CHAPTER XXXV.

THEORY OF EQUATIONS.

534. In Chap. IX. we have established certain relations between the roots and the coefficients of quadratic equations. We shall now investigate similar relations which hold in the case of equations of the n^{th} degree, and we shall then discuss some of the more elementary properties in the general theory of equations.

535. Let $p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots\ldots + p_{n-1}x + p_n$ be a rational integral function of x of n dimensions, and let us denote it by $f(x)$; then $f(x) = 0$ is the general type of a *rational integral equation* of the n^{th} degree. Dividing throughout by p_0, we see that without any loss of generality we may take

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots\ldots + p_{n-1}x + p_n = 0$$

as the type of a rational integral equation of any degree.

Unless otherwise stated the coefficients $p_1, p_2, \ldots p_n$ will always be supposed rational.

536. Any value of x which makes $f(x)$ vanish is called a *root* of the equation $f(x) = 0$.

In Art. 514 it was proved that when $f(x)$ is divided by $x - a$, the remainder is $f(a)$; hence if $f(x)$ is divisible by $x - a$ without remainder, a is a root of the equation $f(x) = 0$.

537. We shall assume that every equation of the form $f(x) = 0$ has a root, real or imaginary. The proof of this proposition will be found in treatises on the *Theory of Equations;* it is beyond the range of the present work.

538. *Every equation of the* nth *degree has* n *roots, and no more.*

Denote the given equation by $f(x) = 0$, where

$$f(x) = p_0 x^n + p_1 x^{n-1} + p_2 x^{n-2} + \dots\dots + p_n.$$

The equation $f(x) = 0$ has a root, real or imaginary; let this be denoted by a_1; then $f(x)$ is divisible by $x - a_1$, so that

$$f(x) = (x - a_1)\,\phi_1(x),$$

where $\phi_1(x)$ is a rational integral function of $n-1$ dimensions. Again, the equation $\phi_1(x) = 0$ has a root, real or imaginary; let this be denoted by a_2; then $\phi_1(x)$ is divisible by $x - a_2$, so that

$$\phi_1(x) = (x - a_2)\,\phi_2(x),$$

where $\phi_2(x)$ is a rational integral function of $n - 2$ dimensions.

Thus $$f(x) = (x - a_1)(x - a_2)\,\phi_2(x).$$

Proceeding in this way, we obtain, as in Art. 309,

$$f(x) = p_0 (x - a_1)(x - a_2) \dots\dots (x - a_n).$$

Hence the equation $f(x) = 0$ has n roots, since $f(x)$ vanishes when x has any of the values $a_1, a_2, a_3, \dots a_n$.

Also the equation cannot have more than n roots; for if x has any value different from any of the quantities $a_1, a_2, a_3, \dots a_n$, all the factors on the right are different from zero, and therefore $f(x)$ cannot vanish for that value of x.

In the above investigation some of the quantities $a_1, a_2, a_3, \dots a_n$ may be equal; in this case, however, we shall suppose that the equation has still n roots, although these are not all different.

539. *To investigate the relations between the roots and the coefficients in any equation.*

Let us denote the equation by

$$x^n + p_1 x^{n-1} + p_2 x^{n-2} + \dots\dots + p_{n-1} x + p_n = 0,$$

and the roots by $a, b, c, \dots\dots k$; then we have identically

$$x^n + p_1 x^{n-1} + p_2 x^{n-2} + \dots\dots + p_{n-1} x + p_n \\ = (x - a)(x - b)(x - c) \dots\dots (x - k);$$

hence, with the notation of Art. 163, we have

$$x^n + p_1 x^{n-1} + p_2 x^{n-2} + \dots\dots + p_{n-1} x + p_n \\ = x^n - S_1 x^{n-1} + S_2 x^{n-2} - \dots\dots + (-1)^{n-1} S_{n-1} x + (-1)^n S_n.$$

Equating coefficients of like powers of x in this identity,

$-p_1 = S_1$ = sum of the roots;

$p_2 = S_2$ = sum of the products of the roots taken two at a time;

$-p_3 = S_3$ = sum of the products of the roots taken three at a time;

....

$(-1)^n p_n = S_n$ = product of the roots.

If the coefficient of x^n is p_0, then on dividing each term by p_0, the equation becomes

$$x^n + \frac{p_1}{p_0} x^{n-1} + \frac{p_2}{p_0} x^{n-2} + \ldots\ldots + \frac{p_{n-1}}{p_0} x + \frac{p_n}{p_0} = 0,$$

and, with the notation of Art. 521, we have

$$\Sigma a = -\frac{p_1}{p_0}, \quad \Sigma ab = \frac{p_2}{p_0}, \quad \Sigma abc = -\frac{p_3}{p_0}, \quad \ldots\ldots, \quad abc\ldots k = (-1)^n \frac{p_n}{p_0}.$$

Example 1. Solve the equations

$$x + ay + a^2 z = a^3, \quad x + by + b^2 z = b^3, \quad x + cy + c^2 z = c^3.$$

From these equations we see that a, b, c are the values of t which satisfy the cubic equation

$$t^3 - zt^2 - yt - x = 0;$$

hence $$z = a + b + c, \quad y = -(bc + ca + ab), \quad x = abc.$$

Example 2. If a, b, c are the roots of the equation $x^3 + p_1 x^2 + p_2 x + p_3 = 0$, form the equation whose roots are a^2, b^2, c^2.

The required equation is $(y - a^2)(y - b^2)(y - c^2) = 0$,

or $$(x^2 - a^2)(x^2 - b^2)(x^2 - c^2) = 0, \text{ if } y = x^2;$$

that is, $$(x - a)(x - b)(x - c)(x + a)(x + b)(x + c) = 0.$$

But $$(x - a)(x - b)(x - c) = x^3 + p_1 x^2 + p_2 x + p_3;$$

hence $$(x + a)(x + b)(x + c) = x^3 - p_1 x^2 + p_2 x - p_3.$$

Thus the required equation is

$$(x^3 + p_1 x^2 + p_2 x + p_3)(x^3 - p_1 x^2 + p_2 x - p_3) = 0,$$

or $$(x^3 + p_2 x)^2 - (p_1 x^2 + p_3)^2 = 0,$$

or $$x^6 + (2p_2 - p_1^2) x^4 + (p_2^2 - 2p_1 p_3) x^2 - p_3^2 = 0;$$

and if we replace x^2 by y, we obtain

$$y^3 + (2p_2 - p_1^2) y^2 + (p_2^2 - 2p_1 p_3) y - p_3^2 = 0.$$

540. The student might suppose that the relations established in the preceding article would enable him to solve any proposed equation; for the number of the relations is equal to the number of the roots. A little reflection will shew that is this not the case; for suppose we eliminate any $n-1$ of the quantities $a, b, c, \dots k$ and so obtain an equation to determine the remaining one; then since these quantities are involved symmetrically in each of the equations, it is clear that we shall always obtain an equation having the same coefficients; this equation is therefore the original equation with some one of the roots $a, b, c, \dots k$ substituted for x.

Let us take for example the equation

$$x^3 + p_1 x^2 + p_2 x + p_3 = 0;$$

and let a, b, c be the roots; then

$$a + b + c = -p_1,$$
$$ab + ac + bc = p_2,$$
$$abc = -p_3.$$

Multiply these equations by a^2, $-a$, 1 respectively and add; thus

$$a^3 = -p_1 a^2 - p_2 a - p_3,$$

that is,
$$a^3 + p_1 a^2 + p_2 a + p_3 = 0,$$

which is the original equation with a in the place of x.

The above process of elimination is quite general, and is applicable to equations of any degree.

541. If two or more of the roots of an equation are connected by an assigned relation, the properties proved in Art. 539 will sometimes enable us to obtain the complete solution.

Example 1. Solve the equation $4x^3 - 24x^2 + 23x + 18 = 0$, having given that the roots are in arithmetical progression.

Denote the roots by $a - b$, a, $a + b$; then the sum of the roots is $3a$; the sum of the products of the roots two at a time is $3a^2 - b^2$; and the product of the roots is $a(a^2 - b^2)$; hence we have the equations

$$3a = 6, \quad 3a^2 - b^2 = \frac{23}{4}, \quad a(a^2 - b^2) = -\frac{9}{2};$$

from the first equation we find $a = 2$, and from the second $b = \pm\frac{5}{2}$, and since these values satisfy the third, the three equations are consistent. Thus the roots are $-\frac{1}{2}$, 2, $\frac{9}{2}$.

Example 2. Solve the equation $24x^3-14x^2-63x+45=0$, one root being double another.

Denote the roots by a, $2a$, b; then we have

$$3a+b=\frac{7}{12},\quad 2a^2+3ab=-\frac{21}{8},\quad 2a^2b=-\frac{15}{8}.$$

From the first two equations, we obtain

$$8a^2-2a-3=0;$$

$$\therefore\ a=\frac{3}{4}\ \text{or}\ -\frac{1}{2}\ \text{and}\ b=-\frac{5}{3}\ \text{or}\ \frac{25}{12}.$$

It will be found on trial that the values $a=-\frac{1}{2}$, $b=\frac{25}{12}$ do not satisfy the third equation $2a^2b=-\frac{15}{8}$; hence we are restricted to the values

$$a=\frac{3}{4},\quad b=-\frac{5}{3}.$$

Thus the roots are $\frac{3}{4}$, $\frac{3}{2}$, $-\frac{5}{3}$.

542. Although we may not be able to find the roots of an equation, we can make use of the relations proved in Art. 539 to determine the values of symmetrical functions of the roots.

Example 1. Find the sum of the squares and of the cubes of the roots of the equation $$x^3-px^2+qx-r=0.$$

Denote the roots by a, b, c; then

$$a+b+c=p,\quad bc+ca+ab=q.$$

Now
$$\begin{aligned}a^2+b^2+c^2&=(a+b+c)^2-2(bc+ca+ab)\\&=p^2-2q.\end{aligned}$$

Again, substitute a, b, c for x in the given equation and add; thus

$$a^3+b^3+c^3-p(a^2+b^2+c^2)+q(a+b+c)-3r=0;$$

$$\begin{aligned}\therefore\ a^3+b^3+c^3&=p(p^2-2q)-pq+3r\\&=p^3-3pq+3r.\end{aligned}$$

Example 2. If a, b, c, d are the roots of

$$x^4+px^3+qx^2+rx+s=0,$$

find the value of Σa^2b.

We have
$$a+b+c+d=-p \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (1),$$

$$ab+ac+ad+bc+bd+cd=q \quad\ldots\ldots\ldots\ldots\ldots\ldots\ (2),$$

$$abc+abd+acd+bcd=-r \quad\ldots\ldots\ldots\ldots\ldots\ldots\ (3).$$

From these equations we have

$$-pq = \Sigma a^2 b + 3\,(abc + abd + acd + bcd)$$
$$= \Sigma a^2 b - 3r;$$
$$\therefore\ \Sigma a^2 b = 3r - pq.$$

EXAMPLES. XXXV. a.

Form the equation whose roots are

1. $\frac{2}{3}, \frac{3}{2}, \pm\sqrt{3}$.

2. $0, 0, 2, 2, -3, -3$.

3. $2, 2, -2, -2, 0, 5$.

4. $a+b, a-b, -a+b, -a-b$.

Solve the equations:

5. $x^4 - 16x^3 + 86x^2 - 176x + 105 = 0$, two roots being 1 and 7.

6. $4x^3 + 16x^2 - 9x - 36 = 0$, the sum of two of the roots being zero.

7. $4x^3 + 20x^2 - 23x + 6 = 0$, two of the roots being equal.

8. $3x^3 - 26x^2 + 52x - 24 = 0$, the roots being in geometrical progression.

9. $2x^3 - x^2 - 22x - 24 = 0$, two of the roots being in the ratio of $3 : 4$.

10. $24x^3 + 46x^2 + 9x - 9 = 0$, one root being double another of the roots.

11. $8x^4 - 2x^3 - 27x^2 + 6x + 9 = 0$, two of the roots being equal but opposite in sign.

12. $54x^3 - 39x^2 - 26x + 16 = 0$, the roots being in geometrical progression.

13. $32x^3 - 48x^2 + 22x - 3 = 0$, the roots being in arithmetical progression.

14. $6x^4 - 29x^3 + 40x^2 - 7x - 12 = 0$, the product of two of the roots being 2.

15. $x^4 - 2x^3 - 21x^2 + 22x + 40 = 0$, the roots being in arithmetical progression.

16. $27x^4 - 195x^3 + 494x^2 - 520x + 192 = 0$, the roots being in geometrical progression.

17. $18x^3 + 81x^2 + 121x + 60 = 0$, one root being half the sum of the other two.

18. If a, b, c are the roots of the equation $x^3-px^2+qx-r=0$, find the value of

(1) $\dfrac{1}{a^2}+\dfrac{1}{b^2}+\dfrac{1}{c^2}$. (2) $\dfrac{1}{b^2c^2}+\dfrac{1}{c^2a^2}+\dfrac{1}{a^2b^2}$.

19. If a, b, c are the roots of $x^3+qx+r=0$, find the value of

(1) $(b-c)^2+(c-a)^2+(a-b)^2$. (2) $(b+c)^{-1}+(c+a)^{-1}+(a+b)^{-1}$.

20. Find the sum of the squares and of the cubes of the roots of

$$x^4+qx^2+rx+s=0.$$

21. Find the sum of the fourth powers of the roots of

$$x^3+qx+r=0.$$

543. *In an equation with real coefficients imaginary roots occur in pairs.*

Suppose that $f(x)=0$ is an equation with real coefficients, and suppose that it has an imaginary root $a+ib$; we shall shew that $a-ib$ is also a root.

The factor of $f(x)$ corresponding to these two roots is

$$(x-a-ib)(x-a+ib), \text{ or } (x-a)^2+b^2.$$

Let $f(x)$ be divided by $(x-a)^2+b^2$; denote the quotient by Q, and the remainder, if any, by $Rx+R'$; then

$$f(x)=Q\{(x-a)^2+b^2\}+Rx+R'.$$

In this identity put $x=a+ib$, then $f(x)=0$ by hypothesis; also $(x-a)^2+b^2=0$; hence $R(a+ib)+R'=0$.

Equating to zero the real and imaginary parts,

$$Ra+R'=0, \quad Rb=0;$$

and b by hypothesis is not zero,

$$\therefore\ R=0 \text{ and } R'=0.$$

Hence $f(x)$ is exactly divisible by $(x-a)^2+b^2$, that is, by

$$(x-a-ib)(x-a+ib);$$

hence $x=a-ib$ is also a root.

544. In the preceding article we have seen that if the equation $f(x)=0$ has a pair of imaginary roots $a\pm ib$, then $(x-a)^2+b^2$ is a factor of the expression $f(x)$.

Suppose that $a \pm ib$, $c \pm id$, $e \pm ig, \ldots$ are the imaginary roots of the equation $f(x)=0$, and that $\phi(x)$ is the product of the quadratic factors corresponding to these imaginary roots; then

$$\phi(x) = \{(x-a)^2+b^2\}\{(x-c)^2+d^2\}\{(x-e)^2+g^2\}\ldots.$$

Now each of these factors is positive for every real value of x; hence $\phi(x)$ is always positive for real values of x.

545. As in Art. 543 we may shew that in an equation with *rational* coefficients, surd roots enter in pairs; that is, if $a+\sqrt{b}$ is a root then $a-\sqrt{b}$ is also a root.

Example 1. Solve the equation $6x^4-13x^3-35x^2-x+3=0$, having given that one root is $2-\sqrt{3}$.

Since $2-\sqrt{3}$ is a root, we know that $2+\sqrt{3}$ is also a root, and corresponding to this pair of roots we have the quadratic factor x^2-4x+1.

Also $\quad 6x^4-13x^3-35x^2-x+3=(x^2-4x+1)(6x^2+11x+3)$;

hence the other roots are obtained from

$$6x^2+11x+3=0, \text{ or } (3x+1)(2x+3)=0;$$

thus the roots are $\quad -\dfrac{1}{3}, \; -\dfrac{3}{2}, \; 2+\sqrt{3}, \; 2-\sqrt{3}.$

Example 2. Form the equation of the fourth degree with rational coefficients, one of whose roots is $\sqrt{2}+\sqrt{-3}$.

Here we must have $\sqrt{2}+\sqrt{-3}$, $\sqrt{2}-\sqrt{-3}$ as one pair of roots, and $-\sqrt{2}+\sqrt{-3}$, $-\sqrt{2}-\sqrt{-3}$ as another pair.

Corresponding to the first pair we have the quadratic factor $x^2-2\sqrt{2}x+5$, and corresponding to the second pair we have the quadratic factor

$$x^2+2\sqrt{2}x+5.$$

Thus the required equation is

$$(x^2+2\sqrt{2}x+5)(x^2-2\sqrt{2}x+5)=0,$$

or $$(x^2+5)^2-8x^2=0,$$

or $$x^4+2x^2+25=0.$$

Example 3. Shew that the equation

$$\frac{A^2}{x-a}+\frac{B^2}{x-b}+\frac{C^2}{x-c}+\ldots+\frac{H^2}{x-h}=k,$$

has no imaginary roots.

If possible let $p+iq$ be a root; then $p-iq$ is also a root. Substitute these values for x and subtract the first result from the second; thus

$$q\left\{\frac{A^2}{(p-a)^2+q^2}+\frac{B^2}{(p-b)^2+q^2}+\frac{C^2}{(p-c)^2+q^2}+\ldots+\frac{H^2}{(p-h)^2+q^2}\right\}=0;$$

which is impossible unless $q=0$.

546. To determine the nature of some of the roots of an equation it is not always necessary to solve it; for instance, the truth of the following statements will be readily admitted.

(i) If the coefficients are all positive, the equation has no positive root; thus the equation $x^5 + x^3 + 2x + 1 = 0$ cannot have a positive root.

(ii) If the coefficients of the even powers of x are all of one sign, and the coefficients of the odd powers are all of the contrary sign, the equation has no negative root; thus the equation

$$x^7 + x^5 - 2x^4 + x^3 - 3x^2 + 7x - 5 = 0$$

cannot have a negative root.

(iii) If the equation contains only even powers of x and the coefficients are all of the same sign, the equation has no real root; thus the equation $2x^8 + 3x^4 + x^2 + 7 = 0$ cannot have a real root.

(iv) If the equation contains only odd powers of x, and the coefficients are all of the same sign, the equation has no real root except $x = 0$; thus the equation $x^9 + 2x^5 + 3x^3 + x = 0$ has no real root except $x = 0$.

All the foregoing results are included in the theorem of the next article, which is known as Descartes' *Rule of Signs*.

547. *An equation* $f(x) = 0$ *cannot have more positive roots than there are changes of sign in* $f(x)$, *and cannot have more negative roots than there are changes of sign in* $f(-x)$.

Suppose that the signs of the terms in a polynomial are $+ + - - + - - - + - + -$; we shall shew that if this polynomial is multiplied by a binomial whose signs are $+ -$, there will be at least one more change of sign in the product than in the original polynomial.

Writing down only the signs of the terms in the multiplication, we have

$$\begin{array}{cccccccccccccc}
+ & + & - & - & + & - & - & - & + & - & + & - & \\
+ & - & & & & & & & & & & & \\
\hline
+ & + & - & - & + & - & - & - & + & - & + & - & \\
 & - & - & + & + & - & + & + & + & - & + & - & + \\
\hline
+ & \pm & - & \mp & + & - & \mp & \mp & + & - & + & - & +
\end{array}$$

Hence we see that in the product

(i) an ambiguity replaces each continuation of sign in the original polynomial;

(ii) the signs before and after an ambiguity or set of ambiguities are unlike;

(iii) a change of sign is introduced at the end.

Let us take the most unfavourable case and suppose that all the ambiguities are replaced by continuations; from (ii) we see that the number of changes of sign will be the same whether we take the upper or the lower signs; let us take the upper; thus the number of changes of sign cannot be less than in

$$+ + - - + - - - + - + - +,$$

and this series of signs is the same as in the original polynomial with an additional change of sign at the end.

If then we suppose the factors corresponding to the negative and imaginary roots to be already multiplied together, each factor $x - a$ corresponding to a positive root introduces at least one change of sign; therefore no equation can have more positive roots than it has changes of sign.

Again, the roots of the equation $f(-x) = 0$ are equal to those of $f(x) = 0$ but opposite to them in sign; therefore the negative roots of $f(x) = 0$ are the positive roots of $f(-x) = 0$; but the number of these positive roots cannot exceed the number of changes of sign in $f(-x)$; that is, the number of negative roots of $f(x) = 0$ cannot exceed the number of changes of sign in $f(-x)$.

Example. Consider the equation $x^9 + 5x^8 - x^3 + 7x + 2 = 0$.

Here there are two changes of sign, therefore there are at most two positive roots.

Again $f(-x) = -x^9 + 5x^8 + x^3 - 7x + 2$, and here there are three changes of sign, therefore the given equation has at most three negative roots, and therefore it must have at least four imaginary roots.

EXAMPLES. XXXV. b.

Solve the equations:

1. $3x^4 - 10x^3 + 4x^2 - x - 6 = 0$, one root being $\dfrac{1 + \sqrt{-3}}{2}$.

2. $6x^4 - 13x^3 - 35x^2 - x + 3 = 0$, one root being $2 - \sqrt{3}$.

3. $x^4 + 4x^3 + 5x^2 + 2x - 2 = 0$, one root being $-1 + \sqrt{-1}$.

4. $x^4+4x^3+6x^2+4x+5=0$, one root being $\sqrt{-1}$.

5. Solve the equation $x^5-x^4+8x^2-9x-15=0$, one root being $\sqrt{3}$ and another $1-2\sqrt{-1}$.

Form the equation of lowest dimensions with rational coefficients, one of whose roots is

6. $\sqrt{3}+\sqrt{-2}$.

7. $-\sqrt{-1}+\sqrt{5}$.

8. $-\sqrt{2}-\sqrt{-2}$.

9. $\sqrt{5+2\sqrt{6}}$.

10. Form the equation whose roots are $\pm 4\sqrt{3}$, $5\pm 2\sqrt{-1}$.

11. Form the equation whose roots are $1\pm\sqrt{-2}$, $2\pm\sqrt{-3}$.

12. Form the equation of the eighth degree with rational coefficients one of whose roots is $\sqrt{2}+\sqrt{3}+\sqrt{-1}$.

13. Find the nature of the roots of the equation

$$3x^4+12x^2+5x-4=0.$$

14. Shew that the equation $2x^7-x^4+4x^3-5=0$ has at least four imaginary roots.

15. What may be inferred respecting the roots of the equation

$$x^{10}-4x^6+x^4-2x-3=0\,?$$

16. Find the least possible number of imaginary roots of the equation $x^9-x^5+x^4+x^2+1=0$.

17. Find the condition that $x^3-px^2+qx-r=0$ may have

(1) two roots equal but of opposite sign;

(2) the roots in geometrical progression.

18. If the roots of the equation $x^4+px^3+qx^2+rx+s=0$ are in arithmetical progression, shew that $p^3-4pq+8r=0$; and if they are in geometrical progression, shew that $p^2s=r^2$.

19. If the roots of the equation $x^n-1=0$ are $1, \alpha, \beta, \gamma, \ldots$, shew that

$$(1-\alpha)(1-\beta)(1-\gamma)\ldots\ldots=n.$$

If a, b, c are the roots of the equation $x^3-px^2+qx-r=0$, find the value of

20. Σa^2b^2.

21. $(b+c)(c+a)(a+b)$.

22. $\Sigma\left(\dfrac{b}{c}+\dfrac{c}{b}\right)$.

23. Σa^2b.

If a, b, c, d are the roots of $x^4+px^3+qx^2+rx+s=0$, find the value of

24. Σa^2bc.

25. Σa^4.

548. *To find the value of* f (x + h), *when* f (x) *is a rational integral function of* x.

Let $f(x) = p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots\ldots + p_{n-1}x + p_n$; then

$$f(x+h) = p_0(x+h)^n + p_1(x+h)^{n-1} + p_2(x+h)^{n-2} + \ldots\ldots$$
$$+ p_{n-1}(x+h) + p_n.$$

Expanding each term and arranging the result in ascending powers of h, we have

$$p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots + p_{n-1}x + p_n$$
$$+ h\{np_0x^{n-1} + (n-1)p_1x^{n-2} + (n-2)p_2x^{n-3} + \ldots + p_{n-1}\}$$
$$+ \frac{h^2}{\underline{|2}}\{n(n-1)p_0x^{n-2} + (n-1)(n-2)p_1x^{n-3} + \ldots + 2p_{n-2}\}$$
$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$+ \frac{h^n}{\underline{|n}}\{n(n-1)(n-2)\ldots2.1p_0\}.$$

This result is usually written in the form

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{\underline{|2}}f''(x) + \frac{h^3}{\underline{|3}}f'''(x) + \ldots + \frac{h^n}{\underline{|n}}f^n(x),$$

and the functions $f'(x)$, $f''(x)$, $f'''(x)$,... are called the *first, second, third,... derived functions* of $f(x)$.

The student who knows the elements of the Differential Calculus will see that the above expansion of $f(x+h)$ is only a particular case of *Taylor's Theorem;* the functions $f'(x)$, $f''(x)$, $f'''(x)$ may therefore be written down at once by the ordinary rules for differentiation: thus to obtain $f'(x)$ from $f(x)$ we multiply each term in $f(x)$ by the index of x in that term and then diminish the index by unity.

Similarly by successive differentiations we obtain $f''(x)$, $f'''(x)$, ...

By writing $-h$ in the place of h, we have

$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{\underline{|2}}f''(x) - \frac{h^3}{\underline{|3}}f'''(x) + \ldots + (-1)^n\frac{h^n}{\underline{|n}}f^n(x).$$

The function $f(x+h)$ is evidently symmetrical with respect to x and h; hence

$$f(x+h) = f(h) + xf'(h) + \frac{x^2}{\underline{|2}}f''(h) + \ldots + \frac{x^n}{\underline{|n}}f^n(h).$$

Here the expressions $f'(h), f''(h), f'''(h), \ldots$ denote the results obtained by writing h in the place of x in the successive derived functions $f'(x)$, $f''(x)$, $f'''(x), \ldots$.

Example. If $f(x)=2x^4-x^3-2x^2+5x-1$, find the value of $f(x+3)$.

Here $\quad f(x)=2x^4-x^3-2x^2+5x-1$, so that $f(3)=131$;

$$f'(x)=8x^3-3x^2-4x+5, \text{ and } f'(3)=182;$$

$$\frac{f''(x)}{\underline{|2}}=12x^2-3x-2, \text{ and } \frac{f''(3)}{\underline{|2}}=97;$$

$$\frac{f'''(x)}{\underline{|3}}=8x-1, \text{ and } \frac{f'''(3)}{\underline{|3}}=23;$$

$$\frac{f^{\text{iv}}(x)}{\underline{|4}}=2.$$

Thus $\quad f(x+3)=2x^4+23x^3+97x^2+182x+131.$

The calculation may, however, be effected more systematically by *Horner's process*, as explained in the next article.

