undamped system vibration.

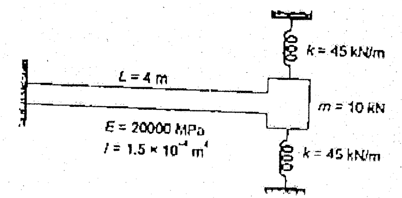
Displacement,	$y = y_0\cos\omega t + \frac{V_0}{\omega}\sin\omega t$
	$= A\sin(\omega t + \theta)$
Velocity,	$\dot{y} = -y_0\omega\sin\omega t + V_0\cos\omega t$
	$= A\omega\cos(\omega t + \theta)$
Acceleration,	$\ddot{y} = -y_0\omega^2\cos\omega t - V_0\omega\sin\omega t$
	$= -A\omega^2\sin(\omega t + \theta)$

where, $\tan\theta = \dfrac{y_0}{\left(\dfrac{V_0}{\omega}\right)}$

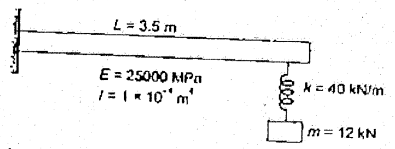
- Amplitude, $A = \sqrt{y_0^2 + \left(\dfrac{V_0}{\omega}\right)^2}$
- Maximum velocity, $V_{max} = A\omega$
- Maximum acceleration, $a_{max} = A\omega^2$
- Angular frequency, $\omega = \sqrt{\dfrac{k}{m}}$
- Time period, $T = \dfrac{2\pi}{\omega}$
- Natural frequency, $f = \dfrac{1}{T} = \dfrac{\omega}{2\pi} = \dfrac{1}{2\pi}\sqrt{\dfrac{k}{m}}$
- Total energy of motion, $E = \dfrac{1}{2}kA^2$

Example 11.1 Evaluate the natural frequency (radian/second) and natural period for the structural systems shown below in figure.

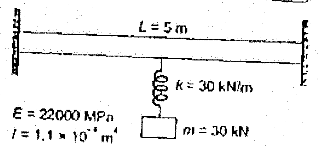
(a)



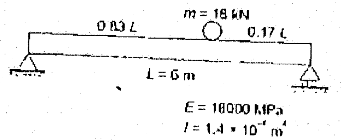
(b)



(c)

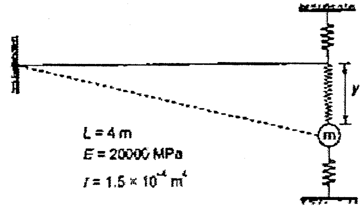


(d)



Solution:

(a) For given system, providing a small displacement 'y' in the beam of mass 'm'



Resistance of beam $= \dfrac{3EI}{L^3}y$

Total force on free end, $F = -k_b y - ky - ky$

$= \dfrac{-3EI}{L^3}y - 2ky$

$= -\left[\dfrac{3\times 20000\times 10^6\times 1.5\times 10^{-4}}{4^3} + 2\times 45\times 10^3\right]y$

$= -230625\, y$

Mass, $m = \dfrac{10000}{9.81} = 1019.37$ kg

Using Newton's 2nd law, $F = ma$

i.e., $101019.37\ddot{y} + 230625y = 0$

Using general solution $y = e^{bt}$

$\Rightarrow$ $1019.37\, b^2 + 230625 = 0$

$\Rightarrow$ $b = \pm 15.04\, i$

Solution, $y = C_1 \cos 15.04\, t + C_2 \sin 15.04\, t$

Angular frequency, $\omega = 15.04$ rad/sec

Natural period, $T = \dfrac{2\pi}{\omega} = \dfrac{2\pi}{15.04} = 0.418$ sec

Alternate Solution (a)

(a) Stiffness of beam $k_b = \dfrac{3EI}{L^3}$

$k_b = \dfrac{3\times 3\times 10^6}{4^3} = 1.406\times 10^5$ N/m

$m = \dfrac{10000}{9.81} = 1019.37$ kg

Now, equivalent spring constant (k_{eq}) is

$k_{eq} = k_b + 2k$

(because spring have the same displacement thus are in parallel connection)

$k_{eq} = 1.406\times 10^5 + 2(45\times 10^3)$

$k_{eq} = 230.6\times 10^3$ N/m

Natural frequency is given by

$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{230.6\times10^3}{1019.37}}$$

$$\omega_n = 15.04 \text{ radians/second}$$

$$\text{Natural period, } T = \frac{2\pi}{\omega_n} = 0.42\text{sec}$$

(b)

$$k_b = \frac{3EI}{L^3} = \frac{3\times2.5\times10^6}{3.5^3} = 174.927\times10^3 \text{ N/m}$$

$$m = \frac{12000}{9.81} = 1223.24 \text{ kg}$$

$$\frac{1}{k_{eq}} = \frac{1}{k_b}+\frac{1}{k} = \frac{1}{174.927\times10^3}+\frac{1}{40\times10^3}$$

$$k_{eq} = 32.56\times10^3 \text{ N/m}$$

(because spring and the beam have same force acting, thus the spring and beam are in series)

$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{32.56\times10^3}{1223.24}} = 5.16 \text{ radian/sec}$$

$$\text{Natural period, } T = \frac{2\pi}{\omega_n} = 1.22 \text{ sec}$$

(c)

$$k_b = \frac{192EI}{L^3} = \frac{192\times2.2\times1.1\times10^6}{5^3} = 3717.12\times10^3 \text{ N/m}$$

$$m = \frac{30000}{9.81} = 3058.104 \text{ kg}$$

Equivalent spring constant, k_{eq}

$$\frac{1}{k_{eq}} = \frac{1}{k_b}+\frac{1}{k} = \frac{1}{3717.12\times10^3}+\frac{1}{30\times10^3}$$

(Because both the springs have unequal displacement, there are in series)

$$k_{eq} = 29.76\times10^3 \text{ N/m}$$

$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{29.76\times10^3}{3058.104}} = 3.12 \text{ rad/sec}$$

$$T = \frac{2\pi}{\omega_n} = 2.014 \text{ sec}$$

(d)

$$k_b = \frac{3LEI}{a^2(L-a)^2} = \frac{3LEI}{(0.83L)^2(0.17L)^2}$$

$$= \frac{3EI}{0.83^2\times0.17^2\times6^3} = \frac{3\times1.8\times1.4\times10^6}{0.83^2\times0.17^2\times6^3}$$

$$= 1757.98 \text{ kN/m}$$

$$m = \frac{18000}{9.81} = 1834.86 \text{ kg}$$

$$k_{eq} = k_b$$

$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{1757.98\times10^3}{1834.86}} = 30.95 \text{ rad/sec}$$

$$T = \frac{2\pi}{\omega_n} = 0.203 \text{ sec}$$

Example 11.2 A single degree of freedom system (SDOF) of mass m and stiffness k is found to vibrate with a period of 0.19 sec. When mass in increased by 4 kg, the period recorded is 0.22 sec. Determine the mass and stiffness for the original system as shown in figure.

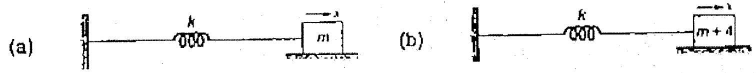


Figure (a) and (b) showing original and modified system.

Solution:

Angular frequency of original system is

$$\omega_n = \frac{2\pi}{T} = \frac{2\pi}{0.19} = 33.07 \text{ rad/sec}$$

Angular frequency of modified system is

$$\omega_m = \frac{2\pi}{T} = \frac{2\pi}{0.22} = 28.56 \text{ rad/sec}$$

Now,

$$\omega_n^2 = \frac{k}{m} \text{ and } \omega_m^2 = \frac{k}{m+4}$$

$$\Rightarrow \quad \left(\frac{\omega_n}{\omega_m}\right)^2 = \left(\frac{m+4}{m}\right) = \left(\frac{33.07}{28.56}\right)^2$$

$$m = 11.74 \text{ kg}$$

Also,

$$\omega_n^2 = \frac{k}{m}$$

$$33.07^2 = \frac{k}{11.74}$$

$$k = 12.84 \text{ kN/m}$$

Example 11.3 A dynamic system has the maximum velocity of 135 mm/s and the natural period of 1.5 sec. If the initial displacement is 5 mm, determine the amplitude, the initial velocity and the maximum acceleration.

Solution:

Natural frequency is,

$$\omega_n = \frac{2\pi}{T} = \frac{2\pi}{1.5}$$

$$\omega_n = 4.189 \text{ rad/sec}$$

Maximum velocity is,

$$\dot{y} = A\omega_n$$

$$\dot{y} = 0.135 = A\times4.189$$

$\Rightarrow$ Amplitude, $A = 0.032\text{ m}$

Now,

$$A = \sqrt{y_0^2 + \left(\frac{\dot{y}_0}{\omega_n}\right)^2}$$

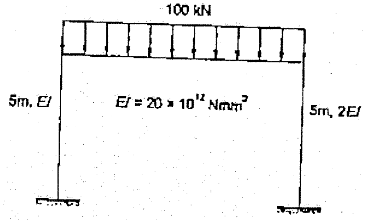
$$0.032 = \sqrt{0.005^2 + \left(\frac{V_0}{4.189}\right)^2}$$

$\Rightarrow$ V_0 (Initial velocity) $= 0.133$ m/s

And maximum acceleration is

$$\ddot{y}_{max} = A\omega_n^2 = 0.032 \times 4.189^2 = 0.561 \text{ m/s}^2$$

Example 11.4 Compute the natural frequency in side sway for the frame is figure below. If the initial displacement is 35 mm and the initial velocity is 20 mm/s, what is the amplitude and displacement at $t = 2$ sec?



