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Vector Algebra

Short Answer Type Questions

Q. 1 Find the unit vector in the direction of sum of vectors $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$ and $\vec{b} = 2\hat{j} + \hat{k}$.

Thinking Process

We know that, unit vector in the direction of a vector $\vec{a}$ *is* $\dfrac{\vec{a}}{|\vec{a}|}$. *So, first we will find the sum of vectors and then we will use this concept.*

Sol. Let $\vec{c}$ denote the sum of $\vec{a}$ and $\vec{b}$.

We have,
$$\vec{c} = \vec{a} + \vec{b}$$
$$= 2\hat{i} - \hat{j} + \hat{k} + 2\hat{j} + \hat{k} = 2\hat{i} + \hat{j} + 2\hat{k}$$

$\therefore$ Unit vector in the direction of $\vec{c} = \dfrac{\vec{c}}{|\vec{c}|} = \dfrac{2\hat{i} + \hat{j} + 2\hat{k}}{\sqrt{2^2 + 1^2 + 2^2}} = \dfrac{2\hat{i} + \hat{j} + 2\hat{k}}{\sqrt{9}}$

$$\hat{c} = \frac{2\hat{i} + \hat{j} + 2\hat{k}}{3}$$

Q. 2 If $\vec{a} = \hat{i} + \hat{j} + 2\hat{k}$ and $\vec{b} = 2\hat{i} + \hat{j} + 2\hat{k}$, then find the unit vector in the direction of

(i) $6\vec{b}$ (ii) $2\vec{a} - \vec{b}$

Sol. Here, $\vec{a} = \hat{i} + \hat{j} + 2\hat{k}$ and $\vec{b} = 2\hat{i} + \hat{j} - 2\hat{k}$

(i) Since, $6\vec{b} = 12\hat{i} + 6\hat{j} - 12\hat{k}$

$\therefore$ Unit vector in the direction of $6\vec{b} = \dfrac{6\vec{b}}{|6\vec{b}|}$

$$= \frac{12\hat{i} + 6\hat{j} - 12\hat{k}}{\sqrt{12^2 + 6^2 + 12^2}} = \frac{6(2\hat{i} + \hat{j} - 2\hat{k})}{\sqrt{324}}$$

$$= \frac{6(2\hat{i} + \hat{j} - 2\hat{k})}{18} = \frac{2\hat{i} + \hat{j} - 2\hat{k}}{3}$$

(ii) Since, $2\vec{a} - \vec{b} = 2(\hat{i} + \hat{j} + 2\hat{k}) - (2\hat{i} + \hat{j} - 2\hat{k})$

$$= 2\hat{i} + 2\hat{j} + 4\hat{k} - 2\hat{i} - \hat{j} + 2\hat{k} = \hat{j} + 6\hat{k}$$

$\therefore$ Unit vector in the direction of $2\vec{a} - \vec{b} = \dfrac{2\vec{a} - \vec{b}}{|2\vec{a} - \vec{b}|} = \dfrac{\hat{j} + 6\hat{k}}{\sqrt{1 + 36}} = \dfrac{1}{\sqrt{37}}(\hat{j} + 6\hat{k})$

Q. 3 Find a unit vector in the direction of $\overrightarrow{PQ}$, where P and Q have coordinates (5, 0, 8) and (3, 3, 2), respectively.

Sol. Since, the coordinates of P and Q are (5, 0, 8) and (3, 3, 2), respectively.

$$\therefore \quad \overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$$

$$= (3\hat{i} + 3\hat{j} + 2\hat{k}) - (5\hat{i} + 0\hat{j} + 8\hat{k})$$

$$= -2\hat{i} + 3\hat{j} - 6\hat{k}$$

$\therefore$ Unit vector in the direction of $\overrightarrow{PQ} = \dfrac{\overrightarrow{PQ}}{|\overrightarrow{PQ}|} = \dfrac{-2\hat{i} + 3\hat{j} - 6\hat{k}}{\sqrt{2^2 + 3^2 + 6^2}}$

$$= \frac{-2\hat{i} + 3\hat{j} - 6\hat{k}}{\sqrt{49}} = \frac{-2\hat{i} + 3\hat{j} - 6\hat{k}}{7}$$

Q. 4 If $\vec{a}$ and $\vec{b}$ are the position vectors of $\vec{A}$ and $\vec{B}$ respectively, then find the position vector of a point $\vec{C}$ in $\vec{BA}$ produced such that $\vec{BC} = 1.5\vec{BA}$.

Sol. Since, $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OB} = \vec{b}$

$$\therefore \quad \overrightarrow{BA} = \overrightarrow{OA} - \overrightarrow{OB} = \vec{a} - \vec{b}$$

and $\quad 1.5\overrightarrow{BA} = 1.5(\vec{a} - \vec{b})$

Since, $\quad \overrightarrow{BC} = 1.5\overrightarrow{BA} = 1.5(\vec{a} - \vec{b})$

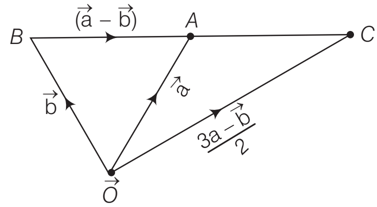
$$\overrightarrow{OC} - \overrightarrow{OB} = 1.5\vec{a} - 1.5\vec{b}$$

$$\overrightarrow{OC} = 1.5\vec{a} - 1.5\vec{b} + \vec{b} \qquad [\because \overrightarrow{OB} = \vec{b}]$$

$$= 1.5\vec{a} - 0.5\vec{b}$$

$$= \frac{3\vec{a} - \vec{b}}{2}$$

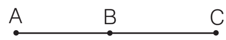
Graphically, explanation of the above solution is given below



Q. 5 Using vectors, find the value of k, such that the points $(k, -10, 3)$, $(1, -1, 3)$ and $(3, 5, 3)$ are collinear.

Thinking Process

Here, use the following stepwise approach first, get the values of $|\vec{\mathbf{AB}}|, |\vec{\mathbf{BC}}|$ *and* $|\vec{\mathbf{AC}}|$ *and then use the concept that three points are collinear, if* $|\vec{\mathbf{AB}}| + |\vec{\mathbf{BC}}| = |\vec{\mathbf{AC}}|$ *such that.*



Sol. Let the points are $A(k, -10, 3)$, $B(1, -1, 3)$ and $C(3, 5, 3)$.

So, $$\vec{\mathbf{AB}} = \vec{\mathbf{OB}} - \vec{\mathbf{OA}}$$
$$= (\hat{\mathbf{i}} - \hat{\mathbf{j}} + 3\hat{\mathbf{k}}) - (k\hat{\mathbf{i}} - 10\hat{\mathbf{j}} + 3\hat{\mathbf{k}})$$
$$= (1-k)\hat{\mathbf{i}} + (-1+10)\hat{\mathbf{j}} + (3-3)\hat{\mathbf{k}}$$
$$= (1-k)\hat{\mathbf{i}} + 9\hat{\mathbf{j}} + 0\hat{\mathbf{k}}$$

$\therefore$ $$|\vec{\mathbf{AB}}| = \sqrt{(1-k)^2 + (9)^2 + 0} = \sqrt{(1-k)^2 + 81}$$

Similarly, $$\vec{\mathbf{BC}} = \vec{\mathbf{OC}} - \vec{\mathbf{OB}}$$
$$= (3\hat{\mathbf{i}} + 5\hat{\mathbf{j}} + 3\hat{\mathbf{k}}) - (\hat{\mathbf{i}} - \hat{\mathbf{j}} + 3\hat{\mathbf{k}})$$
$$= 2\hat{\mathbf{i}} + 6\hat{\mathbf{j}} + 0\hat{\mathbf{k}}$$

$\therefore$ $$|\vec{\mathbf{BC}}| = \sqrt{2^2 + 6^2 + 0} = 2\sqrt{10}$$

and $$\vec{\mathbf{AC}} = \vec{\mathbf{OC}} - \vec{\mathbf{OA}}$$
$$= (3\hat{\mathbf{i}} + 5\hat{\mathbf{j}} + 3\hat{\mathbf{k}}) - (k\hat{\mathbf{i}} - 10\hat{\mathbf{j}} + 3\hat{\mathbf{k}})$$
$$= (3-k)\hat{\mathbf{i}} + 15\hat{\mathbf{j}} + 0\hat{\mathbf{k}}$$