549. Let $f(x)=p_0x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots+p_{n-1}x+p_n$; put $x=y+h$, and suppose that $f(x)$ then becomes

$$q_0y^n+q_1y^{n-1}+q_2y^{n-2}+\ldots+q_{n-1}y+q_n.$$

Now $y=x-h$; hence we have the identity

$$p_0x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots+p_{n-1}x+p_n$$
$$=q_0(x-h)^n+q_1(x-h)^{n-1}+\ldots+q_{n-1}(x-h)+q_n;$$

therefore q_n is the remainder found by dividing $f(x)$ by $x-h$; also the quotient arising from the division is

$$q_0(x-h)^{n-1}+q_1(x-h)^{n-2}+\ldots+q_{n-1}.$$

Similarly q_{n-1} is the remainder found by dividing the last expression by $x-h$, and the quotient arising from the division is

$$q_0(x-h)^{n-2}+q_1(x-h)^{n-3}+\ldots+q_{n-2};$$

and so on. Thus $q_n, q_{n-1}, q_{n-2}, \ldots$ may be found by the rule explained in Art. 515. The last quotient is q_0, and is obviously equal to p_0.

Example. Find the result of changing x into $x+3$ in the expression

$$2x^4-x^3-2x^2+5x-1.$$

Here we divide successively by $x-3$.

$$\begin{array}{rrrrr}
2 & -1 & -2 & 5 & -1 \\
 & 6 & 15 & 39 & 132 \\ \hline
 & 5 & 13 & 44 & \mid 131 = q_4 \\
 & 6 & 33 & 138 & \\ \hline
 & 11 & 46 & \mid 182 = q_3 & \\
 & 6 & 51 & & \\ \hline
 & 17 & \mid 97 = q_2 & & \\
 & 6 & & & \\ \hline
 & 23 = q_1 & & &
\end{array}$$

Or more briefly thus:

$$\begin{array}{rrrrr}
2 & -1 & -2 & 5 & -1 \\
2 & 5 & 13 & 44 & \mid 131 \\
2 & 11 & 46 & \mid 182 & \\
2 & 17 & \mid 97 & & \\
2 & \mid 23 & & & \\
2 & & & &
\end{array}$$

Hence the result is $2x^4+23x^3+97x^2+182x+131$. Compare Art. 548.

It may be remarked that Horner's process is chiefly useful in *numerical* work.

550. *If the variable* x *changes continuously from* a *to* b *the function* f (x) *will change continuously from* f (a) *to* f (b).

Let c and $c+h$ be any two values of x lying between a and b. We have

$$f(c+h)-f(c)=hf'(c)+\frac{h^2}{\lfloor 2}f''(c)+\ldots+\frac{h^n}{\lfloor n}f^n(c);$$

and by taking h small enough the difference between $f(c+h)$ and $f(c)$ can be made as small as we please; hence to a small change in the variable x there corresponds a small change in the function $f(x)$, and therefore as x changes gradually from a to b, the function $f(x)$ changes gradually from $f(a)$ to $f(b)$.

551. It is important to notice that we have not proved that $f(x)$ always increases from $f(a)$ to $f(b)$, or decreases from $f(a)$ to $f(b)$, but that it passes from one value to the other without any sudden change; sometimes it may be increasing and at other times it may be decreasing.

The student who has a knowledge of the elements of Curve-tracing will in any particular example find it easy to follow the gradual changes of value of $f(x)$ by drawing the curve $y=f(x)$.

552. *If* f (a) *and* f (b) *are of contrary signs then one root of the equation* f (x) = 0 *must lie between* a *and* b.

As x changes gradually from a to b, the function $f(x)$ changes gradually from $f(a)$ to $f(b)$, and therefore must pass through all intermediate values; but since $f(a)$ and $f(b)$ have contrary signs the value zero must lie between them; that is, $f(x)=0$ for some value of x between a and b.

It does not follow that $f(x)=0$ has *only one* root between a and b; neither does it follow that if $f(a)$ and $f(b)$ have the *same* sign $f(x)=0$ has no root between a and b.

553. *Every equation of an odd degree has at least one real root whose sign is opposite to that of its last term.*

In the function $f(x)$ substitute for x the values $+\infty$, 0, $-\infty$ successively, then

$$f(+\infty)=+\infty, \quad f(0)=p_n, \quad f(-\infty)=-\infty.$$

If p_n is positive, then $f(x)=0$ has a root lying between 0 and $-\infty$, and if p_n is negative $f(x)=0$ has a root lying between 0 and $+\infty$.

554. *Every equation which is of an even degree and has its last term negative has at least two real roots, one positive and one negative.*

For in this case

$$f(+\infty)=+\infty, \quad f(0)=p_n, \quad f(-\infty)=+\infty;$$

but p_n is negative; hence $f(x)=0$ has a root lying between 0 and $+\infty$, and a root lying between 0 and $-\infty$.

555. *If the expressions* f(a) *and* f(b) *have contrary signs, an odd number of roots of* f(x) = 0 *will lie between* a *and* b; *and if* f(a) *and* f(b) *have the same sign, either no root or an even number of roots will lie between* a *and* b.

Suppose that a is greater than b, and that $\alpha, \beta, \gamma, \ldots \kappa$ represent all the roots of $f(x)=0$ which lie between a and b. Let $\phi(x)$ be the quotient when $f(x)$ is divided by the product $(x-\alpha)(x-\beta)(x-\gamma)\ldots(x-\kappa)$; then

$$f(x)=(x-\alpha)(x-\beta)(x-\gamma)\ldots(x-\kappa)\,\phi(x).$$

Hence $$f(a)=(a-\alpha)(a-\beta)(a-\gamma)\ldots(a-\kappa)\,\phi(a).$$

$$f(b)=(b-\alpha)(b-\beta)(b-\gamma)\ldots(b-\kappa)\,\phi(b).$$

Now $\phi(a)$ and $\phi(b)$ must be of the same sign, for otherwise a root of the equation $\phi(x)=0$, and therefore of $f(x)=0$, would lie between a and b [Art. 552], which is contrary to the hypothesis. Hence if $f(a)$ and $f(b)$ have contrary signs, the expressions

$$(a-\alpha)(a-\beta)(a-\gamma)\ldots(a-\kappa),$$
$$(b-\alpha)(b-\beta)(b-\gamma)\ldots(b-\kappa)$$

must have contrary signs. Also the factors in the first expression are all positive, and the factors in the second are all negative; hence the number of factors must be odd, that is the number of roots $\alpha, \beta, \gamma, \ldots \kappa$ must be odd.

Similarly if $f(a)$ and $f(b)$ have the same sign the number of factors must be even. In this case the given condition is satisfied if $\alpha, \beta, \gamma, \ldots \kappa$ are all greater than a, or less than b; thus it does not necessarily follow that $f(x)=0$ has a root between a and b.

556. If $a, b, c, \ldots k$ are the roots of the equation $f(x)=0$, then

$$f(x)=p_0(x-a)(x-b)(x-c)\ldots(x-k).$$

Here the quantities $a, b, c, \ldots k$ are not necessarily unequal. If r of them are equal to a, s to b, t to c, ..., then

$$f(x)=p_0(x-a)^r(x-b)^s(x-c)^t\ldots.$$

In this case it is convenient still to speak of the equation $f(x)=0$ as having n roots, each of the equal roots being considered a distinct root.

557. *If the equation* $f(x)=0$ *has* r *roots equal to* a, *then the equation* $f'(x)=0$ *will have* $r-1$ *roots equal to* a.

Let $\phi(x)$ be the quotient when $f(x)$ is divided by $(x-a)^r$; then $f(x)=(x-a)^r\phi(x)$.

Write $x+h$ in the place of x; thus

$$f(x+h)=(x-a+h)^r\phi(x+h);$$

$$\therefore f(x)+hf'(x)+\frac{h^2}{\underline{|2}}f''(x)+\ldots$$

$$=\left\{(x-a)^r+r(x-a)^{r-1}h+\ldots\right\}\left\{\phi(x)+h\phi'(x)+\frac{h^2}{\underline{|2}}\phi''(x)+\ldots\right\}.$$

In this identity, by equating the coefficients of h, we have

$$f'(x)=r(x-a)^{r-1}\phi(x)+(x-a)^r\phi'(x).$$

Thus $f'(x)$ contains the factor $x-a$ repeated $r-1$ times; that is, the equation $f'(x)=0$ has $r-1$ roots equal to a.

Similarly we may shew that if the equation $f(x)=0$ has s roots equal to b, the equation $f'(x)=0$ has $s-1$ roots equal to b; and so on.

558. From the foregoing proof we see that if $f(x)$ contains a factor $(x-a)^r$, then $f'(x)$ contains a factor $(x-a)^{r-1}$; and thus $f(x)$ and $f'(x)$ have a common factor $(x-a)^{r-1}$. Therefore if $f(x)$ and $f'(x)$ have no common factor, no factor in $f(x)$ will be repeated; hence *the equation* f(x) = 0 *has or has not equal roots, according as* f(x) *and* f′(x) *have or have not a common factor involving* x.

559. From the preceding article it follows that in order to obtain the equal roots of the equation $f(x)=0$, we must first find the highest common factor of $f(x)$ and $f'(x)$.

Example 1. Solve the equation $x^4-11x^3+44x^2-76x+48=0$, which has equal roots.

Here
$$f(x)=x^4-11x^3+44x^2-76x+48,$$
$$f'(x)=4x^3-33x^2+88x-76;$$

and by the ordinary rule we find that the highest common factor of $f(x)$ and $f'(x)$ is $x-2$; hence $(x-2)^2$ is a factor of $f(x)$; and

$$\begin{aligned} f(x) &= (x-2)^2(x^2-7x+12) \\ &= (x-2)^2(x-3)(x-4); \end{aligned}$$

thus the roots are 2, 2, 3, 4.

Example 2. Find the condition that the equation $ax^3+3bx^2+3cx+d=0$ may have two roots equal.

In this case the equations $f(x)=0$, and $f'(x)=0$, that is

$$ax^3+3bx^2+3cx+d=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1),$$

$$ax^2+2bx+c=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

must have a common root, and the condition required will be obtained by eliminating x between these two equations.

By combining (1) and (2), we have

$$bx^2+2cx+d=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3).$$

From (2) and (3), we obtain

$$\frac{x^2}{2(bd-c^2)}=\frac{x}{bc-ad}=\frac{1}{2(ac-b^2)};$$

thus the required condition is

$$(bc-ad)^2=4(ac-b^2)(bd-c^2).$$

560. We have seen that if the equation $f(x)=0$ has r roots equal to a, the equation $f'(x)=0$ has $r-1$ roots equal to a. But $f''(x)$ is the first derived function of $f'(x)$; hence the equation $f''(x)=0$ must have $r-2$ roots equal to a; similarly the equation $f'''(x)=0$ must have $r-3$ roots equal to a; and so on. These considerations will sometimes enable us to discover the equal roots of $f(x)=0$ with less trouble than the method of Art. 559.

561. *If* a, b, c, ...k *are the roots of the equation* f(x) = 0, *to prove that*

$$\mathrm{f}'(\mathrm{x}) = \frac{\mathrm{f}(\mathrm{x})}{\mathrm{x}-\mathrm{a}} + \frac{\mathrm{f}(\mathrm{x})}{\mathrm{x}-\mathrm{b}} + \frac{\mathrm{f}(\mathrm{x})}{\mathrm{x}-\mathrm{c}} + \ldots + \frac{\mathrm{f}(\mathrm{x})}{\mathrm{x}-\mathrm{k}}.$$

We have $f(x)=(x-a)(x-b)(x-c)\ldots(x-k)$;

writing $x+h$ in the place of x,

$$f(x+h)=(x-a+h)(x-b+h)(x-c+h)\ldots(x-k+h)\ \ldots\ (1).$$

But $$f(x+h)=f(x)+hf'(x)+\frac{h^2}{\underline{|2}}f''(x)+\ldots;$$

hence $f'(x)$ is equal to the coefficient of h in the right-hand member of (1); therefore, as in Art. 163,

$$f'(x)=(x-b)(x-c)\ldots(x-k)+(x-a)(x-c)\ldots(x-k)+\ldots;$$

that is, $$f'(x)=\frac{f(x)}{x-a}+\frac{f(x)}{x-b}+\frac{f(x)}{x-c}+\ldots+\frac{f(x)}{x-k}.$$

562. The result of the preceding article enables us very easily to find the sum of an assigned power of the roots of an equation.

Example. If S_k denote the sum of the k^{th} powers of the roots of the equation $$x^5+px^4+qx^2+t=0,$$
find the value of S_4, S_6 and S_{-4}.

Let $$f(x)=x^5+px^4+qx^2+t;$$
then $$f'(x)=5x^4+4px^3+2qx.$$

Now $$\frac{f(x)}{x-a}=x^4+(a+p)x^3+(a^2+ap)x^2+(a^3+a^2p+q)x+a^4+a^3p+aq;$$

and similar expressions hold for

$$\frac{f(x)}{x-b},\quad \frac{f(x)}{x-c},\quad \frac{f(x)}{x-d},\quad \frac{f(x)}{x-e}.$$

Hence by addition,

$$5x^4+4px^3+2qx=5x^4+(S_1+5p)x^3+(S_2+pS_1)x^2$$
$$+(S_3+pS_2+5q)x+(S_4+pS_3+qS_1).$$

By equating coefficients,

$$S_1+5p=4p, \text{ whence } S_1=-p;$$
$$S_2+pS_1=0, \quad \text{whence } S_2=p^2;$$
$$S_3+pS_2+5q=2q, \text{ whence } S_3=-p^3-3q;$$
$$S_4+pS_3+qS_1=0, \quad \text{whence } S_4=p^4+4pq.$$

To find the value of S_k for other values of k, we proceed as follows.

Multiplying the given equation by x^{k-5},

$$x^k+px^{k-1}+qx^{k-3}+tx^{k-5}=0.$$

Substituting for x in succession the values a, b, c, d, e and adding the results, we obtain $\quad S_k+pS_{k-1}+qS_{k-3}+tS_{k-5}=0.$

Put $k=5$; thus $\quad S_5+pS_4+qS_2+5t=0,$

whence $\quad S_5=-p^5-5p^2q-5t.$

Put $k=6$; thus $\quad S_6+pS_5+qS_3+tS_1=0,$

whence $\quad S_6=p^6+6p^3q+3q^2+6pt.$

To find S_{-4}, put $k=4, 3, 2, 1$ in succession; then

$$S_4+pS_3+qS_1+tS_{-1}=0, \text{ whence } S_{-1}=0;$$
$$S_3+pS_2+5q+tS_{-2}=0, \text{ whence } S_{-2}=-\frac{2q}{t};$$
$$S_2+pS_1+qS_{-1}+tS_{-3}=0, \text{ whence } S_{-3}=0;$$
$$S_1+5p+qS_{-2}+tS_{-4}=0, \text{ whence } S_{-4}=\frac{2q^2}{t^2}-\frac{4p}{t}.$$

563. When the coefficients are numerical we may also proceed as in the following example.

Example. Find the sum of the fourth powers of the roots of

$$x^3-2x^2+x-1=0.$$

Here $\quad f(x)=x^3-2x^2+x-1,$

$$f'(x)=3x^2-4x+1.$$

Also
$$\frac{f'(x)}{f(x)}=\frac{1}{x-a}+\frac{1}{x-b}+\frac{1}{x-c}$$
$$=\Sigma\left(\frac{1}{x}+\frac{a}{x^2}+\frac{a^2}{x^3}+\frac{a^3}{x^4}+\dots\right)$$
$$=\frac{3}{x}+\frac{S_1}{x^2}+\frac{S_2}{x^3}+\frac{S_3}{x^4}+\dots;$$

hence S_4 is equal to the coefficient of $\frac{1}{x^5}$ in the quotient of $f'(x)$ by $f(x)$, which is very conveniently obtained by the method of synthetic division as follows:

$$\begin{array}{r|llllll}
1 & 3 & -4 & +1 & & & \\
2 & & 6 & -3 & +3 & & \\
-1 & & & 4 & -2 & +2 & \\
1 & & & & 4 & -2 & +2 \\
 & & & & & 10 & -5+5 \\
\hline
 & 3 & +2 & +2 & +5 & +10 & +\ldots\ldots
\end{array}$$

Hence the quotient is $\frac{3}{x}+\frac{2}{x^2}+\frac{2}{x^3}+\frac{5}{x^4}+\frac{10}{x^5}+\ldots\ldots;$

thus $$S_4=10.$$

EXAMPLES. XXXV. c.

1. If $f(x)=x^4+10x^3+39x^2+76x+65$, find the value of $f(x-4)$.

2. If $f(x)=x^4-12x^3+17x^2-9x+7$, find the value of $f(x+3)$.

3. If $f(x)=2x^4-13x^2+10x-19$, find the value of $f(x+1)$.

4. If $f(x)=x^4+16x^3+72x^2+64x-129$, find the value of $f(x-4)$.

5. If $f(x)=ax^8+bx^5+cx+d$, find the value of $f(x+h)-f(x-h)$.

6. Shew that the equation $10x^3-17x^2+x+6=0$ has a root between 0 and -1.

7. Shew that the equation $x^4-5x^3+3x^2+35x-70=0$ has a root between 2 and 3 and one between -2 and -3.

8. Shew that the equation $x^4-12x^2+12x-3=0$ has a root between -3 and -4 and another between 2 and 3.

9. Shew that $x^5+5x^4-20x^2-19x-2=0$ has a root between 2 and 3, and a root between -4 and -5.

Solve the following equations which have equal roots:

10. $x^4-9x^2+4x+12=0$.

11. $x^4-6x^3+12x^2-10x+3=0$.

12. $x^5-13x^4+67x^3-171x^2+216x-108=0$.

13. $x^5-x^3+4x^2-3x+2=0$.

14. $8x^4+4x^3-18x^2+11x-2=0$.

15. $x^6-3x^5+6x^3-3x^2-3x+2=0$.

16. $x^6-2x^5-4x^4+12x^3-3x^2-18x+18=0$.

17. $x^4-(a+b)x^3-a(a-b)x^2+a^2(a+b)x-a^3b=0$.

Find the solutions of the following equations which have common roots:

18. $2x^4-2x^3+x^2+3x-6=0$, $4x^4-2x^3+3x-9=0$.

19. $4x^4+12x^3-x^2-15x=0$, $6x^4+13x^3-4x^2-15x=0$.

20. Find the condition that $x^n-px^2+r=0$ may have equal roots.

21. Shew that $x^4+qx^2+s=0$ cannot have three equal roots.

22. Find the ratio of b to a in order that the equations

$$ax^2+bx+a=0 \text{ and } x^3-2x^2+2x-1=0$$

may have (1) one, (2) two roots in common.

23. Shew that the equation

$$x^n+nx^{n-1}+n(n-1)x^{n-2}+\ldots+\underline{|n}=0$$

cannot have equal roots.

24. If the equation $x^5-10a^3x^2+b^4x+c^5=0$ has three equal roots, shew that $ab^4-9a^5+c^5=0$.

25. If the equation $x^4+ax^3+bx^2+cx+d=0$ has three equal roots, shew that each of them is equal to $\dfrac{6c-ab}{3a^2-8b}$.

26. If $x^5+qx^3+rx^2+t=0$ has two equal roots, prove that one of them will be a root of the quadratic

$$15rx^2-6q^2x+25t-4qr=0.$$

27. In the equation $x^3-x-1=0$, find the value of S_6.

28. In the equation $x^4-x^3-7x^2+x+6=0$, find the values of S_4 and S_6.

Transformation of Equations.

564. The discussion of an equation is sometimes simplified by transforming it into another equation whose roots bear some assigned relation to those of the one proposed. Such transformations are especially useful in the solution of cubic equations.

565. *To transform an equation into another whose roots are those of the proposed equation with contrary signs.*

Let $f(x)=0$ be the proposed equation.

Put $-y$ for x; then the equation $f(-y)=0$ is satisfied by every root of $f(x)=0$ with its sign changed; thus the required equation is $f(-y)=0$.

If the proposed equation is

$$p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \dots\dots + p_{n-1}x + p_n = 0,$$

then it is evident that the required equation will be

$$p_0y^n - p_1y^{n-1} + p_2y^{n-2} - \dots\dots + (-1)^{n-1}p_{n-1}y + (-1)^n p_n = 0,$$

which is obtained from the original equation by *changing the sign of every alternate term beginning with the second.*

566. *To transform an equation into another whose roots are equal to those of the proposed equation multiplied by a given quantity.*

Let $f(x) = 0$ be the proposed equation, and let q denote the given quantity. Put $y = qx$, so that $x = \frac{y}{q}$, then the required equation is $f\left(\frac{y}{q}\right) = 0$.

The chief use of this transformation is to clear an equation of fractional coefficients.

Example. Remove fractional coefficients from the equation

$$2x^3 - \frac{3}{2}x^2 - \frac{1}{8}x + \frac{3}{16} = 0.$$

Put $x = \frac{y}{q}$ and multiply each term by q^3; thus

$$2y^3 - \frac{3}{2}qy^2 - \frac{1}{8}q^2y + \frac{3}{16}q^3 = 0.$$

By putting $q = 4$ all the terms become integral, and on dividing by 2, we obtain

$$y^3 - 3y^2 - y + 6 = 0.$$

567. *To transform an equation into another whose roots are the reciprocals of the roots of the proposed equation.*

Let $f(x) = 0$ be the proposed equation; put $y = \frac{1}{x}$, so that $x = \frac{1}{y}$; then the required equation is $f\left(\frac{1}{y}\right) = 0$.

One of the chief uses of this transformation is to obtain the values of expressions which involve symmetrical functions of negative powers of the roots.

Example 1. If a, b, c are the roots of the equation

$$x^3 - px^2 + qx - r = 0,$$

find the value of $$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}.$$

Write $\frac{1}{y}$ for x, multiply by y^3, and change all the signs; then the resulting equation $$ry^3 - qy^2 + py - 1 = 0,$$

has for its roots $$\frac{1}{a}, \ \frac{1}{b}, \ \frac{1}{c};$$

hence $$\Sigma\frac{1}{a} = \frac{q}{r}, \quad \Sigma\frac{1}{ab} = \frac{p}{r};$$

$$\therefore \quad \Sigma\frac{1}{a^2} = \frac{q^2 - 2pr}{r^2}.$$

Example 2. If a, b, c are the roots of

$$x^3 + 2x^2 - 3x - 1 = 0,$$

find the value of $$a^{-3} + b^{-3} + c^{-3}.$$

Writing $\frac{1}{y}$ for x, the transformed equation is

$$y^3 + 3y^2 - 2y - 1 = 0;$$

and the given expression is equal to the value of S_3 in this equation.

Here $$S_1 = -3;$$

$$S_2 = (-3)^2 - 2(-2) = 13;$$

and $$S_3 + 3S_2 - 2S_1 - 3 = 0;$$

whence we obtain $$S_3 = -42.$$

568. If an equation is unaltered by changing x into $\frac{1}{x}$, it is called a *reciprocal equation.*

If the given equation is

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \dots\dots + p_{n-2}x^2 + p_{n-1}x + p_n = 0,$$

the equation obtained by writing $\frac{1}{x}$ for x, and clearing of fractions is

$$p_nx^n + p_{n-1}x^{n-1} + p_{n-2}x^{n-2} + \dots + p_2x^2 + p_1x + 1 = 0.$$

If these two equations are the same, we must have

$$p_1 = \frac{p_{n-1}}{p_n}, \ p_2 = \frac{p_{n-2}}{p_n}, \ \dots, \ p_{n-2} = \frac{p_2}{p_n}, \ p_{n-1} = \frac{p_1}{p_n}, \ p_n = \frac{1}{p_n};$$

from the last result we have $p_n = \pm 1$, and thus we have two classes of reciprocal equations.

(i) If $p_n = 1$, then

$$p_1 = p_{n-1}, \quad p_2 = p_{n-2}, \quad p_3 = p_{n-3}, \ldots\ldots;$$

that is, the coefficients of terms equidistant from the beginning and end are equal.

(ii) If $p_n = -1$, then

$$p_1 = -p_{n-1}, \quad p_2 = -p_{n-2}, \quad p_3 = -p_{n-3}, \ldots\ldots;$$

hence if the equation is of $2m$ dimensions $p_m = -p_m$, or $p_m = 0$. In this case the coefficients of terms equidistant from the beginning and end are equal in magnitude and opposite in sign, and if the equation is of an even degree the middle term is wanting.

569. Suppose that $f(x) = 0$ is a reciprocal equation.

If $f(x) = 0$ is of the first class and of an odd degree it has a root -1; so that $f(x)$ is divisible by $x + 1$. If $\phi(x)$ is the quotient, then $\phi(x) = 0$ is a reciprocal equation of the first class and of an even degree.

If $f(x) = 0$ is of the second class and of an odd degree, it has a root $+1$; in this case $f(x)$ is divisible by $x - 1$, and as before $\phi(x) = 0$ is a reciprocal equation of the first class and of an even degree.

If $f(x) = 0$ is of the second class and of an even degree, it has a root $+1$ and a root -1; in this case $f(x)$ is divisible by $x^2 - 1$, and as before $\phi(x) = 0$ is a reciprocal equation of the first class and of an even degree.

Hence *any reciprocal equation is of an even degree with its last term positive, or can be reduced to this form;* which may therefore be considered as the standard form of reciprocal equations.

570. *A reciprocal equation of the standard form can be reduced to an equation of half its dimensions.*

Let the equation be

$$ax^{2m} + bx^{2m-1} + cx^{2m-2} + \ldots + kx^m + \ldots + cx^2 + bx + a = 0;$$

dividing by x^m and rearranging the terms, we have

$$a\left(x^m + \frac{1}{x^m}\right) + b\left(x^{m-1} + \frac{1}{x^{m-1}}\right) + c\left(x^{m-2} + \frac{1}{x^{m-2}}\right) + \ldots + k = 0.$$

Now

$$x^{p+1}+\frac{1}{x^{p+1}}=\left(x^p+\frac{1}{x^p}\right)\left(x+\frac{1}{x}\right)-\left(x^{p-1}+\frac{1}{x^{p-1}}\right);$$

hence writing z for $x+\frac{1}{x}$, and giving to p in succession the values 1, 2, 3,... we obtain

$$x^2+\frac{1}{x^2}=z^2-2,$$

$$x^3+\frac{1}{x^3}=z(z^2-2)-z=z^3-3z;$$

$$x^4+\frac{1}{x^4}=z(z^3-3z)-(z^2-2)=z^4-4z^2+2;$$

and so on; and generally $x^m+\frac{1}{x^m}$ is of m dimensions in z, and therefore the equation in z is of m dimensions.