Solution:

Equivalent lateral stiffness of columns is

$$k = \frac{12EI}{L^3} + \frac{12(2EI)}{L^3} = \frac{36EI}{L^3}$$

$$k = \frac{36 \times 20 \times 10^{12}}{(5000)^3} = 5760 \text{ N/mm} = 5.76 \times 10^5 \text{ N/m}$$

Angular frequency is,

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{5.76 \times 10^6}{\frac{100000}{9.81}}} = 23.77 \text{ rad/sec}$$

Amplitude is

$$A = \sqrt{y_0^2 + \left(\frac{V_0}{\omega_n}\right)^2} = \sqrt{35^2 + \left(\frac{20}{23.77}\right)^2} = 35.01 \text{ mm}$$

Displacement at $t = 2$s is

$$y = y_0 \cos\omega_n t + \frac{V_0}{\omega_n}\sin\omega_n t$$

$$= 35\cos(23.77 \times 2) + \frac{20}{23.77}\sin(23.77 \times 2)$$

$$= -32.35 \text{ mm}$$

B. Damped Single-degree-of-freedom Systems

In undamped system it was assumed that there is no energy loss in the system during motion but in reality there are always forces acting such as friction, which results in energy losses in the system. Since these energy losses are essentially due to damping out vibration of a structure, this energy loss mechanism is called damping. There is no knowledge of actual energy loss mechanism, so some assumptions are made to model these energy losses.

11.9 Viscous Damping

- Damping forces in dynamic analysis are assumed to be proportional to magnitude of the velocity and opposite to the direction of the motion.
- This type of damping is known as viscous damping as it is equivalent to a force developed in a body restrained in its motion by a surrounding viscous fluid.
- This is an assumption made for simple mathematical analysis.

11.10 Equation of Motion and it's Analysis

- Model of a structural system oscillator with viscous damping is shown in figure below:

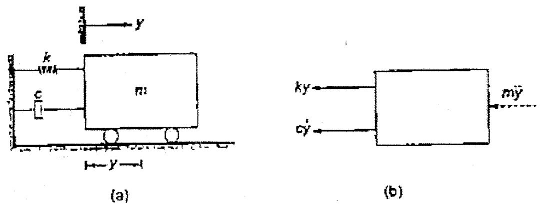


Fig. 11.7 (a) Viscous damped oscillator (b) Free body diagram

- Here m, k and c are the mass, spring constant of oscillator and viscous damping coefficient, respectively
- From figure (b), we have differential equation

$$m\ddot{y} + c\dot{y} + ky = 0 \quad \text{...2 (i)}$$

Taking trial solution as $y = e^{Pt}$ in equation 2(i)

$$mP^2 e^{Pt} + cPe^{Pt} + ke^{Pt} = 0 \quad \text{...2(ii)}$$

After removing common factors,

$$mP^2 + cP + k = 0 \quad \text{...2(iii)}$$

Roots of above quadratic equation are

$$P_1 = \frac{c}{2m} + \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}$$

and $$P_2 = \frac{c}{2m} - \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}$$

The general solution of equation $m\ddot{y} + c\dot{y} + ky = 0$ is obtained by superposing the two possible solution, i.e.,

$$y = C_1 e^{P_1 t} + C_2 e^{P_2 t} \quad \text{...2(iv)}$$

where C_1 and C_2 are constants of integration determined by using boundary conditions.

NOTE

As seen above roots P_1 and P_2 have radical under square root, based on relative values of $\frac{c}{2m}$ and $\frac{k}{m}$, radial under square root may be zero, positive or negative and thus accordingly there are three cases: critically damped system, overdamped systems and underdamped systems.

NOTE

As we know natural angular frequency $\omega_n = \sqrt{\frac{k}{m}}$, so roots of equation $m\ddot{y} + c\dot{y} + ky = 0$ also be written as

$$P_1 = -\frac{c}{2m} + \sqrt{\left(\frac{c}{2m}\right)^2 - \omega_n^2}, \qquad P_2 = -\frac{c}{2m} - \sqrt{\left(\frac{c}{2m}\right)^2 - \omega_n^2}$$

11.11 Critically Damped System

- For a system oscillating with critical damping, the expression under the radical in equation (2(iii)) is equal to zero, that is

$$\left(\frac{c_{cr}}{2m}\right)^2 - \frac{k}{m} = 0 \quad \text{...2 (v)}$$

$$c_{cr} = 2\sqrt{km} = 2m\omega_n = \frac{2k}{\omega_n}$$

where, c_{cr} is critical damping coefficient

NOTE

- Damping ratio (ξ) is the ratio of actual system damping (c) to the critical damping (c_{cr})

$$\Rightarrow \xi = \frac{c}{c_r} = \frac{c}{2m\omega_n} = \frac{c\omega_n}{2k}.$$

- If $\xi = 1$, system is critically damped, meaning there will be no oscillation and the system will reach equilibrium quickly
- If $\xi < 1$, system is under damped, so it will oscillate around the mean value, with decreasing amplitude.
- If $\xi > 1$, system is overdamped and will reach equilibrium slowly with no oscillation.

- Thus the roots of equation: $mP^2 + cP + k = 0$ is

$$P_1 = P_2 = \frac{-c_{cr}}{2m} = \frac{-c_{cr}\omega_n}{2m\omega_n} = -\xi\omega_n$$

- Therefore, the solution is

$$y = (C_1 + C_2 t)e^{(-c_{cr}/2m)t} \quad \text{...2 (vi)}$$

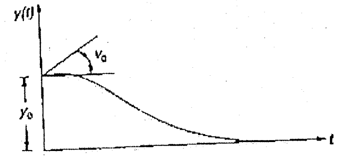


Fig. 11.8 Free vibration response with critical damping

11.12 Overdamped System

- Here, $$\left[\left(\frac{c}{2m}\right)^2 - \omega_n^2\right] > 0 \quad \text{...2 (vii)}$$

$\Rightarrow$ Two roots are real and negative.

$\Rightarrow$ The solution is given by (2(iv)), that is

$$y = C_1 e^{P_1 t} + C_2 e^{P_2 t}$$

NOTE

- In an overdamping system, the damping coefficient is greater than the value for critical damping, that ie $c > c_{cr}$.
- In an over damped system, the magnitude of oscillation decays exponentially with time to zero. Thus motion is not oscillatory.
- The response of the over damped system is similar to the response of the critical damped system (as shown in figure above). But the return to the natural position requires extra time as the damping value is increased.

11.13 Underdamped System

- Here, $$\left[\left(\frac{c}{2m}\right)^2 - \omega_n^2\right] < 0 \quad \text{...2 (viii)}$$

$\Rightarrow$ Two roots are completed conjugates, that is

$$P_1 = \frac{-c}{2m} + i\sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2}$$

$$P_2 = \frac{-c}{2m} - i\sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2} \quad \text{...2 (ix)}$$

where, $i = \sqrt{-1}$

Note: In underdamped system, the value of damping coefficient is less than the critical value that $C < C_r$

- The solution of the underdamped system makes use of Euler's equations

$$\left.\begin{aligned} e^{ix} &= \cos x + i\sin x \\ e^{-ix} &= \cos x - i\sin x \end{aligned}\right\} \qquad \text{...2 (x)}$$

and substitution of the roots from (2(ix)) into (2(iv)), along with the use of (2(x)).
The general solution is

$$y = e^{-(c/2m)t}(A\cos\omega_D t + B\sin\omega_D t) \qquad \text{...2 (xi)}$$

where, A and B are constants of integration and ω_D is the damped frequency of system

$$\omega_D = \sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2}$$

$$\omega_D = \omega_n\sqrt{1-\xi^2}$$

where,

$$\omega_n = \sqrt{\frac{k}{m}} \text{ and } \xi = \frac{c}{c_{cr}}$$

- Now, initial conditions of displacement (y_0) and velocity (V_0) are used. and constants of integration are evaluated and substituted into equation (2(xi))

$$y(t) = e^{-\xi\omega_n t}\left(y_0\cos\omega_D t + \frac{V_0 + y_0\xi\omega}{\omega_D}\sin\omega_D t\right) \qquad \text{...2 (xii)}$$

- Alternatively, the expression can be written as

$$y(t) = Ce^{-\xi\omega_n t}\cos(\omega_D t - \alpha) \qquad \text{...2(xiii)}$$

$\Rightarrow$ Amplitude,

$$C = \sqrt{y_0^2 + \frac{(V_0 + y_0\xi\omega_n)^2}{\omega_D^{\,2}}}$$

and

$$\tan\alpha = \frac{(V_0 + y_0\xi\omega_n)}{\omega_D y_0}$$

- A graphical record of the response of an underdamped system is shown below:

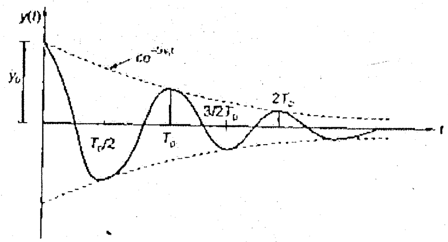


Fig. 11.9 Free vibration response for underdamped system

$\Rightarrow$ Motion is oscillatory, not periodic

$$T_D = \frac{2\pi}{\omega_D} = \frac{2\pi}{\omega_n\left(\sqrt{1-\xi^2}\right)}$$

NOTE

- Real structures damping coefficient is much less than the critical damping coefficient. It is generally between 2 to 20% of the critical damping value.
- Structure with damping coefficient as much as 20% is essentially equal to the undamped natural frequency system.

- Figure below shows comparative study of damped systems.