$\therefore$ $$|\vec{\mathbf{AC}}| = \sqrt{(3-k)^2 + 225}$$

If A, B and C are collinear, then sum of modulus of any two vectors will be equal to the modulus of third vectors

For $|\vec{\mathbf{AB}}| + |\vec{\mathbf{BC}}| = |\vec{\mathbf{AC}}|$,

$$\sqrt{(1-k)^2 + 81} + 2\sqrt{10} = \sqrt{(3-k)^2 + 225}$$

$\Rightarrow$ $$\sqrt{(3-k)^2 + 225} - \sqrt{(1-k)^2 + 81} = 2\sqrt{10}$$

$\Rightarrow$ $$\sqrt{9 + k^2 - 6k + 225} - \sqrt{1 + k^2 - 2k + 81} = 2\sqrt{10}$$

$\Rightarrow$ $$\sqrt{k^2 - 6k + 234} - 2\sqrt{10} = \sqrt{k^2 - 2k + 82}$$

$\Rightarrow$ $$k^2 - 6k + 234 + 40 - 2\sqrt{k^2 - 6k + 234}\cdot 2\sqrt{10} = k^2 - 2k + 82$$

$\Rightarrow$ $$k^2 - 6k + 234 + 40 - k^2 + 2k - 82 = 4\sqrt{10}\sqrt{k^2 + 234 - 6k}$$

$\Rightarrow$ $$-4k + 192 = 4\sqrt{10}\sqrt{k^2 + 234 - 6k}$$

$\Rightarrow$ $$-k + 48 = \sqrt{10}\sqrt{k^2 + 234 - 6k}$$

On squaring both sides, we get

$$48 \times 48 + k^2 - 96k = 10(k^2 + 234 - 6k)$$

$\Rightarrow$ $$k^2 - 96k - 10k^2 + 60k = -48 \times 48 + 2340$$

$\Rightarrow$ $$-9k^2 - 36k = -48 \times 48 + 2340$$

$\Rightarrow \quad (k^2 + 4k) = + 16 \times 16 - 260$ [dividing by 9 in both sides]

$\Rightarrow \quad k^2 + 4k = -4$

$k^2 + 4k + 4 = 0$

$\Rightarrow \quad (k+2)^2 = 0$

$\therefore \quad k = -2$

Q. 6 A vector $\vec{\mathbf{r}}$ is inclined at equal angles to the three axes. If the magnitude of $\vec{\mathbf{r}}$ is $2\sqrt{3}$ units, then find the value of $\vec{\mathbf{r}}$.

Thinking Process

If a vector $\vec{\mathbf{r}}$ is inclined at equal angles to the three axes, then direction cosines of vector, $\vec{\mathbf{r}}$ will be same and then use, $\vec{\mathbf{r}} = \vec{\mathbf{r}} \cdot |\vec{\mathbf{r}}|$.

Sol. We have, $|\vec{\mathbf{r}}| = 2\sqrt{3}$

Since, $\vec{\mathbf{r}}$ is equally inclined to the three axes, $\vec{\mathbf{r}}$ so direction cosines of the unit vector $\vec{\mathbf{r}}$ will be same. *i.e.*, $l = m = n$.

We know that,

$$l^2 + m^2 + n^2 = 1$$

$$\Rightarrow \quad l^2 + l^2 + l^2 = 1$$

$$\Rightarrow \quad l^2 = \frac{1}{3}$$

$$\Rightarrow \quad l = \pm\left(\frac{1}{\sqrt{3}}\right)$$

So, $$\hat{\mathbf{r}} = \pm\frac{1}{\sqrt{3}}\hat{\mathbf{i}} \pm \frac{1}{\sqrt{3}}\hat{\mathbf{j}} \pm \frac{1}{\sqrt{3}}\hat{\mathbf{k}}$$

$$\therefore \quad \vec{\mathbf{r}} = \hat{\mathbf{r}}|\vec{\mathbf{r}}| \qquad \left[\because \hat{\mathbf{r}} = \frac{\vec{\mathbf{r}}}{|\vec{\mathbf{r}}|}\right]$$

$$= \left[\pm\frac{1}{\sqrt{3}}\hat{\mathbf{i}} \pm \frac{1}{\sqrt{3}}\hat{\mathbf{j}} \pm \frac{1}{\sqrt{3}}\hat{\mathbf{k}}\right] 2\sqrt{3} \qquad [\because |r| = 2\sqrt{3}]$$

$$= \pm 2\hat{\mathbf{i}} \pm 2\hat{\mathbf{j}} \pm 2\hat{\mathbf{k}} = \pm 2(\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}})$$

Q. 7 If a vector $\vec{\mathbf{r}}$ has magnitude 14 and direction ratios 2, 3 and − 6. Then, find the direction cosines and components of $\vec{\mathbf{r}}$, given that $\vec{\mathbf{r}}$ makes an acute angle with *X*-axis.

Sol. Here, $|\vec{\mathbf{r}}| = 14$, $\vec{\mathbf{a}} = 2k$, $\vec{\mathbf{b}} = 3k$ and $\vec{\mathbf{c}} = -6k$

$\therefore$ Direction cosines l, m and n are

$$l = \frac{\vec{\mathbf{a}}}{|\vec{\mathbf{r}}|} = \frac{2k}{14} = \frac{k}{7}$$

$$m = \frac{\vec{\mathbf{b}}}{|\vec{\mathbf{r}}|} = \frac{3k}{14}$$

and $$n = \frac{\vec{\mathbf{c}}}{|\vec{\mathbf{r}}|} = \frac{-6k}{14} = \frac{-3k}{7}$$

Also, we know that

$$l^2 + m^2 + n^2 = 1$$

$$\Rightarrow \quad \frac{k^2}{49} + \frac{9k^2}{196} + \frac{9k^2}{49} = 1$$

$$\Rightarrow \quad \frac{4k^2 + 9k^2 + 36k^2}{196} = 1$$

$$\Rightarrow \quad k^2 = \frac{196}{49} = 4$$

$$\Rightarrow \quad k = \pm 2$$

So, the direction cosines (l, m, n) are $\frac{2}{7}, \frac{3}{7}$ and $\frac{-6}{7}$.

[since, $\vec{\mathbf{r}}$ makes an acute angle with X-axis]

$$\because \quad \vec{\mathbf{r}} = \hat{\mathbf{r}} \cdot |\vec{\mathbf{r}}|$$

$$\therefore \quad \vec{\mathbf{r}} = (l\hat{\mathbf{i}} + m\hat{\mathbf{j}} + n\hat{\mathbf{k}})|\vec{\mathbf{r}}|$$

$$= \left(\frac{+2}{7}\hat{\mathbf{i}} + \frac{3}{7}\hat{\mathbf{j}} - \frac{6}{7}\hat{\mathbf{k}}\right) \cdot 14$$

$$= +4\hat{\mathbf{i}} + 6\hat{\mathbf{j}} - 12\hat{\mathbf{k}}$$

Q. 8 **Find a vector of magnitude 6, which is perpendicular to both the vectors $2\hat{\mathbf{i}} - \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$ and $4\hat{\mathbf{i}} - \hat{\mathbf{j}} + 3\hat{\mathbf{k}}$.**

Thinking Process

First, we will use this concept any vector perpendicular to both the vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ is given by $\vec{\mathbf{a}} \times \vec{\mathbf{b}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$ *and then we will find the vector with magnitude 6.*

Sol. Let $\vec{\mathbf{a}} = 2\hat{\mathbf{i}} - \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$ and $\vec{\mathbf{b}} = 4\hat{\mathbf{i}} - \hat{\mathbf{j}} + 3\hat{\mathbf{k}}$

So, any vector perpendicular to both the vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ is given by

$$\vec{\mathbf{a}} \times \vec{\mathbf{b}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & -1 & 2 \\ 4 & -1 & 3 \end{vmatrix}$$

$$= \hat{\mathbf{i}}(-3 + 2) - \hat{\mathbf{j}}(6 - 8) + \hat{\mathbf{k}}(-2 + 4)$$

$$= -\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}} = \vec{\mathbf{r}} \quad \text{[say]}$$

A vector of magnitude 6 in the direction of $\vec{r}$

$$= \frac{\vec{\mathbf{r}}}{|\vec{\mathbf{r}}|} . 6 = \frac{-\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}}}{\sqrt{1^2 + 2^2 + 2^2}} . 6$$

$$= \frac{-6}{3}\hat{\mathbf{i}} + \frac{12}{3}\hat{\mathbf{j}} + \frac{12}{3}\hat{\mathbf{k}}$$

$$= -2\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$$

Q. 9 **Find the angle between the vectors $2\hat{i} - \hat{j} + \hat{k}$ and $3\hat{i} + 4\hat{j} - \hat{k}$.**

Thinking Process

If $\vec{a}$ and $\vec{b}$ are two vectors, making angle θ with each other, then $\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$, using this concept we will find θ.