571. *To find the equation whose roots are the squares of those of a proposed equation.*

Let $f(x)=0$ be the given equation; putting $y=x^2$, we have $x=\sqrt{y}$; hence the required equation is $f(\sqrt{y})=0$.

Example. Find the equation whose roots are the squares of those of the equation $$x^3+p_1x^2+p_2x+p_3=0.$$

Putting $x=\sqrt{y}$, and transposing, we have

$$(y+p_2)\sqrt{y}=-(p_1y+p_3);$$

whence $$(y^2+2p_2y+p_2^2)\,y=p_1^2y^2+2p_1p_3y+p_3^2,$$

or $$y^3+(2p_2-p_1^2)\,y^2+(p_2^2-2p_1p_3)\,y-p_3^2=0.$$

Compare the solution given in Ex. 2, Art. 539.

572. *To transform an equation into another whose roots exceed those of the proposed equation by a given quantity.*

Let $f(x)=0$ be the proposed equation, and let h be the given quantity; put $y=x+h$, so that $x=y-h$; then the required equation is $f(y-h)=0$.

Similarly $f(y+h)=0$ is an equation whose roots are less by h than those of $f(x)=0$.

Example. Find the equation whose roots exceed by 2 the roots of the equation $4x^4 + 32x^3 + 83x^2 + 76x + 21 = 0.$

The required equation will be obtained by substituting $x - 2$ for x in the proposed equation; hence in Horner's process we employ $x + 2$ as divisor, and the calculation is performed as follows:

$$\begin{array}{ccccc} 4 & 32 & 83 & 76 & 21 \\ 4 & 24 & 35 & 6 & |\,9 \\ 4 & 16 & 3 & |\,0 & \\ 4 & 8 & |\,-13 & & \\ 4 & |\,0 & & & \\ 4 & & & & \end{array}$$

Thus the transformed equation is

$$4x^4 - 13x^2 + 9 = 0, \quad \text{or} \quad (4x^2 - 9)(x^2 - 1) = 0.$$

The roots of this equation are $+\frac{3}{2}$, $-\frac{3}{2}$, $+1$, -1; hence the roots of the proposed equation are

$$-\frac{1}{2}, \quad -\frac{7}{2}, \quad -1, \quad -3.$$

573. The chief use of the substitution in the preceding article is to remove some assigned term from an equation.

Let the given equation be

$$p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots + p_{n-1}x + p_n = 0;$$

then if $y = x - h$, we obtain the new equation

$$p_0(y+h)^n + p_1(y+h)^{n-1} + p_2(y+h)^{n-2} + \ldots + p_n = 0,$$

which, when arranged in descending powers of y, becomes

$$p_0y^n + (np_0h + p_1)y^{n-1} + \left\{\frac{n(n-1)}{\lfloor 2} p_0h^2 + (n-1)p_1h + p_2\right\}y^{n-2} + \ldots = 0.$$

If the term to be removed is the second, we put $np_0h + p_1 = 0$, so that $h = -\frac{p_1}{np_0}$; if the term to be removed is the third we put

$$\frac{n(n-1)}{\lfloor 2} p_0h^2 + (n-1)p_1h + p_2 = 0,$$

and so obtain a quadratic to find h; and similarly we may remove any other assigned term.

Sometimes it will be more convenient to proceed as in the following example.

Example. Remove the second term from the equation

$$px^3+qx^2+rx+s=0.$$

Let α, β, γ be the roots, so that $\alpha+\beta+\gamma=-\frac{q}{p}$. Then if we increase each of the roots by $\frac{q}{3p}$, in the transformed equation the sum of the roots will be equal to $-\frac{q}{p}+\frac{q}{p}$; that is, the coefficient of the second term will be zero.

Hence the required transformation will be effected by substituting $x-\frac{q}{3p}$ for x in the given equation.

574. From the equation $f(x)=0$ we may form an equation whose roots are connected with those of the given equation by some assigned relation.

Let y be a root of the required equation and let $\phi(x, y)=0$ denote the assigned relation; then the transformed equation can be obtained either by expressing x as a function of y by means of the equation $\phi(x, y)=0$ and substituting this value of x in $f(x)=0$; or by eliminating x between the equations $f(x)=0$ and $\phi(x, y)=0$.

Example 1. If a, b, c are the roots of the equation $x^3+px^2+qx+r=0$, form the equation whose roots are

$$a-\frac{1}{bc},\quad b-\frac{1}{ca},\quad c-\frac{1}{ab}.$$

When $x=a$ in the given equation, $y=a-\frac{1}{bc}$ in the transformed equation;

but
$$a-\frac{1}{bc}=a-\frac{a}{abc}=a+\frac{a}{r};$$

and therefore the transformed equation will be obtained by the substitution

$$y=x+\frac{x}{r},\quad \text{or}\quad x=\frac{ry}{1+r};$$

thus the required equation is

$$r^2y^3+pr(1+r)y^2+q(1+r)^2y+(1+r)^3=0.$$

Example 2. Form the equation whose roots are the squares of the differences of the roots of the cubic

$$x^3+qx+r=0.$$

Let a, b, c be the roots of the cubic; then the roots of the required equation are $\quad (b-c)^2,\quad (c-a)^2,\quad (a-b)^2.$

Now $$(b-c)^2 = b^2+c^2-2bc = a^2+b^2+c^2-a^2-\frac{2abc}{a}$$

$$=(a+b+c)^2-2(bc+ca+ab)-a^2-\frac{2abc}{a}$$

$$=-2q-a^2+\frac{2r}{a};$$

also when $x=a$ in the given equation, $y=(b-c)^2$ in the transformed equation;

$$\therefore \quad y=-2q-x^2+\frac{2r}{x}.$$

Thus we have to eliminate x between the equations

$$x^3+qx+r=0,$$

and $$x^3+(2q+y)x-2r=0.$$

By subtraction $(q+y)x=3r$; or $x=\dfrac{3r}{q+y}$.

Substituting and reducing, we obtain

$$y^3+6qy^2+9q^2y+27r^2+4q^3=0.$$

Cor. If a, b, c are real, $(b-c)^2$, $(c-a)^2$, $(a-b)^2$ are all positive; therefore $27r^2+4q^3$ is negative.

Hence in order that the equation $x^3+qx+r=0$ may have all its roots real $27r^2+4q^3$ must be negative, that is $\left(\frac{r}{2}\right)^2+\left(\frac{q}{3}\right)^3$ must be negative.

If $27r^2+4q^3=0$ the transformed equation has one root zero, therefore the original equation has two equal roots.

If $27r^2+4q^3$ is positive, the transformed equation has a negative root [Art. 553], therefore the original equation must have two imaginary roots, since it is only such a pair of roots which can produce a negative root in the transformed equation.

EXAMPLES. XXXV. d.

1. Transform the equation $x^3-4x^2+\frac{1}{4}x-\frac{1}{9}=0$ into another with integral coefficients, and unity for the coefficient of the first term.

2. Transform the equation $3x^4-5x^3+x^2-x+1=0$ into another the coefficient of whose first term is unity.

Solve the equations:

3. $2x^4+x^3-6x^2+x+2=0.$

4. $x^4-10x^3+26x^2-10x+1=0.$

5. $x^5-5x^4+9x^3-9x^2+5x-1=0.$

6. $4x^6-24x^5+57x^4-73x^3+57x^2-24x+4=0.$

7. Solve the equation $3x^3 - 22x^2 + 48x - 32 = 0$, the roots of which are in harmonical progression.

8. The roots of $x^3 - 11x^2 + 36x - 36 = 0$ are in harmonical progression; find them.

9. If the roots of the equation $x^3 - ax^2 + x - b = 0$ are in harmonical progression, shew that the mean root is $3b$.

10. Solve the equation $40x^4 - 22x^3 - 21x^2 + 2x + 1 = 0$, the roots of which are in harmonical progression.

Remove the second term from the equations:

11. $x^3 - 6x^2 + 10x - 3 = 0$.

12. $x^4 + 4x^3 + 2x^2 - 4x - 2 = 0$.

13. $x^5 + 5x^4 + 3x^3 + x^2 + x - 1 = 0$.

14. $x^6 - 12x^5 + 3x^2 - 17x + 300 = 0$.

15. Transform the equation $x^3 - \frac{x}{4} - \frac{3}{4} = 0$ into one whose roots exceed by $\frac{3}{2}$ the corresponding roots of the given equation.

16. Diminish by 3 the roots of the equation

$$x^5 - 4x^4 + 3x^2 - 4x + 6 = 0.$$

17. Find the equation each of whose roots is greater by unity than a root of the equation $x^3 - 5x^2 + 6x - 3 = 0$.

18. Find the equation whose roots are the squares of the roots of

$$x^4 + x^3 + 2x^2 + x + 1 = 0.$$

19. Form the equation whose roots are the cubes of the roots of

$$x^3 + 3x^2 + 2 = 0.$$

If a, b, c are the roots of $x^3 + qx + r = 0$, form the equation whose roots are

20. ka^{-1}, kb^{-1}, kc^{-1}.

21. b^2c^2, c^2a^2, a^2b^2.

22. $\frac{b+c}{a^2}$, $\frac{c+a}{b^2}$, $\frac{a+b}{c^2}$.

23. $bc + \frac{1}{a}$, $ca + \frac{1}{b}$, $ab + \frac{1}{c}$.

24. $a(b+c)$, $b(c+a)$, $c(a+b)$.

25. a^3, b^3, c^3.

26. $\frac{b}{c} + \frac{c}{b}$, $\frac{c}{a} + \frac{a}{c}$, $\frac{a}{b} + \frac{b}{a}$.

27. Shew that the cubes of the roots of $x^3 + ax^2 + bx + ab = 0$ are given by the equation $x^3 + a^3x^2 + b^3x + a^3b^3 = 0$.

28. Solve the equation $x^5 - 5x^4 - 5x^3 + 25x^2 + 4x - 20 = 0$, whose roots are of the form a, $-a$, b, $-b$, c.

29. If the roots of $x^3 + 3px^2 + 3qx + r = 0$ are in harmonical progression, shew that $2q^3 = r(3pq - r)$.

CUBIC EQUATIONS.

575. The general type of a cubic equation is

$$x^3 + Px^2 + Qx + R = 0,$$

but as explained in Art. 573 this equation can be reduced to the simpler form $\quad x^3 + qx + r = 0,$
which we shall take as the standard form of a cubic equation.

576. To solve the equation $x^3 + qx + r = 0$.

Let $x = y + z$; then

$$x^3 = y^3 + z^3 + 3yz(y+z) = y^3 + z^3 + 3yzx,$$

and the given equation becomes

$$y^3 + z^3 + (3yz + q)x + r = 0.$$

At present y, z are any two quantities subject to the condition that their sum is equal to one of the roots of the given equation; if we further suppose that they satisfy the equation $3yz + q = 0$, they are completely determinate. We thus obtain

$$y^3 + z^3 = -r, \quad y^3z^3 = -\frac{q^3}{27};$$

hence y^3, z^3 are the roots of the quadratic

$$t^2 + rt - \frac{q^3}{27} = 0.$$

Solving this equation, and putting

$$y^3 = -\frac{r}{2} + \sqrt{\frac{r^2}{4} + \frac{q^3}{27}} \quad \text{.......................} (1),$$

$$z^3 = -\frac{r}{2} - \sqrt{\frac{r^2}{4} + \frac{q^3}{27}} \quad \text{.......................} (2),$$

we obtain the value of x from the relation $x = y + z$; thus

$$x = \left\{-\frac{r}{2} + \sqrt{\frac{r^2}{4} + \frac{q^3}{27}}\right\}^{\frac{1}{3}} + \left\{-\frac{r}{2} - \sqrt{\frac{r^2}{4} + \frac{q^3}{27}}\right\}^{\frac{1}{3}}.$$

The above solution is generally known as *Cardan's Solution*, as it was first published by him in the *Ars Magna*, in 1545. Cardan obtained the solution from Tartaglia; but the solution of the cubic seems to have been due originally to Scipio Ferreo, about

1505. An interesting historical note on this subject will be found at the end of Burnside and Panton's *Theory of Equations.*

577. By Art. 110, each of the quantities on the right-hand side of equations (1) and (2) of the preceding article has three cube roots, hence it would appear that x has *nine* values; this, however, is not the case. For since $yz = -\frac{q}{3}$, the cube roots are to be taken in pairs so that the product of each pair is rational. Hence if y, z denote the values of any pair of cube roots which fulfil this condition, the only other admissible pairs will be ωy, $\omega^2 z$ and $\omega^2 y$, ωz, where ω, ω^2 are the imaginary cube roots of unity. Hence the roots of the equation are

$$y + z, \quad \omega y + \omega^2 z, \quad \omega^2 y + \omega z.$$

Example. Solve the equation $x^3 - 15x = 126$.

Put $y + z$ for x, then

$$y^3 + z^3 + (3yz - 15)\,x = 126;$$

put $$3yz - 15 = 0,$$

then $$y^3 + z^3 = 126;$$

also $$y^3 z^3 = 125;$$

hence y^3, z^3 are the roots of the equation

$$t^2 - 126t + 125 = 0;$$

$$\therefore \quad y^3 = 125, \quad z^3 = 1;$$

$$\therefore \quad y = 5, \quad z = 1.$$

Thus $$y + z = 5 + 1 = 6;$$

$$\omega y + \omega^2 z = \frac{-1 + \sqrt{-3}}{2} \cdot 5 + \frac{-1 - \sqrt{-3}}{2}$$

$$= -3 + 2\sqrt{-3};$$

$$\omega^2 y + \omega z = -3 - 2\sqrt{-3};$$

and the roots are $$6, \quad -3 + 2\sqrt{-3}, \quad -3 - 2\sqrt{-3}.$$

578. To explain the reason why we apparently obtain *nine* values for x in Art. 576, we observe that y and z are to be found from the equations $y^3 + z^3 + r = 0$, $yz = -\frac{q}{3}$; but in the process of solution the second of these was changed into $y^3 z^3 = -\frac{q^3}{27}$, which

would also hold if $yz = -\frac{\omega q}{3}$, or $yz = -\frac{\omega^2 q}{3}$; hence the other six values of x are solutions of the cubics

$$x^3 + \omega q x + r = 0, \quad x^3 + \omega^2 q x + r = 0.$$

579. We proceed to consider more fully the roots of the equation $x^3 + qx + r = 0$.

(i) If $\frac{r^2}{4} + \frac{q^3}{27}$ is positive, then y^3 and z^3 are both real; let y and z represent their arithmetical cube roots, then the roots are

$$y + z, \quad \omega y + \omega^2 z, \quad \omega^2 y + \omega z.$$

The first of these is real, and by substituting for ω and ω^2 the other two become

$$-\frac{y+z}{2} + \frac{y-z}{2}\sqrt{-3}, \quad -\frac{y+z}{2} - \frac{y-z}{2}\sqrt{-3}.$$

(ii) If $\frac{r^2}{4} + \frac{q^3}{27}$ is zero, then $y^3 = z^3$; in this case $y = z$, and the roots become $2y$, $y(\omega + \omega^2)$, $y(\omega + \omega^2)$, or $2y$, $-y$, $-y$.

(iii) If $\frac{r^2}{4} + \frac{q^3}{27}$ is negative, then y^3 and z^3 are imaginary expressions of the form $a + ib$ and $a - ib$. Suppose that the cube roots of these quantities are $m + in$ and $m - in$; then the roots of the cubic become

$$\begin{aligned} &m + in + m - in, && \text{or } 2m; \\ &(m + in)\omega + (m - in)\omega^2, && \text{or } -m - n\sqrt{3}; \\ &(m + in)\omega^2 + (m - in)\omega, && \text{or } -m + n\sqrt{3}; \end{aligned}$$

which are all real quantities. As however there is no general arithmetical or algebraical method of finding the exact value of the cube root of imaginary quantities [Compare Art. 89], the solution obtained in Art. 576 is of little practical use when the roots of the cubic are all real and unequal.

This case is sometimes called the *Irreducible Case* of Cardan's solution.

580. In the *irreducible case* just mentioned the solution may be completed by Trigonometry as follows. Let the solution be

$$x = (a + ib)^{\frac{1}{3}} + (a - ib)^{\frac{1}{3}};$$

put $a = r\cos\theta$, $b = r\sin\theta$, so that $r^2 = a^2 + b^2$, $\tan\theta = \dfrac{b}{a}$;

then $$(a+ib)^{\frac{1}{3}} = \{r(\cos\theta + i\sin\theta)\}^{\frac{1}{3}}.$$

Now by De Moivre's theorem the three values of this expression are

$$r^{\frac{1}{3}}\left(\cos\frac{\theta}{3} + i\sin\frac{\theta}{3}\right),\quad r^{\frac{1}{3}}\left(\cos\frac{\theta+2\pi}{3} + i\sin\frac{\theta+2\pi}{3}\right),$$

and $$r^{\frac{1}{3}}\left(\cos\frac{\theta+4\pi}{3} + i\sin\frac{\theta+4\pi}{3}\right),$$

where $r^{\frac{1}{3}}$ denotes the arithmetical cube root of r, and θ the smallest angle found from the equation $\tan\theta = \dfrac{b}{a}$.

The three values of $(a-ib)^{\frac{1}{3}}$ are obtained by changing the sign of i in the above results; hence the roots are

$$2r^{\frac{1}{3}}\cos\frac{\theta}{3},\quad 2r^{\frac{1}{3}}\cos\frac{\theta+2\pi}{3},\quad 2r^{\frac{1}{3}}\cos\frac{\theta+4\pi}{3}.$$

Biquadratic Equations.

581. We shall now give a brief discussion of some of the methods which are employed to obtain the general solution of a biquadratic equation. It will be found that in each of the methods we have first to solve an auxiliary cubic equation; and thus it will be seen that as in the case of the cubic, the general solution is not adapted for writing down the solution of a given numerical equation.

582. The solution of a biquadratic equation was first obtained by Ferrari, a pupil of Cardan, as follows.

Denote the equation by

$$x^4 + 2px^3 + qx^2 + 2rx + s = 0;$$

add to each side $(ax+b)^2$, the quantities a and b being determined so as to make the left side a perfect square; then

$$x^4 + 2px^3 + (q+a^2)x^2 + 2(r+ab)x + s + b^2 = (ax+b)^2.$$

Suppose that the left side of the equation is equal to $(x^2+px+k)^2$; then by comparing the coefficients, we have

$$p^2 + 2k = q + a^2,\quad pk = r + ab,\quad k^2 = s + b^2;$$

by eliminating a and b from these equations, we obtain

$$(pk-r)^2 = (2k+p^2-q)(k^2-s),$$

or
$$2k^3 - qk^2 + 2(pr-s)k + p^2s - qs - r^2 = 0.$$

From this cubic equation one real value of k can always be found [Art. 553]; thus a and b are known. Also

$$(x^2+px+k)^2 = (ax+b)^2;$$

$$\therefore\ x^2+px+k = \pm(ax+b);$$

and the values of x are to be obtained from the two quadratics

$$x^2 + (p-a)x + (k-b) = 0,$$

and
$$x^2 + (p+a)x + (k+b) = 0.$$

Example. Solve the equation

$$x^4 - 2x^3 - 5x^2 + 10x - 3 = 0.$$

Add $a^2x^2 + 2abx + b^2$ to each side of the equation, and assume

$$x^4 - 2x^3 + (a^2-5)x^2 + 2(ab+5)x + b^2 - 3 = (x^2 - x + k)^2;$$

then by equating coefficients, we have

$$a^2 = 2k+6, \quad ab = -k-5, \quad b^2 = k^2+3;$$

$$\therefore\quad (2k+6)(k^2+3) = (k+5)^2;$$

$$\therefore\quad 2k^3 + 5k^2 - 4k - 7 = 0.$$

By trial, we find that $k = -1$; hence $a^2 = 4$, $b^2 = 4$, $ab = -4$.

But from the assumption, it follows that

$$(x^2 - x + k)^2 = (ax+b)^2.$$

Substituting the values of k, a and b, we have the two equations

$$x^2 - x - 1 = \pm(2x-2);$$

that is,
$$x^2 - 3x + 1 = 0, \quad \text{and} \quad x^2 + x - 3 = 0;$$

whence the roots are
$$\frac{3 \pm \sqrt{5}}{2}, \quad \frac{-1 \pm \sqrt{13}}{2}.$$

583. The following solution was given by Descartes in 1637.

Suppose that the biquadratic equation is reduced to the form

$$x^4 + qx^2 + rx + s = 0;$$

assume
$$x^4 + qx^2 + rx + s = (x^2 + kx + l)(x^2 - kx + m);$$

then by equating coefficients, we have

$$l+m-k^2=q, \quad k(m-l)=r, \quad lm=s.$$

From the first two of these equations, we obtain

$$2m=k^2+q+\frac{r}{k}, \qquad 2l=k^2+q-\frac{r}{k};$$

hence substituting in the third equation,

$$(k^3+qk+r)(k^3+qk-r)=4sk^2,$$

or

$$k^6+2qk^4+(q^2-4s)k^2-r^2=0.$$

This is a cubic in k^2 which always has one real positive solution [Art. 553]; thus when k^2 is known the values of l and m are determined, and the solution of the biquadratic is obtained by solving the two quadratics

$$x^2+kx+l=0, \text{ and } x^2-kx+m=0.$$

Example. Solve the equation

$$x^4-2x^2+8x-3=0.$$

Assume $\quad x^4-2x^2+8x-3=(x^2+kx+l)(x^2-kx+m);$

then by equating coefficients, we have

$$l+m-k^2=-2, \quad k(m-l)=8, \quad lm=-3;$$

whence we obtain $\quad (k^3-2k+8)(k^3-2k-8)=-12k^2,$

or

$$k^6-4k^4+16k^2-64=0.$$

This equation is clearly satisfied when $k^2-4=0$, or $k=\pm 2$. It will be sufficient to consider one of the values of k; putting $k=2$, we have

$$m+l=2, \quad m-l=4; \text{ that is, } l=-1, \quad m=3.$$

Thus $\quad x^4-2x^2+8x-3=(x^2+2x-1)(x^2-2x+3);$

hence $\quad x^2+2x-1=0, \text{ and } x^2-2x+3=0;$

and therefore the roots are $\quad -1\pm\sqrt{2}, \quad 1\pm\sqrt{-2}.$

584. The general algebraical solution of equations of a degree higher than the fourth has not been obtained, and Abel's demonstration of the impossibility of such a solution is generally accepted by Mathematicians. If, however, the coefficients of an equation are numerical, the value of any real root may be found to any required degree of accuracy by Horner's Method of approximation, a full account of which will be found in treatises on the *Theory of Equations.*

585. We shall conclude with the discussion of some miscellaneous equations.

Example 1. Solve the equations:

$$x+y+z+u=0,$$
$$ax+by+cz+du=0,$$
$$a^2x+b^2y+c^2z+d^2u=0,$$
$$a^3x+b^3y+c^3z+d^3u=k.$$

Multiply these equations, beginning from the lowest, by 1, p, q, r respectively; p, q, r being quantities which are at present undetermined. Assume that they are such that the coefficients of y, z, u vanish; then

$$x(a^3+pa^2+qa+r)=k,$$

whilst b, c, d are the roots of the equation

$$t^3+pt^2+qt+r=0.$$

Hence $$a^3+pa^2+qa+r=(a-b)(a-c)(a-d);$$

and therefore $$(a-b)(a-c)(a-d)x=k.$$

Thus the value x is found, and the values of y, z, u can be written down by symmetry.

Cor. If the equations are

$$x+y+z+u=1,$$
$$ax+by+cz+du=k,$$
$$a^2x+b^2y+c^2z+d^2u=k^2,$$
$$a^3x+b^3y+c^3z+d^3u=k^3,$$

by proceeding as before, we have

$$x(a^3+pa^2+qa+r)=k^3+pk^2+qk+r;$$
$$\therefore\quad (a-b)(a-c)(a-d)x=(k-b)(k-c)(k-d).$$

Thus the value of x is found, and the values of y, z, u can be written down by symmetry.

The solution of the above equations has been facilitated by the use of *Undetermined Multipliers.*

Example 2. Shew that the roots of the equation

$$(x-a)(x-b)(x-c)-f^2(x-a)-g^2(x-b)-h^2(x-c)+2fgh=0$$

are all real.

From the given equation, we have

$$(x-a)\{(x-b)(x-c)-f^2\}-\{g^2(x-b)+h^2(x-c)-2fgh\}=0.$$

Let p, q be the roots of the quadratic

$$(x-b)(x-c)-f^2=0,$$

and suppose p to be not less than q. By solving the quadratic, we have

$$2x = b + c \pm \sqrt{(b-c)^2 + 4f^2} \quad \text{.........................(1)};$$

now the value of the surd is greater than $b \sim c$, so that p is greater than b or c, and q is less than b or c.

In the given equation substitute for x successively the values

$$+\infty, \quad p, \quad q, \quad -\infty;$$

the results are respectively

$$+\infty, \quad -(g\sqrt{p-b} - h\sqrt{p-c})^2, \quad +(g\sqrt{b-q} - h\sqrt{c-q})^2, \quad -\infty,$$

since $$(p-b)(p-c) = f^2 = (b-q)(c-q).$$

Thus the given equation has three real roots, one greater than p, one between p and q, and one less than q.

If $p = q$, then from (1) we have $(b-c)^2 + 4f^2 = 0$ and therefore $b = c$, $f = 0$. In this case the given equation becomes

$$(x-b)\{(x-a)(x-b) - g^2 - h^2\} = 0;$$

thus the roots are all real.

If p is a root of the given equation, the above investigation fails; for it only shews that there is *one* root between q and $+\infty$, namely p. But as before, there is a second real root less than q; hence the third root must also be real. Similarly if q is a root of the given equation we can shew that all the roots are real.

The equation here discussed is of considerable importance; it occurs frequently in Solid Geometry, and is there known as the *Discriminating Cubic*.

586. The following system of equations occurs in many branches of Applied Mathematics.

Example. Solve the equations:

$$\frac{x}{a+\lambda} + \frac{y}{b+\lambda} + \frac{z}{c+\lambda} = 1,$$

$$\frac{x}{a+\mu} + \frac{y}{b+\mu} + \frac{z}{c+\mu} = 1,$$

$$\frac{x}{a+\nu} + \frac{y}{b+\nu} + \frac{z}{c+\nu} = 1.$$

Consider the following equation in θ,

$$\frac{x}{a+\theta} + \frac{y}{b+\theta} + \frac{z}{c+\theta} = 1 - \frac{(\theta-\lambda)(\theta-\mu)(\theta-\nu)}{(a+\theta)(b+\theta)(c+\theta)};$$

x, y, z being for the present regarded as known quantities.

This equation when cleared of fractions is of the *second* degree in θ, and is satisfied by the *three* values $\theta=\lambda$, $\theta=\mu$, $\theta=\nu$, in virtue of the given equations; hence it must be an identity. [Art. 310.]