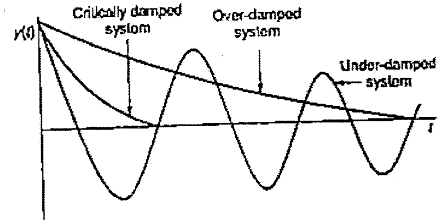


Fig. 11.10 The motion of damped dynamic systems

11.14 Logarithmic Decrement

- The decay is expressed as the logarithmic decrement (δ), which is used to find out damping ratio of underdamped system in time domain is natural logarithm of the ratio of any two successive peak amplitudes (y_1 and y_2) in free damped vibration

$$\delta = \ln\frac{y_1}{y_2} \qquad \text{...2 (xiv)}$$

$$\delta = 2\pi\xi\sqrt{1-\xi^2} \qquad \text{...2 (xv)}$$

- For small value of damping ratio,

$$\delta = 2\pi\xi \qquad \text{...2 (xvi)}$$

Example 11.5 A vibrating system of mass 20 kg and spring with stiffness 32 N/mm is damped so that the ratio of two consecutive amplitudes is 1.00 to 0.91. Determine

(a) the natural frequency of the undamped system
(b) the logarithmic decrement
(c) the damping ratio
(d) the damped coefficient
(e) the damped natural frequency

Solution:

(a) Undamped natural frequency (f).

$$k = 32 \text{ N/mm} = 32000 \text{ N/m}$$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{32000}{20}} = 40 \text{ rad/sec}$$

$$f = \frac{\omega}{2\pi} = 6.37 \text{ cycles/sec}$$

(b) The logarithmic decrement

$$\delta = \ln\frac{y_1}{y_2} = \ln\frac{1.00}{0.91} = 0.094$$

(c) The damping ratio is

$$\xi = \frac{\delta}{2\pi} = \frac{0.094}{2\pi} = 0.015$$

(d) The damped coefficient

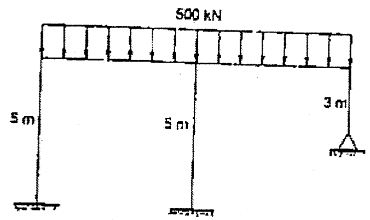
$$c = \xi c_{cr}$$
$$= 0.015 \times 2\sqrt{km}$$
$$= 0.015 \times 2\sqrt{32000 \times 20}$$
$$= 24 \text{ Ns/m}$$

(e) The natural frequency of the damped system

$$\omega_D = \omega_n\sqrt{1-\xi^2} = 40\sqrt{1-0.015^2}$$
$$= 39.99 \text{ rad/esc} \simeq \omega_n$$

Example 11.6 For the frame shown below, the initial displacement is 25 mm and the initial velocity is 13 mm/s. Damping is 10% of critical. Cross-sectional dimension of each column is 250 mm × 300 mm and elastic modulus is 22.8 GPa. Determine in rad/sec

(a) the natural frequency (rad/s)
(b) the amplitude
(c) the expression for the displacement



Solution:

$$I_{column} \text{ (moment of inertia of each column)} = \frac{250 \times 300^3}{12} = 562.5 \times 10^6 \text{ mm}^4$$

Flexural rigidity,

$$EI = 22.8 \times 10^3 \times 562.5 \times 10^6$$
$$= 12825 \times 10^9 \text{ N-mm}^2$$

Equivalent lateral stiffness of columns (in parallel) is

$$k = \frac{2 \times 12EI}{L_1^3} + \frac{3 \times EI}{L_2^3}$$

$$= \frac{2 \times 12 \times 1.2825 \times 10^{13}}{(5000)^3} + \frac{3 \times 1.2825 \times 10^{13}}{(3000)^3}$$

$$= 3887.4 \text{ N/mm}$$

(a) Natural frequency,

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{3887.4}{\frac{500 \times 10^3}{9810}}} = 8.73 \text{ rad/sec}$$

Damped natural frequency,

$$\omega_D = \omega_n\sqrt{1-\xi^2}$$

$$\omega_D = 8.73\sqrt{1-0.1^2} = 8.69 \text{ rad/sec}$$

(b) Amplitude,

$$A = \sqrt{y_0^2 + \left(\frac{\dot{y}_0 + \omega_n \xi y_0}{\omega_D}\right)^2}$$

$$A = \sqrt{25^2 + \left(\frac{13 + 8.73 \times 0.1 \times 25}{8.69}\right)^2}$$

$$A = 25.32 \text{ mm}$$

(c) Expression for displacement

$$y = e^{-\omega_n \xi t}\left[y_0 \cos\omega_D t + \frac{\dot{y}_0 + \omega_n \xi y_0}{\omega_D}\sin\omega_D t\right]$$

$$y = e^{-8.73 \times 0.1t}\left[25\cos 8.69t + \frac{13 + 8.73 \times 0.1 \times 25}{8.69}\sin 8.69t\right]$$

$$y = e^{-0.873t}\left[25\cos 8.69t + 4.01\sin 8.69t\right] \text{ mm}$$

C. Response of One-degree-of-freedom System to Harmonic Loading

- Structures supporting the reciprocating or rotating machines are subjected to harmonic excitations. Such excitations are functions of sine or cosine with respect to time.

11.15 Undamped Harmonic Excitation

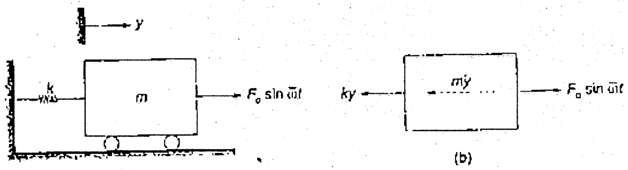


Fig. 11.11 (a) Undamped oscillator harmonically excited (b) Free body diagram

- $F_0 \sin\bar{\omega}t$ is harmonic force; where F_0 is peak amplitude and $\bar{\omega}$ is the frequency of the force in radians per second.
- The differential equation: $m\ddot{y} + ky = F_0 \sin\bar{\omega}t$ ...3 (i)
- Solution of equation $m\ddot{y} + ky = F_0 \sin\bar{\omega}t$ is

$$y(t) = y_c(t) + y_p(t) \qquad ...3\text{ (ii)}$$

where, $y_c(t)$ is complementary solution satisfying the homogeneous equation that is left hand side of equation (3(i)) set equal to zero.

$y_p(t)$ is particular solution based on the solution satisfying the non-homogeneous differential equation (3(i))

- The complementary solution,

$$y_c(t) = A\cos\omega t + B\sin\omega t \qquad ...3\text{ (iii)}$$

where, $\omega = \sqrt{\dfrac{k}{m}}$, A and B are constant of integration

- Expression of forcing function in (3(i)) decides the expression of particular solution, $y_p(t)$

$$y_p(t) = b\sin\bar{\omega}t \qquad ...3\text{ (iv)}$$

where, b is peak value of particular solution.

- Substitute solution (3(iv)) in (3(i)) and on cancelling the common factors; we have

$$-m\bar{\omega}^2 b + kb = F_0$$

$$\Rightarrow \quad b = \frac{F_0}{k - m\bar{\omega}^2} = \frac{F_0/k}{1-r^2} \qquad ...3\text{ (v)}$$

where, r is frequency ratio of the applied frequency to the natural frequency of vibration of the system, i.e.,

$$r = \frac{\bar{\omega}}{\omega}$$

- Thus, solution for equation (3(i)) is

$$y(t) = A\cos\omega t + B\sin\omega t + \frac{F_0/k}{1-r^2}\sin\bar{\omega}t \qquad ...3\text{ (vi)}$$

- Assume initial condition (at $t = 0$) taking y_0 and V_0 equal to zero, and applying to 3 (vi)

$$\Rightarrow \quad A = 0$$

$$B = -\frac{rF_0/k}{1-r^2}$$

- Thus finally, we have solution of equation (3(i)) as

$$y(t) = \frac{F_0/k}{1-r^2}(\sin\bar{\omega}t - r\sin\omega t) \qquad ...3\text{ (vii)}$$

NOTE

- From solution (3(vii)) it is clear that the resulting motion due to superposition of two harmonic terms of different frequencies is not harmonic.
- The last term in solution (3(vii)) is transient response because damping forces will always present and causes the last term to vanish eventually. However the first term is

$y(t) = \dfrac{F_0/k}{1-r^2}\sin\bar{\omega}t$ is known as the steady state response.

- If $r = 1$ i.e., when applied frequency is equal to natural frequency, then amplitude becomes infinitely large. The phenomena is called resonance.

11.16 Damped Harmonic Excitation

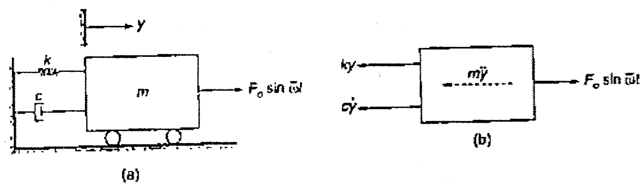


Fig. 11.12 (a) Damped oscillator harmonically excited (b) Free body diagram

- The differential equation of motion

$$m\ddot{y} + c\dot{y} + ky = F_0\sin\bar{\omega}t \qquad ...3\text{ (viii)}$$

- The solution is given as

$$y(t) = y_c(t) + y_p(t)$$

- For underdamped case ($c < c_c$), as given in equation (2(xi))

$$y_c(t) = e^{-\xi\omega t}\left(A\cos\omega_D t + B\sin\omega_D t\right) \qquad ...3\text{ (ix)}$$

- For determination of $y_p(t)$:

Assume $y_p(t)$ of the form

$$y_p(t) = C_1\sin\bar{\omega}t + C_2\cos\bar{\omega}t$$

Substitute $y_p(t)$ above expression in equation (3(viii)) and equate the coefficients of sine are cosine functions.

