Sequences and Series

- **Sequence:** A sequence is an arrangement of numbers in definite order according to some rule.
  Also, we define a sequence as a function whose domain is the set of natural numbers or some subset of the type $\{1, 2, 3 \dots k\}$.
    - A sequence containing finite number of terms is called a finite sequence.
    - sequence containing infinite number of terms is called an infinite sequence.

- A general sequence can be written as

$a_1, a_2, a_3 \dots a_{n-1}, a_n, \dots$
Here, $a_1, a_2 \dots$ etc. are called the terms of the sequence and a_n is called the general term or nth of the sequence.

- **Fibonacci sequence:** An arrangement of numbers such as 1, 2, 4, 6, 10 … has no visible pattern. However, the sequence is generated by the recurrence relation given by
  $a_1 = 1, a_2 = 2, a_3 = 4$
  $a_n = a_{n-2} + a_{n-1}, n > 3$
  This sequence is called the Fibonacci sequence.

- Let $a_1, a_2, \dots a_n, \dots$ be a given sequence. Accordingly, the sum of this sequence is given by the expression $a_1 + a_2 + \dots + a_n + \dots$
  This is called the series associated with the given sequence.
  The series is finite or infinite according as the given sequence.
  A series is usually represented in a compact form using sigma notation (∑).
  This means the series $a_1 + a_2 + \dots + a_{n-1} + a_n \dots$ can be written as $\sum_{k=1}^{n} a_k$.

- ***n*^th^ term of an AP**

  The n^{th} term (a_n) of an AP with first term a and common difference d is given by $a_n = a + (n-1)\,d$.
  Here, a_n is called the general term of the AP.

- **n^{th} term from the end of an AP**

  The n^{th} term from the end of an AP with last term l and common difference d is given by $l - (n - 1)\, d$.

  Example:
  Find the 12^{th} term of the AP 5, 9, 13 …
  Solution:
  Here, $a = 5$, $d = 9 - 5 = 4$, $n = 12$

  $$\begin{aligned} a_{12} &= a + (n-1)\,d \\ &= 5 + (12-1)\,4 \\ &= 5 + 11 \times 4 \\ &= 5 + 44 \\ &= 49 \end{aligned}$$

- **Sum of *n* terms of an AP**
  - The sum of the first n terms of an AP is given by Sn=n22a+n-1d , where a is the first term and d is the common difference.

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  - If there are only n terms in an AP, then Sn=n2a+l, where $l = a_n$ is the last term.

Example :

Find the value of 2 + 10 + 18 + …. + 802.

Solution:

2, 10, 18… 802 is an AP where $a = 2$, $d = 8$, and $l = 802$.

Let there be n terms in the series. Then,

$a_n = 802$

$\Rightarrow a + (n - 1)\, d = 802$

$\Rightarrow 2 + (n - 1)\, 8 = 802$

$\Rightarrow 8(n - 1) = 800$

$\Rightarrow n - 1 = 100$

$\Rightarrow n = 101$

Thus, required sum = n2a+l = 10122+802 = 40602

- **Properties of an Arithmetic progression**
  - If a constant is added or subtracted or multiplied to each term of an A.P. then the resulting sequence is also an A.P.
  - If each term of an A.P. is divided by a non-zero constant then the resulting sequence is also an A.P.

- **Arithmetic mean**
  - For any two numbers a and b, we can insert a number A between them such that a, A, b is an A.P. Such a number i.e., A is called the arithmetic mean (A.M) of numbers a and b and it is given by $A = \frac{a+b}{2}$.
  - For any two given numbers a and b, we can insert as many numbers between them as we want such that the resulting sequence becomes an A.P.

Let $A_1, A_2 \ldots A_n$ be n numbers between a and b such that $a, A_1, A_2 \ldots A_n, b$ is an A.P.

Here, common difference (d) is given by $\frac{b-a}{n+1}$.

Example:
Insert three numbers between –2 and 18 such that the resulting sequence is an A.P.
Solution:
Let A_1, A_2, and A_3 be three numbers between – 2 and 18 such that – 2, A_1, A_2, A_3, 18 are in an A.P.
Here, $a = -2$, $b = 18$, $n = 5$
$\therefore 18 = -2 + (5 - 1)\, d$
$\Rightarrow 20 = 4\, d$
$\Rightarrow d = 5$
Thus, $A_1 = a + d = -2 + 5 = 3$
$A_2 = a + 2d = -2 + 10 = 8$
$A_3 = a + 3d = -2 + 15 = 13$
Hence, the required three numbers between –2 and 18 are 3, 8, and 13.

- **Geometric Progression:** A sequence is said to be a geometric progression (G.P.) if the ratio of any term to its preceding term is the same throughout. This constant factor is called the common ratio and it is denoted by r.

- In standard form, the G.P. is written as $a, ar, ar^2 \ldots$ where, a is the first term and r is the common ratio.

- **General Term of a G.P.:** The n^{th} term (or general term) of a G.P. is given by $a_n = ar^{n-1}$

Example: Find the number of terms in G.P. 5, 20, 80 ... 5120.

Solution: Let the number of terms be n.