To find the value of x, multiply up by $a+\theta$, and then put $a+\theta=0$;

thus $$x=-\frac{(-a-\lambda)(-a-\mu)(-a-\nu)}{(b-a)(c-a)};$$

that is, $$x=\frac{(a+\lambda)(a+\mu)(a+\nu)}{(a-b)(a-c)}.$$

By symmetry, we have

$$y=\frac{(b+\lambda)(b+\mu)(b+\nu)}{(b-c)(b-a)},$$

and $$z=\frac{(c+\lambda)(c+\mu)(c+\nu)}{(c-a)(c-b)}.$$

EXAMPLES. XXXV. e.

Solve the following equations:

1. $x^3-18x=35$.
2. $x^3+72x-1720=0$.
3. $x^3+63x-316=0$.
4. $x^3+21x+342=0$.
5. $28x^3-9x^2+1=0$.
6. $x^3-15x^2-33x+847=0$.
7. $2x^3+3x^2+3x+1=0$.

8. Prove that the real root of the equation $x^3+12x-12=0$ is $2\sqrt[3]{2}-\sqrt[3]{4}$.

Solve the following equations:

9. $x^4-3x^2-42x-40=0$.
10. $x^4-10x^2-20x-16=0$.
11. $x^4+8x^3+9x^2-8x-10=0$.
12. $x^4+2x^3-7x^2-8x+12=0$.
13. $x^4-3x^2-6x-2=0$.
14. $x^4-2x^3-12x^2+10x+3=0$.
15. $4x^4-20x^3+33x^2-20x+4=0$.
16. $x^5-6x^4-17x^3+17x^2+6x-1=0$.
17. $x^4+9x^3+12x^2-80x-192=0$, which has equal roots.

18. Find the relation between q and r in order that the equation $x^3+qx+r=0$ may be put into the form $x^4=(x^2+ax+b)^2$.

Hence solve the equation

$$8x^3-36x+27=0.$$

19. If $x^3+3px^2+3qx+r$ and $x^2+2px+q$ have a common factor, shew that

$$4(p^2-q)(q^2-pr)-(pq-r)^2=0.$$

If they have two common factors, shew that

$$p^2-q=0, \quad q^2-pr=0.$$

20. If the equation $ax^3+3bx^2+3cx+d=0$ has two equal roots, shew that each of them is equal to $\dfrac{bc-ad}{2(ac-b^2)}$.

21. Shew that the equation $x^4+px^3+qx^2+rx+s=0$ may be solved as a quadratic if $r^2=p^2s$.

22. Solve the equation

$$x^6-18x^4+16x^3+28x^2-32x+8=0,$$

one of whose roots is $\sqrt{6}-2$.

23. If $\alpha, \beta, \gamma, \delta$ are the roots of the equation

$$x^4+qx^2+rx+s=0,$$

find the equation whose roots are $\beta+\gamma+\delta+(\beta\gamma\delta)^{-1}$, &c.

24. In the equation $x^4-px^3+qx^2-rx+s=0$, prove that if the sum of two of the roots is equal to the sum of the other two $p^3-4pq+8r=0$; and that if the product of two of the roots is equal to the product of the other two $r^2=p^2s$.

25. The equation $x^5-209x+56=0$ has two roots whose product is unity: determine them.

26. Find the two roots of $x^5-409x+285=0$ whose sum is 5.

27. If $a, b, c, \ldots k$ are the roots of

$$x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots\ldots+p_{n-1}x+p_n=0,$$

shew that

$$(1+a^2)(1+b^2)\ldots\ldots(1+k^2)=(1-p_2+p_4-\ldots)^2+(p_1-p_3+p_5-\ldots)^2.$$

28. The sum of two roots of the equation

$$x^4-8x^3+21x^2-20x+5=0$$

is 4; explain why on attempting to solve the equation from the knowledge of this fact the method fails.

MISCELLANEOUS EXAMPLES.

1. If s_1, s_2, s_3 are the sums of n, $2n$, $3n$ terms respectively of an arithmetical progression, shew that $s_3 = 3(s_2 - s_1)$.

2. Find two numbers such that their difference, sum and product, are to one another as 1, 7, 24.

3. In what scale of notation is 25 doubled by reversing the digits?

4. Solve the equations:

(1) $(x+2)(x+3)(x-4)(x-5)=44.$

(2) $x(y+z)+2=0, \quad y(z-2x)+21=0, \quad z(2x-y)=5.$

5. In an A. P., of which a is the first term, if the sum of the first p terms $=0$, shew that the sum of the next q terms

$$= -\frac{a(p+q)q}{p-1}.$$

[R. M. A. WOOLWICH.]

6. Solve the equations:

(1) $(a+b)(ax+b)(a-bx)=(a^2x-b^2)(a+bx).$

(2) $x^{\frac{1}{3}}+(2x-3)^{\frac{1}{3}}=\{12(x-1)\}^{\frac{1}{3}}.$ [INDIA CIVIL SERVICE.]

7. Find an arithmetical progression whose first term is unity such that the second, tenth and thirty-fourth terms form a geometric series.

8. If α, β are the roots of $x^2+px+q=0$, find the values of

$$\alpha^2+\alpha\beta+\beta^2, \quad \alpha^3+\beta^3, \quad \alpha^4+\alpha^2\beta^2+\beta^4.$$

9. If $2x=a+a^{-1}$ and $2y=b+b^{-1}$, find the value of

$$xy+\sqrt{(x^2-1)(y^2-1)}.$$

10. Find the value of

$$\frac{(4+\sqrt{15})^{\frac{3}{2}}+(4-\sqrt{15})^{\frac{3}{2}}}{(6+\sqrt{35})^{\frac{3}{2}}-(6-\sqrt{35})^{\frac{3}{2}}}.$$

[R. M. A. WOOLWICH.]

11. If α and β are the imaginary cube roots of unity, shew that

$$\alpha^4+\beta^4+\alpha^{-1}\beta^{-1}=0.$$

12. Shew that in any scale, whose radix is greater than 4, the number 12432 is divisible by 111 and also by 112.

13. A and B run a mile race. In the first heat A gives B a start of 11 yards and beats him by 57 seconds; in the second heat A gives B a start of 81 seconds and is beaten by 88 yards: in what time could each run a mile?

14. Eliminate x, y, z between the equations:

$$x^2-yz=a^2,\quad y^2-zx=b^2,\quad z^2-xy=c^2,\quad x+y+z=0.$$

[R. M. A. WOOLWICH.]

15. Solve the equations:

$$ax^2+bxy+cy^2=bx^2+cxy+ay^2=d.$$

[MATH. TRIPOS.]

16. A waterman rows to a place 48 miles distant and back in 14 hours: he finds that he can row 4 miles with the stream in the same time as 3 miles against the stream: find the rate of the stream.

17. Extract the square root of

(1) $(a^2+ab+bc+ca)(bc+ca+ab+b^2)(bc+ca+ab+c^2)$.

(2) $1-x+\sqrt{22x-15-8x^2}$.

18. Find the coefficient of x^6 in the expansion of $(1-3x)^{\frac{10}{3}}$, and the term independent of x in $\left(\frac{4}{3}x^2-\frac{3}{2x}\right)^9$.

19. Solve the equations:

(1) $\dfrac{2x-3}{x-1}-\dfrac{3x-8}{x-2}+\dfrac{x+3}{x-3}=0$.

(2) $x^2-y^2=xy-ab,\quad (x+y)(ax+by)=2ab(a+b)$.

[TRIN. COLL. CAMB.]

20. Shew that if $a(b-c)x^2+b(c-a)xy+c(a-b)y^2$ is a perfect square, the quantities a, b, c are in harmonical progression.

[ST CATH. COLL. CAMB.]

21. If

$$(y-z)^2+(z-x)^2+(x-y)^2=(y+z-2x)^2+(z+x-2y)^2+(x+y-2z)^2,$$

and x, y, z are real, shew that $x=y=z$. ST CATH. COLL. CAMB.]

22. Extract the square root of $3e58261$ in the scale of twelve, and find in what scale the fraction $\frac{1}{5}$ would be represented by $\cdot\dot{1}\dot{7}$.

23. Find the sum of the products of the integers 1, 2, 3, ... n taken two at a time, and shew that it is equal to half the excess of the sum of the cubes of the given integers over the sum of their squares.

24. A man and his family consume 20 loaves of bread in a week. If his wages were raised 5 per cent., and the price of bread were raised $2\frac{1}{2}$ per cent., he would gain $6d.$ a week. But if his wages were lowered $7\frac{1}{2}$ per cent., and bread fell 10 per cent., then he would lose $1\frac{1}{2}d.$ a week: find his weekly wages and the price of a loaf.

25. The sum of four numbers in arithmetical progression is 48 and the product of the extremes is to the product of the means as 27 to 35: find the numbers.

26. Solve the equations:

(1) $a(b-c)x^2+b(c-a)x+c(a-b)=0.$

(2) $\dfrac{(x-a)(x-b)}{x-a-b}=\dfrac{(x-c)(x-d)}{x-c-d}.$ [MATH. TRIPOS.]

27. If $\sqrt{a-x}+\sqrt{b-x}+\sqrt{c-x}=0$, shew that

$$(a+b+c+3x)(a+b+c-x)=4(bc+ca+ab);$$

and if $\sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c}=0$, shew that $(a+b+c)^3=27abc$.

28. A train, an hour after starting, meets with an accident which detains it an hour, after which it proceeds at three-fifths of its former rate and arrives 3 hours after time: but had the accident happened 50 miles farther on the line, it would have arrived $1\frac{1}{2}$ hrs. sooner: find the length of the journey.

29. Solve the equations:

$$2x+y=2z, \quad 9z-7x=6y, \quad x^3+y^3+z^3=216.$$

[R. M. A. WOOLWICH.]

30. Six papers are set in examination, two of them in mathematics: in how many different orders can the papers be given, provided only that the two mathematical papers are not successive?

31. In how many ways can £5. 4s. 2d. be paid in exactly 60 coins, consisting of half-crowns, shillings and fourpenny-pieces?

32. Find a and b so that $x^3+ax^2+11x+6$ and $x^3+bx^2+14x+8$ may have a common factor of the form x^2+px+q.

[LONDON UNIVERSITY.]

33. In what time would A, B, C together do a work if A alone could do it in six hours more, B alone in one hour more, and C alone in twice the time?

34. If the equations $ax+by=1$, $cx^2+dy^2=1$ have only one solution prove that $\frac{a^2}{c}+\frac{b^2}{d}=1$, and $x=\frac{a}{c}$, $y=\frac{b}{d}$. [MATH. TRIPOS.]

35. Find by the Binomial Theorem the first five terms in the expansion of $(1-2x+2x^2)^{-\frac{1}{2}}$.

36. If one of the roots of $x^2+px+q=0$ is the square of the other, shew that $p^3-q(3p-1)+q^2=0$.

[PEMB. COLL. CAMB.]

37. Solve the equation

$$x^4-5x^2-6x-5=0.$$

[QUEEN'S COLL. OX.]

38. Find the value of a for which the fraction

$$\frac{x^3-ax^2+19x-a-4}{x^3-(a+1)x^2+23x-a-7}$$

admits of reduction. Reduce it to its lowest terms. [MATH. TRIPOS.]

39. If a, b, c, x, y, z are real quantities, and

$$(a+b+c)^2=3(bc+ca+ab-x^2-y^2-z^2),$$

shew that $a=b=c$, and $x=0$, $y=0$, $z=0$.

[CHRIST'S COLL. CAMB.]

40. What is the greatest term in the expansion of $\left(1-\frac{2}{3}x\right)^{-\frac{3}{2}}$ when the value of x is $\frac{6}{7}$? [EMM. COLL. CAMB.]

41. Find two numbers such that their sum multiplied by the sum of their squares is 5500, and their difference multiplied by the difference of their squares is 352. [CHRIST'S COLL. CAMB.]

42. If $x=\lambda a$, $y=(\lambda-1)b$, $z=(\lambda-3)c$, $\lambda=\frac{1+b^2+3c^2}{a^2+b^2+c^2}$, express $x^2+y^2+z^2$ in its simplest form in terms of a, b, c.

[SIDNEY COLL. CAMB.]

43. Solve the equations:

(1) $x^4+3x^2=16x+60$.

(2) $y^2+z^2-x=z^2+x^2-y=x^2+y^2-z=1$.

[CORPUS COLL. OX.]

44. If x, y, z are in harmonical progression, shew that

$$\log(x+z)+\log(x-2y+z)=2\log(x-z).$$

45. Shew that

$$\frac{1}{2}+\frac{1.3}{2.4}\left(\frac{1}{4}\right)+\frac{1.3.5}{2.4.6}\left(\frac{1}{4}\right)^2+\ldots\ldots=\frac{4}{3}(2-\sqrt{3})\sqrt{3}.$$

[EMM. COLL. CAMB.]

46. If
$$\frac{3x+2y}{3a-2b}=\frac{3y+2z}{3b-2c}=\frac{3z+2x}{3c-2a},$$

then will $$5(x+y+z)(5c+4b-3a)=(9x+8y+13z)(a+b+c).$$

[CHRIST'S COLL. CAMB.]

47. With 17 consonants and 5 vowels, how many words of four letters can be formed having 2 different vowels in the middle and 1 consonant (repeated or different) at each end?

48. A question was lost on which 600 persons had voted; the same persons having voted again on the same question, it was carried by twice as many as it was before lost by, and the new majority was to the former as 8 to 7: how many changed their minds? [ST JOHN'S COLL. CAMB.]

49. Shew that

$$\log\frac{(1+x)^{\frac{1-x}{2}}}{(1-x)^{\frac{1+x}{2}}}=x+\frac{5x^3}{2.3}+\frac{9x^5}{4.5}+\frac{13x^7}{6.7}+\ldots$$

[CHRIST'S COLL. CAMB.]

50. A body of men were formed into a hollow square, three deep, when it was observed, that with the addition of 25 to their number a solid square might be formed, of which the number of men in each side would be greater by 22 than the square root of the number of men in each side of the hollow square: required the number of men.

51. Solve the equations:

(1) $\sqrt[m]{(a+x)^2}+2\sqrt[m]{(a-x)^2}=3\sqrt[m]{a^2-x^2}.$

(2) $(x-a)^{\frac{1}{2}}(x-b)^{\frac{1}{2}}-(x-c)^{\frac{1}{2}}(x-d)^{\frac{1}{2}}=(a-c)^{\frac{1}{2}}(b-d)^{\frac{1}{2}}.$

52. Prove that

$$\sqrt[3]{4}=1+\frac{2}{6}+\frac{2.5}{6.12}+\frac{2.5.8}{6.12.18}+\ldots\ldots$$

[SIDNEY COLL. CAMB.]

53. Solve $\sqrt[3]{6(5x+6)}-\sqrt[3]{5(6x-11)}=1.$

[QUEENS' COLL. CAMB.]

54. A vessel contains a gallons of wine, and another vessel contains b gallons of water: c gallons are taken out of each vessel and transferred to the other; this operation is repeated any number of times: shew that if $c(a+b)=ab$, the quantity of wine in each vessel will always remain the same after the first operation.

55. The arithmetic mean between m and n and the geometric mean between a and b are each equal to $\dfrac{ma+nb}{m+n}$: find m and n in terms of a and b.

56. If x, y, z are such that their sum is constant, and if

$$(z+x-2y)(x+y-2z)$$

varies as yz, prove that $2(y+z)-x$ varies as yz.

[EMM. COLL. CAMB.]

57. Prove that, if n is greater than 3,

$$1 . 2 . {}^nC_r - 2 . 3 . {}^nC_{r-1} + 3 . 4 . {}^nC_{r-2} - \ldots\ldots + (-1)^r (r+1)(r+2) = 2 . {}^{n-3}C_r.$$

[CHRIST'S COLL. CAMB.]

53. Solve the equations:

(1) $\sqrt{2x-1}+\sqrt{3x-2}=\sqrt{4x-3}+\sqrt{5x-4}$.

(2) $4\{(x^2-16)^{\frac{3}{4}}+8\}=x^2+16(x^2-16)^{\frac{1}{4}}$.

[ST JOHN'S COLL. CAMB.]

59. Prove that two of the quantities x, y, z must be equal to one another, if

$$\frac{y-z}{1+yz}+\frac{z-x}{1+zx}+\frac{x-y}{1+xy}=0.$$

60. In a certain community consisting of p persons, a per cent. can read and write; of the males alone b per cent., and of the females alone c per cent. can read and write: find the number of males and females in the community.

61. If $x=\left(\dfrac{a}{b}\right)^{\frac{2ab}{a^2-b^2}}$, shew that $\dfrac{ab}{a^2+b^2}\left(x^{\frac{a}{b}}+x^{\frac{b}{a}}\right)=\left(\dfrac{a}{b}\right)^{\frac{a^2+b^2}{a^2-b^2}}$.

[EMM. COLL. CAMB.]

62. Shew that the coefficient of x^{4n} in the expansion of $(1-x+x^2-x^3)^{-1}$ is unity.

63. Solve the equation

$$\frac{x-a}{b}+\frac{x-b}{a}=\frac{b}{x-a}+\frac{a}{x-b}.$$

[LONDON UNIVERSITY.]

64. Find (1) the arithmetical series, (2) the harmonical series of n terms of which a and b are the first and last terms; and shew that the product of the r^{th} term of the first series and the $(n-r+1)^{\text{th}}$ term of the second series is ab.

65. If the roots of the equation

$$\left(1-q+\frac{p^2}{2}\right)x^2+p(1+q)x+q(q-1)+\frac{p^2}{2}=0$$

are equal, shew that $p^2=4q$. [R. M. A. WOOLWICH.]

66. If $a^2+b^2=7ab$, shew that

$$\log\left\{\frac{1}{3}(a+b)\right\}=\frac{1}{2}(\log a+\log b).$$

[QUEEN'S COLL. OX.]

67. If n is a root of the equation

$$x^2(1-ac)-x(a^2+c^2)-(1+ac)=0,$$

and if n harmonic means are inserted between a and c, shew that the difference between the first and last mean is equal to $ac(a-c)$.

[WADHAM COLL. OX.]

68. If ${}^{n+2}C_8 : {}^{n-2}P_4=57 : 16$, find n.

69. A person invests a certain sum in a $6\frac{1}{2}$ per cent. Government loan: if the price had been £3 less he would have received $\frac{1}{5}$ per cent. more interest on his money; at what price was the loan issued?

70. Solve the equation:

$$\{(x^2+x+1)^3-(x^2+1)^3-x^3\}\{(x^2-x+1)^3-(x^2+1)^3+x^3\}$$
$$=3\{(x^4+x^2+1)^3-(x^4+1)^3-x^6\}.$$

[MERTON COLL. OX.]

71. If by eliminating x between the equations

$$x^2+ax+b=0 \quad\text{and}\quad xy+l(x+y)+m=0,$$

a quadratic in y is formed whose roots are the same as those of the original quadratic in x, then either $a=2l$, and $b=m$, or $b+m=al$.

[R. M. A. WOOLWICH.]

72. Given $\log 2=\cdot30103$, and $\log 3=\cdot47712$, solve the equations:

(1) $6^x=\dfrac{10}{3}-6^{-x}$. (2) $\sqrt{5^x}+\sqrt{5^{-x}}=\dfrac{29}{10}$.

73. Find two numbers such that their sum is 9, and the sum of their fourth powers 2417. [LONDON UNIVERSITY.]

74. A set out to walk at the rate of 4 miles an hour; after he had been walking $2\frac{3}{4}$ hours, B set out to overtake him and went $4\frac{1}{2}$ miles the first hour, $4\frac{3}{4}$ miles the second, 5 the third, and so gaining a quarter of a mile every hour. In how many hours would he overtake A?

75. Prove that the integer next above $(\sqrt{3}+1)^{2m}$ contains 2^{m+1} as a factor.

76. The series of natural numbers is divided into groups 1; 2, 3, 4; 5, 6, 7, 8, 9; and so on: prove that the sum of the numbers in the n^{th} group is $(n-1)^3+n^3$.

77. Shew that the sum of n terms of the series

$$\frac{1}{2}+\frac{1}{\underline{|2}}\left(\frac{1}{2}\right)^2+\frac{1.3}{\underline{|3}}\left(\frac{1}{2}\right)^3+\frac{1.3.5}{\underline{|4}}\left(\frac{1}{2}\right)^4+\ldots\ldots$$

$$\text{is equal to } 1-\frac{1.3.5.7\ldots\ldots(2n-1)}{2^n\,\underline{|n}}.$$

[R. M. A. Woolwich.]

78. Shew that the coefficient of x^n in the expansion of $\dfrac{1+2x}{1-x+x^2}$ is

$$(-1)^{\frac{n}{3}},\ 3(-1)^{\frac{n-1}{3}},\ 2(-1)^{\frac{n-2}{3}},$$

according as n is of the form $3m$, $3m+1$, $3m+2$.

79. Solve the equations:

$$(1)\quad \frac{x}{a}=\frac{y}{b}=\frac{z}{c}=\frac{xyz}{x+y+z}.$$

$$(2)\quad \frac{x}{y}+\frac{y}{z}+\frac{z}{x}=\frac{y}{x}+\frac{z}{y}+\frac{x}{z}=x+y+z=3.$$

[Univ. Coll. Ox.]

80. The value of xyz is $7\frac{1}{2}$ or $3\frac{3}{5}$ according as the series a, x, y, z, b is arithmetic or harmonic: find the values of a and b assuming them to be positive integers. [Merton Coll. Ox.]

81. If $ay-bx=c\sqrt{(x-a)^2+(y-b)^2}$, shew that no real values of x and y will satisfy the equation unless $c^2<a^2+b^2$.

82. If $(x+1)^2$ is greater than $5x-1$ and less than $7x-3$, find the integral value of x.

83. If P is the number of integers whose logarithms have the characteristic p, and Q the number of integers the logarithms of whose reciprocals have the characteristic $-q$, shew that

$$\log_{10}P-\log_{10}Q=p-q+1.$$

84. In how many ways may 20 shillings be given to 5 persons so that no person may receive less than 3 shillings?

85. A man wishing his two daughters to receive equal portions when they came of age bequeathed to the elder the accumulated interest of a certain sum of money invested at the time of his death in 4 per cent. stock at 88; and to the younger he bequeathed the accumulated interest of a sum less than the former by £3500 invested at the same time in the 3 per cents. at 63. Supposing their ages at the time of their father's death to have been 17 and 14, what was the sum invested in each case, and what was each daughter's fortune?

86. A number of three digits in scale 7 when expressed in scale 9 has its digits reversed in order: find the number.

[ST JOHN'S COLL. CAMB.]

87. If the sum of m terms of an arithmetical progression is equal to the sum of the next n terms, and also to the sum of the next p terms; prove that $(m+n)\left(\frac{1}{m}-\frac{1}{p}\right)=(m+p)\left(\frac{1}{m}-\frac{1}{n}\right)$.

[ST JOHN'S COLL. CAMB.]

88. Prove that

$$\frac{1}{(y-z)^2}+\frac{1}{(z-x)^2}+\frac{1}{(x-y)^2}=\left(\frac{1}{y-z}+\frac{1}{z-x}+\frac{1}{x-y}\right)^2.$$

[R. M. A. WOOLWICH.]

89. If m is negative, or positive and greater than 1, shew that

$$1^m+3^m+5^m+\ldots\ldots+(2n-1)^m > n^{m+1}.$$

[EMM. COLL. CAMB.]

90. If each pair of the three equations

$$x^2-p_1x+q_1=0,\quad x^2-p_2x+q_2=0,\quad x^2-p_3x+q_3=0,$$

have a common root, prove that

$$p_1{}^2+p_2{}^2+p_3{}^2+4(q_1+q_2+q_3)=2(p_2p_3+p_3p_1+p_1p_2).$$

[ST JOHN'S COLL. CAMB.]

91. A and B travelled on the same road and at the same rate from Huntingdon to London. At the 50th milestone from London, A overtook a drove of geese which were proceeding at the rate of 3 miles in 2 hours; and two hours afterwards met a waggon, which was moving at the rate of 9 miles in 4 hours. B overtook the same drove of geese at the 45th milestone, and met the waggon exactly 40 minutes before he came to the 31st milestone. Where was B when A reached London?

[ST JOHN'S COLL. CAMB.]

92. If $a+b+c+d=0$, prove that

$$abc+bcd+cda+dab=\sqrt{(bc-ad)(ca-bd)(ab-cd)}.$$

[R. M. A. WOOLWICH.]

93. An A. P., a G. P., and an H. P. have a and b for their first two terms: shew that their $(n+2)^{\text{th}}$ terms will be in G. P. if

$$\frac{b^{2n+2}-a^{2n+2}}{ba(b^{2n}-a^{2n})}=\frac{n+1}{n}.$$

[MATH. TRIPOS.]

94. Shew that the coefficient of x^n in the expansion of $\dfrac{x}{(x-a)(x-b)}$ in ascending power of x is $\dfrac{a^n-b^n}{a-b}\cdot\dfrac{1}{a^nb^n}$; and that the coefficient of x^{2n} in the expansion of $\dfrac{(1+x^2)^n}{(1-x)^3}$ is $2^{n-1}(n^2+4n+2)$.

[EMM. COLL. CAMB.]

95. Solve the equations:

$$\sqrt{x-y}+\frac{1}{2}\sqrt{x+y}=\frac{x-1}{\sqrt{x-y}}, \quad x^2+y^2 : xy=34 : 15.$$

[ST JOHN'S COLL. CAMB.]

96. Find the value of $1+\frac{1}{3+}\,\frac{1}{2+}\,\frac{1}{3+}\,\frac{1}{2+}\ldots$ in the form of a quadratic surd. [R. M. A. WOOLWICH.]

97. Prove that the cube of an integer may be expressed as the difference of two squares; that the cube of every odd integer may be so expressed in two ways; and that the difference of the cubes of any two consecutive integers may be expressed as the difference of two squares. [JESUS COLL. CAMB.]

98. Find the value of the infinite series

$$\frac{1}{\underline{|3}}+\frac{2}{\underline{|5}}+\frac{3}{\underline{|7}}+\frac{4}{\underline{|9}}+\ldots$$

[EMM. COLL. CAMB.]

99. If

$$x=\frac{a}{b+}\,\frac{c}{d+}\,\frac{a}{b+}\,\frac{c}{d+}\,\ldots\ldots,$$

and

$$y=\frac{c}{d+}\,\frac{a}{b+}\,\frac{c}{d+}\,\frac{a}{b+}\,\ldots\ldots,$$

then

$$bx-dy=a-c.$$

[CHRIST'S COLL. CAMB.]

100. Find the generating function, the sum to n terms, and the n^{th} term of the recurring series $1+5x+7x^2+17x^3+31x^4+\ldots\ldots$

101. If a, b, c are in H. P., then

(1) $\frac{a+b}{2a-b}+\frac{c+b}{2c-b}>4.$

(2) $b^2(a-c)^2=2\{c^2(b-a)^2+a^2(c-b)^2\}.$ [PEMB. COLL. CAMB.]