- **Alternatively, we use Euler's relation (preferred)**

$$e^{i\bar{\omega}t} = \cos\bar{\omega}t + i\sin\bar{\omega}t$$

Thus, $\quad m\ddot{y} + c\dot{y} + ky = F_0\sin\omega t$

Can be written as

$$m\ddot{y} + c\dot{y} + ky = F_0 e^{i\bar{\omega}t} \qquad ...3\text{(x)}$$

but here, only imaginary component of $F_0 e^{i\bar{\omega}t}$ that is the imaginary force component $F_0\sin\bar{\omega}t$ is acting. It means response consists only of the imaginary part of the total solution of equation (3(x))

- Thus the particular solution of equation (3(x)) will be

$$y_p = be^{i\bar{\omega}t} \quad ...3(xi)$$

Substituting equation (3(xi)) into equation (3(x)) we obtain

$$-m\bar{\omega}^2 b + ic\bar{\omega}b + kb = F_0$$

$$\Rightarrow \quad b = \frac{F_0}{k - m\bar{\omega}^2 + ic\bar{\omega}}$$

and

$$y_p = \frac{F_0 e^{i\bar{\omega}t}}{k - m\bar{\omega}^2 + ic\bar{\omega}} \quad ...3(xii)$$

- The complex denominator (3(xii)) can be written using polar coordinate

$$y_p = \frac{F_0 e^{i\bar{\omega}t}}{\left(\sqrt{(k - m\bar{\omega}^2)^2 + (c\bar{\omega})^2}\right)e^{i\theta}}$$

or

$$y_p = \frac{F_0 e^{i(\bar{\omega}t - \theta)}}{\sqrt{(k - m\bar{\omega}^2)^2 + (c\bar{\omega})^2}} \quad ...3(xiii)$$

where,

$$\tan\theta = \frac{c\bar{\omega}}{k - m\bar{\omega}^2} \quad ...3(xiv)$$

- The response of the force $F_0 \sin\bar{\omega}t$ (imaginary component of $F_0 e^{i\bar{\omega}t}$) is the imaginary component of equation (3(xiii)), that is

$$y_p = \frac{F_0 \sin(\bar{\omega}t - \theta)}{\sqrt{(k - m\bar{\omega}^2)^2 + (c\bar{\omega})^2}} \quad ...3(xv)$$

or

$$y_p = y_1 \sin(\bar{\omega}t - \theta) \quad ...3(xvi)$$

where,

Amplitude of steady state motion, $y_1 = \dfrac{F_0}{\sqrt{(k - m\bar{\omega}^2)^2 + (c\bar{\omega})^2}}$

- Equation (3(xv) and (3(xvi)) can be easily written in terms of dimensionless ratios

$$y_p = \frac{y_{st}\sin(\bar{\omega}t - \theta)}{\sqrt{(1 - r^2)^2 + (2\xi r)^2}}$$

and

$$\tan\theta = \frac{2\xi r}{1 - r^2}$$

where, y_{st} (static deflection of the spring) $= F_0/k$

- Therefore, the overall response is

$$y(t) = e^{-\xi\omega t}(A\cos\omega_D t + B\sin\omega_D t) + \frac{y_{st}\sin(\bar{\omega}t - \theta)}{\sqrt{(1 - r^2)^2 + (2r\xi)^2}} \quad ...3(xvii)$$

- Dynamic magnification factor, D, is given as

$$D = \frac{y_1 \text{(steady state amplitude)}}{y_{st} \text{(static deflection)}} \quad ...3(xviii)$$

$$= \frac{1}{\sqrt{(1 - r^2)^2 + (2r\xi)^2}} \quad ...3(xix)$$

- Figure below shows that for light damped system, peak amplitude occurs at r (frequency ratio) close to one i.e., D(dynamic magnification factor) has maximum value virtually at resonance ($r = 1$)
- It can be observed from equation (3(xix)) that at $r = 1$, we obtain

$$D = \frac{1}{2\xi} \quad ...3(xx)$$

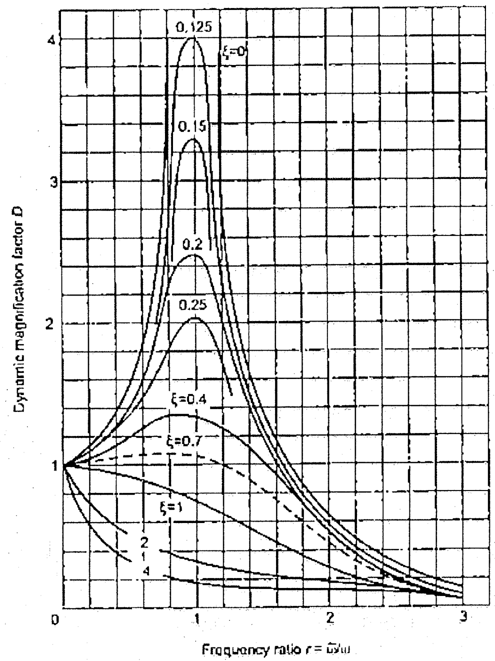


Fig.11.13 Dynamic magnification factor as a function of the frequency ratio for various amounts of damping

Example 11.7 Calculate damping coefficient of a viscously damped system having mass of 4 kg undergoing resonant amplitude of 1.2 m with a period of 0.3 s, when subjected to harmonic force of peak amplitude 270 N.

Solution:

At resonance,
$$r = \frac{\bar{\omega}}{\omega_n} = 1$$

$$\bar{\omega} = \omega_n = \frac{2\pi}{T} = \frac{2\pi}{0.3} = 20.94 \text{ rad/sec}$$

$\Rightarrow$
$$k = \bar{\omega}^2 \times m = 20.94^2 \times 4 \simeq 1754 \text{ N/m}$$

Now,
$$\frac{\text{Amplitude } (y_{max})}{y_{st}} = \frac{1}{2\xi} \quad \left(\text{because } D = \frac{1}{2\xi} \text{ at resonance}\right)$$

$$\xi = \frac{F_0/k}{2y_{max}} = \frac{270/1754}{2 \times 1.2} = 0.0641$$

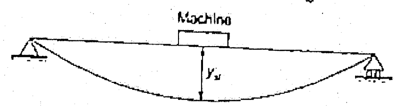
or
$$\xi = 6.41\%$$

Now,
$$\xi = \frac{c}{c_{cr}} = \frac{c}{2m\omega_n}$$

$$0.0641 = \frac{c}{2 \times 4 \times 20.94}$$

$$c = 10.74 \text{ Ns/m}$$

Example 11.8 Determine the magnification factor of forced vibration produced by a machine operating at a speed of 1000 rpm, installed at midspan of the beam. The static deflection at midspan due to weight of machine is 0.5 mm and the weight of machine is 7000 N. The viscous damping force is 400 N at a velocity of 30 mm/s. Assume negligible beam weight.



Solution:

Applied frequency,
$$\bar{\omega} = 1000 \text{ rpm}$$

$$\bar{\omega} = \frac{1000}{60} \times 2\pi = 104.72 \text{ rad/sec}$$

Damping force,
$$c\dot{y} = 400 \text{ N}$$

$$c = \frac{400}{\dot{y}} = \frac{400}{30} = 13.33 \text{ Ns/mm}$$

Stiffness,
$$k = \frac{F_0}{y_{st}} = \frac{7000}{0.5} = 14000 \text{ N/mm}$$

Natural frequency,
$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{14000}{\frac{7000}{9810}}}$$

$$\omega_n = 140.07 \text{ rad/sec}$$

Critical damping coefficient, c_{cr}

$$c_{cr} = 2m\omega_n = 2 \times \frac{7000}{9810} \times 140.07 = 199.9 \text{ Ns/mm}$$

$$\xi \text{ (damping ratio)} = \frac{c}{c_{cr}} = \frac{13.33}{199.9} = 0.067$$

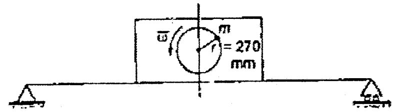
$\Rightarrow$
$$\xi = 6.7\%$$

$$r \text{ (frequency ratio)} = \frac{\bar{\omega}}{\omega} = \frac{104.72}{140.07} = 0.75$$

$$D \text{ (dynamic magnification factor)} = \frac{1}{\sqrt{(1-r^2)^2 + (2r\xi)^2}} = \frac{1}{\sqrt{(1-0.75^2)^2 + (2 \times 0.75 \times 0.067)^2}}$$

$$= 2.23$$

Example 11.9 A simply supported beam having a span of 4 m supports a machine having weight of 50 kN at it midspan as shown in figure below. The motor runs at 300 rpm and the rotor is out of balance to the extent of w(weight of rotor) = 400 N at a radius r(eccentricity) = 270 mm. Calculate the amplitude at steady state response. Assume viscous damping of system to be 10% of critical damping, elastic modulus is 2×10^5 N/mm² and moment of inertia is 50×10^8 mm⁴.



Solution:

Applied/excitation frequency,
$$\bar{\omega} = 300 \text{ rpm} = \frac{300}{60} \times 2\pi = 10\pi = 31.42 \text{ rad/sec}$$

Amplitude of excitation force,
$$F_0 = mr\bar{\omega}^2 = \frac{400}{9810} \times 270 \times 31.42^2 = 10868.44 \text{ N}$$

Stiffness of the beam,
$$k = \frac{48EI}{L^3}$$

$$k = \frac{48 \times 2 \times 10^5 \times 50 \times 10^6}{4000^3} = 7500 \text{ N/mm}$$

Mass of the motor,
$$m_m = \frac{50000}{9.81} = 5096.84 \text{ kg}$$

Natural frequency,
$$\omega_n = \sqrt{\frac{k}{m_m}} = \sqrt{\frac{7500 \times 10^3}{5096.84}} = 38.36 \text{ rad/sec}$$

Frequency ratio,
$$r = \frac{\bar{\omega}}{\omega_n} = \frac{31.42}{38.36} = 0.819$$

Since, damping ratio,
$$\xi = 0.10$$

Amplitude of steady state response motion is

$$y_{max} = \frac{F_0/k}{\sqrt{\left(1-r^2\right)^2+(2r\xi)^2}}$$

$$= \frac{10868.44/7500}{\sqrt{\left(1-0.819^2\right)^2+(2\times0.819\times0.1)^2}}$$

$$= 3.94 \text{ mm}$$

11.17 Transmissibility of Force

- When a machine is kept on a floor, vibration due to machine is transferred to floor or vice-versa. The ratio of maximum force on the floor as a result of vibration of a machine to maximum machine force.
- Figure below shows the support of the simple oscillator structure subjected to harmonic motion.

It is given by

$$y_s(t) = y_0 \sin\bar{\omega}t \qquad ...3\ (xxi)$$

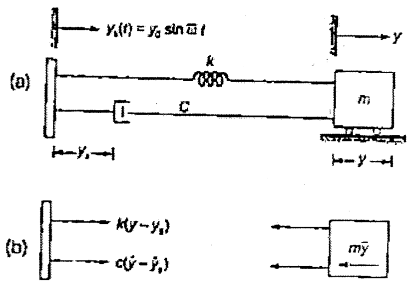


Fig.11.14 (a) damped oscillator harmonically excited through its support. (b) free body diagram.