Sol. Let $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$ and $\vec{b} = 3\hat{i} + 4\hat{j} - \hat{k}$

We know that, angle between two vectors $\vec{a}$ and $\vec{b}$ is given by

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$$

$$= \frac{(2\hat{i} - \hat{j} + \hat{k})(3\hat{i} + 4\hat{j} - \hat{k})}{\sqrt{4+1+1}\sqrt{9+16+1}}$$

$$= \frac{6-4-1}{\sqrt{6}\sqrt{26}} = \frac{1}{2\sqrt{39}}$$

$$\therefore \quad \theta = \cos^{-1}\left(\frac{1}{2\sqrt{39}}\right)$$

Q. 10 **If $\vec{a} + \vec{b} + \vec{c} = 0$, then show that $\vec{a}\times\vec{b} = \vec{b}\times\vec{c} = \vec{c}\times\vec{a}$. Interpret the result geometrically.**

Sol. Since, $\vec{a} + \vec{b} + \vec{c} = 0$

$\Rightarrow \quad \vec{b} = -\vec{c} - \vec{a}$

Now, $\vec{a}\times\vec{b} = \vec{a}\times(-\vec{c} - \vec{a})$

$= \vec{a}\times(-\vec{c}) + \vec{a}\times(-\vec{a}) = -\vec{a}\times\vec{c}$

$\Rightarrow \quad \vec{a}\times\vec{b} = \vec{c}\times\vec{a}$ ...(i)

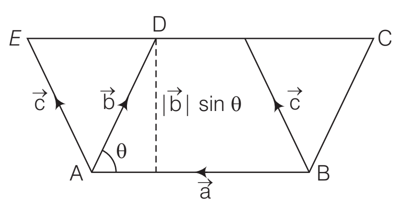
Also, $\vec{b}\times\vec{c} = (-\vec{c} - \vec{a})\times\vec{c}$

$= (-\vec{c}\times\vec{c}) + (-\vec{a}\times\vec{c}) = -\vec{a}\times\vec{c}$

$\Rightarrow \quad \vec{b}\times\vec{c} = \vec{c}\times\vec{a}$ ...(ii)

From Eqs. (i) and (ii), $\vec{a}\times\vec{b} = \vec{b}\times\vec{c} = \vec{c}\times\vec{a}$

Geometrical interpretation of the result



If *ABCD* is a parallelogram such that $\vec{AB} = \vec{a}$ and $\vec{AD} = \vec{b}$ and these adjacent sides are making angle θ between each other, then we say that

Area of parallelogram $ABCD = |\vec{a}|\,|\vec{b}||\sin\theta| = |\vec{a}\times\vec{b}|$

Since, parallelogram on the same base and between the same parallels are equal in area.

We can say that, $|\vec{a} \times \vec{b}| = |\vec{a} \times \vec{c}| = |\vec{b} \times \vec{c}|$

This also implies that, $\vec{a} \times \vec{b} = \vec{a} \times \vec{c} = \vec{b} \times \vec{c}$

So, area of the parallelograms formed by taking any two sides represented by $\vec{a}$, $\vec{b}$ and $\vec{c}$ as adjacent are equal.

Q. 11 Find the sine of the angle between the vectors $\vec{a} = 3\hat{i} + \hat{j} + 2\hat{k}$ and $\vec{b} = 2\hat{i} - 2\hat{j} + 4\hat{k}$.

Thinking Process

We know that, if $\vec{a}$ *and* $\vec{b}$ *are in their component form, then* $\cos\theta = \dfrac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}}$. *After getting* $\cos\theta$, *we shall find the sine of the angle.*

Sol. Here, $a_1 = 3$, $a_2 = 1$, $a_3 = 2$ and $b_1 = 2$, $b_2 = -2$, $b_3 = 4$

We know that,

$$\cos\theta = \frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}}$$

$$= \frac{3 \times 2 + 1 \times (-2) + 2 \times 4}{\sqrt{3^2 + 1^2 + 2^2}\sqrt{2^2 + (-2)^2 + 4^2}}$$

$$= \frac{6 - 2 + 8}{\sqrt{14}\sqrt{24}} = \frac{12}{2\sqrt{14}\sqrt{6}} = \frac{6}{\sqrt{84}} = \frac{6}{2\sqrt{21}} = \frac{3}{\sqrt{21}}$$

$\therefore$

$$\sin\theta = \sqrt{1 - \cos^2\theta}$$

$$= \sqrt{1 - \frac{9}{21}} = \sqrt{\frac{12}{21}} = \frac{2\sqrt{3}}{\sqrt{3}\sqrt{7}} = \frac{2}{\sqrt{7}}$$

Q. 12 If *A, B, C* and *D* are the points with position vectors $\hat{i} - \hat{j} + \hat{k}$, $2\hat{i} - \hat{j} + 3\hat{k}$, $2\hat{i} - 3\hat{k}$ and $3\hat{i} - 2\hat{j} + \hat{k}$ respectively, then find the projection of $\vec{AB}$ along $\vec{CD}$.

Thinking Process

We shall use the concept that projection of $\vec{a}$ *along* $\vec{b}$ *is* $\dfrac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$.

Sol. Here, $\vec{OA} = \hat{i} + \hat{j} - \hat{k}$, $\vec{OB} = 2\hat{i} - \hat{j} + 3\hat{k}$, $\vec{OC} = 2\hat{i} - 3\hat{k}$ and $\vec{OD} = 3\hat{i} - 2\hat{j} + \hat{k}$

$\therefore$

$$\vec{AB} = \vec{OB} - \vec{OA} = (2-1)\hat{i} + (-1-1)\hat{j} + (3+1)\hat{k}$$

$$= \hat{i} - 2\hat{j} + 4\hat{k}$$

and

$$\vec{CD} = \vec{OD} - \vec{OC} = (3-2)\hat{i} + (-2-0)\hat{j} + (1+3)\hat{k}$$

$$= \hat{i} - 2\hat{j} + 4\hat{k}$$

So, the projection of $\vec{AB}$ along $\vec{CD} = \vec{AB} \cdot \dfrac{\vec{CD}}{|\vec{CD}|}$

$$= \frac{(\hat{i} - 2\hat{j} + 4\hat{k}) \cdot (\hat{i} - 2\hat{j} + 4\hat{k})}{\sqrt{1^2 + 2^2 + 4^2}}$$

$$= \frac{1 + 4 + 16}{\sqrt{21}} = \frac{21}{\sqrt{21}}$$

$$= \sqrt{21} \text{ units}$$

Q.13 Using vectors, find the area of the ΔABC with vertices $A(1, 2, 3)$, $B(2, -1, 4)$ and $C(4, 5, -1)$.

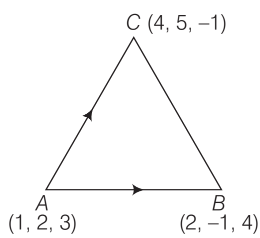
Thinking Process

We know that,

Area of $\Delta ABC = \frac{1}{2}|\vec{AB} \times \vec{AC}|$. *So, here we shall use this concept.*

Sol. Here,
$$\vec{AB} = (2-1)\hat{i} + (-1-2)\hat{j} + (4-3)\hat{k}$$
$$= \hat{i} - 3\hat{j} + \hat{k}$$

and
$$\vec{AC} = (4-1)\hat{i} + (5-2)\hat{j} + (-1-3)\hat{k}$$
$$= 3\hat{i} + 3\hat{j} - 4\hat{k}$$



$\therefore$
$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -3 & 1 \\ 3 & 3 & -4 \end{vmatrix}$$
$$= \hat{i}(12 - 3) - \hat{j}(-4 - 3) + \hat{k}(3 + 9)$$
$$= 9\hat{i} + 7\hat{j} + 12\hat{k}$$

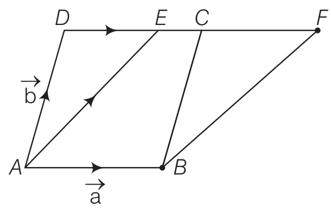
and
$$|\vec{AB} \times \vec{AC}| = \sqrt{9^2 + 7^2 + 12^2}$$
$$= \sqrt{81 + 49 + 144}$$
$$= \sqrt{274}$$

$\therefore$
$$\text{Area of } \Delta ABC = \frac{1}{2}|\vec{AB} \times \vec{AC}|$$
$$= \frac{1}{2}\sqrt{274} \text{ sq units}$$

Q. 14 Using vectors, prove that the parallelogram on the same base and between the same parallels are equal in area.

Sol. Let *ABCD* and *ABFE* are parallelograms on the same base *AB* and between the same parallel lines *AB* and *DF*.