102. If a, b, c are all real quantities, and $x^3-3b^2x+2c^3$ is divisible by $x-a$ and also by $x-b$; prove that either $a=b=c$, or $a=-2b=-2c$. [JESUS COLL. OX.]

103. Shew that the sum of the squares of three consecutive odd numbers increased by 1 is divisible by 12, but not by 24.

104. Shew that $\frac{ac-b^2}{a}$ is the greatest or least value of $ax^2+2bx+c$, according as a is negative or positive.

If $x^4+y^4+z^4+y^2z^2+z^2x^2+x^2y^2=2xyz(x+y+z)$, and x, y, z are all real, shew that $x=y=z$. [ST JOHN'S COLL. CAMB.]

105. Shew that the expansion of $\sqrt{\dfrac{1-\sqrt{1-x^2}}{2}}$

is
$$\frac{x}{2}+\frac{1.3}{2.4}\cdot\frac{x^3}{6}+\frac{1.3.5.7}{2.4.6.8}\cdot\frac{x^5}{10}+\ldots\ldots$$

106. If α, β are roots of the equations
$$x^2+px+q=0, \quad x^{2n}+p^nx^n+q^n=0,$$
where n is an even integer, shew that $\dfrac{\alpha}{\beta}$, $\dfrac{\beta}{\alpha}$ are roots of
$$x^n+1+(x+1)^n=0.$$ [Pemb. Coll. Camb.]

107. Find the difference between the squares of the infinite continued fractions
$$a+\frac{b}{2a+}\,\frac{b}{2a+}\,\frac{b}{2a+}\ldots\ldots, \quad\text{and}\quad c+\frac{d}{2c+}\,\frac{d}{2c+}\,\frac{d}{2c+}\ldots\ldots$$
[Christ's Coll. Camb.]

108. A sum of money is distributed amongst a certain number of persons. The second receives 1*s.* more than the first, the third 2*s.* more than the second, the fourth 3*s.* more than the third, and so on. If the first person gets 1*s.* and the last person £3. 7*s.*, what is the number of persons and the sum distributed?

109. Solve the equations:

(1) $\dfrac{x}{a}+\dfrac{y+z}{b+c}=\dfrac{y}{b}+\dfrac{z+x}{c+a}=\dfrac{z}{c}+\dfrac{x+y}{a+b}=2.$

(2) $\dfrac{x^2+y^2}{xy}+x^2+y^2=13\frac{1}{3}, \qquad \dfrac{xy}{x^2+y^2}+xy=3\frac{3}{10}.$

110. If a and b are positive and unequal, prove that
$$a^n-b^n>n(a-b)(ab)^{\frac{n-1}{2}}.$$
[St Cath. Coll. Camb.]

111. Express $\dfrac{763}{396}$ as a continued fraction; hence find the least values of x and y which satisfy the equation $396x-763y=12$.

112. To complete a certain work, a workman A alone would take m times as many days as B and C working together; B alone would take n times as many days as A and C together; C alone would take p times as many days as A and B together: shew that the numbers of days in which each would do it alone are as $m+1 : n+1 : p+1$.

Prove also $\dfrac{m}{m+1}+\dfrac{n}{n+1}+\dfrac{p}{p+1}=2.$ [R. M. A. Woolwich.]

113. The expenses of a hydropathic establishment are partly constant and partly vary with the number of boarders. Each boarder pays £65 a year, and the annual profits are £9 a head when there are 50 boarders, and £10. 13*s*. 4*d*. when there are 60: what is the profit on each boarder when there are 80?

114. If $x^2y = 2x - y$, and x^2 is not greater than 1, shew that

$$4\left(x^2 + \frac{x^6}{3} + \frac{x^{10}}{5} + \dots\right) = y^2 + \frac{y^4}{2} + \frac{y^6}{3} + \dots$$

[PETERHOUSE, CAMB.]

115. If $\dfrac{x}{a^2 - y^2} = \dfrac{y}{a^2 - x^2} = \dfrac{1}{b}$, and $xy = c^2$, shew that when a and c are unequal,

$$(a^2 - c^2)^2 - b^2c^2 = 0, \text{ or } a^2 + c^2 - b^2 = 0.$$

116. If $$(1 + x + x^2)^{3r} = 1 + k_1x + k_2x^2 + \dots,$$

and $$(x - 1)^{3r} = x^{3r} - c_1x^{3r-1} + c_2x^{3r-2} - \dots;$$

prove that (1) $1 - k_1 + k_2 - \dots\dots = 1$,

(2) $1 - k_1c_1 + k_2c_2 - \dots\dots = \pm \dfrac{\lfloor 3r}{\lfloor r \lfloor 2r}$.

[R. M. A. WOOLWICH.]

117. Solve the equations:

(1) $(x - y)^2 + 2ab = ax + by, \quad xy + ab = bx + ay.$

(2) $x^2 - y^2 + z^2 = 6, \quad 2yz - zx + 2xy = 13, \quad x - y + z = 2.$

118. If there are n positive quantities $a_1, a_2, \dots a_n$, and if the square roots of all their products taken two together be found, prove that

$$\sqrt{a_1a_2} + \sqrt{a_1a_3} + \dots\dots < \frac{n-1}{2}(a_1 + a_2 + \dots\dots + a_n);$$

hence prove that the arithmetic mean of the square roots of the products two together is less than the arithmetic mean of the given quantities. [R. M. A. WOOLWICH.]

119. If $b^2x^4 + a^2y^4 = a^2b^2$, and $a^2 + b^2 = x^2 + y^2 = 1$, prove that

$$b^4x^6 + a^4y^6 = (b^2x^4 + a^2y^4)^2.$$ [INDIA CIVIL SERVICE.]

120. Find the sum of the first n terms of the series whose r^{th} terms are (1) $\dfrac{2r+1}{r^2(r+1)^2}$, (2) $(a + r^2b)\,x^{n-r}$.

[ST JOHN'S COLL. CAMB.]

121. Find the greatest value of $\dfrac{x+2}{2x^2 + 3x + 6}$.

122. Solve the equations:

(1) $1+x^4=7(1+x)^4$.

(2) $3xy+2z=xz+6y=2yz+3x=0$.

123. If a_1, a_2, a_3, a_4 are any four consecutive coefficients of an expanded binomial, prove that

$$\frac{a_1}{a_1+a_2}+\frac{a_3}{a_3+a_4}=\frac{2a_2}{a_2+a_3}.$$ [QUEENS' COLL. CAMB.]

124. Separate $\dfrac{x^3+7x^2-x-8}{(x^2+x+1)(x^2-3x-1)}$ into partial fractions; and find the general term when $\dfrac{3x-8}{x^2-4x-4}$ is expanded in ascending powers of x.

125. In the recurring series

$$\frac{5}{4}-\frac{1}{2}x+2x^2+lx^3+5x^4+7x^5+\ldots\ldots$$

the scale of relation is a quadratic expression; determine the unknown coefficient of the fourth term and the scale of relation, and give the general term of the series. [R. M. A. WOOLWICH.]

126. If x, y, z are unequal, and if

$$2a-3y=\frac{(z-x)^2}{y}, \text{ and } 2a-3z=\frac{(x-y)^2}{z},$$

then will $$2a-3x=\frac{(y-z)^2}{x}, \text{ and } x+y+z=a.$$ [MATH. TRIPOS.]

127. Solve the equations:

(1) $xy+6=2x-x^2$, $xy-9=2y-y^2$.

(2) $(ax)^{\log a}=(by)^{\log b}$, $b^{\log x}=a^{\log y}$.

128. Find the limiting values of

(1) $x\sqrt{x^2+a^2}-\sqrt{x^4+a^4}$, when $x=\infty$.

(2) $\dfrac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}}$, when $x=a$. [LONDON UNIVERSITY.]

129. There are two numbers whose product is 192, and the quotient of the arithmetical by the harmonical mean of their greatest common measure and least common multiple is $3\frac{25}{48}$: find the numbers. [R. M. A. WOOLWICH.]

130. Solve the following equations:

(1) $\sqrt[3]{13x+37}-\sqrt[3]{13x-37}=\sqrt[3]{2}$.

(2) $$\left.\begin{array}{l} b\sqrt{1-z^2}+c\sqrt{1-y^2}=a, \\ c\sqrt{1-x^2}+a\sqrt{1-z^2}=b, \\ a\sqrt{1-y^2}+b\sqrt{1-x^2}=c. \end{array}\right\}$$

131. Prove that the sum to infinity of the series

$$\frac{1}{2^3\,\underline{|3}}-\frac{1.3}{2^4\,\underline{|4}}+\frac{1.3.5}{2^5\,\underline{|5}}-\dots \text{ is } \frac{23}{24}-\frac{2}{3}\sqrt{2}.$$ [MATH. TRIPOS.]

132. A number consisting of three digits is doubled by reversing the digits; prove that the same will hold for the number formed by the first and last digits, and also that such a number can be found in only one scale of notation out of every three. [MATH. TRIPOS.]

133. Find the coefficients of x^{12} and x^r in the product of

$$\frac{1+x^3}{(1-x^2)(1-x)} \text{ and } 1-x+x^2.$$ [R. M. A. WOOLWICH.]

134. A purchaser is to take a plot of land fronting a street; the plot is to be rectangular, and three times its frontage added to twice its depth is to be 96 yards. What is the greatest number of square yards he may take? [LONDON UNIVERSITY.]

135. Prove that

$$\begin{aligned}&(a+b+c+d)^4+(a+b-c-d)^4+(a-b+c-d)^4+(a-b-c+d)^4\\ &-(a+b+c-d)^4-(a+b-c+d)^4-(a-b+c+d)^4-(-a+b+c+d)^4\\ &=192abcd.\end{aligned}$$

[TRIN. COLL. CAMB.]

136. Find the values of a, b, c which will make each of the expressions $x^4+ax^3+bx^2+cx+1$ and $x^4+2ax^3+2bx^2+2cx+1$ a perfect square. [LONDON UNIVERSITY.]

137. Solve the equations:

(1) $$\frac{\sqrt[3]{x+y}-\sqrt[3]{x-y}}{\sqrt[3]{x+y}+\sqrt[3]{x-y}}=3,\quad x^2+y^2=65.$$

(2) $$\sqrt{2x^2+1}+\sqrt{2x^2-1}=\frac{2}{\sqrt{3-2x^2}}.$$

138. A farmer sold 10 sheep at a certain price and 5 others at 10s. less per head; the sum he received for each lot was expressed in pounds by the same two digits: find the price per sheep.

139. Sum to n terms:

(1) $(2n-1)+2(2n-3)+3(2n-5)+\dots$.

(2) The squares of the terms of the series 1, 3, 6, 10, 15....

(3) The odd terms of the series in (2). [TRIN. COLL. CAMB.]

140. If α, β, γ are the roots of the equation $x^3+qx+r=0$ prove that $3(\alpha^2+\beta^2+\gamma^2)(\alpha^5+\beta^5+\gamma^5)=5(\alpha^3+\beta^3+\gamma^3)(\alpha^4+\beta^4+\gamma^4)$.

[ST JOHN'S COLL. CAMB.]

141. Solve the equations:

(1) $\left.\begin{aligned} x(3y-5)&=4 \\ y(2x+7)&=27 \end{aligned}\right\}$. (2) $\left.\begin{aligned} x^3+y^3+z^3&=495 \\ x+y+z&=15 \\ xyz&=105 \end{aligned}\right\}$.

[TRIN. COLL. CAMB.]

142. If a, b, c are the roots of the equation $x^3+qx^2+r=0$, form the equation whose roots are $a+b-c$, $b+c-a$, $c+a-b$.

143. Sum the series:

(1) $n+(n-1)x+(n-2)x^2+\dots+2x^{n-2}+x^{n-1}$;

(2) $3-x-2x^2-16x^3-28x^4-676x^5+\dots$ to infinity;

(3) $6+9+14+23+40+\dots$ to n terms.

[OXFORD MODS.]

144. Eliminate x, y, z from the equations

$$x^{-1}+y^{-1}+z^{-1}=a^{-1}, \quad x+y+z=b.$$

$$x^2+y^2+z^2=c^2, \quad x^3+y^3+z^3=d^3,$$

and shew that if x, y, z are all finite and numerically unequal, b cannot be equal to d. [R. M. A. WOOLWICH.]

145. The roots of the equation $3x^2(x^2+8)+16(x^3-1)=0$ are not all unequal: find them. [R. M. A. WOOLWICH.]

146. A traveller set out from a certain place, and went 1 mile the first day, 3 the second, 5 the next, and so on, going every day 2 miles more than he had gone the preceding day. After he had been gone three days, a second sets out, and travels 12 miles the first day, 13 the second, and so on. In how many days will the second overtake the first? Explain the double answer.

147. Find the value of

$$\frac{1}{3+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{3+}\,\frac{1}{2+}\,\frac{1}{1+}\cdots$$

148. Solve the equation

$$x^3+3ax^2+3(a^2-bc)x+a^3+b^3+c^3-3abc=0.$$

[INDIA CIVIL SERVICE.]

149. If n is a prime number which will divide neither a, b, nor $a+b$, prove that $a^{n-2}b-a^{n-3}b^2+a^{n-4}b^3-\ldots+ab^{n-2}$ exceeds by 1 a multiple of n. [ST JOHN'S COLL. CAMB.]

150. Find the n^{th} term and the sum to n terms of the series whose sum to infinity is $(1-abx^2)(1-ax)^{-2}(1-bx)^{-2}$.

[OXFORD MODS.]

151. If a, b, c are the roots of the equation $x^3+px+q=0$, find the equation whose roots are $\dfrac{b^2+c^2}{a}$, $\dfrac{c^2+a^2}{b}$, $\dfrac{a^2+b^2}{c}$.

[TRIN. COLL. CAMB.]

152. Prove that

$$(y+z-2x)^4+(z+x-2y)^4+(x+y-2z)^4=18(x^2+y^2+z^2-yz-zx-xy)^2.$$

[CLARE COLL. CAMB.]

153. Solve the equations:

(1) $x^3-30x+133=0$, by Cardan's method.

(2) $x^5-4x^4-10x^3+40x^2+9x-36=0$, having roots of the form $\pm a$, $\pm b$, c.

154. It is found that the quantity of work done by a man in an hour varies directly as his pay per hour and inversely as the square root of the number of hours he works per day. He can finish a piece of work in six days when working 9 hours a day at 1*s.* per hour. How many days will he take to finish the same piece of work when working 16 hours a day at 1*s.* 6*d.* per hour?

155. If s_n denote the sum to n terms of the series

$$1.2+2.3+3.4+\ldots,$$

and σ_{n-1} that to $n-1$ terms of the series

$$\frac{1}{1.2.3.4}+\frac{1}{2.3.4.5}+\frac{1}{3.4.5.6}+\ldots,$$

shew that $$18s_n\sigma_{n-1}-s_n+2=0.$$

[MAGD. COLL. OX.]

156. Solve the equations:

(1) $(12x-1)(6x-1)(4x-1)(3x-1)=5$.

(2) $\dfrac{1}{5}\dfrac{(x+1)(x-3)}{(x+2)(x-4)}+\dfrac{1}{9}\dfrac{(x+3)(x-5)}{(x+4)(x-6)}-\dfrac{2}{13}\dfrac{(x+5)(x-7)}{(x+6)(x-8)}=\dfrac{92}{585}$.

[ST JOHN'S COLL. CAMB.]

157. A cottage at the beginning of a year was worth £250, but it was found that by dilapidations at the end of each year it lost ten per cent. of the value it had at the beginning of each year: after what number of years would the value of the cottage be reduced below £25? Given $\log_{10} 3 = \cdot 4771213$. [R. M. A. WOOLWICH.]

158. Shew that the infinite series

$$1 + \frac{1}{4} + \frac{1.4}{4.8} + \frac{1.4.7}{4.8.12} + \frac{1.4.7.10}{4.8.12.16} + \ldots,$$

$$1 + \frac{2}{6} + \frac{2.5}{6.12} + \frac{2.5.8}{6.12.18} + \frac{2.5.8.11}{6.12.18.24} + \ldots,$$

are equal. [PETERHOUSE, CAMB.]

159. Prove the identity

$$\left\{1 - \frac{x}{a} + \frac{x(x-a)}{a\beta} - \frac{x(x-a)(x-\beta)}{a\beta\gamma} + \ldots\right\}$$
$$\left\{1 + \frac{x}{a} + \frac{x(x+a)}{a\beta} + \frac{x(x+a)(x+\beta)}{a\beta\gamma} + \ldots\right\}$$
$$= 1 - \frac{x^2}{a^2} + \frac{x^2(x^2-a^2)}{a^2\beta^2} - \frac{x^2(x^2-a^2)(x^2-\beta^2)}{a^2\beta^2\gamma^2} + \ldots.$$

[TRIN. COLL. CAMB.]

160. If n is a positive integer greater than 1, shew that

$$n^5 - 5n^3 + 60n^2 - 56n$$

is a multiple of 120. [WADHAM COLL. OX.]

161. A number of persons were engaged to do a piece of work which would have occupied them 24 hours if they had commenced at the same time; but instead of doing so, they commenced at equal intervals and then continued to work till the whole was finished, the payment being proportional to the work done by each: the first comer received eleven times as much as the last; find the time occupied.

162. Solve the equations:

(1) $$\frac{x}{y^2-3} = \frac{y}{x^2-3} = \frac{-7}{x^3+y^3}.$$

(2) $$y^2 + z^2 - x(y+z) = a^2,$$
$$z^2 + x^2 - y(z+x) = b^2,$$
$$x^2 + y^2 - z(x+y) = c^2.$$

[PEMB. COLL. CAMB.]

163. Solve the equation

$$a^3(b-c)(x-b)(x-c)+b^3(c-a)(x-c)(x-a)+c^3(a-b)(x-a)(x-b)=0;$$

also shew that if the two roots are equal

$$\frac{1}{\sqrt{a}} \pm \frac{1}{\sqrt{b}} \pm \frac{1}{\sqrt{c}} = 0.$$ [ST JOHN'S COLL. CAMB.]

164. Sum the series:

(1) $1.2.4+2.3.5+3.4.6+\ldots$ to n terms.

(2) $\dfrac{1^2}{\lfloor 3} + \dfrac{2^2}{\lfloor 4} + \dfrac{3^2}{\lfloor 5} + \ldots$ to inf.

165. Shew that, if a, b, c, d be four positive unequal quantities and $s=a+b+c+d$, then

$$(s-a)(s-b)(s-c)(s-d) > 81abcd.$$

[PETERHOUSE, CAMB.]

166. Solve the equations:

(1) $\sqrt{x+a} - \sqrt{y-a} = \dfrac{5}{2}\sqrt{a}$, $\sqrt{x-a} - \sqrt{y+a} = \dfrac{3}{2}\sqrt{a}$.

(2) $x+y+z=x^2+y^2+z^2=\dfrac{1}{2}(x^3+y^3+z^3)=3$.

[MATH. TRIPOS.]

167. Eliminate l, m, n from the equations:

$$lx+my+nz=mx+ny+lz=nx+ly+mz=k^2(l^2+m^2+n^2)=1.$$

168. Simplify

$$\frac{a(b+c-a)^2+\ldots+\ldots+(b+c-a)(c+a-b)(a+b-c)}{a^2(b+c-a)+\ldots+\ldots-(b+c-a)(c+a-b)(a+b-c)}.$$

[MATH. TRIPOS.]

169. Shew that the expression

$$(x^2-yz)^3+(y^2-zx)^3+(z^2-xy)^3-3(x^2-yz)(y^2-zx)(z^2-xy)$$

is a perfect square, and find its square root. [LONDON UNIVERSITY.]

170. There are three towns A, B, and C; a person by walking from A to B, driving from B to C, and riding from C to A makes the journey in $15\frac{1}{2}$ hours; by driving from A to B, riding from B to C, and walking from C to A he could make the journey in 12 hours. On foot he could make the journey in 22 hours, on horseback in $8\frac{1}{4}$ hours, and driving in 11 hours. To walk a mile, ride a mile, and drive a mile he takes altogether half an hour: find the rates at which he travels, and the distances between the towns.

171. Shew that $n^7-7n^5+14n^3-8n$ is divisible by 840, if n is an integer not less than 3.

172. Solve the equations:

(1) $\sqrt{x^2+12y}+\sqrt{y^2+12x}=33, \quad x+y=23.$

(2) $\dfrac{u(y-x)}{z-u}=a, \quad \dfrac{z(y-x)}{z-u}=b, \quad \dfrac{y(u-z)}{x-y}=c, \quad \dfrac{x(u-z)}{x-y}=d.$

[MATH. TRIPOS.]

173. If s be the sum of n positive unequal quantities $a, b, c\ldots$, then

$$\frac{s}{s-a}+\frac{s}{s-b}+\frac{s}{s-c}+\ldots>\frac{n^2}{n-1}.$$ [MATH. TRIPOS.]

174. A merchant bought a quantity of cotton; this he exchanged for oil which he sold. He observed that the number of cwt. of cotton, the number of gallons of oil obtained for each cwt., and the number of shillings for which he sold each gallon formed a descending geometrical progression. He calculated that if he had obtained one cwt. more of cotton, one gallon more of oil for each cwt., and 1*s.* more for each gallon, he would have obtained £508. 9*s.* more; whereas if he had obtained one cwt. less of cotton, one gallon less of oil for each cwt., and 1*s.* less for each gallon, he would have obtained £483. 13*s.* less: how much did he actually receive?

175. Prove that

$$\Sigma(b+c-a-x)^4(b-c)(a-x)=16(b-c)(c-a)(a-b)(x-a)(x-b)(x-c).$$

[JESUS COLL. CAMB.]

176. If α, β, γ are the roots of the equation $x^3-px^2+r=0$, find the equation whose roots are $\dfrac{\beta+\gamma}{\alpha}, \dfrac{\gamma+\alpha}{\beta}, \dfrac{\alpha+\beta}{\gamma}$. [R. M. A. WOOLWICH.]

177. If any number of factors of the form a^2+b^2 are multiplied together, shew that the product can be expressed as the sum of two squares.

Given that $(a^2+b^2)(c^2+d^2)(e^2+f^2)(g^2+h^2)=p^2+q^2$, find p and q in terms of a, b, c, d, e, f, g, h. [LONDON UNIVERSITY.]

178. Solve the equations

$$x^2+y^2=61, \quad x^3-y^3=91.$$ [R. M. A. WOOLWICH.]

179. A man goes in for an Examination in which there are four papers with a maximum of m marks for each paper; shew that the number of ways of getting $2m$ marks on the whole is

$$\frac{1}{3}(m+1)(2m^2+4m+3).$$ [MATH. TRIPOS.]

180. If α, β are the roots of $x^2+px+1=0$, and γ, δ are the roots of $x^2+qx+1=0$; shew that $(\alpha-\gamma)(\beta-\gamma)(\alpha+\delta)(\beta+\delta)=q^2-p^2$.

[R. M. A. WOOLWICH.]

181. Shew that if a_m be the coefficient of x^m in the expansion of $(1+x)^n$, then whatever n be,

$$a_0-a_1+a_2-\ldots+(-1)^{m-1}a_{m-1}=\frac{(n-1)(n-2)\ldots(n-m+1)}{\underline{|m-1}}(-1)^{m-1}.$$

[NEW COLL. OX.]

182. A certain number is the product of three prime factors, the sum of whose squares is 2331. There are 7560 numbers (including unity) which are less than the number and prime to it. The sum of its divisors (including unity and the number itself) is 10560. Find the number. [CORPUS COLL. CAMB.]

183. Form an equation whose roots shall be the products of every two of the roots of the equation $x^3-ax^2+bx+c=0$.

Solve completely the equation

$$2x^5+x^4+x+2=12x^3+12x^2.$$

[R. M. A. WOOLWICH.]

184. Prove that if n is a positive integer,

$$n^n-n(n-2)^n+\frac{n(n-1)}{\underline{|2}}(n-4)^n-\ldots\ldots=2^n\underline{|n}.$$

185. If $(6\sqrt{6}+14)^{2n+1}=N$, and if F be the fractional part of N, prove that $NF=20^{2n+1}$. [EMM. COLL. CAMB.]

186. Solve the equations:

(1) $x+y+z=2$, $x^2+y^2+z^2=0$, $x^3+y^3+z^3=-1$.

(2) $x^2-(y-z)^2=a^2$, $y^2-(z-x)^2=b^2$, $z^2-(x-y)^2=c^2$.

[CHRIST'S COLL. CAMB.]

187. At a general election the whole number of Liberals returned was 15 more than the number of English Conservatives, the whole number of Conservatives was 5 more than twice the number of English Liberals. The number of Scotch Conservatives was the same as the number of Welsh Liberals, and the Scotch Liberal majority was equal to twice the number of Welsh Conservatives, and was to the Irish Liberal majority as 2 : 3. The English Conservative majority was 10 more than the whole number of Irish members. The whole number of members was 652, of whom 60 were returned by Scotch constituencies. Find the numbers of each party returned by England, Scotland, Ireland, and Wales, respectively. [ST JOHN'S COLL. CAMB.]

188. Shew that $a^5(c-b)+b^5(a-c)+c^5(b-a)$

$$=(b-c)(c-a)(a-b)(\Sigma a^3+\Sigma a^2b+abc).$$

189. Prove that $\begin{vmatrix} a^3 & 3a^2 & 3a & 1 \\ a^2 & a^2+2a & 2a+1 & 1 \\ a & 2a+1 & a+2 & 1 \\ 1 & 3 & 3 & 1 \end{vmatrix} = (a-1)^6.$ [BALL. COLL. OX.]

190. If $\frac{1}{a}+\frac{1}{c}+\frac{1}{a-b}+\frac{1}{c-b}=0$, prove that a, b, c are in harmonical progression, unless $b=a+c$. [TRIN. COLL. CAMB.]

191. Solve the equations:

(1) $x^3-13x^2+15x+189=0$, having given that one root exceeds another root by 2.

(2) $x^4-4x^2+8x+35=0$, having given that one root is $2+\sqrt{-3}$. [R. M. A. WOOLWICH.]

192. Two numbers a and b are given; two others a_1, b_1 are formed by the relations $3a_1=2a+b$, $3b_1=a+2b$; two more a_2, b_2 are formed from a_1, b_1 in the same manner, and so on; find a_n, b_n in terms of a and b, and prove that when n is infinite, $a_n=b_n$. [R. M. A. WOOLWICH.]

193. If $x+y+z+w=0$, shew that

$$wx(w+x)^2+yz(w-x)^2+wy(w+y)^2$$
$$+zx(w-y)^2+wz(w+z)^2+xy(w-z)^2+4xyzw=0.$$

[MATH. TRIPOS.]

194. If $a+\frac{bc-a^2}{a^2+b^2+c^2}$ be not altered in value by interchanging a pair of the letters a, b, c not equal to each other, it will not be altered by interchanging any other pair; and it will vanish if $a+b+c=1$. [MATH. TRIPOS.]