- From figure, the differential equation is

$$m\ddot{y}+c(\dot{y}-\dot{y}_s)+k(y-y_s) = 0 \qquad ...3(xxii)$$

- Solving (3(xxi)) and (3(xxii)) equations simultaneously, we have solution

$$y(t) = \frac{F_0/k\sin(\bar{\omega}t+\beta-\theta)}{\sqrt{\left(1-r^2\right)^2+(2r\xi)^2}} \qquad ...3(xxiii)$$

where,

$$F_0 = y_0\sqrt{k^2+(c\bar{\omega})^2} = y_0k\sqrt{1+(2r\xi)^2}$$

and

$$\tan\beta = \frac{c\bar{\omega}}{k} = 2r\xi$$

- Also

$$\frac{y(t)}{y_0} = \frac{\sqrt{1+(2r\xi)^2}}{\sqrt{\left(1-r^2\right)^2+(2r\xi)^2}}\sin(\bar{\omega}t+\beta-\theta) \qquad ...3\ (xxiv)$$

This expression is for the relative transmission of the support motion of the oscillator.

- Equation (3(xxiv)) is important for vibration isolating equipment for protecting the structure. The degree of relative isolation is called transmissibility, defined as

Transmissibility,

$$T_r = \frac{y \text{ (amplitude of motion of the oscillator)}}{y_0}$$

$$= \left(\frac{(1+2r\xi)^2}{(1-r)^2+(2r\xi)^2}\right)^{1/2}$$

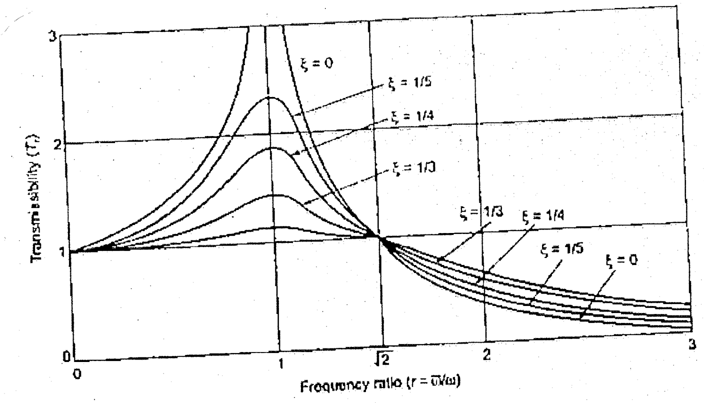


Fig. 11.15 Transmissibility versus frequency ratio for vibration isolation

NOTE

From Fig. 11.15 we can see that, when frequency ratio

(i) $r = 0$ or $\sqrt{2}$, $T_r = 1$

(ii) $0 < r < \sqrt{2}$, $T_r > 1$

(iii) $r > \sqrt{2}$, $T_r < 1$

(iv) $r = 1$, T_r is very high

- The steady state response in terms of relative motion between the mass m and the support is

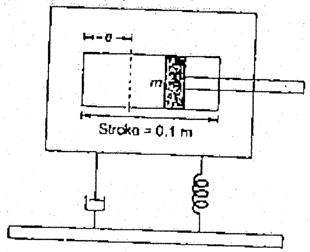
$$\frac{U(t)}{y_0} = \frac{r^2\sin(\bar{\omega}t-\theta)}{\sqrt{\left(1-r^2\right)^2+(2r\xi)^2}}$$

where,

$$\tan\theta = \frac{2\xi r}{1-r^2}$$

Note: Maximum force (F_T) transmitted to the foundation is; $F_T = F_0T_r$

Example 11.10 Figure shown below illustrates a machine weighing 90 kg mounted on a spring of stiffness k = 18 x 10⁵ N/m with damping factor 15%. A 3 kg piston within machine has reciprocating motion, with a stroke of 0.1 m and speed 3500 cpm. Assume piston motion to be harmonic. Calculate the steady state amplitude of vibration of machine and vibrating force transmitted to the foundation.



Solution:

$$\bar{\omega} = 3500 \text{ rpm} = \frac{3500}{60} \times 2\pi = 366.52 \text{ rad/sec}$$

Amplitude of excitation force, $F_0 = me\bar{\omega}^2$

$$F_0 = 3 \times 0.05 \times 366.52^2 = 20150.54 \text{ N} \quad \left(e = \frac{0.1}{2} = 0.05 \text{ m}\right)$$

Also,

stiffness $(k) = 18 \times 10^5$ N/m

Mass $(m) = 90$ kg

$\Rightarrow$ Natural frequency $(\omega_n) = \sqrt{\frac{k}{m}} = \sqrt{\frac{18 \times 10^5}{90}} = 141.42$ rad/sec

Frequency ratio $(r) = \frac{\bar{\omega}}{\omega_n} = \frac{366.52}{141.42} = 2.592$

Damping ratio, $\xi = 0.15$

Amplitude of steady state motion,

$$y_{max} = \frac{F_0/k}{\sqrt{(1-r^2)^2 + (2r\xi)^2}}$$

$$= \frac{20150.54 / 18 \times 10^5}{\sqrt{(1-2.592^2)^2 + (2 \times 2.592 \times 0.15)^2}}$$

$$= 1.94 \times 10^{-3} \text{ m} = 1.94 \text{ mm}$$

$$\text{Transmissibility } (T_r) = \frac{\sqrt{1+(2r\xi)^2}}{\sqrt{(1-r^2)^2 + (2r\xi)^2}} = 0.22$$

Force transmitted $(F_T) = F_0 \times T_r$

$= 20150.54 \times 0.22$

$= 4433.12$ N

D. Response of One-degree Freedom of System to Forced General Dynamic Loading

- Real structures are subjected to loads which are not harmonic.
- For such cases, response could be obtained through integration technique.

11.18 Impulsive Loading and Duhamel's Integral

- The impulsive loading is applied for short duration and corresponding impulse is product of the force and duration time i.e., $dP = F(t) \cdot dt$
- By Newton's law of motion

$$\frac{\partial P}{\partial t} = F(t)$$

$$m\frac{dv}{dT} = F(\tau)$$

Rearranging: $$dv = \frac{F(\tau)d\tau}{m}$$

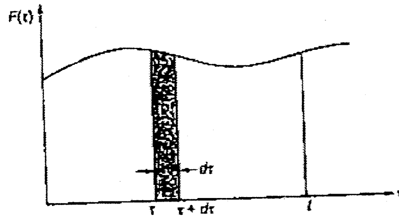


Fig. 11.16 General load history as impulsive loading

- Now consider this impulse $F(\tau)\,d\tau$ acting on the undamped oscillator structure.

Using equation of motion: $y = y_0 \cos\omega t + \frac{v_0}{\omega}\sin\omega t$ and applying initial conditions: $y_0 = 0$, initial velocity $= V_0$ at time τ; we have

$$dy(t) = \frac{F(\tau)d\tau}{m\omega}\sin\omega(t-\tau)$$

Now, $$y(t) = \frac{1}{m\omega}\int_0^t F(\tau)\sin\omega(t-\tau)d\tau \qquad \text{....4 (i)}$$

In above equation, the integral part is Duhamel's Integral

- Equation (4(i)) gives total displacement produced by exciting force $F(\tau)$ acting on undamped oscillator structure (including both steady state and the transient components of the motion)
- On including initial conditions at $t = 0$; initial displacement (y_0); initial velocity (v_0); the total displacement of an undamped single degree of freedom system is

$$y(t) = y_0\cos\omega t + \frac{v_0}{\omega}\sin\omega t + \frac{1}{m\omega}\int_0^t F(\tau)\sin\omega(t-\tau)d\tau \quad \text{...4(ii)}$$

11.19 Constant Force

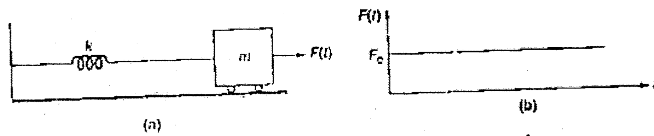


Fig. 11.17 Undamped oscillator acted upon by a constant force

- Assuming initial conditions at $t = 0$,

$$y_0 = 0$$
$$V_0 = 0; \text{ we have}$$
$$y(t) = \frac{1}{m\omega}\int_0^t F_0 \sin\omega(t-\tau)d\tau$$

- On integration

$$y(t) = \frac{F_0}{m\omega^2}\cos\omega(t-\tau)\Big|_0^t$$
$$y(t) = \frac{F_0}{k}(1-\cos\omega t) = y_{st}(1-\cos\omega t) \quad \text{...4 (iii)}$$
$$\dot{y}(t) = \frac{F_0\omega}{k}\sin\omega t$$

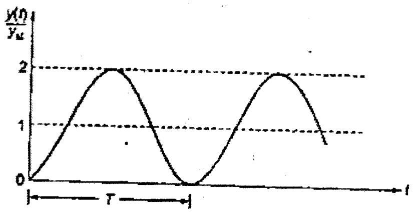


Fig. 11.18 Response of an undamped single degree-of-freedom system to a suddenly applied constant force

11.20 Rectangular Load

- A constant force F_0 is suddenly applied for limited duration as shown in figure.
- Till t_d duration, equation (4(iii)) in applicable
- At t_d time, we obtain

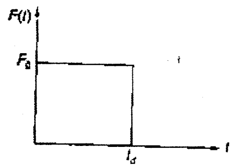


Fig. 11.19

$$y_d = \frac{F_0}{k}(1-\cos\omega t_d)$$
$$V_d = \frac{F_0}{k}\omega\sin\omega t_d$$

For the response (after t_d), we use y_d and t_d at $t = t_d$ as initial conditions and replace t by $t - t_d$ in equation motion i.e., $y = y_0\cos\omega t + \frac{V_0}{\omega}\sin\omega t$, we obtain

$$y(t) = \frac{F_0}{k}\left[(1-\cos\omega t_d)\cos\omega(t-t_d) + \sin\omega t_d \sin\omega(t-t_d)\right]$$

or

$$y(t) = \frac{F_0}{k}\{\cos\omega(t-t_d) - \cos\omega t\} \quad \text{...4 (iv)}$$

E. Fourier Analysis and Response in the Frequency Domain

- Application of Fourier series:
  (i) the response of a system to periodic forces
  (ii) and the response of a system to non-periodic forces in a frequency domain

11.21 Fourier Analysis

- Figure below shows periodic loading on a single degree of freedom system.