Here, $AB \| CD$ and $AE \| BF$



Let $\vec{\mathbf{AB}} = \vec{\mathbf{a}}$ and $\vec{\mathbf{AD}} = \vec{\mathbf{b}}$

$\therefore$ Area of parallelogram $ABCD = \vec{\mathbf{a}} \times \vec{\mathbf{b}}$

Now, area of parallelogram $ABFF = \vec{\mathbf{AB}} \times \vec{\mathbf{AE}}$

$= \vec{\mathbf{AB}} \times (\vec{\mathbf{AD}} + \vec{\mathbf{DE}})$

$= \vec{\mathbf{AB}} \times (\vec{\mathbf{b}} + k\vec{\mathbf{a}})$ [let $\vec{\mathbf{DE}} = k\vec{\mathbf{a}}$, where k is a scalar]

$= \vec{\mathbf{a}} \times (\vec{\mathbf{b}} + k\vec{\mathbf{a}})$

$= (\vec{\mathbf{a}} \times \vec{\mathbf{b}}) + (\vec{\mathbf{a}} \times k\vec{\mathbf{a}})$

$= (\vec{\mathbf{a}} \times \vec{\mathbf{b}}) + k(\vec{\mathbf{a}} \times \vec{\mathbf{a}})$

$= (\vec{\mathbf{a}} \times \vec{\mathbf{b}})$ [$\because \vec{\mathbf{a}} \times \vec{\mathbf{a}} = 0$]

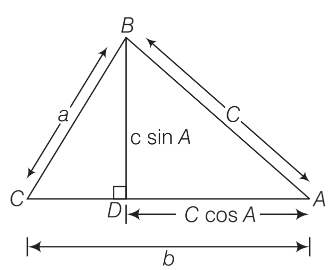
= Area of parallelogram *ABCD*

Hence proved.

Long Answer Type Questions

Q. 15 Prove that in any ΔABC, $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$, where a, b and c are the magnitudes of the sides opposite to the vertices A, B and C, respectively.

Sol. Here, components of C are $c \cos A$ and $c \sin A$ is drawn.



Since, $\qquad \vec{\mathbf{CD}} = b - c\cos A$

In ΔBDC,

$$a^2 = (b - c\cos A)^2 + (c\sin A)^2$$

$\Rightarrow \qquad a^2 = b^2 + c^2\cos^2 A - 2bc\cos A + c^2\sin^2 A$

$\Rightarrow \qquad 2bc\cos A = b^2 - a^2 + c^2(\cos^2 A + \sin^2 A)$

$\therefore \qquad \cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

Q. 16 If $\vec{\mathbf{a}}, \vec{\mathbf{b}}$ and $\vec{\mathbf{c}}$ determine the vertices of a triangle, show that $\frac{1}{2}[\vec{\mathbf{b}} \times \vec{\mathbf{c}} + \vec{\mathbf{c}} \times \vec{\mathbf{a}} + \vec{\mathbf{a}} \times \vec{\mathbf{b}}]$ gives the vector area of the triangle. Hence, deduce the condition that the three points $\vec{\mathbf{a}}, \vec{\mathbf{b}}$ and $\vec{\mathbf{c}}$ are collinear. Also, find the unit vector normal to the plane of the triangle.

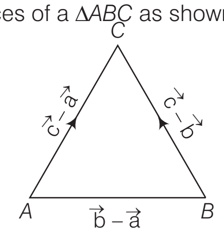
Thinking Process

Here, we shall use the following two concepts.

(i) *If* $\vec{\mathbf{a}}, \vec{\mathbf{b}}$ *and* $\vec{\mathbf{c}}$ *are collinear, then the area of the triangle formed by the vectors will be zero.*

(ii) *We know that,* $\vec{\mathbf{a}} \times \vec{\mathbf{b}} = |\vec{\mathbf{a}}||\vec{\mathbf{b}}|\sin\theta\hat{\mathbf{n}}$.

Sol. Since, $\vec{a}, \vec{b}$ and $\vec{c}$ are the vertices of a ΔABC as shown.



$\therefore \qquad$ Area of $\Delta ABC = \frac{1}{2}|\vec{\mathbf{AB}} \times \vec{\mathbf{AC}}|$

Now, $\qquad \vec{\mathbf{AB}} = \vec{\mathbf{b}} - \vec{\mathbf{a}}$ and $\vec{\mathbf{AC}} = \vec{\mathbf{c}} - \vec{\mathbf{a}}$

$\therefore \qquad$ Area of $\Delta ABC = \frac{1}{2}|\vec{b} - \vec{a} \times \vec{c} - \vec{a}|$

$$= \frac{1}{2}|\vec{b} \times \vec{c} - \vec{b} \times \vec{a} - \vec{a} \times \vec{c} + \vec{a} \times \vec{a}|$$

$$= \frac{1}{2}|\vec{b} \times \vec{c} + \vec{a} \times \vec{b} + \vec{c} \times \vec{a} + \vec{0}|$$

$$= \frac{1}{2}|\vec{b} \times \vec{c} + \vec{a} \times \vec{b} + \vec{c} \times \vec{a}| \qquad \text{...(i)}$$

For three points to be collinear, area of the Δ ABC should be equal to zero.

$\Rightarrow \qquad \frac{1}{2}[\vec{b} \times \vec{c} + \vec{c} \times \vec{a} + \vec{a} \times \vec{b}] = 0$

$\Rightarrow \qquad \vec{b} \times \vec{c} + \vec{c} \times \vec{a} + \vec{a} \times \vec{b} = 0 \qquad \text{...(ii)}$

This is the required condition for collinearity of three points $\vec{a}, \vec{b}$ and $\vec{c}$.

Let $\hat{n}$ be the unit vector normal to the plane of the Δ *ABC*.

$$\therefore \qquad \hat{\mathbf{n}} = \frac{\vec{\mathbf{AB}} \times \vec{\mathbf{AC}}}{|\vec{\mathbf{AB}} \times \vec{\mathbf{AC}}|}$$

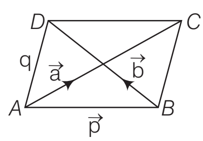
$$= \frac{\vec{\mathbf{a}} \times \vec{\mathbf{b}} + \vec{\mathbf{b}} \times \vec{\mathbf{c}} + \vec{\mathbf{c}} \times \vec{\mathbf{a}}}{|\vec{\mathbf{a}} \times \vec{\mathbf{b}} + \vec{\mathbf{b}} \times \vec{\mathbf{c}} + \vec{\mathbf{c}} \times \vec{\mathbf{a}}|}$$

Q. 17 Show that area of the parallelogram whose diagonals are given by $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ is $\frac{|\vec{\mathbf{a}} \times \vec{\mathbf{b}}|}{2}$. Also, find the area of the parallelogram, whose diagonals are $2\hat{\mathbf{i}} - \hat{\mathbf{j}} + k$ and $\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - \hat{\mathbf{k}}$.

Thinking Process

If $\vec{\mathbf{p}}$ and $\vec{\mathbf{q}}$ are adjacent sides of a parallelogram, then the area formed by parallelogram $= |\vec{\mathbf{p}} \times \vec{\mathbf{q}}|$ and then we shall obtained the desired result.