195. On a quadruple line of rails between two termini A and B, two down trains start at 6.0 and 6.45, and two up trains at 7.15 and 8.30. If the four trains (regarded as points) all pass one another simultaneously, find the following equations between x_1, x_2, x_3, x_4, their rates in miles per hour,

$$\frac{3x_2}{x_2-x_1}=\frac{4m+5x_3}{x_1+x_3}=\frac{4m+10x_4}{x_1+x_4},$$

where m is the number of miles in AB. [TRIN. COLL. CAMB.]

196. Prove that, rejecting terms of the third and higher orders,

$$\frac{(1-x)^{-\frac{1}{2}}+(1-y)^{-\frac{1}{2}}}{1+\sqrt{(1-x)(1-y)}}=1+\frac{1}{2}(x+y)+\frac{1}{8}(3x^2+xy+3y^2).$$

[TRIN. COLL. CAMB.]

197. Shew that the sum of the products of the series

$$a,\ a-b,\ a-2b,\ \ldots\ldots,\ a-(n-1)b,$$

taken two and two together vanishes when n is of the form $3m^2-1$, and $2a=(3m-2)(m+1)b$.

198. If n is even, and $a+\beta$, $a-\beta$ are the middle pair of terms, shew that the sum of the cubes of an arithmetical progression is

$$na\{a^2+(n^2-1)\beta^2\}.$$

199. If a, b, c are real positive quantities, shew that

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}<\frac{a^8+b^8+c^8}{a^3b^3c^3}.$$

[TRIN. COLL. CAMB.]

200. A, B, and C start at the same time for a town a miles distant; A walks at a uniform rate of u miles an hour, and B and C drive at a uniform rate of v miles an hour. After a certain time B dismounts and walks forward at the same pace as A, while C drives back to meet A; A gets into the carriage with C and they drive after B entering the town at the same time that he does: shew that the whole time occupied was $\frac{a}{v}\cdot\frac{3v+u}{3u+v}$ hours. [PETERHOUSE, CAMB.]

201. The streets of a city are arranged like the lines of a chess-board. There are m streets running north and south, and n east and west. Find the number of ways in which a man can travel from the N.W. to the S.E. corner, going the shortest possible distance.

[OXFORD MODS.]

202. Solve the equation $\sqrt[4]{x+27}+\sqrt[4]{55-x}=4$.

[BALL. COLL. OX.]

203. Shew that in the series

$$ab+(a+x)(b+x)+(a+2x)(b+2x)+\ldots\ldots \text{ to } 2n \text{ terms},$$

the excess of the sum of the last n terms over the sum of the first n terms is to the excess of the last term over the first as n^2 to $2n-1$.

204. Find the n^{th} convergent to

(1) $\dfrac{1}{2-}\,\dfrac{1}{2-}\,\dfrac{1}{2-}\ldots\ldots$

(2) $\dfrac{4}{3+}\,\dfrac{4}{3+}\,\dfrac{4}{3+}\ldots\ldots$

205. Prove that

$$\begin{aligned}(a-x)^4(y-z)^4+(a-y)^4(z-x)^4+(a-z)^4(x-y)^4\\ =2\{(a-y)^2(a-z)^2(x-y)^2(x-z)^2+(a-z)^2(a-x)^2(y-z)^2(y-x)^2\\ +(a-x)^2(a-y)^2(z-x)^2(z-y)^2\}.\end{aligned}$$

[PETERHOUSE, CAMB.]

206. If α, β, γ are the roots of $x^3+qx+r=0$, find the value of

$$\frac{m\alpha+n}{m\alpha-n}+\frac{m\beta+n}{m\beta-n}+\frac{m\gamma+n}{m\gamma-n}$$

in terms of m, n, q, r. [QUEENS' COLL. CAMB.]

207. In England one person out of 46 is said to die every year, and one out of 33 to be born. If there were no emigration, in how many years would the population double itself at this rate? Given

$$\log 2 = \cdot 3010300,\ \log 1531 = 3\cdot 1849752,\ \log 1518 = 3\cdot 1812718.$$

208. If $(1+x+x^2)^n = a_0 + a_1 x + a_2 x^2 + \ldots\ldots$, prove that

$$a_r - n a_{r-1} + \frac{n(n-1)}{1.2} a_{r-2} - \ldots\ldots + (-1)^r \frac{n!}{r!\,(n-r)!} a_0 = 0,$$

unless r is a multiple of 3. What is its value in this case?

[ST JOHN'S COLL. CAMB.]

209. In a mixed company consisting of Poles, Turks, Greeks, Germans and Italians, the Poles are one less than one-third of the number of Germans, and three less than half the number of Italians. The Turks and Germans outnumber the Greeks and Italians by 3; the Greeks and Germans form one less than half the company; while the Italians and Greeks form seven-sixteenths of the company: determine the number of each nation.

210. Find the sum to infinity of the series whose n^{th} term is

$$(n+1)\, n^{-1} (n+2)^{-1} (-x)^{n+1}.$$ [OXFORD MODS.]

211. If n is a positive integer, prove that

$$n - \frac{n(n^2-1)}{\underline{|2}} + \frac{n(n^2-1)(n^2-2^2)}{\underline{|2}\,\underline{|3}} - \ldots\ldots$$
$$+(-1)^r \frac{n(n^2-1)(n^2-2^2)\ldots\ldots(n^2-r^2)}{\underline{|r}\,\underline{|r+1}} + \ldots\ldots = (-1)^{n+1}.$$

[PEMB. COLL. CAMB.]

212. Find the sum of the series:

(1) 6, 24, 60, 120, 210, 336,...... to n terms.

(2) $4 - 9x + 16x^2 - 25x^3 + 36x^4 - 49x^5 + \ldots\ldots$ to inf.

(3) $\frac{1.3}{2} + \frac{3.5}{2^2} + \frac{5.7}{2^3} + \frac{7.9}{2^4} + \ldots\ldots$ to inf.

213. Solve the equation $\begin{vmatrix} 4x & 6x+2 & 8x+1 \\ 6x+2 & 9x+3 & 12x \\ 8x+1 & 12x & 16x+2 \end{vmatrix} = 0.$

[KING'S COLL. CAMB.]

214. Shew that

(1) $a^2(1+b^2)+b^2(1+c^2)+c^2(1+a^2)>6abc$,

(2) $n(a^{p+q}+b^{p+q}+c^{p+q}+\ldots)>(a^p+b^p+c^p+\ldots)(a^q+b^q+c^q+\ldots)$,

the number of quantities a, b, c,... being n.

215. Solve the equations

$$\left.\begin{aligned} yz&=a(y+z)+\alpha \\ zx&=a(z+x)+\beta \\ xy&=a(x+y)+\gamma \end{aligned}\right\}.$$

[TRIN. COLL. CAMB.]

216. If n be a prime number, prove that

$$1(2^{n-1}+1)+2\left(3^{n-1}+\frac{1}{2}\right)+3\left(4^{n-1}+\frac{1}{3}\right)+\ldots+(n-1)\left(n^{n-1}+\frac{1}{n-1}\right)$$

is divisible by n. [QUEEN'S COLL. OX.]

217. In a shooting competition a man can score 5, 4, 3, 2, or 0 points for each shot: find the number of different ways in which he can score 30 in 7 shots. [PEMB. COLL. CAMB.]

218. Prove that the expression $x^5-bx^3+cx^2+dx-e$ will be the product of a complete square and a complete cube if

$$\frac{12b}{5}=\frac{9d}{b}=\frac{5e}{c}=\frac{d^2}{c^2}.$$

219. A bag contains 6 black balls and an unknown number, not greater than six, of white balls; three are drawn successively and not replaced and are all found to be white; prove that the chance that a black ball will be drawn next is $\dfrac{677}{909}$. [JESUS COLL. CAMB.]

220. Shew that the sum of the products of every pair of the squares of the first n whole numbers is $\dfrac{1}{360}n(n^2-1)(4n^2-1)(5n+6)$.

[CAIUS COLL. CAMB.]

221. If $\dfrac{\alpha^2(b-c)}{x-a}+\dfrac{\beta^2(c-a)}{x-b}+\dfrac{\gamma^2(a-b)}{x-c}=0$ has equal roots, prove that $\alpha(b-c)\pm\beta(c-a)\pm\gamma(a-b)=0$.

222. Prove that when n is a positive integer,

$$n=2^{n-1}-\frac{n-2}{1}2^{n-3}+\frac{(n-3)(n-4)}{\underline{|2}}2^{n-5}$$
$$-\frac{(n-4)(n-5)(n-6)}{\underline{|3}}2^{n-7}+\ldots.$$

[CLARE COLL. CAMB.]

223. Solve the equations:

(1) $x^2+2yz=y^2+2zx=z^2+2xy+3=76$.

(2) $$\left.\begin{aligned} x+y+z&=a+b+c \\ \frac{x}{a}+\frac{y}{b}+\frac{z}{c}&=3 \\ ax+by+cz&=bc+ca+ab \end{aligned}\right\}.$$

[CHRIST'S COLL. CAMB.]

224. Prove that if each of m points in one straight line be joined to each of n in another by straight lines terminated by the points, then, excluding the given points, the lines will intersect $\frac{1}{4}mn(m-1)(n-1)$ times. [MATH. TRIPOS.]

225. Having given $y=x+x^2+x^5$, expand x in the form

$$y+ay^2+by^3+cy^4+dy^5+\ldots\ldots;$$

and shew that $a^2d-3abc+2b^3=-1$. [BALL. COLL. OX.]

226. A farmer spent three equal sums of money in buying calves, pigs, and sheep. Each calf cost £1 more than a pig and £2 more than a sheep; altogether he bought 47 animals. The number of pigs exceeded that of the calves by as many sheep as he could have bought for £9: find the number of animals of each kind.

227. Express log 2 in the form of the infinite continued fraction

$$\frac{1}{1+}\,\frac{1}{1+}\,\frac{2^2}{1+}\,\frac{3^2}{1+}\ldots\ldots\frac{n^2}{1+}\ldots.$$ [EULER.]

228. In a certain examination six papers are set, and to each are assigned 100 marks as a maximum. Shew that the number of ways in which a candidate may obtain forty per cent. of the whole number of marks is

$$\frac{\lfloor 1}{\lfloor 5}\left\{\frac{\lfloor 245}{\lfloor 240}-6\cdot\frac{\lfloor 144}{\lfloor 139}+15\cdot\frac{\lfloor 43}{\lfloor 38}\right\}.$$ [OXFORD MODS.]

229. Test for convergency

$$\frac{x}{2}+\frac{1.3}{2.4}\cdot\frac{x^3}{6}+\frac{1.3.5.7}{2.4.6.8}\cdot\frac{x^5}{10}+\frac{1.3.5.7.9.11}{2.4.6.8.10.12}\cdot\frac{x^7}{14}+\ldots\ldots.$$

230. Find the scale of relation, the n^{th} term, and the sum of n terms of the recurring series $1+6+40+288+\ldots\ldots$.

Shew also that the sum of n terms of the series formed by taking for its r^{th} term the sum of r terms of this series is

$$\frac{2}{3^2}(2^{2n}-1)+\frac{4}{7^2}(2^{3n}-1)-\frac{5n}{21}.$$ [CAIUS COLL. CAMB.]

231. It is known that at noon at a certain place the sun is hidden by clouds on an average two days out of every three: find the chance that at noon on at least four out of five specified future days the sun will be shining. [QUEEN'S COLL. OX.]

232. Solve the equations

$$\left.\begin{array}{l} x^2+(y-z)^2=a^2 \\ y^2+(z-x)^2=b^2 \\ z^2+(x-y)^2=c^2 \end{array}\right\}.$$

[EMM. COLL. CAMB.]

233. Eliminate x, y, z from the equations:

$$\frac{x^2-xy-xz}{a}=\frac{y^2-yz-yx}{b}=\frac{z^2-zx-zy}{c}, \text{ and } ax+by+cz=0.$$

[MATH. TRIPOS.]

234. If two roots of the equation $x^3+px^2+qx+r=0$ be equal and of opposite signs, shew that $pq=r$. [QUEENS' COLL. CAMB.]

235. Sum the series:

(1) $1+2^3x+3^3x^2+\ldots\ldots+n^3x^{n-1}$,

(2) $\dfrac{25}{1^2.2^3.3^3}+\dfrac{52}{2^2.3^3.4^3}+\ldots\ldots+\dfrac{5n^2+12n+8}{n^2(n+1)^3(n+2)^3}$.

[EMM. COLL. CAMB.]

236. If $(1+a^3x^4)(1+a^5x^8)(1+a^9x^{16})(1+a^{17}x^{32})\ldots\ldots$

$$=1+A_4x^4+A_8x^8+A_{12}x^{12}+\ldots\ldots$$

prove that $A_{8n+4}=a^3A_{8n}$, and $A_{8n}=a^{2n}A_{4n}$; and find the first ten terms of the expansion. [CORPUS COLL. CAMB.]

237. On a sheet of water there is no current from A to B but a current from B to C; a man rows down stream from A to C in 3 hours, and up stream from C to A in $3\frac{1}{2}$ hours; had there been the same current all the way as from B to C, his journey down stream would have occupied $2\frac{3}{4}$ hours; find the length of time his return journey would have taken under the same circumstances.

238. Prove that the n^{th} convergent to the continued fraction

$$\frac{3}{2+}\,\frac{3}{2+}\,\frac{3}{2+}\ldots\ldots \text{ is } \frac{3^{n+1}+3(-1)^{n+1}}{3^{n+1}-(-1)^{n+1}}.$$

[EMM. COLL. CAMB.]

239. If all the coefficients in the equation

$$x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots\ldots+p_n=f(x)=0,$$

be whole numbers, and if $f(0)$ and $f(1)$ be each odd integers, prove that the equation cannot have a commensurable root.

[LONDON UNIVERSITY.]

240. Shew that the equation

$$\sqrt{ax+\alpha}+\sqrt{bx+\beta}+\sqrt{cx+\gamma}=0$$

reduces to a simple equation if $\sqrt{a}\pm\sqrt{b}\pm\sqrt{c}=0$.

Solve the equation

$$\sqrt{6x^2-15x-7}+\sqrt{4x^2-8x-11}-\sqrt{2x^2-5x+5}=2x-3.$$

241. A bag contains 3 red and 3 green balls, and a person draws out 3 at random. He then drops 3 blue balls into the bag, and again draws out 3 at random. Shew that he may just lay 8 to 3 with advantage to himself against the 3 latter balls being all of different colours. [PEMB. COLL. CAMB.]

242. Find the sum of the fifth powers of the roots of the equation $x^4-7x^2+4x-3=0$. [LONDON UNIVERSITY.]

243. A Geometrical and Harmonical Progression have the same p^{th}, q^{th}, r^{th} terms a, b, c respectively: shew that

$$a(b-c)\log a+b(c-a)\log b+c(a-b)\log c=0.$$

[CHRIST'S COLL. CAMB.]

244. Find four numbers such that the sum of the first, third and fourth exceeds that of the second by 8; the sum of the squares of the first and second exceeds the sum of the squares of the third and fourth by 36; the sum of the products of the first and second, and of the third and fourth is 42; the cube of the first is equal to the sum of the cubes of the second, third, and fourth.

245. If T_n, T_{n+1}, T_{n+2} be 3 consecutive terms of a recurring series connected by the relation $T_{n+2}=aT_{n+1}-bT_n$, prove that

$$\frac{1}{b^n}\{T^2_{n+1}-aT_nT_{n+1}+bT_n^2\}=\text{a constant}.$$

246. Eliminate x, y, z from the equations:

$$\left.\begin{array}{ll}\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}=\dfrac{1}{a}, & x^2+y^2+z^2=b^2,\\ x^3+y^3+z^3=c^3, & xyz=d^3.\end{array}\right\}$$

[EMM. COLL. CAMB.]

247. Shew that the roots of the equation

$$x^4-px^3+qx^2-rx+\frac{r^2}{p^2}=0$$

are in proportion. Hence solve $x^4-12x^3+47x^2-72x+36=0$.

248. A can hit a target four times in 5 shots; B three times in 4 shots; and C twice in 3 shots. They fire a volley: what is the probability that two shots at least hit? And if two hit what is the probability that it is C who has missed? [ST CATH. COLL. CAMB.]

249. Sum each of the following series to n terms:

(1) $1+0-1+0+7+28+79+\ldots\ldots;$

(2) $-\dfrac{2.2}{1.2.3.4}+\dfrac{1.2^2}{2.3.4.5}+\dfrac{6.2^3}{3.4.5.6}+\dfrac{13.2^4}{4.5.6.7}+\ldots\ldots;$

(3) $3+x+9x^2+x^3+33x^4+x^5+129x^6+\ldots\ldots$

[SECOND PUBLIC EXAM. OX.]

250. Solve the equations:

(1) $\left.\begin{aligned} y^2+yz+z^2&=ax,\\ z^2+zx+x^2&=ay,\\ x^2+xy+y^2&=az.\end{aligned}\right\}$ (2) $\left.\begin{aligned} x(y+z-x)&=a,\\ y(z+x-y)&=b,\\ z(x+y-z)&=c.\end{aligned}\right\}$

[PETERHOUSE, CAMB.]

251. If $\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}=\dfrac{1}{a+b+c}$, and n is an odd integer, shew that

$$\frac{1}{a^n}+\frac{1}{b^n}+\frac{1}{c^n}=\frac{1}{a^n+b^n+c^n}.$$

If $u^6-v^6+5u^2v^2(u^2-v^2)+4uv(1-u^4v^4)=0$, prove that

$$(u^2-v^2)^6=16u^2v^2(1-u^8)(1-v^8).$$ [PEMB. COLL. CAMB.]

252. If $x+y+z=3p$, $yz+zx+xy=3q$, $xyz=r$, prove that

$$(y+z-x)(z+x-y)(x+y-z)=-27p^3+36pq-8r,$$

and $$(y+z-x)^3+(z+x-y)^3+(x+y-z)^3=27p^3-24r.$$

253. Find the factors, linear in x, y, z, of

$$\{a(b+c)x^2+b(c+a)y^2+c(a+b)z^2\}^2-4abc(x^2+y^2+z^2)(ax^2+by^2+cz^2).$$

[CAIUS COLL. CAMB.]

254. Shew that $\left(\dfrac{x^2+y^2+z^2}{x+y+z}\right)^{x+y+z}>x^xy^yz^z>\left(\dfrac{x+y+z}{3}\right)^{x+y+z}.$

[ST JOHN'S COLL. CAMB.]

255. By means of the identity $\left\{1-\dfrac{4x}{(1+x)^2}\right\}^{-\frac{1}{2}}=\dfrac{1+x}{1-x}$, prove that

$$\sum_{r=1}^{r=n}(-1)^{n-r}\frac{(n+r-1)!}{r!\,(r-1)!\,(n-r)!}=1.$$

[PEMB. COLL. CAMB.]

256. Solve the equations:

(1) $ax+by+z=zx+ay+b=yz+bx+a=0.$

(2) $$\left.\begin{aligned} x+y+z-u &= 12,\\ x^2+y^2-z^2-u^2 &= 6,\\ x^3+y^3-z^3+u^3 &= 218,\\ xy+zu &= 45. \end{aligned}\right\}$$

257. If $p=q$ nearly, and $n>1$, shew that

$$\frac{(n+1)p+(n-1)q}{(n-1)p+(n+1)q}=\left(\frac{p}{q}\right)^{\frac{1}{n}}.$$

If $\frac{p}{q}$ agree with unity as far as the r^{th} decimal place, to how many places will this approximation in general be correct? [MATH. TRIPOS.]

258. A lady bought 54 lbs. of tea and coffee; if she had bought five-sixths of the quantity of tea and four-fifths of the quantity of coffee she would have spent nine-elevenths of what she had actually spent; and if she had bought as much tea as she did coffee and *vice-versâ*, she would have spent 5*s.* more than she did. Tea is more expensive than coffee, and the price of 6 lbs. of coffee exceeds that of 2 lbs. of tea by 5*s.*; find the price of each.

259. If s_n represent the sum of the products of the first n natural numbers taken two at a time, then

$$\frac{2}{3\,!}+\frac{11}{4\,!}+\ldots\ldots+\frac{s_{n-1}}{n\,!}+\ldots\ldots=\frac{11}{24}e.$$

[CAIUS COLL. CAMB.]

260. If $\dfrac{P}{pa^2+2qab+rb^2}=\dfrac{Q}{pac+q(bc-a^2)-rab}=\dfrac{R}{pc^2-2qca+ra^2}$,

prove that P, p; Q, q; and R, r may be interchanged without altering the equalities. [MATH. TRIPOS.]

261. If $\alpha+\beta+\gamma=0$, shew that

$$\alpha^{n+3}+\beta^{n+3}+\gamma^{n+3}=\alpha\beta\gamma(\alpha^n+\beta^n+\gamma^n)+\frac{1}{2}(\alpha^2+\beta^2+\gamma^2)(\alpha^{n+1}+\beta^{n+1}+\gamma^{n+1}).$$

[CAIUS COLL. CAMB.]

262. If α, β, γ, δ be the roots of the equation

$$x^4+px^3+qx^2+rx+s=0,$$

find in terms of the coefficients the value of $\Sigma(\alpha-\beta)^2(\gamma-\delta)^2$.

[LONDON UNIVERSITY.]

263. A farmer bought a certain number of turkeys, geese, and ducks, giving for each bird as many shillings as there were birds of that kind; altogether he bought 23 birds and spent £10. 11*s.*; find the number of each kind that he bought.

264. Prove that the equation

$$(y+z-8x)^{\frac{1}{3}}+(z+x-8y)^{\frac{1}{3}}+(x+y-8z)^{\frac{1}{3}}=0,$$

is equivalent to the equation

$$x(y-z)^2+y(z-x)^2+z(x-y)^2=0.$$

[St John's Coll. Camb.]

265. If the equation $\dfrac{a}{x+a}+\dfrac{b}{x+b}=\dfrac{c}{x+c}+\dfrac{d}{x+d}$ have a pair of equal roots, then either one of the quantities a or b is equal to one of the quantities c or d, or else $\dfrac{1}{a}+\dfrac{1}{b}=\dfrac{1}{c}+\dfrac{1}{d}$. Prove also that the roots are then $-a,\ -a,\ 0$; $-b,\ -b,\ 0$; or $0,\ 0,\ -\dfrac{2ab}{a+b}$.

[Math. Tripos.]

266. Solve the equations:

(1) $x+y+z=ab,\ x^{-1}+y^{-1}+z^{-1}=a^{-1}b,\ xyz=a^3.$

(2) $ayz+by+cz=bzx+cz+ax=cxy+ax+by=a+b+c.$

[Second Public Exam. Oxford.]

267. Find the simplest form of the expression

$$\frac{a^3}{(a-\beta)(a-\gamma)(a-\delta)(a-\epsilon)}+\frac{\beta^3}{(\beta-a)(\beta-\gamma)(\beta-\delta)(\beta-\epsilon)}+\dots$$
$$+\frac{\epsilon^3}{(\epsilon-a)(\epsilon-\beta)(\epsilon-\gamma)(\epsilon-\delta)}.$$

[London University.]

268. In a company of Clergymen, Doctors, and Lawyers it is found that the sum of the ages of all present is 2160; their average age is 36; the average age of the Clergymen and Doctors is 39; of the Doctors and Lawyers $32\frac{8}{11}$; of the Clergymen and Lawyers $36\frac{2}{3}$. If each Clergyman had been 1 year, each Lawyer 7 years, and each Doctor 6 years older, their average age would have been greater by 5 years: find the number of each profession present and their average ages.

269. Find the condition, among its coefficients, that the expression

$$a_0x^4+4a_1x^3y+6a_2x^2y^2+4a_3xy^3+a_4y^4$$

should be reducible to the sum of the fourth powers of two linear expressions in x and y. [London University.]

270. Find the real roots of the equations

$$x^2+v^2+w^2=a^2, \quad vw+u(y+z)=bc,$$
$$y^2+w^2+u^2=b^2, \quad wu+v(z+x)=ca,$$
$$z^2+u^2+v^2=c^2, \quad uv+w(x+y)=ab.$$

[MATH. TRIPOS.]

271. It is a rule in Gaelic that no consonant or group of consonants can stand immediately between a strong and a weak vowel; the strong vowels being *a*, *o*, *u*; and the weak vowels *e* and *i*. Shew that the whole number of Gaelic words of $n+3$ letters each, which can be formed of n consonants and the vowels *aeo* is $\dfrac{2\,\underline{|n+3}}{n+2}$ where no letter is repeated in the same word. [CAIUS COLL. CAMB.]

272. Shew that if $x^2+y^2=2z^2$, where x, y, z are integers, then

$$2x=r(l^2+2lk-k^2), \quad 2y=r(k^2+2lk-l^2), \quad 2z=r(l^2+k^2)$$

where r, l, and k are integers. [CAIUS COLL. CAMB.]

273. Find the value of $\dfrac{1}{1+}\;\dfrac{1}{1+}\;\dfrac{2}{3+}\;\dfrac{4}{5+}\;\dfrac{6}{7+}\ldots\ldots$ to inf.

[CHRIST'S COLL. CAMB.]

274. Sum the series:

(1) $\dfrac{x^2}{2\,.\,3}+\dfrac{2x^3}{3\,.\,4}+\dfrac{3x^4}{4\,.\,5}+\ldots\ldots$ to inf.

(2) $\dfrac{\underline{|1}}{a+1}+\dfrac{\underline{|2}}{(a+1)(a+2)}+\ldots\ldots+\dfrac{\underline{|n}}{(a+1)(a+2)\ldots(a+n)}$.

275. Solve the equations:

(1) $2xyz+3=(2x-1)(3y+1)(4z-1)+12$
$$=(2x+1)(3y-1)(4z+1)+80=0.$$

(2) $3ux-2vy=vx+uy=3u^2+2v^2=14$; $xy=10uv$.

276. Shew that
$$\begin{vmatrix} a^2+\lambda & ab & ac & ad \\ ab & b^2+\lambda & bc & bd \\ ac & bc & c^2+\lambda & cd \\ ad & bd & cd & d^2+\lambda \end{vmatrix}$$

is divisible by λ^3 and find the other factor. [CORPUS COLL. CAMB.]

277. If $a, b, c, \ldots$ are the roots of the equation

$$x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots + p_{n-1} x + p_n = 0;$$

find the sum of $a^3 + b^3 + c^3 + \ldots$, and shew that

$$\frac{a^2}{b} + \frac{b^2}{a} + \frac{a^2}{c} + \frac{c^2}{a} + \frac{b^2}{c} + \frac{c^2}{b} + \ldots = p_1 - \frac{p_{n-1}(p_1^2 - 2p_2)}{p_n}.$$

[ST JOHN'S COLL. CAMB.]

