11. QUADRATIC EQUATIONS

1. Quadratic Equation : $a x^2 + b x + c = 0,\ a \neq 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$, The expression $b^2 - 4ac \equiv D$ is called discriminant of quadratic equation.

If α, β are the roots, then (a) $\alpha + \beta = -\frac{b}{a}$ (b) $\alpha\beta = \frac{c}{a}$

A quadratic equation whose roots are α & β, is $(x - \alpha)(x - \beta) = 0$ i.e. $x^2 - (\alpha + \beta)x + \alpha\beta = 0$

2. Nature of Roots:

Consider the quadratic equation, $a x^2 + b x + c = 0$ having α, β as its roots; $D \equiv b^2 - 4ac$

$D = 0$
Roots are equal $\alpha = \beta = -b/2a$

$D \neq 0$
Roots are unequal

$a, b, c \in R$ & $D > 0$
Roots are real

$a, b, c \in R$ & $D < 0$
Roots are imaginary $\alpha = p + iq,\ \beta = p - iq$

$a, b, c \in Q$ &
D is a perfect square
$\Rightarrow$ Roots are rational
$\downarrow$
$a = 1,\ b, c \in I$ & D is a perfect square
$\Rightarrow$ Roots are integral.

$a, b, c \in Q$ &
D is not a perfect square
$\Rightarrow$ Roots are irrational
i.e. $\alpha = p + \sqrt{q}$, $\beta = p - \sqrt{q}$

3. Common Roots:

Consider two quadratic equations $a_1x^2 + b_1x + c_1 = 0$ & $a_2x^2 + b_2x + c_2 = 0$.

(i) If two quadratic equations have both roots common, then $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$.

(ii) If only one root α is common, then $\alpha = \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1} = \frac{b_1c_2 - b_2c_1}{c_1a_2 - c_2a_1}$

4. Range of Quadratic Expression f (x) = $a x^2 + b x + c$.

Range in restricted domain: Given $x \in [x_1, x_2]$

(a) If $-\frac{b}{2a} \notin [x_1, x_2]$ then, $f(x) \in \left[\min\{f(x_1), f(x_2)\},\ \max\{f(x_1), f(x_2)\}\right]$

(b) If $-\frac{b}{2a} \in [x_1, x_2]$ then, $f(x) \in \left[\min\left\{f(x_1), f(x_2), -\frac{D}{4a}\right\},\ \max\left\{f(x_1), f(x_2), -\frac{D}{4a}\right\}\right]$

5. Location of Roots:

Let $f(x) = ax^2 + bx + c$, where $a > 0$ & $a, b, c \in R$.

(i) Conditions for both the roots of $f(x) = 0$ to be greater than a specified number 'x_0' are $b^2 - 4ac \ge 0$; $f(x_0) > 0$ & $(-b/2a) > x_0$.

(ii) Conditions for both the roots of $f(x) = 0$ to be smaller than a specified number 'x_0' are $b^2 - 4ac \ge 0$; $f(x_0) > 0$ & $(-b/2a) < x_0$.

(iii) Conditions for both roots of $f(x) = 0$ to lie on either side of the number 'x_0' (in other words the number 'x_0' lies between the roots of $f(x) = 0$), is $f(x_0) < 0$.

(iv) Conditions that both roots of $f(x) = 0$ to be confined between the numbers x_1 and x_2, $(x_1 < x_2)$ are $b^2 - 4ac \ge 0$; $f(x_1) > 0$; $f(x_2) > 0$ & $x_1 < (-b/2a) < x_2$.

(v) Conditions for exactly one root of $f(x) = 0$ to lie in the interval (x_1, x_2) i.e.
$x_1 < x < x_2$ is $f(x_1) . f(x_2) < 0$.