Sol. Let *ABCD* be a parallelogram such that



$$\vec{\mathbf{AB}} = \vec{\mathbf{p}},\ \overrightarrow{\mathbf{AD}} = \vec{\mathbf{q}} \Rightarrow \vec{\mathbf{BC}} = \vec{\mathbf{q}}$$

By triangle law of addition, we get

$$\vec{\mathbf{AC}} = \vec{\mathbf{p}} + \vec{\mathbf{q}} = \vec{\mathbf{a}} \qquad \text{[say] ...(i)}$$

Similarly,

$$\vec{\mathbf{BD}} = -\vec{\mathbf{p}} + \vec{\mathbf{q}} = \vec{\mathbf{b}} \qquad \text{[say] ...(ii)}$$

On adding Eqs. (i) and (ii), we get

$$\vec{\mathbf{a}} + \vec{\mathbf{b}} = 2\vec{\mathbf{q}} \Rightarrow \vec{\mathbf{q}} = \frac{1}{2}(\vec{\mathbf{a}} + \vec{\mathbf{b}})$$

On subtracting Eq. (ii) from Eq. (i), we get

$$\vec{\mathbf{a}} - \vec{\mathbf{b}} = 2\vec{\mathbf{p}} \Rightarrow \vec{\mathbf{p}} = \frac{1}{2}(\vec{\mathbf{a}} - \vec{\mathbf{b}})$$

Now,

$$\vec{\mathbf{p}} \times \vec{\mathbf{q}} = \frac{1}{4}(\vec{\mathbf{a}} - \vec{\mathbf{b}}) \times (\vec{\mathbf{a}} + \vec{\mathbf{b}})$$

$$= \frac{1}{4}(\vec{\mathbf{a}} \times \vec{\mathbf{a}} + \vec{\mathbf{a}} \times \vec{\mathbf{b}} - \vec{\mathbf{b}} \times \vec{\mathbf{a}} - \vec{\mathbf{b}} \times \vec{\mathbf{b}})$$

$$= \frac{1}{4}[\vec{\mathbf{a}} \times \vec{\mathbf{b}} + \vec{\mathbf{a}} \times \vec{\mathbf{b}}]$$

$$= \frac{1}{2}(\vec{\mathbf{a}} \times \vec{\mathbf{b}})$$

So, area of a parallelogram $ABCD = |\vec{\mathbf{p}} \times \vec{\mathbf{q}}| = \frac{1}{2}|\vec{\mathbf{a}} \times \vec{\mathbf{b}}|$

Now, area of a parallelogram, whose diagonals are $2\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - \hat{\mathbf{k}}$.

$$= \frac{1}{2}|(2\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}) \times (\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - \hat{\mathbf{k}})|$$

$$= \frac{1}{2}\left\| \begin{matrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & -1 & 1 \\ 1 & 3 & -1 \end{matrix} \right\|$$

$$= \frac{1}{2}|[\hat{\mathbf{i}}(1-3) - \hat{\mathbf{j}}(-2-1) + \hat{\mathbf{k}}(6+1)]|$$

$$= \frac{1}{2}|-2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 7\hat{\mathbf{k}}|$$

$$= \frac{1}{2}\sqrt{4+9+49}$$

$$= \frac{1}{2}\sqrt{62} \text{ sq units}$$

Q. 18 If $\vec{\mathbf{a}} = \hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\vec{\mathbf{b}} = \hat{\mathbf{j}} - \hat{\mathbf{k}}$, then find a vector $\vec{\mathbf{c}}$ such that $\vec{\mathbf{a}} \times \vec{\mathbf{c}} = \vec{\mathbf{b}}$ and $\vec{\mathbf{a}} \cdot \vec{\mathbf{c}} = 3$.

Thinking Process

We know that, for any two vectors

$$\vec{\mathbf{a}} \times \vec{\mathbf{b}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

and $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = a_1b_1 + a_2b_2 + a_3b_3$, *where* $\vec{\mathbf{a}} = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}$ *and* $\vec{\mathbf{b}} = b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + b_3\hat{\mathbf{k}}$.

So, we shall use this concept.

Sol. Let $\vec{\mathbf{c}} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$

Also, $\vec{\mathbf{a}} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\vec{\mathbf{b}} = \hat{\mathbf{j}} - \hat{\mathbf{k}}$

For $\vec{\mathbf{a}} \times \vec{\mathbf{c}} = \vec{\mathbf{b}}$,

$$\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 1 & 1 \\ x & y & z \end{vmatrix} = \hat{\mathbf{j}} - \hat{\mathbf{k}}$$

$\Rightarrow$ $\hat{\mathbf{i}}(z-y) - \hat{\mathbf{j}}(z-x) + \hat{\mathbf{k}}(y-x) = \hat{\mathbf{j}} - \hat{\mathbf{k}}$

$\therefore$ $z - y = 0$ ...(i)

$x - z = 1$ ...(ii)

$x - y = 1$ ...(iii)

Also, $\vec{\mathbf{a}} \cdot \vec{\mathbf{c}} = 3$

$(\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}) \cdot (x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}) = 3$

$\Rightarrow$ $x + y + z = 3$ ...(iv)

On adding Eqs. (ii) and (iii), we get

$2x - y - z = 2$ ...(v)

On solving Eqs. (iv) and (v), we get

$$x = \frac{5}{3}$$

$\therefore$ $$y = \frac{5}{3} - 1 = \frac{2}{3} \text{ and } z = \frac{2}{3}$$

Now, $$\vec{\mathbf{c}} = \frac{5}{3}\hat{\mathbf{i}} + \frac{2}{3}\hat{\mathbf{j}} + \frac{2}{3}\hat{\mathbf{k}}$$

$$= \frac{1}{3}(5\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}})$$

Objective Type Questions

Q. 19 The vector in the direction of the vector $\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$ that has magnitude 9 is

(a) $\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$

(b) $\dfrac{\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}}}{3}$

(c) $3(\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}})$

(d) $9(\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}})$

Sol. *(c)* Let $\vec{\mathbf{a}} = \hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$

Any vector in the direction of a vector $\vec{\mathbf{a}}$ is given by $\dfrac{\vec{\mathbf{a}}}{|\vec{\mathbf{a}}|}$.

$$= \frac{\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}}}{\sqrt{1^2 + 2^2 + 2^2}} = \frac{\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}}}{3}$$

$\therefore$ Vector in the direction of $\vec{\mathbf{a}}$ with magnitude $9 = 9 \cdot \dfrac{\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}}}{3}$

$$= 3(\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}})$$

Q. 20 The position vector of the point which divides the join of points $2\vec{\mathbf{a}} - 3\vec{\mathbf{b}}$ and $\vec{\mathbf{a}} + \vec{\mathbf{b}}$ in the ratio 3 : 1, is

(a) $\dfrac{3\vec{\mathbf{a}} - 2\vec{\mathbf{b}}}{2}$

(b) $\dfrac{7\vec{\mathbf{a}} - 8\vec{\mathbf{b}}}{4}$

(c) $\dfrac{3\vec{\mathbf{a}}}{4}$

(d) $\dfrac{5\vec{\mathbf{a}}}{4}$

Sol. *(d)* Let the position vector of the point R divides the join of points $2\vec{\mathbf{a}} - 3\vec{\mathbf{b}}$ and $\vec{\mathbf{a}} + \vec{\mathbf{b}}$.

$\therefore$ Position vector $$R = \frac{3(\vec{\mathbf{a}} + \vec{\mathbf{b}}) + 1(2\vec{\mathbf{a}} - 3\vec{\mathbf{b}})}{3 + 1}$$

Since, the position vector of a point R dividing the line segment joining the points P and Q, whose position vectors are $\vec{\mathbf{p}}$ and $\vec{\mathbf{q}}$ in the ratio $m : n$ internally, is given by $\dfrac{m\vec{\mathbf{q}} + n\vec{\mathbf{p}}}{m + n}$.

$\therefore$ $$R = \frac{5\vec{\mathbf{a}}}{4}$$

Q. 21 The vector having initial and terminal points as (2, 5, 0) and (–3, 7, 4), respectively is

(a) $-\hat{\mathbf{i}} + 12\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$
(b) $5\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 4\hat{\mathbf{k}}$
(c) $-5\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$
(d) $\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$

Sol. *(c)* Required vector $= (-3-2)\hat{\mathbf{i}} + (7-5)\hat{\mathbf{j}} + (4-0)\hat{\mathbf{k}}$

$$= -5\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$$

Similarly, we can say that for having initial and terminal points as

(i) (4, 1, 1) and (3, 13, 5), respectively.
(ii) (1, 1, 9) and (6, 3, 5), respectively.
(iii) (1, 2, 3) and (2, 3, 4), respectively, we shall get (a), (b) and (d) as its correct options.