278. By the expansion of $\frac{1+2x}{1-x^3}$, or otherwise, prove that

$$1 - 3n + \frac{(3n-1)(3n-2)}{1 \cdot 2} - \frac{(3n-2)(3n-3)(3n-4)}{1 \cdot 2 \cdot 3}$$
$$+ \frac{(3n-3)(3n-4)(3n-5)(3n-6)}{1 \cdot 2 \cdot 3 \cdot 4} - \&c. = (-1)^n,$$

when n is an integer, and the series stops at the first term that vanishes.

[MATH. TRIPOS.]

279. Two sportsmen A and B went out shooting and brought home 10 birds. The sum of the squares of the number of shots was 2880, and the product of the numbers of shots fired by each was 48 times the product of the numbers of birds killed by each. If A had fired as often as B and B as often as A, then B would have killed 5 more birds than A: find the number of birds killed by each.

280. Prove that $8(a^3+b^3+c^3)^2 > 9(a^2+bc)(b^2+ca)(c^2+ab)$.

[PEMB. COLL. CAMB.]

281. Shew that the n^{th} convergent to

$$\frac{2}{3-}\,\frac{4}{4-}\,\frac{6}{5-}\,\ldots \text{ is } 2 - \frac{2^{n+1}}{\Sigma_0^n 2^r (n-r)!}.$$

What is the limit of this when n is infinite? [KING'S COLL. CAMB.]

282. If $\frac{p_n}{q_n}$ is the n^{th} convergent to the continued fraction

$$\frac{1}{a+}\,\frac{1}{b+}\,\frac{1}{c+}\,\frac{1}{a+}\,\frac{1}{b+}\,\frac{1}{c+}\,\ldots$$

shew that $p_{3n+3} = bp_{3n} + (bc+1)\,q_{3n}$. [QUEENS' COLL. CAMB.]

283. Out of n straight lines whose lengths are 1, 2, 3, ... n inches respectively, the number of ways in which four may be chosen which will form a quadrilateral in which a circle may be inscribed is

$$\frac{1}{48}\{2n(n-2)(2n-5) - 3 + 3(-1)^n\}.$$

[MATH. TRIPOS.]

284. If u_2, u_3 are respectively the arithmetic means of the squares and cubes of all numbers less than n and prime to it, prove that $n^3-6nu_2+4u_3=0$, unity being counted as a prime.

[St John's Coll. Camb.]

285. If n is of the form $6m-1$ shew that $(y-z)^n+(z-x)^n+(x-y)^n$ is divisible by $x^2+y^2+z^2-yz-zx-xy$; and if n is of the form $6m+1$, shew that it is divisible by

$$(x^2+y^2+z^2-yz-zx-xy)^2.$$

286. If S is the sum of the m^{th} powers, P the sum of the products m together of the n quantities $a_1, a_2, a_3, \dots a_n$, shew that

$$\underline{|n-1} \,.\, S > \underline{|n-m} \,.\, \underline{|m} \,.\, P.$$

[Caius Coll. Camb.]

287. Prove that if the equations

$$x^3+qx-r=0 \text{ and } rx^3-2q^2x^2-5qrx-2q^3-r^2=0$$

have a common root, the first equation will have a pair of equal roots; and if each of these is a, find all the roots of the second equation.

[India Civil Service.]

288. If $\quad x\sqrt{2a^2-3x^2}+y\sqrt{2a^2-3y^2}+z\sqrt{2a^2-3z^2}=0,$

where a^2 stands for $x^2+y^2+z^2$, prove that

$$(x+y+z)(-x+y+z)(x-y+z)(x+y-z)=0.$$

[Trin. Coll. Camb.]

289. Find the values of $x_1, x_2, \dots x_n$ which satisfy the following system of simultaneous equations:

$$\frac{x_1}{a_1-b_1}+\frac{x_2}{a_1-b_2}+\dots+\frac{x_n}{a_1-b_n}=1,$$

$$\frac{x_1}{a_2-b_1}+\frac{x_2}{a_2-b_2}+\dots+\frac{x_n}{a_2-b_n}=1,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\frac{x_1}{a_n-b_1}+\frac{x_2}{a_n-b_2}+\dots+\frac{x_n}{a_n-b_n}=1.$$

[London University.]

290. Shew that $\begin{vmatrix} yz-x^2 & zx-y^2 & xy-z^2 \\ zx-y^2 & xy-z^2 & yz-x^2 \\ xy-z^2 & yz-x^2 & zx-y^2 \end{vmatrix} = \begin{vmatrix} r^2 & u^2 & u^2 \\ u^2 & r^2 & u^2 \\ u^2 & u^2 & r^2 \end{vmatrix},$

where $r^2=x^2+y^2+z^2$, and $u^2=yz+zx+xy$.

[Trin. Coll. Camb.]

291. A piece of work was done by A, B, C; at first A worked alone, but after some days was joined by B, and these two after some days were joined by C. The whole work could have been done by B and C, if they had each worked twice the number of days that they actually did. The work could also have been completed without B's help if A had worked two-thirds and C four times the number of days they actually did; or if A and B had worked together for 40 days without C; or if all three had worked together for the time that B had worked. The number of days that elapsed before B began to work was to the number that elapsed before C began to work as 3 to 5: find the number of days that each man worked.

292. Shew that if S_r is the sum of the products r together of

$$1,\ x,\ x^2,\ x^3, \dots x^{n-1},$$

then

$$S_{n-r} = S_r \,.\, x^{\frac{1}{2}(n-1)(n-2r)}.$$

[St John's Coll. Camb.]

293. If a, b, c are positive and the sum of any two greater than the third, prove that

$$\left(1+\frac{b-c}{a}\right)^a \left(1+\frac{c-a}{b}\right)^b \left(1+\frac{a-b}{c}\right)^c < 1.$$

[St John's Coll. Camb.]

294. Resolve into factors

$$(a+b+c)(b+c-a)(c+a-b)(a+b-c)(a^2+b^2+c^2)-8a^2b^2c^2.$$

Prove that

$$4\{\alpha^4+\beta^4+\gamma^4+(\alpha+\beta+\gamma)^4\} = (\beta+\gamma)^4+(\gamma+\alpha)^4+(\alpha+\beta)^4$$
$$+6(\beta+\gamma)^2(\gamma+\alpha)^2+6(\gamma+\alpha)^2(\alpha+\beta)^2+6(\alpha+\beta)^2(\beta+\gamma)^2.$$

[Jesus Coll. Camb.]

295. Prove that the sum of the homogeneous products of r dimensions of the numbers 1, 2, 3, ... n, and their powers is

$$\frac{(-1)^{n-1}}{\underline{|n-1}}\left\{1^{n+r-1}-\frac{n-1}{1}\,.\,2^{n+r-1}+\frac{(n-1)(n-2)}{1\,.\,2}\,.\,3^{n+r-1}-\dots \text{ to } n \text{ terms}\right\}.$$

[Emm. Coll. Camb.]

296. Prove that, if n be a positive integer,

$$1-3n+\frac{3n(3n-3)}{1\,.\,2}-\frac{3n(3n-4)(3n-5)}{1\,.\,2\,.\,3}+\dots = 2(-1)^n.$$

[Oxford Mods.]

297. If $x(2a-y)=y(2a-z)=z(2a-u)=u(2a-x)=b^2$, shew that $x=y=z=u$ unless $b^2=2a^2$, and that if this condition is satisfied the equations are not independent. [Math. Tripos.]

298. Shew that if a, b, c are positive and unequal, the equations

$$ax+yz+z=0, \quad zx+by+z=0, \quad yz+zx+c=0,$$

give three distinct triads of real values for x, y, z; and the ratio of the products of the three values of x and y is $b(b-c) : a(c-a)$.

[OXFORD MODS.]

299. If

$$A=ax-by-cz, \quad D=bz+cy,$$
$$B=by-cz-ax, \quad E=cx+az,$$
$$C=cz-ax-by, \quad F=ay+bx,$$

prove that $ABC-AD^2-BE^2-CF^2+2DEF$

$$=(a^2+b^2+c^2)(ax+by+cz)(x^2+y^2+z^2).$$

[SECOND PUBLIC EXAM. OXFORD.]

300. A certain student found it necessary to decipher an old manuscript. During previous experiences of the same kind he had observed that the number of words he could read daily varied jointly as the number of miles he walked and the number of hours he worked during the day. He therefore gradually increased the amount of daily exercise and daily work at the rate of 1 mile and 1 hour per day respectively, beginning the first day with his usual quantity. He found that the manuscript contained 232000 words, that he counted 12000 on the first day, and 72000 on the last day; and that by the end of half the time he had counted 62000 words: find his usual amount of daily exercise and work.

ANSWERS.

I. Pages 10—12.

1. (1) $54b : a$. (2) $9 : 7$. (3) $bx : ay$. 2. 18. 3. 385, 660.
4. 11. 5. $5 : 13$. 6. $5 : 6$ or $-3 : 5$.
10. $\frac{x}{4} = \frac{y}{2} = \frac{z}{3}$, or $\frac{x}{1} = \frac{y}{-1} = \frac{z}{0}$. 17. $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$.
20. 3, 4, 1. 21. -3, 4, 1. 22. 7, 3, 2. 23. 3, 4, 1.
25. $\pm a(b^2 - c^2)$, $\pm b(c^2 - a^2)$, $\pm c(a^2 - b^2)$.
26. $bc(b - c)$, $ca(c - a)$, $ab(a - b)$.

II. Pages 19, 20.

1. 45. 2. (1) 12. (2) $300a^3b$. 3. $\dfrac{x^3}{y(x^2 + y^2)}$.
13. 0, 5, $\frac{8}{7}$. 14. 0, 3, 8. 15. $\dfrac{a(b + c)}{cm - bm - 2an}$.
18. 8. 19. 6, 9, 10, 15. 20. 3 gallons from A; 8 gallons from B.
21. 45 gallons. 23. $17 : 3$. 24. $a = 4b$.
25. 64 per cent. copper and 36 per cent. zinc. 3 parts of brass are taken to 5 parts of bronze. 26. 63 or 12 minutes.

III. Pages 26, 27.

1. $5\frac{1}{3}$. 2. 9. 3. $1\frac{1}{3}$. 4. 2. 7. 60.
9. $y = 2x - \frac{8}{x}$. 10. $y = 5x + \frac{36}{x^2}$. 11. 4.
12. $x = \frac{22}{15}z + \frac{2}{15z}$. 14. 36. 15. 1610 feet; 305·9 feet.
16. $22\frac{1}{24}$ cubic feet. 17. $4 : 3$.
18. The regatta lasted 6 days; 4th, 5th, 6th days.
20. 16, 25 years; £200, £250. 21. 1 day 18 hours 28 minutes.
22. The cost is least when the rate is 12 miles an hour; and then the cost per mile is £$\frac{3}{32}$, and for the journey is £9. 7*s*. 6*d*.

IV. a. Pages 31, 32.

1. $277\frac{1}{2}$. **2.** 153. **3.** 0. **4.** $\frac{n(10-n)}{3}$. **5.** 30.
6. -42. **7.** -185. **8.** $1325\sqrt{3}$. **9.** $75\sqrt{5}$.
10. $820a-1680b$. **11.** $n(n+1)a-n^2b$. **12.** $\frac{21}{2}(11a-9b)$.
13. $-\frac{1}{4}, -\frac{3}{4}, \ldots, -9\frac{1}{4}$. **14.** $1, -1\frac{1}{2}, \ldots, -39$. **15.** $-33x, -31x, \ldots, x$.
16. $x^2-x+1, x^2-2x+2, \ldots, x$. **17.** n^2. **18.** 3. **19.** 5.
20. 612. **21.** 4, 9, 14. **22.** 1, 4, 7. **23.** 495. **24.** 160.
25. $\frac{p(p+1)}{2a}+pb$. **26.** $n(n+1)a-\frac{n^2}{a}$.

IV. b. Pages 35, 36.

1. 10 or -8. **2.** 8 or -13. **3.** 2, 5, 8,...
4. First term 8, number of terms 59.
5. First term $7\frac{1}{2}$, number of terms 54.
6. Instalments £51, £53, £55,... **7.** 12. **8.** 25.
9. $\frac{n}{2(1-x)}(2+\overline{n-3}\,.\sqrt{x})$. **10.** n^2. **12.** $-(p+q)$.
13. 3, 5, 7, 9. [Assume for the numbers $a-3d$, $a-d$, $a+d$, $a+3d$.]
14. 2, 4, 6, 8. **15.** $p+q-m$. **16.** 12 or -17. **17.** $6r-1$.
20. $10p-8$. **21.** 8 terms. Series $1\frac{1}{2}, 3, 4\frac{1}{2}, \ldots\ldots$
22. 3, 5, 7; 4, 5, 6. **23.** $ry=(n+1-r)x$.

V. a. Pages 41, 42.

1. $\frac{2059}{1458}$. **2.** $\frac{1281}{512}$. **3.** $191\frac{1}{4}$. **4.** -682.
5. $\frac{1093}{45}$. **6.** $\frac{1}{4}(5^p-1)$. **7.** $\frac{9}{7}\left\{1-\left(\frac{4}{3}\right)^{2n}\right\}$. **8.** $364(\sqrt{3}+1)$.
9. $\frac{1}{2}(585\sqrt{2}-292)$. **10.** $-\frac{463}{192}$. **11.** $\frac{3}{2}, 1, \frac{2}{3}$.
12. $\frac{16}{3}, 8, \ldots, 27$. **13.** $-7, \frac{7}{2}, \ldots, \frac{7}{32}$. **14.** $\frac{64}{65}$.
15. $\frac{27}{58}$. **16.** $\cdot 999$. **17.** $\frac{1}{2}$. **18.** $\frac{3(3+\sqrt{3})}{2}$.
19. $7(7+\sqrt{42})$. **20.** 2. **21.** 16, 24, 36,... **22.** 2.
23. 2. **24.** 8, 12, 18. **25.** 2, 6, 18. **28.** $6, -3, 1\frac{1}{2}, \ldots\ldots$

V. b. Pages 45, 46.

1. $\frac{1-a^n}{(1-a)^2}-\frac{na^n}{1-a}$. 2. $\frac{8}{3}$. 3. $\frac{1+x}{(1-x)^2}$.

4. $4-\frac{1}{2^{n-2}}-\frac{n}{2^{n-1}}$. 5. 6. 6. $\frac{1}{(1-x)^3}$.

9. $\frac{1}{(1-r)(1-br)}$. 10. 40, 20, 10. 11. $4, 1, \frac{1}{4}, \ldots$

12. $\frac{x(x^n-1)}{x-1}+\frac{n(n+1)a}{2}$. 13. $\frac{x^2(x^{2n}-1)}{x^2-1}+\frac{xy(x^ny^n-1)}{xy-1}$.

14. $4p^2a+\frac{2}{9}\left(1-\frac{1}{2^{2p}}\right)$. 15. $1\frac{1}{8}$. 16. $\frac{23}{48}$. 19. $n \cdot 2^{n+2}-2^{n+1}+2$.

20. $\frac{(1+a)(a^nc^n-1)}{ac-1}$. 21. $\frac{a}{r-1}\left\{\frac{r(r^{2n}-1)}{r^2-1}-n\right\}$.

VI. a. Pages 52, 53.

1. (1) 5. (2) $3\frac{1}{2}$. (3) $3\frac{23}{32}$. 2. $6\frac{1}{9}$, $7\frac{6}{7}$. 3. $\frac{2}{5}, \frac{2}{7}, \frac{2}{9}, \frac{2}{11}$.

4. 6 and 24. 5. $4:9$. 10. $n^2(n+1)$.

11. $\frac{1}{4}n(n+1)(n^2+n+3)$. 12. $\frac{1}{6}n(n+1)(2n+7)$.

13. $\frac{1}{2}n(n+1)(n^2+3n+1)$. 14. $\frac{1}{2}(3^{n+1}+1)-2^{n+1}$.

15. $4^{n+1}-4-n(n+1)(n^2-n-1)$.

18. The n^{th} term $=b+c(2n-1)$, for all values of n greater than 1. The first term is $a+b+c$; the other terms form the A.P. $b+3c$, $b+5c$, $b+7c, \ldots$

19. n^4. 22. $\frac{n}{2}(2a+\overline{n-1}d)\left\{a^2+(n-1)ad+\frac{n(n-1)}{2}d^2\right\}$.

VI. b. Page 56.

1. 1240. 2. 1140. 3. 16646. 4. 2470. 5. 21321.
6. 52. 7. 11879. 8. 1840. 9. 11940. 10. 190.
11. 300. 12. 18296. 14. Triangular 364; Square 4900.
15. 120. 16. $n-1$.

VII. a. Page 59.

1. 333244. 2. 728626. 3. 1740137. 4. *e*7074. 5. 112022.
6. 334345. 7. 17832126. 8. 1625. 9. 2012. 10. 342.
11. *ttt*90001. 12. 231. 13. 1456. 14. 7071. 15. *eee*.
16. (1) 121. (2) 122000.

VII. b. Pages 65, 66.

1. 20305. 2. 4444. 3. 11001110. 4. 2000000. 5. 7338.
6. 34402. 7. 6587. 8. 8978. 9. 26011. 10. 37214.

11. 30034342. **12.** 710*te*3. **13.** 2714687. **14.** $\cdot\dot{2}04\dot{6}$. **15.** 15·1*t*6.

16. 20·73. **17.** $125\cdot01\dot{2}\dot{5}$. **18.** $\frac{5}{8}$. **19.** $\frac{2}{3}$, $\frac{5}{8}$.

20. Nine. **21.** Four. **22.** Twelve. **23.** Eight. **24.** Eleven.

25. Twelve. **26.** Ten. **30.** $2^{11}+2^7+2^6$.

31. $3^9-3^8-3^7-3^6-3^5+3^3+3^2+1$.

VIII. a. Pages 72, 73.

1. $\dfrac{2+\sqrt{2}+\sqrt{6}}{4}$. **2.** $\dfrac{3+\sqrt{6}+\sqrt{15}}{6}$.

3. $\dfrac{a\sqrt{b}+b\sqrt{a}-\sqrt{ab\,(a+b)}}{2ab}$. **4.** $\dfrac{a-1+\sqrt{a^2-1}+\sqrt{2a\,(a-1)}}{a-1}$.

5. $\dfrac{3\sqrt{30}+5\sqrt{15}-12-10\sqrt{2}}{7}$. **6.** $\dfrac{\sqrt{2}+\sqrt{3}+\sqrt{5}}{2}$.

7. $3^{\frac{5}{3}}+3^{\frac{4}{3}}\cdot2^{\frac{1}{2}}+3\cdot2+3^{\frac{2}{3}}\cdot2^{\frac{3}{2}}+3^{\frac{1}{3}}\cdot2^2+2^{\frac{5}{2}}$.

8. $5^{\frac{5}{6}}-5^{\frac{4}{6}}\cdot2^{\frac{1}{3}}+5^{\frac{3}{6}}\cdot2^{\frac{2}{3}}-5^{\frac{2}{6}}\cdot2+5^{\frac{1}{6}}\cdot2^{\frac{4}{3}}-2^{\frac{5}{3}}$.

9. $a^{\frac{11}{6}}-a^{\frac{10}{6}}b^{\frac{1}{4}}+a^{\frac{9}{6}}b^{\frac{1}{2}}-\ldots+a^{\frac{1}{6}}b^{\frac{10}{4}}-b^{\frac{11}{4}}$. **10.** $3^{\frac{2}{3}}+3^{\frac{1}{3}}+1$.

11. $2^3-2^2\cdot7^{\frac{1}{4}}+2\cdot7^{\frac{1}{2}}-7^{\frac{3}{4}}$.

12. $5^{\frac{11}{3}}+5^{\frac{10}{3}}\cdot3^{\frac{1}{4}}+5^3\cdot3^{\frac{2}{4}}+\ldots+5^{\frac{1}{3}}\cdot3^{\frac{10}{4}}+3^{\frac{11}{4}}$. **13.** $\dfrac{1-3^{\frac{2}{3}}+3^{\frac{1}{3}}}{2}$.

14. $17-3^{\frac{5}{3}}\cdot2^{\frac{3}{2}}+3^{\frac{4}{3}}\cdot2^2-3\cdot2^{\frac{5}{2}}+3^{\frac{2}{3}}\cdot2^3-3^{\frac{1}{3}}\cdot2^{\frac{7}{2}}$.

15. $3^2\cdot2^{\frac{1}{2}}-3^{\frac{5}{3}}\cdot2+3^{\frac{4}{3}}\cdot2^{\frac{3}{2}}-3\cdot2^2+3^{\frac{2}{3}}\cdot2^{\frac{5}{2}}-3^{\frac{1}{3}}\cdot2^3$.

16. $\dfrac{1}{2}\left(3^{\frac{5}{6}}-3^{\frac{4}{6}}+3^{\frac{3}{6}}-3^{\frac{2}{6}}+3^{\frac{1}{6}}-1\right)$. **17.** $\dfrac{2^5+2^{\frac{31}{6}}+2^{\frac{26}{6}}+2^{\frac{21}{6}}+2^{\frac{16}{6}}+2^{\frac{11}{6}}+1}{31}$.

18. $\dfrac{3^{\frac{3}{2}}+3^{\frac{5}{6}}+3^{\frac{1}{6}}}{8}$. **19.** $\sqrt{5}+\sqrt{7}-2$.

20. $\sqrt{5}-\sqrt{7}+2\sqrt{3}$. **21.** $1+\sqrt{3}-\sqrt{2}$. **22.** $1+\sqrt{\dfrac{3}{2}}-\sqrt{\dfrac{5}{2}}$.

23. $2+\sqrt{a}-\sqrt{3b}$. **24.** $3-\sqrt{7}+\sqrt{2}-\sqrt{3}$. **25.** $1+\sqrt{3}$.

26. $2+\sqrt{5}$. **27.** $3-2\sqrt{2}$. **28.** $\sqrt{14}-2\sqrt{2}$.

29. $2\sqrt{3}+\sqrt{5}$. **30.** $3\sqrt{3}-\sqrt{6}$. **31.** $\sqrt{\dfrac{2a+x}{2}}+\sqrt{\dfrac{x}{2}}$.

32. $\sqrt{\dfrac{3a+b}{2}}-\sqrt{\dfrac{a-b}{2}}$. **33.** $\sqrt{\dfrac{1+a+a^2}{2}}+\sqrt{\dfrac{1-a+a^2}{2}}$.

34. $\dfrac{1}{\sqrt[4]{1-a^2}}\left(\sqrt{\dfrac{1+a}{2}}+\sqrt{\dfrac{1-a}{2}}\right)$.

35. $11+56\sqrt{3}$. **36.** 289. **37.** $\frac{1}{3}\sqrt{3}$.

38. $3\sqrt{3}+5$. 39. 3. 40. $8\sqrt{3}$.

41. $3+\sqrt{5}=5\cdot23607$. 42. $x^2+1+\sqrt[3]{4}+x-x\sqrt[3]{2}+\sqrt[3]{2}$.

43. $3a+\sqrt{b^2-3a^2}$. 44. $\dfrac{a-1}{2}$.

VIII. b. Pages 81, 82.

1. $6-2\sqrt{6}$. 2. -13. 3. $e^{2\sqrt{-1}}-e^{-2\sqrt{-1}}$.

4. x^2-x+1. 5. $\dfrac{3+\sqrt{-2}}{11}$. 6. $-19-6\sqrt{10}$.

7. $-\dfrac{8}{29}$. 8. $\dfrac{4ax\sqrt{-1}}{a^2+x^2}$. 9. $\dfrac{2(3x^2-1)\sqrt{-1}}{x^2+1}$.

10. $\dfrac{3a^2-1}{2a}$. 11. $\sqrt{-1}$. 12. 100.

13. $\pm(2+3\sqrt{-1})$. 14. $\pm(5-6\sqrt{-1})$. 15. $\pm(1+4\sqrt{-3})$.

16. $\pm2(1-\sqrt{-1})$. 17. $\pm(a+\sqrt{-1})$. 18. $\pm\{(a+b)-(a-b)\sqrt{-1}\}$.

19. $-\dfrac{9}{13}+\dfrac{19}{13}i$. 20. $\dfrac{4}{7}-\dfrac{\sqrt{6}}{14}i$. 21. i.

22. $-\dfrac{1}{5}+\dfrac{3}{5}i$. 23. $\dfrac{2b(3a^2-b^2)}{a^2+b^2}i$.

IX. a. Pages 88—90.

1. $35x^2+13x-12=0$. 2. $mnx^2+(n^2-m^2)x-mn=0$.

3. $(p^2-q^2)x^2+4pqx-p^2+q^2=0$. 4. $x^2-14x+29=0$.

5. $x^2+10x+13=0$. 6. $x^2+2px+p^2-8q=0$.

7. $x^2+6x+34=0$. 8. $x^2+2ax+a^2+b^2=0$.

9. $x^2+a^2-\;ab+b^2=0$. 10. $6x^3+11x^2-19x+6=0$.

11. $2ax^3+(4-a^2)x^2-2ax=0$. 12. $x^3-8x^2+17x-4=0$.

14. $3, 5$. 15. $2, -\dfrac{10}{9}$. 16. $\dfrac{a-b}{a+b}$.

18. $\dfrac{b^2-2ac}{c^2}$. 19. $\dfrac{bc^4(3ac-b^2)}{a^7}$. 20. $\dfrac{b^2(b^2-4ac)}{a^2c^2}$.

21. 7. 22. -15. 23. 0. 24. $x^2-2(p^2-2q)x+p^2(p^2-4q)=0$.

26. (1) $\dfrac{b^2-2ac}{a^2c^2}$. (2) $\dfrac{b(b^2-3ac)}{a^3c^3}$. 27. $nb^2=(1+n)^2ac$.

28. $a^2c^2x^2-(b^2-2ac)(a^2+c^2)x+(b^2-2ac)^2=0$.

29. $x^2-4mnx-(m^2-n^2)^2=0$.

IX. b. Pages 92, 93.

1. 2 and -2. 5. $bx^2-2ax+a=0$.

6. (1) $\dfrac{p(p^2-4q)(p^2-q)}{q}$. (2) $\dfrac{p^4-4p^2q+2q^2}{q^4}$. 11. $\dfrac{1}{3}$.

IX. c. Page 96.

1. -2. **2.** ± 7. **5.** $(ln' - l'n)^2 = (lm' - l'm)(mn' - m'n)$.
7. $(aa' - bb')^2 + 4(ha' + h'b)(hb' + h'a) = 0$.
10. $(bb' - 2ac' - 2a'c)^2 = (b^2 - 4ac)(b'^2 - 4a'c')$; which reduces to $(ac' - a'c)^2 = (ab' - a'b)(bc' - b'c)$.