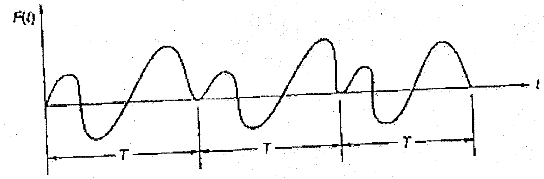


Fig. 11.20 Arbitrary periodic function

- Fourier series is summation of an infinite number of sine and cosine terms
- Fourier series can be written as

$$F(t) = a_0 + a_1\cos\bar{\omega}t + a_2\cos 2\bar{\omega}t + a_3\cos 3\bar{\omega}t + \ldots$$
$$+ a_n\cos n\bar{\omega}t + \ldots + b_1\sin\bar{\omega}t + b_2\sin 2\bar{\omega}t + \ldots b_n\sin n\bar{\omega}t + \ldots$$

or

$$F(t) = a_0 + \sum_{n=1}^{\infty}\{a_n\cos n\bar{\omega}t + b_n\sin n\bar{\omega}t\} \quad \text{...5 (i)}$$

where,

$$\bar{\omega} = \frac{2\pi}{T}$$

- The evaluation of the coefficient is done as

$$a_0 = \frac{1}{T}\int_{t_1}^{t_1+T} F(t)dt$$
$$a_n = \frac{2}{T}\int_{t_1}^{t_1+T} F(t)\cos n\bar{\omega}t\,dt \quad \text{(For } n \geq 1\text{)}$$
$$b_n = \frac{2}{T}\int_{t_1}^{t_1+T} F(t)\sin n\bar{\omega}t\,dt \quad \text{(For } n \geq 1\text{)}$$

Where, t_1 may be any value of time but usually taken as either zero or $-\frac{T}{2}$.

Note:
- a_0 represents average of the periodic function $F(t)$
- a_j and b_j are amplitudes of j^{th} harmonic of frequencies.

11.22 Response of Single Degree of Freedom Undamped System to a Periodic Force

- The response represented by Fourier series is obtained by the superposition of the response of each component of the series.
- When the transient is neglected, the response of an undamped system (is steady state response)

i.e., $y(t) = \dfrac{F_0/k}{1-r^2}\sin\bar{\omega}t$ for any sine term is given as (from equation 3(vii)):

$$y_n(t) = \frac{b_n/k}{1-r_n^2}\sin n\bar{\omega}t \qquad \text{where, } r_n = \frac{n\bar{\omega}}{\omega}$$

Also, for cosine term

$$y_n(t) = \frac{a_n/k}{1-r_n^2}\cos n\bar{\omega}t$$

- Total response:

$$y(t) = \frac{a_0}{k} + \sum_{n=1}^{\infty}\frac{1}{1-r_n^2}\left(\frac{a_n}{k}\cos n\bar{\omega}t + \frac{b_n}{k}\sin n\bar{\omega}t\right) \qquad \text{...5 (ii)}$$

where, $\dfrac{a_0}{k}$ is steady state response.

F. Multi-degree of Freedom System (MDOF)

- Practically, the multi-storey buildings are modelled and analysed as MDOF system.
- For building transformation into a discrete number of degree of freedom with lumped masses at the floor level, assumptions are required.

  There are:

  (i) Entire mass is concentrated at floor levels

  (ii) Axial forces and deformations are neglected

  (iii) The floor comprising slabs and beams are infinitely rigid as compared to the columns and remain horizontal without rotation.
- The shear building can be modelled as MDOF system. Its behaviour resembles the cantilever beam subjected to the shear forces only.

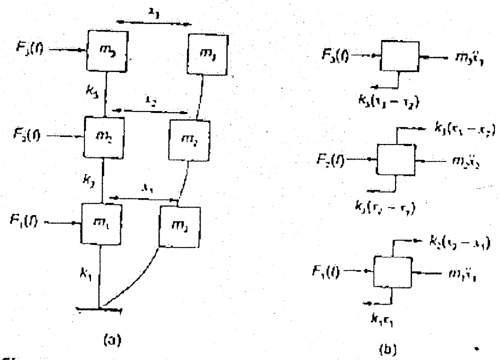


Fig. 11.21 (a) Model of the shear building and (b) Free-body diagram.

- Equations of motion, for free vibration of undapmed MDOF system are

$$\left.\begin{aligned} m_1\ddot{x}_1 + k_1x_1 - k_2(x_2 - x_1) &= 0 \\ m_2\ddot{x}_2 + k_2(x_2 - x_1) - k_3(x_3 - x_2) &= 0 \\ m_3\ddot{x}_3 + k_3(x_3 - x_2) &= 0 \end{aligned}\right\} \qquad \text{...6 (i)}$$

where, $\quad k_i = \sum_{i=1}^{n}\dfrac{12EI_{column}}{h^3}$

where, h is length and n is number of columns.

- On rearranging equations

$$\left.\begin{aligned} m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2x_2 &= 0 \\ m_2\ddot{x}_2 - k_2x_1 + (k_2 + k_3)x_2 - k_3x_3 &= 0 \\ m_3\ddot{x}_3 - k_3x_2 + k_3x_3 &= 0 \end{aligned}\right\} \qquad \text{...6 (ii)}$$

Representing equation (6(ii)) in matrix form

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow \qquad [M]\{\ddot{x}\} + [k]\{x\} = \{0\} \qquad \text{...6 (iii)}$$

where,

$[M]$ is mass matrix

$[k]$ is stiffness matrix

- Using solution form as

$$x(t) = \{\phi\}\sin(\omega_n t + \alpha)$$

the equation of motion reduced to the following Eigen value problem:

$$[[k] - \omega_n^2[M]]\{\phi\} = \{0\} \qquad \text{...6 (iv)}$$

- For non-trivial solution, determinant of $([k] - \omega_n^2[M] = 0)$ is called as the characteristic equation (an eigen value problem) in degree n, with n eigen values ω_n^2. Where ω_n is natural frequency.
- For each eigen value ω_n^2, there is eigen vector known to be the mode shape (which is a characteristic deflected shape, ϕ).

11.23 Orthogonality of Modes

- Modes corresponding to different natural frequencies (ω_n) satisfy the following orthogonality conditions:

$$\{\phi\}_s^T[M]\{\phi\}_r = 0 \quad \text{for } s \neq r$$

and $\quad \{\phi\}_s^T[M]\{\phi\}_r \neq 1 \quad \text{for } s = r \qquad \text{...6 (v)}$

Note: Model orthogonality property states that work done by inertial forces corresponding to n^{th} mode is going through the displacements corresponding to r^{th} mode is zero.

Also $\{\phi\}_s^T [M] \{\phi\}_s = 0 \quad \text{if } r \neq s$

and $\{\phi\}_s^T [K] \{\phi\}_r = \omega_r^2; \quad \text{for } r = s$

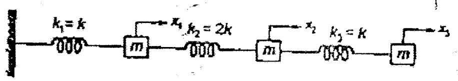
⇒ Two modes $\{\phi\}_r$ and $\{\phi\}_s$ are orthogonal to each other. ...6 (vi)

Example 11.11 For figure shown below:

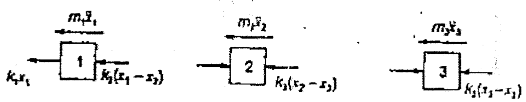
(a) Write the equation of motion

(b) Determine the natural frequencies

(c) Determine the modes for the system



Solution:



Differential equation for dynamic equilibrium of mass m_1 is

$$m_1\ddot{x}_1 + k_1x_1 + k_2(x_1 - x_2) = 0$$

On rearranging:

$$m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2x_2 = 0$$

$$\Rightarrow \quad m_1\ddot{x}_1 + 3kx_1 - 2kx_2 = 0$$

For mass m_2, the differential equation of motion is

$$m_2\ddot{x}_2 - k_2(x_1 - x_2) + k_3(x_2 - x_3) = 0$$

$$m_2\ddot{x}_2 - k_2x_1 + (k_2 + k_3)x_2 - k_3x_3 = 0$$

$$\Rightarrow \quad m_2\ddot{x}_2 - 2kx_1 + 3kx_2 - kx_3 = 0$$

For mass m_3, the differential equation of motion is

$$m_3\ddot{x}_3 + k_3(x_3 - x_2) = 0$$

On rearranging:

$$m_3\ddot{x}_3 - k_3x_2 + k_3x_3 = 0$$

$$\Rightarrow \quad m_3\ddot{x}_3 - kx_2 + kx_3 = 0$$

Representing equation of motion in the matrix form

$$\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + \begin{bmatrix} 3k & -2k & 0 \\ -2k & 3k & -k \\ 0 & -k & k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Also, eigen value equation is

$$|[K] - \omega_n^2 [M]| \{\phi\} = \{0\}$$

$$\Rightarrow \begin{bmatrix} 3k - m\omega_n^2 & -2k & 0 \\ -2k & 3k - m\omega_n^2 & -k \\ 0 & -k & k - m\omega_n^2 \end{bmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