Q. 22 The angle between two vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ with magnitudes $\sqrt{3}$ and 4, respectively and $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = 2\sqrt{3}$ is

(a) $\frac{\pi}{6}$
(b) $\frac{\pi}{3}$
(c) $\frac{\pi}{2}$
(d) $\frac{5\pi}{2}$

Sol. *(b)* Here, $|\vec{\mathbf{a}}| = \sqrt{3}, |\vec{\mathbf{b}}| = 4$ and $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = 2\sqrt{3}$ [given]

We know that, $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = |\vec{\mathbf{a}}||\vec{\mathbf{b}}|\cos\theta$

$\Rightarrow$ $2\sqrt{3} = \sqrt{3} \cdot 4 \cdot \cos\theta$

$\Rightarrow$ $\cos\theta = \frac{2\sqrt{3}}{4\sqrt{3}} = \frac{1}{2}$

$\therefore$ $\theta = \frac{\pi}{3}$

Q. 23 Find the value of λ such that the vectors $\vec{\mathbf{a}} = 2\hat{\mathbf{i}} + \lambda\hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\vec{\mathbf{b}} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$ are orthogonal.

(a) 0
(b) 1
(c) $\frac{3}{2}$
(d) $\frac{-5}{2}$

Thinking Process

Two non-zero vectors are orthogonal, if their dot product is zero. So, by using this concept, we shall get the value of λ.

Sol. *(d)* Since, two non-zero vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ are orthogonal *i.e.*, $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = 0$.

$\therefore$ $(2\hat{\mathbf{i}} + \lambda\hat{\mathbf{j}} + \hat{\mathbf{k}}) \cdot (\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}) = 0$

$\Rightarrow$ $2 + 2\lambda + 3 = 0$

$\therefore$ $\lambda = \frac{-5}{2}$

Q. 24 The value of λ for which the vectors $3\hat{\mathbf{i}} - 6\hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $2\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + \lambda\hat{\mathbf{k}}$ are parallel, is

(a) $\frac{2}{3}$ (b) $\frac{3}{2}$ (c) $\frac{5}{2}$ (d) $\frac{2}{5}$

Sol. *(a)* Since, two vectors are parallel *i.e.,* angle between them is zero.

$\therefore \quad (3\hat{\mathbf{i}} - 6\hat{\mathbf{j}} + \hat{\mathbf{k}})\cdot(2\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + \lambda\hat{\mathbf{k}}) = |3\hat{\mathbf{i}} - 6\hat{\mathbf{j}} + \hat{\mathbf{k}}| \cdot |2\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + \lambda\hat{\mathbf{k}}|$

$[\because \vec{\mathbf{a}}\cdot\vec{\mathbf{b}} = |\mathbf{a}|\,|\mathbf{b}|\cos 0° \Rightarrow \vec{\mathbf{a}}\cdot\vec{\mathbf{b}} = |\vec{\mathbf{a}}||\vec{\mathbf{b}}|]$

$$\Rightarrow \quad 6 + 24 + \lambda = \sqrt{9 + 36 + 1}\sqrt{4 + 16 + \lambda^2}$$

$$\Rightarrow \quad 30 + \lambda = \sqrt{46}\sqrt{20 + \lambda^2}$$

$$\Rightarrow \quad 900 + \lambda^2 + 60\lambda = 46(20 + \lambda^2) \quad \text{[on squaring both sides]}$$

$$\Rightarrow \quad \lambda^2 + 60\lambda - 46\lambda^2 = 920 - 900$$

$$\Rightarrow \quad -45\lambda^2 + 60\lambda - 20 = 0$$

$$\Rightarrow \quad -45\lambda^2 + 30\lambda + 30\lambda - 20 = 0$$

$$\Rightarrow \quad -15\lambda(3\lambda - 2) + 10(3\lambda - 2) = 0$$

$$\Rightarrow \quad (10 - 15\lambda)(3\lambda - 2) = 0$$

$$\therefore \quad \lambda = \frac{2}{3}, \frac{2}{3}$$

Alternate Method

Let $\vec{\mathbf{a}} = 3\hat{\mathbf{i}} - 6\hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\vec{\mathbf{b}} = 2\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + \lambda\hat{\mathbf{k}}$

Since, $\vec{\mathbf{a}} \,||\, \vec{\mathbf{b}}$

$$\Rightarrow \quad \frac{3}{2} = \frac{-6}{-4} = \frac{1}{\lambda} \Rightarrow \lambda = \frac{2}{3}$$

Q. 25 The vectors from origin to the points A and B are $\vec{\mathbf{a}} = 2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$ and $\vec{\mathbf{b}} = 2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + \hat{\mathbf{k}}$ respectively, then the area of ΔOAB is equal to

(a) 340 (b) $\sqrt{25}$

(c) $\sqrt{229}$ (d) $\frac{1}{2}\sqrt{229}$

Sol. *(d)* $\therefore$ Area of $\Delta OAB = \frac{1}{2}|\vec{\mathbf{OA}} \times \vec{\mathbf{OB}}|$

$$= \frac{1}{2}|(2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 2\hat{\mathbf{k}}) \times (2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + \hat{\mathbf{k}})|$$

$$= \frac{1}{2}\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & -3 & 2 \\ 2 & 3 & 1 \end{vmatrix}$$

$$= \frac{1}{2}|[\hat{\mathbf{i}}(-3 - 6) - \hat{\mathbf{j}}(2 - 4) + \hat{\mathbf{k}}(6 + 6)]|$$

$$= \frac{1}{2}|-9\mathbf{i} + 2\hat{\mathbf{j}} + 12\hat{\mathbf{k}}|$$

$$\therefore \quad \text{Area of } \Delta OAB = \frac{1}{2}\sqrt{(81 + 4 + 144)} = \frac{1}{2}\sqrt{229}$$

Q. 26 For any vector $\vec{a}$, the value of $(\vec{a} \times \hat{i})^2 + (\vec{a} \times \hat{j})^2 + (\vec{a} \times \hat{k})^2$ is

(a) $\vec{a}^2$ (b) $3\vec{a}^2$

(c) $4\vec{a}^2$ (d) $2\vec{a}^2$

Sol. *(d)* Let $\vec{a} = x\hat{i} + y\hat{j} + z\hat{k}$

$\therefore \quad \vec{a}^2 = x^2 + y^2 + z^2$

$$\therefore \quad \vec{a} \times \hat{i} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ 1 & 0 & 0 \end{vmatrix}$$

$$= \hat{i}[0] - \hat{j}[-z] + \hat{k}[-y]$$

$$= z\hat{j} - y\hat{k}$$

$$\therefore \quad (\vec{a} \times \hat{i})^2 = (z\hat{j} - y\hat{k})(z\hat{j} - y\hat{k})$$

$$= y^2 + z^2$$

Similarly, $(\vec{a} \times \hat{j})^2 = x^2 + z^2$

and $(\vec{a} \times \hat{k})^2 = x^2 + y^2$

$$\therefore \quad (\vec{a} \times \hat{i})^2 + (\vec{a} \times \hat{j})^2 + (\vec{a} \times \hat{k})^2 = y^2 + z^2 + x^2 + z^2 + x^2 + y^2$$

$$= 2(x^2 + y^2 + z^2) = 2\vec{a}^2$$

Q. 27 If $|\vec{a}| = 10, |\vec{b}| = 2$ and $\vec{a} \cdot \vec{b} = 12$, then the value of $|\vec{a} \times \vec{b}|$ is

(a) 5 (b) 10 (c) 14 (d) 16

Thinking Process

We know that, $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}||\sin\theta|\hat{n}$ *and* $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$. *So, we shall use these formulae to get the value of* $|\vec{a} \times \vec{b}|$.