Here $a = 5$, $r = 4$ and $t_n = 5120$

n^{th} term of G.P. $= ar^{n-1}$

$$\therefore 5(4)^{n-1} = 5120$$
$$\Rightarrow 4^{n-1} = \frac{5120}{5} = 1024$$
$$\Rightarrow (2)^{2n-2} = (2)^{10}$$
$$\Rightarrow 2n - 2 = 10$$
$$\Rightarrow 2n = 12$$
$$\therefore n = 6$$

- **Sum of n Term of a G.P.:** The sum of n terms (S_n) of a G.P. is given by

$$S_n = \begin{cases} \frac{a(1-r^n)}{1-r}, \text{ if } r < 1 \quad \text{or} \quad \frac{a(r^n-1)}{r-1}, \text{ if } r > 1 \\ na, \text{ if } r = 1 \end{cases}$$

Example: Find the sum of the series 1 + 3 + 9 + 27 + ... to 10 terms.

Solution: The sequence 1, 3, 9, 27, ... is a G.P.

Here, $a = 1$, $r = 3$.

Sum of n terms of G.P. $= \frac{a(r^n-1)}{r-1}$ $[r > 1]$

$S_{10} = 1 + 3 + 9 + 27 + \ldots$ to 10 terms

$$= \frac{1 \times \left[(3)^{10} - 1\right]}{(3-1)}$$

$$= \frac{59049-1}{2}$$

$$= \frac{59048}{2}$$

$$= 29524$$

- Three consecutive terms can be taken as ar, a, ar. Here, common ratio is *r*.
- Four consecutive terms can be taken as ar3, ar, ar, ar3. Here, common ratio is *r2*.

- **Geometric Mean:** For any two positive numbers *a* and *b*, we can insert a number G between them such that *a*, *G*, *b* is *a* G.P. Such a number i.e., G is called a geometric mean (G.M.) and is given by $G = \sqrt{ab}$

In general, if $G_1, G_2, \ldots, G_n$ be *n* numbers between positive numbers *a* and *b* such that *a*, $G_1, G_2, \ldots, G_n$, *b* is a G.P., then $G_1, G_2, \ldots, G_n$ are given by

$G_1 = ar$, $G_2 = ar^2, \ldots, G_n = ar^n$

Where, *r* is calculated from the relation $b = ar^{n+1}$, that is $r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$.

Example: Insert three geometric means between 2 and 162.
Solution:
Let G_1, G_2, G_3 be 3 G.M.'s between 2 and 162.
Therefor 2, G_1, G_2, G_3, 162 are in G.P.
Let *r* be the common ratio of G.P.
Here, $a = 2$, $b = 162$ and $n = 3$

$$r = \left(\frac{162}{2}\right)^{\frac{1}{3+1}} = (81)^{\frac{1}{4}} = (3^4)^{\frac{1}{4}} = 3$$

$G_1 = ar = 2 \times 3 = 6$

$G_2 = ar^2 = 2 \times (3)^2 = 2 \times 9 = 18$

$G_3 = ar^3 = 2 \times (3)^3 = 2 \times 27 = 54$

Thus, the required three geometric means between 2 and 162 are 6, 18, and 54.

- **Relation between A.M. and G.M.:** Let A and G be the respective A.M. and G.M. of two given positive real numbers a and b. Accordingly, $A = \frac{a+b}{2}$ and $G = \sqrt{ab}$.

Then, we will always have the following relationship between the A.M. and G.M.: $A \geq G$

- **Sum of *n*-terms of some special series:**
  - Sum of first n natural numbers
    $1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$
  - Sum of squares of the first n natural numbers

    $1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$
  - Sum of cubes of the first n natural numbers

    $1^3 + 2^3 + 3^3 + \ldots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$

Example: Find the sum of n terms of the series whose n^{th} term is $n(n+1)(n-2)$.

Solution: It is given that

$$\begin{aligned} a_n &= n(n+1)(n-2) \\ &= n(n^2 + n - 2n - 2) \\ &= n(n^2 - n - 2) \\ &= n^3 - n^2 - 2n \end{aligned}$$

Thus, the sum of n terms is given by

$$S_n = \sum_{k=1}^{n} k^3 - \sum_{k=1}^{n} k^2 - 2\sum_{k=1}^{n} k$$

$$= \left[\frac{n(n+1)}{2}\right]^2 - \frac{n(n+1)(2n+1)}{6} - \frac{2n(n+1)}{2}$$

$$= \frac{n(n+1)}{2}\left[\frac{n(n+1)}{2} - \frac{2n+1}{3} - 2\right]$$

$$= \frac{n(n+1)}{2}\left[\frac{3n(n+1)-2(2n+1)-12}{6}\right]$$

$$= \frac{n(n+1)}{2}\left[\frac{3n^2+3n-4n-2-12}{6}\right]$$

$$= \frac{n(n+1)}{2}\left[\frac{3n^2-n-14}{6}\right]$$

$$= \frac{n(n+1)(3n^2-n-14)}{12}$$

$$= \frac{n(n+1)(3n^2-7n+6n-14)}{12}$$

$$= \frac{n(n+1)[n(3n-7)+2(3n-7)]}{12}$$

$$= \frac{n(n+1)(n+2)(3n-7)}{12}$$