The solution is

$$|[K] - \omega_n^2 [M]| = 0$$

$$\Rightarrow \begin{vmatrix} 3k - m\omega_n^2 & -2k & 0 \\ -2k & 3k - m\omega_n^2 & -k \\ 0 & -k & k - m\omega_n^2 \end{vmatrix} = 0$$

$$\Rightarrow \quad m^3\omega_n^6 - 7km^2\omega_n^4 + 10k^2m\omega_n^2 - 2k^3 = 0$$

$$\Rightarrow \quad \lambda^3 - 7\left(\frac{k}{m}\right)\lambda^2 + 10\left(\frac{k}{m}\right)^2\lambda - 2\left(\frac{k}{m}\right)^3 = 0$$

where $\lambda = \omega_n^2$

On solving, we obtain

$$\lambda_1 = 5.12\left(\frac{k}{m}\right) \quad \Rightarrow \quad \omega_{n1} = 2.26\sqrt{\frac{k}{m}}$$

$$\lambda_2 = 0.238\left(\frac{k}{m}\right) \quad \Rightarrow \quad \omega_{n2} = 0.488\sqrt{\frac{k}{m}}$$

$$\lambda_3 = 1.64\left(\frac{k}{m}\right) \quad \Rightarrow \quad \omega_{n3} = 1.28\sqrt{\frac{k}{m}}$$

(a) For $\omega_{n1} = 2.26\sqrt{\frac{k}{m}}$, the mode shape is $\begin{Bmatrix} 1.00 \\ -1.06 \\ 0.257 \end{Bmatrix}$

(b) For $\omega_{n2} = 0.488\sqrt{\frac{k}{m}}$, the mode shape is $\begin{Bmatrix} 1.000 \\ 0.685 \\ -1.07 \end{Bmatrix}$

(c) For the frequency $\omega_{n3} = 1.28\sqrt{\frac{k}{m}}$, the mode shape is $\begin{Bmatrix} 1.000 \\ 1.381 \\ 1.813 \end{Bmatrix}$

11.24 Normal Mode Method

- Normal mode method is used for the system matrices diagonal as degrees of freedom gets separated so they can be treated as single degree of freedom.
- Evaluation of normal modes requires the use of eigen values (ω^2) and eigen vectors $\{\phi\}$.

11.24.1 Normalization

- Normalization to transform to modal co-ordinates of system matrices.

$$x(t) = [\phi]\{z(t)\}$$

where, $x(t)$ is the vector of displacement of individual masses.

$\{z(t)\}$ is vector of displacement at global coordinates

$[\phi]$ is transformation matrix

- Normalization of mass matrix

  $[M]_{Normalized} = [\phi]^T [M] [\phi]$
- $[K]_{Normalized} = [\phi]^T [K] [\phi]$
- $F(t)_{Normalized} = [\phi]^T \{F(t)\}$
- The elements the normalized mass, stiffness and force matrices are called as modal masses., modal stiffness and modal forces respectively.

Note: The ratio of modal values is equal to square of corresponding natural frequencies i.e., $k_{11}/m_{11} = \omega_1^2$, $k_{22}/m_{22} = w_2^2$ etc.

11.25 Response of MDOF System

- The equation of motion for the forced MDOF system is

$$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [k]x = \{F(t)\}$$

where $[M]$ = mass matrix

$[k]$ = stiffness matrix

$[C]$ = damping matrix

$\{F(t)\}$ = excitation force matrix

Summary

- The equation of motion of undamped free vibration of single degree of freedom system is, $m\ddot{y} + ky = 0$; where, y is displacement and $\ddot{y}$ is acceleration in y-direction.
- The equation motion can also be obtained by D'Alembert's principle.
- Expression of the motion of simple undamped oscillator modelling structures with SDOF system is $y = y_0 \cos\omega t + \frac{V_0}{\omega}\sin\omega t$.
- The equation of motion of damped single degree of freedom system is $m\ddot{y} + c\dot{y} + ky = 0$.
- There are three types of damping: critically damped, overdamped and underdamped.
- For critically damped system, the solution is:

  $y(t) = (C_1 + C_2 t)e^{-(C_c/2m)t}$

  For overdamped case, the solution is

  $y(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$

  For underdamped case, the solution is

  $y(t) = Ce^{-\xi\omega t}\cos(\omega_D t - \alpha)$

  where, $C(\text{amplitude}) = \sqrt{y_0^2 + \frac{(V_0 + y_0\xi\omega_n)^2}{\omega_D^2}}$

  and $\tan\alpha = \frac{(V_0 + y_0\xi\omega_n)}{\omega_D y_0}$

- The decay is expressed as logarithmic decrement (δ).
- The equation of motion of undamped harmonic excitation is: $m\ddot{y} + ky = F_0 \sin\bar{\omega}t$

  The solution of this differential equation is

  $y(t) = \frac{F_0/k}{1-r^2}[\sin\bar{\omega}t - r\sin\omega t]$
- The equation of motion of damped harmonic excitation is:

  $m\ddot{y} + c\dot{y} + ky = F_0 \sin\bar{\omega}t$

  The solution is:

  $y(t) = e^{-\xi\omega t}(A\cos\omega_D t + B\sin\omega_D t) + \frac{y_{st}\sin(\bar{\omega}t - \theta)}{\sqrt{(1-r^2)^2 + (2r\xi)^2}}$

  where $y_{st} = F_0/k$
- Dynamic magnification factor,

  $D = \frac{y_{(\text{steady state amplitude})}}{y_{st(\text{static deflection})}} = \frac{1}{\sqrt{(1-r^2)^2 + (2r\xi)^2}}$

  Also, $D = \frac{1}{2\xi}$ if $r = 1$ (that is case of resonance)
- Transmissibility is degree of relative isolation.

  $T_r = \frac{y_{(\text{amplitude of motion of the oscillator})}}{y_0} = \frac{\sqrt{1 + (2r\xi)^2}}{\sqrt{(1-r^2)^2 + (2r\xi)^2}}$

  $\Rightarrow$ Maximum force transmitted is : $F_T = F_0 T_r$
- Duhamel's integral is: $\int_0^t F(\tau)\sin\omega(t-\tau)d\tau$
- Response of an undamped oscillator acted upon by a constant force is

  $y(t) = \frac{F_0}{k}(1-\cos\omega t) = y_{st}(1-\cos\omega t)$
- Response of an undamped oscillator acted upon a rectangular load is

  $y(t) = \frac{F_0}{k}\{\cos\omega(t - t_d) - \cos\omega t\}$
- Application of Fourier series is to know the response of a system to periodic forces and non-periodic forces in a frequency domain.
- Fourier series is summation of an infinite number of sine and cosine terms

  $F(t) = a_0 + \sum_{n=1}^{\infty}(a_n \cos n\bar{\omega}t + b_n \sin n\bar{\omega}t)$

  where $\bar{\omega} = \frac{2\pi}{T}$
- Response of SDOF undamped system to a periodic force is

  $y(t) = \frac{a_0}{k} + \sum_{n=1}^{\infty}\frac{1}{1-r_n^2}\left(\frac{a_n}{k}\cos n\bar{\omega}t + \frac{b_n}{k}\sin n\bar{\omega}t\right)$

where, $\frac{a_0}{k}$ is a steady state response

- The equation of motion for MDOF system is

$$[M]\{\ddot{x}\}+[k]\{x\}=0$$

where, $[M]$ = mass matrix
$[k]$ = stiffness matrix

Objective Brain Teasers

Q.1 In which case the dynamic system has no oscillatory motion and returns to equilibrium position at a slower rate
(a) Critically damped (b) Overdamped
(c) Underdamped (d) Any of the above

Q.2 The damping in a dynamic system is assumed to be _____ for simple mathematical analysis.
(a) Viscous damping (b) Coulomb damping
(c) Negative damping (d) Friction damping

Q.3 Transmissibility is not significantly affected by damping in the region
(a) $\frac{\bar{\omega}}{\omega_n}<0.2$ (b) $\frac{\bar{\omega}}{\omega_n}\simeq 1.5$
(c) $\frac{\bar{\omega}}{\omega_n}>1$ (d) $\frac{\bar{\omega}}{\omega_n}<1$

Q.4 Transmissibility is infinity at frequency ratio very close to
(a) 0 (b) 1
(c) 1.2 (d) 2

Q.5 The response is greatly affected by damping in the region
(a) $\frac{\bar{\omega}}{\omega_n}<0.7$ (b) $\frac{\bar{\omega}}{\omega_n}>0.3$
(c) $\frac{\bar{\omega}}{\omega_n}\simeq 1$ (d) $\frac{\bar{\omega}}{\omega_n}=0$

Q.6 Transmissibility is more than 1, when the frequency ratio is
(a) < 0.5 (b) 1.0
(c) $>\sqrt{2}$ (d) $<\sqrt{2}$

Q.7 The transient motion lasts for
(a) the entire duration of excitation force
(b) the short duration in the beginning of the vibration
(c) the short duration in the end of the vibration
(d) arbitrary duration period that is it may occur or end any instant during the vibration

Q.8 The steady state motion depends mainly upon
(a) natural frequency
(b) damped natural frequency
(c) resonant frequency
(d) excitation frequency

Q.9 At resonance, transmissibility only depends upon the
(a) damping ratio
(b) frequency ratio
(c) excitation frequency
(d) resonant frequency

Q.10 In multi-degree-of-freedom system, the modes satisfy the following orthogonality relation
(a) $\{\phi\}_1^T[M]\{\phi\}_1=0$ (b) $\{\phi\}_1^T[M]\{\phi\}_2=0$
(c) $\{\phi\}_1^T[k]\{\phi\}_1=0$ (d) $\{\phi\}_1[M]\{\phi\}_2=0$

Q.11 Normalized force matrix is
(a) $\{\phi\}^T[F]\{\phi\}$ (b) $\{\phi\}[F]\{\phi\}^T$
(c) $\{\phi\}^T[F]$ (d) $\{\phi\}[F]$

Q.12 In normal mode method, after normalization
(a) $[k]$ becomes diagonal
(b) $[M]$ becomes diagonal
(c) $[k]$, $[M]$ and $[F(t)]$ becomes diagonal
(d) $[k]$ and $[M]$ become diagonal

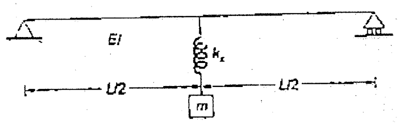
Q.13 A system having natural frequency 20 rad/sec and damping ratio of 10% is subjected to a harmonic force of [50 sin(100 t)] kN. The frequency ratio is
(a) 0.2 (b) 2
(c) 5 (d) data insufficient