Sol. *(d)* Here, $|\vec{a}| = 10, |\vec{b}| = 2$ and $\vec{a} \cdot \vec{b} = 12$ [given]

$$\therefore \quad \vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$$

$$12 = 10 \times 2\cos\theta$$

$$\Rightarrow \quad \cos\theta = \frac{12}{20} = \frac{3}{5}$$

$$\Rightarrow \quad \sin\theta = \sqrt{1 - \cos^2\theta} = \sqrt{1 - \frac{9}{25}}$$

$$\sin\theta = \pm\frac{4}{5}$$

$$\therefore \quad |\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}||\sin\theta|$$

$$= 10 \times 2 \times \frac{4}{5}$$

$$= 16$$

Q. 28 The vectors $\lambda\hat{i} + \hat{j} + 2\hat{k}$, $\hat{i} + \lambda\hat{j} - \hat{k}$ and $2\hat{i} - \hat{j} + \lambda\hat{k}$ are coplanar, if

(a) $\lambda = -2$
(b) $\lambda = 0$
(c) $\lambda = 1$
(d) $\lambda = -1$

Sol. *(a)* Let $\vec{a} = \lambda\hat{i} + \hat{j} + 2\hat{k}$, $\vec{b} = \hat{i} + \lambda\hat{j} - \hat{k}$ and $\vec{c} = 2\hat{i} - \hat{j} + \lambda\hat{k}$

For $\vec{a}$, $\vec{b}$ and $\vec{c}$ to be coplanar,

$$\begin{vmatrix} \lambda & 1 & 2 \\ 1 & \lambda & -1 \\ 2 & -1 & \lambda \end{vmatrix} = 0$$

$\Rightarrow \quad \lambda(\lambda^2 - 1) - 1(\lambda + 2) + 2(-1 - 2\lambda) = 0$

$\Rightarrow \quad \lambda^3 - \lambda - \lambda - 2 - 2 - 4\lambda = 0$

$\Rightarrow \quad \lambda^3 - 6\lambda - 4 = 0$

$\Rightarrow \quad (\lambda + 2)(\lambda^2 - 2\lambda - 2) = 0$

$\Rightarrow \quad \lambda = -2 \text{ or } \lambda = \dfrac{2 \pm \sqrt{12}}{2}$

$\Rightarrow \quad \lambda = -2 \text{ or } \lambda = \dfrac{2 \pm 2\sqrt{3}}{2} = 1 \pm \sqrt{3}$

Q. 29 If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are unit vectors such that $\vec{a} + \vec{b} + \vec{c} = \vec{0}$, then the value of $\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a}$ is

(a) 1
(b) 3
(c) $-\dfrac{3}{2}$
(d) None of these

Sol. *(c)* We have, $\vec{a} + \vec{b} + \vec{c} = 0$ and $\vec{a}^2 = 1, \vec{b}^2 = 1, \vec{c}^2 = 1$

$\because \quad (\vec{a} + \vec{b} + \vec{c})(\vec{a} + \vec{b} + \vec{c}) = 0$

$\Rightarrow \quad \vec{a}^2 + \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c} + \vec{b}\cdot\vec{a} + \vec{b}^2 + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a} + \vec{c}\cdot\vec{b} + \vec{c}^2 = 0$

$\Rightarrow \quad \vec{a}^2 + \vec{b}^2 + \vec{c}^2 + 2(\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a}) = 0$

$[\because \vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}, \vec{b}\cdot\vec{c} = \vec{c}\cdot\vec{b} \text{ and } \vec{c}\cdot\vec{a} = \vec{a}\cdot\vec{c}]$

$\Rightarrow \quad 1 + 1 + 1 + 2(\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a}) = 0$

$\Rightarrow \quad \vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a} = -\dfrac{3}{2}$

Q. 30 The projection vector of $\vec{a}$ on $\vec{b}$ is

(a) $\left(\dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}\right)\vec{b}$
(b) $\dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$
(c) $\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$
(d) $\left(\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\right)\hat{b}$

Sol. *(a)* Projection vector of $\vec{a}$ on $\vec{b}$ is given by $= \vec{a}\cdot\dfrac{\vec{b}}{|\vec{b}|}\vec{b} = \left(\vec{a}\cdot\dfrac{\vec{b}}{|\vec{b}|}\right)\cdot\vec{b}$

Q. 31 If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are three vectors such that $\vec{a}+\vec{b}+\vec{c}=\vec{0}$ and $|\vec{a}|=2$, $|\vec{b}|=3$ and $|\vec{c}|=5$, then the value of $\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}$ is

(a) 0 (b) 1 (c) −19 (d) 38

Sol. *(c)* Here, $\vec{a}+\vec{b}+\vec{c}=\vec{0}$ and $\vec{a}^2=4, \vec{b}^2=9, \vec{c}^2=25$

$\therefore \quad (\vec{a}+\vec{b}+\vec{c})\cdot(\vec{a}+\vec{b}+\vec{c})=\vec{0}$

$\Rightarrow \vec{a}^2+\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}+\vec{b}\cdot\vec{a}+\vec{b}^2+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}+\vec{c}\cdot\vec{b}+\vec{c}^2=\vec{0}$

$\Rightarrow \quad \vec{a}^2+\vec{b}^2+\vec{c}^2+2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a})=0 \qquad [\because \vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{a}]$

$\Rightarrow \quad 4+9+25+2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a})=0$

$\Rightarrow \quad \vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}=\frac{-38}{2}=-19$

Q. 32 If $|\vec{a}|=4$ and $-3\le\lambda\le 2$, then the range of $|\lambda\vec{a}|$ is

(a) [0, 8] (b) [−12, 8]

(c) [0, 12] (d) [8, 12]

Sol. *(c)* We have, $|\vec{a}|=4$ and $-3\le\lambda\le 2$

$\therefore \quad |\lambda\vec{a}|=|\lambda||\vec{a}|=\lambda|4|$

$\Rightarrow \quad |\lambda\vec{a}|=|-3|4=12$, at $\lambda=-3$

$|\lambda\vec{a}|=|0|4=0$, at $\lambda=0$

and $\quad |\lambda\vec{a}|=|2|4=8$, at $\lambda=2$

So, the range of $|\lambda\vec{a}|$ is [0, 12].

Alternate Method

Since, $\quad -3\le\lambda\le 2$

$0\le|\lambda|\le 3$

$\Rightarrow \quad 0\le 4|\lambda|\le 12$

$|\lambda\vec{a}|\in[0, 12]$

Q. 33 The number of vectors of unit length perpendicular to the vectors $\vec{a}=2\hat{i}+\hat{j}+2\hat{k}$ and $\vec{b}=\hat{j}+\hat{k}$ is

(a) one (b) two

(c) three (d) infinite

Sol. *(b)* The number of vectors of unit length perpendicular to the vectors $\vec{a}$ and $\vec{b}$ is $\vec{c}$ (say)

i.e., $\quad \vec{c}=\pm(\vec{a}\times\vec{b})$.

So, there will be two vectors of unit length perpendicular to the vectors $\vec{a}$ and $\vec{b}$.

Fillers

Q. 34 The vector $\vec{a} + \vec{b}$ bisects the angle between the non-collinear vectors $\vec{a}$ and $\vec{b}$, if...... .

Sol. If vector $\vec{a} + \vec{b}$ bisects the angle between the non-collinear vectors, then

$$\vec{a} \cdot (\vec{a} + \vec{b}) = |\vec{a}| \cdot |\vec{a} + \vec{b}| \cos \theta$$

$$\vec{a} \cdot (\vec{a} + \vec{b}) = a\sqrt{a^2 + b^2} \cos \theta$$

$$\Rightarrow \quad \cos \theta = \frac{\vec{a} \cdot (\vec{a} + \vec{b})}{a\sqrt{a^2 + b^2}} \quad \ldots(i)$$

and

$$\vec{b} \cdot (\vec{a} + \vec{b}) = |\vec{b}| \cdot |\vec{a} + \vec{b}| \cos \theta$$

$$\vec{b} \cdot (\vec{a} + \vec{b}) = b\sqrt{a^2 + b^2} \cos \theta \quad \text{[since, } \theta \text{ should be same]}$$

$$\Rightarrow \quad \cos \theta = \frac{\vec{b} \cdot (\vec{a} + \vec{b})}{b\sqrt{a^2 + b^2}} \quad \ldots(ii)$$

From Eqs. (i) and (ii),

$$\frac{\vec{a} \cdot (\vec{a} + \vec{b})}{a\sqrt{a^2 + b^2}} = \frac{\vec{b} \cdot (\vec{a} + \vec{b})}{b\sqrt{a^2 + b^2}} \Rightarrow \frac{\vec{a}}{|\vec{a}|} = \frac{\vec{b}}{|\vec{b}|}$$

$\therefore$ $\hat{a} = \hat{b} \Rightarrow$ $\vec{a}$ and $\vec{b}$ are equal vectors.