Q.14 Which of the following is correct assumption regarding shear building
(a) The floor slab is infinitely rigid
(b) Axial force and deformation are accounted
(c) The columns are infinitely rigid
(d) All the elements are infinitely rigid

Q.15 Viscous damping is
(a) proportional to displacement
(b) proportional to velocity
(c) proportional to acceleration
(d) None of the above

Q.16 Damping ratio is defined as the ratio of
(a) critical damping to system damping
(b) system damping to critical damping
(c) natural damping to system damping
(d) None of the above

Q.17 The equivalent stiffness for the system shown below is _____ units. If flexural rigidity of beam is unity and length of beam is 2 m. The stiffness of spring is 12 unit



(a) 1 (b) 2
(c) 3 (d) 4

Q.18 Equivalent stiffness of a beam system is 10^5 N/m and mass of the object resting on the beam system is 1000 kg. The natural frequency is
(a) 2 rad/sec (b) 5 rad/sec
(c) 10 rad/sec (d) 25 rad/sec

Q.19 In single-degree-of-freedom system, mass is increased by 1 kg and the ratio of modified frequency to the natural frequency (due to mass increment) is 1.1. The mass of single degree of freedom system is
(a) 2 kg (b) 5 kg
(c) 8 kg (d) 10 kg

Q.20 In an undamped free vibration of single degree of freedom, the maximum velocity is 0.3 m/s and natural frequency is 10 rad/sec. The amplitude of vibration is
(a) 30 mm (b) 90 mm
(c) 100 m (d) 210 mm

Q.21 In an undamped free vibration of single degree of freedom, the initial displacement and velocity are 0.01 m and 0.50 m/s respectively. The natural frequency is 5 rad/sec. The amplitude of vibration is
(a) 0.005 m (b) 0.1 m
(c) 0.2 m (d) 1 m

Q.22 A vibrating SDOF system is viscously damped such that ratio of two consecutive amplitudes is 1.00 to 0.60. The logarithmic decrement is
(a) 0.510 (b) 0.693
(c) 1.099 (d) 1.609

Q.23 The logarithmic decrement for a damped freely vibrating SDOF system is 0.08. The damping ratio is in percentage is
(a) 0.1 (b) 1
(c) 1.3 (d) 10

Q.24 In a damped freely vibrating SDOF system, the damping ratio is 0.012 and critical damping coefficient is 1600 Ns/m. Then damped coefficient is _______ Ns/m.
(a) 8.6 (b) 10.8
(c) 19.2 (d) 22.9

Q.25 A damped freely vibrating SDOF system weighs 25 kg, spring with stiffness 20 N/mm and damping ratio is 0.08. The damped coefficient of system is _______ Ns/m.
(a) 91.18 (b) 103.64
(c) 113.14 (d) 156.92

Q.26 A SDOF system has undamped natural frequency 30000 rad/sec and the damping ratio equal to

0.5. The natural frequency of damped system is ______ rad/sec.

(a) 12160 (b) 18660

(c) 25980 (d) 35000

Q.27 A viscously damped system having resonant amplitude of 1.5 m, spring of stiffness of 1600 N/m is subjected to harmonic loading of 300 N. The damping coefficient in percentage is

(a) 4.75 (b) 5.25

(c) 6.25 (d) 9.00

Q.28 Viscous damping force acting upon SDOF system is 1000 N at a velocity of 50 mm/s. The damping coefficient is ______ $\times 10^3$ Ns/m

(a) 20 (b) 50

(c) 80 (d) 100

Q.29 In SDOF system, the frequency ratio is 0.50 and damping ratio is 10%. The dynamic magnificator factor is

(a) 0.68 (b) 1.32

(c) 1.99 (d) 2.58

Q.30 The expression for transmissibility of force is

(a) $\dfrac{1}{\sqrt{\left(1-r^2\right)^2+\left(2r\xi\right)^2}}$

(b) $\dfrac{F_0/k}{\sqrt{\left(1-r^2\right)^2+\left(2r\xi\right)^2}}$

(c) $\dfrac{1}{\sqrt{\left(1-r^2\right)^3+\left(2r\xi\right)^3}}$

(d) $\dfrac{\sqrt{1+\left(2r\xi\right)^2}}{\sqrt{\left(1-r^2\right)^2+\left(2r\xi\right)^2}}$

Q.31 For SDOF system, the amplitude of excitation force is 1000 N and the transmissibility of force is 0.25. The transmitted force is

(a) 250 N (b) 500 N

(c) 750 N (d) 1000 N

Q.32 Expression of the response of undamped oscillator acted upon by a constant force is

(a) $\dfrac{F_0}{k}(1-\tan\omega t)$ (b) $\dfrac{F_0}{k}$

(c) $\dfrac{F_0}{k}(1-\sin\omega t)$ (d) $\dfrac{F_0}{k}(1-\cos\omega t)$

Q.33 Response of an undamped oscillator acting upon by a rectangular load is

(a) $\dfrac{F_0}{k}$

(b) $\dfrac{F_0}{k}\cos\omega t(t-t_d)$

(c) $\dfrac{F_0}{k}(\cos\omega t-\cos\omega t_d)$

(d) $\dfrac{F_0}{k}\{\cos\omega(t-t_d)-\cos\omega t\}$

Q.34 Response of SDOF undamped system to a periodic force is

(a) $\dfrac{a_0}{k}$

(b) $\dfrac{a_0}{k}+\sum_{n=1}^{\infty}\dfrac{1}{1-r_n^2}\dfrac{a_n}{k}\cos n\bar{\omega}t$

(c) $\dfrac{a_0}{k}+\sum_{n=1}^{\infty}\dfrac{1}{1-r_n^2}\dfrac{b_n}{k}\sin n\bar{\omega}t$

(d) $\dfrac{a_0}{k}+\sum_{n=1}^{\infty}\dfrac{1}{1-r_n^2}\left(\dfrac{a_n}{k}\cos n\bar{\omega}t+\dfrac{b_n}{k}\sin n\bar{\omega}t\right)$

Answers

1. (b)	2. (a)	3. (a)	4. (b)	5. (c)
6. (d)	7. (b)	8. (d)	9. (a)	10. (b)
11. (c)	12. (d)	13. (c)	14. (a)	15. (b)
16. (b)	17. (d)	18. (c)	19. (b)	20. (a)
21. (b)	22. (a)	23. (c)	24. (c)	25. (c)
26. (c)	27. (c)	28. (a)	29. (b)	30. (d)
31. (a)	32. (d)	33. (d)	34. (d)	

Hints and Explanations:

12. (d)

Excitation force matrix may not be diagonal.

13. (c)

$$r=\frac{\bar{\omega}}{\omega}=\frac{100}{20}=5$$

17. (d)

Stiffness of beam,

$$k_b = \frac{48EI}{L^3}$$

$$k_b = \frac{48\times 1}{2^3}=6 \text{ unit}$$

$$k_s = 12 \text{ unit}$$

Equivalent stiffness, k_{eq}

$$\frac{1}{k_{eq}} = \frac{1}{6}+\frac{1}{12}=\frac{3}{12}$$

(because spring are in series)

$$k_{eq} = 4 \text{ unit}$$

18. (c)

$$\omega_n=\sqrt{\frac{k_{eq}}{m}}=\sqrt{\frac{10^5}{10^3}}=10 \text{ rad/s}$$

19. (b)

$$\omega_i^2=\frac{k}{m_i}$$

$$\Rightarrow \left(\frac{\omega_{n(\text{natural frequency})}}{\omega_{m(\text{Modified frequency})}}\right)^2=\frac{m+1}{m}$$

$$(1.1)^2=\frac{m+1}{m}$$

$$0.21\,m=1$$

$$m=4.76 \text{ kg}$$

20. (a)

$V=A\omega$

$$A=\frac{0.3}{10}=0.03 \text{ m}=30 \text{ mm}$$

21. (b)

$$A_{(\text{amplitude})}=\sqrt{y_0^2+\left(\frac{V_0}{\omega_n}\right)^2}$$

$$=\sqrt{0.01^2+\left(\frac{0.50}{5}\right)^2}=0.1 \text{ m}$$

22. (a)

The logarithmic decrement (δ) is

$$\delta=\ln\left(\frac{y_1}{y_2}\right)=\ln\left(\frac{1.00}{0.60}\right)=0.51$$

23. (c)

$$\xi(\text{damping ratio})=\frac{\delta}{2\pi}=\frac{0.08}{2\pi}=0.013=1.3\%$$

24. (c)

C(damped coefficient)

$=\xi c_{cr}=0.012\times 1600$

$=1.2\times 16=19.2$ Ns/m

25. (c)

$c=\xi c_{cr}=\xi\times 2\sqrt{km}$

$=0.08\times 2\times\sqrt{20\times 10^3\times 25}$

$=113.14$ Ns/m

26. (c)

ω_D (Natural frequency for damped system)

$$=\omega\sqrt{1-\xi^2}=30000\sqrt{1-0.5^2}$$

$$=25980.76 \text{ rad/sec}$$

27. (c)

$$D\text{ (damping coefficient)}=\frac{1}{2\xi}$$

(because at resonance, $D=\dfrac{1}{2\xi}$)

$$\frac{y_{max}}{y_{st}}=\frac{1}{2\xi}$$

$$\xi=\frac{F_0/k}{2y_{max}} \quad \text{(because } y_{st}=F_0/k\text{)}$$

$$=\frac{300/1600}{2\times 1.5}=0.0625$$

$$\Rightarrow \xi=6.25\%$$

28. (a)

$$c\dot{y}=F$$

$$c\times 50\times 10^{-3}=1000$$

$$C=\frac{10^6}{50}=20000 \text{ Ns/m}$$

$$=20\times 10^3 \text{ Ns/m}$$

29. (b)

$$D=\frac{1}{\sqrt{\left(1-r^2\right)^2+\left(2r\xi\right)^2}}$$

$$=\frac{1}{\sqrt{\left(1-0.5^2\right)^2+\left(2\times 0.5\times 0.1\right)^2}}=1.32$$

31. (a)

F_T (transmitted force)

$=F_0\times T_r$

$=1000\times 0.25=250$ N

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