Q. 35 If $\vec{r} \cdot \vec{a} = 0$, $\vec{r} \cdot \vec{b} = 0$ and $\vec{r} \cdot \vec{c} = 0$ for some non-zero vector $\vec{r}$, then the value of $\vec{a} \cdot (\vec{b} \times \vec{c})$ is...... .

Sol. Since, $\vec{r}$ is a non-zero vector. So, we can say that $\vec{a}$, $\vec{b}$ and $\vec{c}$ are in a same plane.

$$\therefore \quad \vec{a} \cdot (\vec{b} \times \vec{c}) = 0$$

[since, angle between $\vec{a}$, $\vec{b}$ and $\vec{c}$ are zero *i.e.,* $\theta = 0$]

Q. 36 The vectors $\vec{a} = 3\hat{i} - 2\hat{j} + 2\hat{k}$ and $\vec{b} = -\hat{i} - 2\hat{k}$ are the adjacent sides of a parallelogram. The angle between its diagonals is...... .

Sol. We have, $\vec{a} = 3\hat{i} - 2\hat{j} + 2\hat{k}$ and $\vec{b} = -\hat{i} - 2\hat{k}$

$\therefore$ $\vec{a} + \vec{b} = 2\hat{i} - 2\hat{j}$ and $\vec{a} - \vec{b} = 4\hat{i} - 2\hat{j} + 4\hat{k}$

Now, let θ is the acute angle between the diagonals $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$.

$$\therefore \quad \cos\theta = \frac{(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b})}{|\vec{a} + \vec{b}||\vec{a} - \vec{b}|}$$

$$= \frac{(2\hat{i} - 2\hat{j}) \cdot (4\hat{i} - 2\hat{j} + 4\hat{k})}{\sqrt{8}\sqrt{16 + 4 + 16}} = \frac{8 + 4}{2\sqrt{2} \cdot 6} = \frac{1}{\sqrt{2}}$$

$$\therefore \quad \theta = \frac{\pi}{4} \quad \left[\because \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}\right]$$

Q. 37 The values of k, for which $|k\vec{a}| < |\vec{a}|$ and $k\vec{a} + \frac{1}{2}\vec{a}$ is parallel to $\vec{a}$ holds true are ....

Sol. We have, $|k\vec{a}| < |\vec{a}|$ and $k\vec{a} + \frac{1}{2}\vec{a}$ is parallel to $\vec{a}$.

$\therefore \quad |k\vec{a}| < |\vec{a}| \Rightarrow |k||\vec{a}| < |\vec{a}|$

$\Rightarrow \quad |k| < 1 \Rightarrow -1 < k < 1$

Also, since $k\vec{a} + \frac{1}{2}\vec{a}$ is parallel to $\vec{a}$, then we see that at $k = \frac{-1}{2}$, $k\vec{a} + \frac{1}{2}\vec{a}$ becomes a null vector and then it will not be parallel to $\vec{a}$.

So, $k\vec{a} + \frac{1}{2}\vec{a}$ is parallel to $\vec{a}$ holds true when $k \in]-1, 1[\ k \neq \frac{-1}{2}$.

Q. 38 The value of the expression $|\vec{a} \times \vec{b}|^2 + (\vec{a} \cdot \vec{b})^2$ is ....

Sol.
$$|\vec{a} \times \vec{b}|^2 + (\vec{a} \cdot \vec{b})^2 = |\vec{a}|^2|\vec{b}|^2 \sin^2\theta + (\vec{a} \cdot \vec{b})^2$$
$$= |\vec{a}|^2|\vec{b}|^2(1 - \cos^2\theta) + (\vec{a} \cdot \vec{b})^2$$
$$= |\vec{a}|^2|\vec{b}|^2 - |\vec{a}|^2|\vec{b}|^2\cos^2\theta + (\vec{a} \cdot \vec{b})^2$$
$$= |\vec{a}|^2|\vec{b}|^2 - (\vec{a} \cdot \vec{b})^2 + (\vec{a} \cdot \vec{b})^2$$
$$|\vec{a} \times \vec{b}|^2 + (\vec{a} \cdot \vec{b})^2 = |\vec{a}|^2|\vec{b}|^2$$

Q. 39 If $|\vec{a} \times \vec{b}|^2 + |\vec{a} \cdot \vec{b}|^2 = 144$ and $|\vec{a}| = 4$, then $|\vec{b}|$ is equal to ......

Thinking Process

We know that, $|\vec{a} \times \vec{b}|^2 + |\vec{a} \cdot \vec{a}|^2 = |\vec{a}|^2 |\vec{b}|^2$. So, we shall use this concept here to find the value of $|\vec{b}|$.

Sol. $\because \quad |\vec{a} \times \vec{b}|^2 + |\vec{a} \cdot \vec{b}|^2 = 144 = |\vec{a}|^2 \cdot |\vec{b}|^2$

$\Rightarrow \quad |\vec{a}|^2 |\vec{b}|^2 = 144$

$\Rightarrow \quad |\vec{b}|^2 = \frac{144}{|\vec{a}|^2} = \frac{144}{16} = 9$

$\therefore \quad |\vec{b}| = 3$

Q. 40 If $\vec{a}$ is any non-zero vector, then $(\vec{a} \cdot \hat{i}) \cdot \hat{i} + (\vec{a} \cdot \hat{j}) \cdot \hat{j} + (\vec{a} \cdot \hat{k})\hat{k}$ is equal to ......

Sol. Let $\quad \vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$\therefore \quad \vec{a} \cdot \hat{i} = a_1, \vec{a} \cdot \hat{j} = a_2$ and $\vec{a} \cdot \hat{k} = a_3$

$\therefore \quad (\vec{a} \cdot \hat{i})\hat{i} + (\vec{a} \cdot \hat{j})\hat{j} + (\vec{a} \cdot \hat{k})\hat{k} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k} = \vec{a}$

True/False

Q. 41 If $|\vec{\mathbf{a}}| = |\vec{\mathbf{b}}|$, then necessarily it implies $\vec{\mathbf{a}} = \pm \vec{\mathbf{b}}$.

Sol. ***True***

If $\qquad |\vec{a}| = |\vec{b}| \Rightarrow \vec{a} = \pm \vec{b}$

So, it is a true statement.

Q. 42 Position vector of a point $\vec{\mathbf{P}}$ is a vector whose initial point is origin.

Sol. ***True***

Since, $\vec{P} = \overrightarrow{OP}$ = displacement of vector $\vec{P}$ from origin

Q. 43 If $|\vec{\mathbf{a}} + \vec{\mathbf{b}}| = |\vec{\mathbf{a}} - \vec{\mathbf{b}}|$, then the vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ are orthogonal

Sol. ***True***

Since, $\qquad |\vec{a} + \vec{b}| = |\vec{a} - \vec{b}|$

$\Rightarrow \qquad |\vec{a} + \vec{b}|^2 = |\vec{a} - \vec{b}|^2$

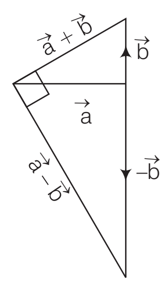
$\Rightarrow \qquad 2|\vec{a}||\vec{b}| = -2|\vec{a}||\vec{b}|$

$\Rightarrow \qquad 4|\vec{a}||\vec{b}| = 0$

$\Rightarrow \qquad |\vec{a}||\vec{b}| = 0$

Hence, $\vec{a}$ and $\vec{b}$ are orthogonal. $\qquad [\because \vec{a} \cdot \vec{b} = |\vec{a}| \cdot |\vec{b}| \cos 90^\circ = 0]$

Q. 44 The formula $(\vec{\mathbf{a}} + \vec{\mathbf{b}})^2 = \vec{\mathbf{a}}^2 + \vec{\mathbf{b}}^2 + 2\,\vec{\mathbf{a}} \times \vec{\mathbf{b}}$ is valid for non-zero vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$.



Sol. ***False***

$$(\vec{a} + \vec{b})^2 = (\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b})$$

$$= \vec{a}^2 + \vec{b}^2 + 2\vec{a} \cdot \vec{b}$$

Q. 45 If $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ are adjacent sides of a rhombus, then $\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = 0$.

Sol. ***False***

If $\vec{a} \cdot \vec{b} = 0$, then $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos 90^\circ$

Hence, angle between $\vec{a}$ and $\vec{b}$ is 90°, which is not possible in a rhombus.
Since, angle between adjacent sides in a rhombus is not equal to 90°.