X. a. Pages 101, 102.

1. $\frac{1}{4}, -\frac{1}{2}$. **2.** $\pm\frac{1}{3}, \pm 1$. **3.** $4, \frac{1}{4}$. **4.** $\frac{4}{9}, \frac{1}{4}$.
5. $3^n, 2^n$. **6.** $1, 2^{2n}$. **7.** $27, \frac{25}{147}$. **8.** $\frac{9}{13}, \frac{4}{13}$.
9. $\frac{1}{9}, \frac{25}{4}$. **10.** $-1, -\frac{1}{32}$. **11.** $2, 0$. **12.** ± 1.
13. -4. **14.** ± 3. **15.** 0. **16.** $\frac{1}{8}, 450$.
17. $9, -7, 1 \pm \sqrt{-24}$. **18.** $2, -4, -1 \pm \sqrt{71}$. **19.** $3, -\frac{3}{2}, \frac{3 \pm \sqrt{-47}}{4}$.
20. $4, -\frac{7}{2}, \frac{1 \pm \sqrt{65}}{4}$. **21.** $2, -8, -3 \pm 3\sqrt{5}$. **22.** $3, -\frac{5}{3}, \frac{2 \pm \sqrt{70}}{3}$.
23. $5, \frac{1}{3}, \frac{8 \pm \sqrt{148}}{3}$. **24.** $7, -\frac{14}{3}, \frac{7 \pm \sqrt{37}}{6}$. **25.** $2, \frac{1}{2}, \frac{5 \pm \sqrt{201}}{4}$.
26. $5, -\frac{7}{3}, \frac{8 \pm \sqrt{415}}{6}$. **27.** $1, 3$. **28.** $5, \frac{1}{2}$.
29. $1, 9, -\frac{18}{5}$. **30.** $a, \frac{a}{2}, -\frac{a}{3}$. **31.** $2, -\frac{9}{2}$.
32. $4, -\frac{10}{3}$. **33.** $0, 5$. **34.** $6, -\frac{5}{2}$.
35. $1, \frac{-3 \pm \sqrt{5}}{2}$. **36.** $3, \frac{1}{3}, \frac{-1 \pm \sqrt{-35}}{6}$. **37.** $2 \pm \sqrt{3}, \frac{-1 \pm \sqrt{-3}}{2}$.
38. $2, -\frac{1}{2}, 5, -\frac{1}{5}$. **39.** $3a, -4a$. **40.** $\pm\frac{2a}{5}$.
41. $0, 1, 3$. **42.** $\frac{-1 \pm \sqrt{17}}{2}, \frac{-1 \pm \sqrt{2}}{2}$.
43. $\frac{3}{2}, \frac{2}{3}$. **44.** $3, -1$. **45.** ± 1.
46. 13. **47.** 4. **48.** $0, \frac{63a}{65}$.
49. $1, \frac{(\sqrt{a} - \sqrt{b})^2 + 4}{(\sqrt{a} + \sqrt{b})^2 - 4}$. **50.** ± 5. **51.** $5, -4, \frac{1 \pm \sqrt{-75}}{2}$.
52. $-\frac{1}{3}, \frac{1 \pm \sqrt{-31}}{6}$.

X. b. Pages 106, 107.

1. $x=5,\ -\frac{8}{3};\ y=4,\ -\frac{15}{2}$.

2. $x=2,\ -\frac{8}{19};\ y=7,\ -\frac{97}{19}$.

3. $x=1,\ -\frac{53}{88};\ y=1,\ -\frac{25}{22}$.

4. $x=\pm 5,\ \pm 3;\ y=\pm 3,\ \pm 5$.

5. $x=8,\ 2;\ y=2,\ 8$.

6. $x=45,\ 5;\ y=5,\ 45$.

7. $x=9,\ 4;\ y=4,\ 9$.

8. $x=\pm 2,\ \pm 3;\ y=\pm 1,\ \pm 2$.

9. $x=\pm 2,\ \pm 3;\ y=\pm 3,\ \pm 4$.

10. $x=\pm 5,\ \pm 3;\ y=\pm 3,\ \pm 4$.

11. $x=\pm 2,\ \pm 1;\ y=\pm 1,\ \pm 3$.

12. $x=\pm\sqrt{3},\ \pm\sqrt{\frac{3}{19}};\ y=0,\ \pm 6\sqrt{\frac{3}{19}}$.

13. $x=5,\ 3,\ 4\pm\sqrt{-97};\ y=3,\ 5,\ 4\mp\sqrt{-97}$.

14. $x=4,\ -2,\ \pm\sqrt{-15}+1;\ y=2,\ -4,\ \pm\sqrt{-15}-1$.

15. $x=4,\ -2,\ \pm\sqrt{-11}+1;\ y=2,\ -4,\ \pm\sqrt{-11}-1$.

16. $x=\frac{4}{5},\ \frac{1}{5};\ y=20,\ 5$.

17. $x=2,\ 1;\ y=1,\ 2$.

18. $x=6,\ 4;\ y=10,\ 15$.

19. $x=729,\ 343;\ y=343,\ 729$.

20. $x=16,\ 1;\ y=1,\ 16$.

21. $x=9,\ 4;\ y=4,\ 9$.

22. $x=5;\ y=\pm 4$.

23. $x=1,\ \frac{5}{3};\ y=2,\ \frac{2}{3}$.

24. $x=9,\ 1;\ y=1,\ 9$.

25. $x=\pm 25;\ y=\pm 9$.

26. $x=6,\ 2,\ 4,\ 3;\ y=1,\ 3,\ \frac{3}{2},\ 2$.

27. $x=\pm 5,\ \pm 4,\ \pm\frac{5}{2},\ \pm 2;\ y=\pm 5,\ \pm 4,\ \pm 10,\ \pm 8$.

28. $x=4,\ \frac{107}{13};\ y=1,\ \frac{48}{13}$.

29. $x=-6,\ \frac{1\pm\sqrt{-143}}{2};\ y=-3,\ \frac{1\pm 3\sqrt{-143}}{4}$.

30. $x=0,\ 9,\ 3;\ y=0,\ 3,\ 9$.

31. $x=0,\ 1,\ \frac{15}{22};\ y=0,\ 2,\ \frac{9}{22}$.

32. $x=5,\ \frac{10}{23},\ 0;\ y=3,\ -\frac{6}{23},\ -\frac{4}{7}$.

33. $x=2,\ \sqrt[3]{4},\ 2;\ y=2,\ 2\sqrt[3]{4},\ 6$.

34. $x=1,\ \sqrt[3]{\frac{1}{2}};\ y=2,\ 3\sqrt[3]{\frac{1}{2}}$.

35. $x=\pm 3,\ \pm\sqrt{-18};\ y=\pm 3,\ \mp\sqrt{-18}$.

36. $x=y=\pm 2$.

37. $x=0,\ \frac{b\sqrt{a}}{\sqrt{a}+\sqrt{b}},\ \frac{b\sqrt{a}}{\sqrt{a}-\sqrt{b}};\ y=0,\ \frac{a\sqrt{b}}{\sqrt{a}+\sqrt{b}},\ -\frac{a\sqrt{b}}{\sqrt{a}-\sqrt{b}}$.

38. $x=b,\ \frac{b(-1\pm\sqrt{3})}{2};\ y=a,\ a(1\mp\sqrt{3})$.

39. $x=\frac{a^2}{b},\ \frac{a(2b-a)}{b};\ y=\frac{b^2}{a},\ \frac{b(2a-b)}{a}$.

40. $x=0,\ \pm a\sqrt{7},\ \pm a\sqrt{13},\ \pm 3a,\ \pm a;\ y=0,\ \mp b\sqrt{7},\ \pm b\sqrt{13},\ \mp b,\ \mp 3b.$

41. $x=\pm 1,\ \pm\dfrac{2a^2}{\sqrt{16a^4-a^2-1}};\ y=\pm 2a,\ \mp\dfrac{a}{\sqrt{16a^4-a^2-1}}.$

X. c. Pages 109, 110.

1. $x=\pm 3;\ y=\pm 5;\ z=\pm 4.$ **2.** $x=5;\ y=-1;\ z=7.$

3. $x=5,\ -1;\ y=1,\ -5;\ z=2.$ **4.** $x=8,\ -3;\ y=3;\ z=3,\ -8.$

5. $x=4,\ 3,\ \dfrac{2\pm\sqrt{151}}{3};\ y=3,\ 4,\ \dfrac{2\mp\sqrt{151}}{3};\ z=2,\ -\dfrac{11}{3}.$

6. $x=\pm 3;\ y=\mp 2;\ z=\pm 5.$ **7.** $x=\pm 5;\ y=\pm 1;\ z=\pm 1.$

8. $x=8,\ -8;\ y=5,\ -5;\ z=3,\ -3.$ **9.** $x=3;\ y=4;\ z=\dfrac{1}{2};\ u=\dfrac{1}{3}.$

10. $x=1;\ y=2;\ z=3.$ **11.** $x=5,\ -7;\ y=3,\ -5;\ z=6,\ -8.$

12. $x=1,\ -2;\ y=7,\ -3;\ z=3,\ -\dfrac{11}{3}.$

13. $x=4,\ \dfrac{60}{7};\ y=6,\ \dfrac{66}{7};\ z=2,\ -6.$ **14.** $x=a,\ 0,\ 0;\ y=0,\ a,\ 0;\ z=0,\ 0,\ a.$

15. $x=\dfrac{a}{\sqrt{3}},\ \dfrac{\sqrt{3}\pm\sqrt{-9}}{6}a;\ y=\dfrac{a}{\sqrt{3}},\ \dfrac{-5\sqrt{3}\pm\sqrt{-9}}{6}a;$

$z=\dfrac{a}{\sqrt{3}},\ -\dfrac{\sqrt{3}\pm\sqrt{-9}}{3}a.$

16. $x=a,\ -2a,\ \dfrac{7\pm\sqrt{-15}}{2}a;\ y=4a,\ a,\ \dfrac{-11\pm\sqrt{-15}}{2}a;$

$z=2a,\ -4a,\ (1\pm\sqrt{-15})\,a.$

X. d. Page 113.

1. $x=29,\ 21,\ 13,\ 5;\ y=2,\ 5,\ 8,\ 11.$

2. $x=1,\ 3,\ 5,\ 7,\ 9;\ y=24,\ 19,\ 14,\ 9,\ 4.$

3. $x=20,\ 8;\ y=1,\ 8.$ **4.** $x=9,\ 20,\ 31;\ y=27,\ 14,\ 1.$

5. $x=30,\ 5;\ y=9,\ 32.$ **6.** $x=50,\ 3;\ y=3,\ 44.$

7. $x=7p-5,\ 2;\ y=5p-4,\ 1.$ **8.** $x=13p-2,\ 11;\ y=6p-1,\ 5.$

9. $x=21p-9,\ 12;\ y=8p-5,\ 3.$ **10.** $x=17p,\ 17;\ y=13p,\ 13.$

11. $x=19p-16,\ 3;\ y=23p-19,\ 4.$ **12.** $x=77p-74,\ 3;\ y=30p-25,\ 5.$

13. 11 horses, 15 cows. **14.** 101. **15.** 56, 25 or 16, 65.

16. To pay 3 guineas and receive 21 half-crowns.

17. 1147; an infinite number of the form $1147+39\times 56p$.

18. To pay 17 florins and receive 3 half-crowns.

19. 37, 99; 77, 59; 117, 19.

20. 28 rams, 1 pig, 11 oxen; or 13 rams, 14 pigs, 13 oxen.

21. 3 sovereigns, 11 half-crowns, 13 shillings.

XI. a. Pages 122—124.

1. 12. **2.** 224. **3.** 40320, 6375600, 10626, 11628.
4. 6720. **5.** 15. **6.** 40320; 720. **7.** 15, 360.
8. 6. **9.** 120. **10.** 720. **11.** 10626, 1771.
12. 1440. **13.** 6375600. **14.** 360, 144. **15.** 230300.
16. 1140, 231. **17.** 144. **18.** 224, 896. **19.** 848.
20. 56. **21.** 360000. **22.** 2052000. **23.** 369600.
24. 21600. **25.** $\dfrac{\lfloor 15}{\lfloor 10\,\lfloor 15\,\lfloor 20}$. **26.** 2520. **27.** 5760.
28. 3456. **29.** 2903040. **30.** 25920. **32.** 41.
33. 1956. **34.** 7.

XI. b. Pages 131, 132.

1. (1) 1663200. (2) 129729600. (3) 3326400. **2.** 4084080.
3. 151351200. **4.** 360. **5.** 72. **6.** 125.
7. n^r. **8.** 531441. **9.** p^n. **10.** 30.
11. 1260. **12.** 3374. **13.** 455. **14.** $\dfrac{\lfloor a+2b+3c+d}{\lfloor a\,(\lfloor b)^2\,(\lfloor c)^3}$.
15. 4095. **16.** 57760000. **17.** 1023. **18.** 720; 3628800.
19. 127. **20.** 315. **21.** $\dfrac{\lfloor mn}{(\lfloor m)^n\,\lfloor n}$. **22.** 64; 325. **23.** 42.
24. (1) $\dfrac{p(p-1)}{2}-\dfrac{q(q-1)}{2}+1$; (2) $\dfrac{p(p-1)(p-2)}{6}-\dfrac{q(q-1)(q-2)}{6}$.
25. $\dfrac{p(p-1)(p-2)}{6}-\dfrac{q(q-1)(q-2)}{6}+1$. **26.** $(p+1)^n-1$.
27. 113; 2190. **28.** 2454. **29.** 6666600. **30.** 5199960.

XIII. a. Pages 142, 143.

1. $x^5-15x^4+90x^3-270x^2+405x-243$.
2. $81x^4+216x^3y+216x^2y^2+96xy^3+16y^4$.
3. $32x^5-80x^4y+80x^3y^2-40x^2y^3+10xy^4-y^5$.
4. $1-18a^2+135a^4-540a^6+1215a^8-1458a^{10}+729a^{12}$.
5. $x^{10}+5x^9+10x^8+10x^7+5x^6+x^5$.
6. $1-7xy+21x^2y^2-35x^3y^3+35x^4y^4-21x^5y^5+7x^6y^6-x^7y^7$.
7. $16-48x^2+54x^4-27x^6+\dfrac{81x^8}{16}$.
8. $729a^6-972a^5+540a^4-160a^3+\dfrac{80a^2}{3}-\dfrac{64a}{27}+\dfrac{64}{729}$.
9. $1+\dfrac{7x}{2}+\dfrac{21x^2}{4}+\dfrac{35x^3}{8}+\dfrac{35x^4}{16}+\dfrac{21x^5}{32}+\dfrac{7x^6}{64}+\dfrac{x^7}{128}$.

10. $\frac{64x^6}{729}-\frac{32x^4}{27}+\frac{20x^2}{3}-20+\frac{135}{4x^2}-\frac{243}{8x^4}+\frac{729}{64x^6}$.

11. $\frac{1}{256}+\frac{a}{16}+\frac{7a^2}{16}+\frac{7a^3}{4}+\frac{35a^4}{8}+7a^5+7a^6+4a^7+a^8$.

12. $1-\frac{10}{x}+\frac{45}{x^2}-\frac{120}{x^3}+\frac{210}{x^4}-\frac{252}{x^5}+\frac{210}{x^6}-\frac{120}{x^7}+\frac{45}{x^8}-\frac{10}{x^9}+\frac{1}{x^{10}}$.

13. $-35750x^{10}$. 14. $-112640x^9$. 15. $-312x^2$.

16. $\frac{\lfloor 30}{\lfloor 27 \lfloor 3}(5x)^3(8y)^{27}$. 17. $40a^7b^3$. 18. $\frac{1120}{81}a^4b^4$.

19. $\frac{10500}{x^3}$. 20. $\frac{70x^6y^{10}}{a^2b^6}$. 21. $2x^4+24x^2+8$.

22. $2x(16x^4-20x^2a^2+5a^4)$. 23. $140\sqrt{2}$.

24. $2(365-363x+63x^2-x^3)$. 25. 252. 26. $-\frac{429}{16}x^{14}$.

27. $110565a^4$. 28. $84a^3b^6$. 29. 1365, -1365.

30. $\frac{189a^{17}}{8}$, $-\frac{21}{16}a^{19}$. 31. $\frac{7}{18}$. 32. 18564.

33. $\frac{\lfloor n}{\lfloor \frac{1}{2}(n-r)\ \lfloor \frac{1}{2}(n+r)}$. 34. $(-1)^n\frac{\lfloor 3n}{\lfloor n\ \lfloor 2n}$.

XIII. b. Pages 147, 148.

1. The 9th. 2. The 12th. 3. The 6th. 4. The 10th and 11th.

5. The 3rd $=6\frac{2}{3}$. 6. The 4th and 5th $=\frac{7}{144}$. 9. $x=2$, $y=3$, $n=5$.

10. $1+8x+20x^2+8x^3-26x^4-8x^5+20x^6-8x^7+x^8$.

11. $27x^6-54ax^5+117a^2x^4-116a^3x^3+117a^4x^2-54a^5x+27a^6$.

12. $\frac{\lfloor n}{\lfloor r-1\ \lfloor n-r+1}x^{r-1}a^{n-r+1}$. 13. $(-1)^p\frac{\lfloor 2n+1}{\lfloor p+1\ \lfloor 2n-p}x^{2p-2n+1}$.

14. 14. 15. $2r=n$.

XIV. a. Page 155.

1. $1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3$. 2. $1+\frac{3}{2}x+\frac{3}{8}x^2-\frac{1}{16}x^3$.

3. $1-\frac{2}{5}x-\frac{3}{25}x^2-\frac{8}{125}x^3$. 4. $1-2x^2+3x^4-4x^6$.

5. $1-x-x^2-\frac{5}{3}x^3$. 6. $1+x+2x^2+\frac{14}{3}x^3$.

7. $1-x+\frac{3}{2}x^2-\frac{5}{2}x^3$. 8. $1-x+\frac{2}{3}x^2-\frac{10}{27}x^3$.

9. $1+x+\frac{x^2}{6}-\frac{x^3}{54}$. 10. $1-2a+\frac{5}{2}a^2-\frac{5}{2}a^3$.

11. $\frac{1}{8}\left(1-\frac{3}{2}x+\frac{3}{2}x^2-\frac{5}{4}x^3\right)$.
12. $3\left(1+\frac{x}{9}-\frac{1}{162}x^2+\frac{1}{1458}x^3\right)$.

13. $4\left(1+a-\frac{1}{4}a^2+\frac{1}{6}a^3\right)$.
14. $\frac{1}{27}\left(1+x+\frac{5}{6}x^2+\frac{35}{54}x^3\right)$.

15. $\frac{1}{2a^{\frac{1}{2}}}\left(1+\frac{x}{a}+\frac{3}{2}\cdot\frac{x^2}{a^2}+\frac{5}{2}\cdot\frac{x^3}{a^3}\right)$.
16. $-\frac{429}{16}x^7$.
17. $\frac{77}{256}x^{30}$.

18. $-\frac{1040}{81}a^{18}$.
19. $\frac{16b^4}{243a^5}$.
20. $(r+1)x^r$.

21. $\frac{(r+1)(r+2)(r+3)}{1.2.3}x^r$.
22. $(-1)^{r-1}\frac{1.3.5\ldots(2r-3)}{2^r\,\lfloor r}x^r$.

23. $(-1)^{r-4}\frac{11.8.5.2.1.4\ldots(3r-14)}{3^r\,\lfloor r}x^r$.

24. $-1848x^{13}$.
25. $-\frac{19712}{3}x^6$.

XIV. b. Pages 161, 162.

1. $(-1)^r\frac{1.3.5.7\ldots(2r-1)}{2^r\,\lfloor r}x^r$.
2. $\frac{(r+1)(r+2)(r+3)(r+4)}{\lfloor 4}x^r$.

3. $(-1)^{r-1}\frac{1.2.5\ldots(3r-4)}{\lfloor r}x^r$.
4. $(-1)^r\frac{2.5.8\ldots(3r-1)}{3^r\,\lfloor r}x^r$.

5. $(-1)^r\frac{(r+1)(r+2)}{\lfloor 2}x^{2r}$.
6. $\frac{3.5.7\ldots(2r+1)}{\lfloor r}x^r$.

7. $(-1)^r\frac{b^r}{a^{r+1}}\cdot x^r$.
8. $\frac{r+1}{2^{r+2}}x^r$.

9. $-\frac{2.1.4\ldots(3r-5)}{3^r\,\lfloor r}\cdot\frac{x^{3r}}{a^{3r-2}}$.
10. $(-1)^r\frac{1.3.5\ldots(2r-1)}{\lfloor r}x^r$.

11. $\frac{2.5.8\ldots(3r-1)}{\lfloor r}x^r$.
12. $\frac{(n+1)(2n+1)\ldots(\overline{r-1}.n+1)}{\lfloor r}\cdot\frac{x^r}{a^{nr+1}}$.

13. The 3rd.
14. The 5th.
15. The 13th.
16. The 7th.

17. The 4th and 5th.
18. The 3rd.
19. 9·89949.

20. 9·99333.
21. 10·00999.
22. 6·99927.
23. ·19842.

24. 1·00133.
25. ·00795.
26. 5·00096.
27. $1-\frac{23x}{6}$.

28. $\frac{2}{3}\left(1+\frac{x}{24}\right)$.
29. $1-\frac{5x}{8}$.
30. $\frac{1}{4}-\frac{5}{6}x$.
31. $1-\frac{343}{120}x$.

32. $\frac{1}{3}-\frac{71}{360}x$.
35. $1-4x+13x^2$.
36. $2+\frac{29}{4}x+\frac{297}{32}x^2$.

XIV. c. Pages 167—169.

1. -197.
2. 142.
3. $(-1)^{n-1}$.

4. $(-1)^n(n^2+2n+2)$.
6. $\sqrt{8}=\left(1-\frac{1}{2}\right)^{-\frac{3}{2}}$.

7. $\left(1-\frac{2}{3}\right)^{-n}=2^n\left(1-\frac{1}{3}\right)^{-n}$. 12. $\frac{\lfloor 2n}{\lfloor n \lfloor n}$.

14. Deduced from $(1-x^3)-(1-x)^3=3x-3x^2$. 16. (1) 45. (2) 6561.

18. (1) Equate coefficients of x^r in $(1+x)^n(1+x)^{-1}=(1+x)^{n-1}$.

(2) Equate absolute terms in $(1+x)^n\left(1+\frac{1}{x}\right)^{-2}=x^2(1+x)^{n-2}$.

20. Series on the left $+(-1)^n q_n{}^2=$ coefficient of x^{2n} in $(1-x^2)^{-\frac{1}{2}}$.

21. $2^{2n-1}-\frac{1}{2}\cdot\frac{\lfloor 2n}{\lfloor n \lfloor n}$.

[Use $(c_0+c_1+c_2+\ldots c_n)^2-2(c_0c_1+c_1c_2+\ldots)=c_0{}^2+c_1{}^2+c_2{}^2+\ldots c_n{}^2$].

XV. Pages 173, 174.

1. -12600. 2. -168. 3. 3360. 4. $-1260a^2b^3c^4$.
5. -9. 6. 8085. 7. 30. 8. 1905.
9. -10. 10. $-\frac{3}{2}$. 11. -1. 12. $-\frac{4}{81}$.
13. $\frac{59}{16}$. 14. -1. 15. $\frac{211}{3}$. 16. $1-\frac{1}{2}x-\frac{7}{8}x^2$.
17. $1-2x^2+4x^3+5x^4-20x^5$. 18. $16\left(1-\frac{3}{2}x^3+3x^4+\frac{9}{32}x^6-\frac{9}{8}x^7+\frac{9}{8}x^8\right)$.

XVI. a. Pages 178, 179.

1. 8, 6. 2. 2, -1. 3. $-\frac{16}{3}$, $-\frac{1}{2}$. 4. -4, $-\frac{3}{2}$.
5. $\frac{4}{3}$, $-\frac{4}{5}$. 6. $\frac{2}{5}$, $-\frac{1}{2}$, $-\frac{5}{2}$. 7. $\frac{7}{3}$, -3, $-\frac{4}{3}$, $\frac{2}{3}$.
8. $6\log a+9\log b$. 9. $\frac{2}{3}\log a+\frac{3}{2}\log b$.
10. $-\frac{4}{9}\log a+\frac{1}{3}\log b$. 11. $-\frac{2}{3}\log a-\frac{1}{2}\log b$.
12. $-\frac{7}{12}\log a-\log b$. 13. $\frac{1}{2}\log a$. 14. $-5\log c$. 16. $\log 3$.
18. $\frac{\log c}{\log a-\log b}$. 19. $\frac{5\log c}{2\log a+3\log b}$.
20. $\frac{\log a+\log b}{2\log c-\log a+\log b}$. 21. $x=\frac{4\log m}{\log a}$, $y=-\frac{\log m}{\log b}$.
22. $\log x=\frac{1}{5}(a+3b)$, $\log y=\frac{1}{5}(a-2b)$. 24. $\frac{\log(a-b)}{\log(a+b)}$.

XVI. b. Pages 185, 186.

1. $4, 1, 2, \bar{2}, \bar{1}, \bar{1}, \bar{1}$.

2. ·8821259, 2·8821259, $\bar{3}$·8821259, 5·8821259, $\bar{6}$·8821259.

3. 5, 2, 4, 1.

4. Second decimal place; units' place; fifth decimal place.

5. 1·8061800. 6. 1·9242793. 7. $\bar{1}$·1072100. 8. $\bar{2}$·0969100.
9. 1·1583626. 10. ·6690067. 11. ·3597271. 12. ·0563520.
13. $\bar{1}$·5052973. 14. ·44092388. 15. 1·948445. 16. 191563·1.
17. 1·1998692. 18. $\bar{1}$·0039238. 19. 9·076226. 20. 178·141516.
21. 9. 23. 301. 24. 3·46. 25. 4·29. 26. 1·206. 27. 14·206.

28. 4·562. 29. $x=\dfrac{\log 3}{\log 3-\log 2}$; $y=\dfrac{\log 2}{\log 3-\log 2}$.

30. $x=\dfrac{3\log 3-2\log 2}{4(\log 3-\log 2)}$; $y=\dfrac{\log 3}{4(\log 3-\log 2)}$. 31. 1·64601.

32. $\dfrac{\log 2}{2\log 7}=\cdot1781$; $\dfrac{2\log 7}{\log 2}=5\cdot614$.

XVII. Pages 195—197.

1. $\log_e 2$. 2. $\log_e 3-\log_e 2$. 6. ·0020000006666670.

9. $e^{x^2}-e^{y^2}$. 10. ·8450980; 1·0413927; 1·1139434. In Art. 225 put $n=50$ in (2); $n=10$ in (1); and $n=1000$ in (1) respectively.

12. $(-1)^{r-1}\cdot\dfrac{2^r+1}{r}x^r$. 13. $\dfrac{(-1)^{r-1}3^r+2^r}{r}x^r$.

14. $2\left\{1+\dfrac{(2x)^2}{\lfloor 2}+\dfrac{(2x)^4}{\lfloor 4}+\ldots+\dfrac{(2x)^{2r}}{\lfloor 2r}+\ldots\right\}$.

15. $1-\dfrac{x^2}{\lfloor 2}+\dfrac{x^4}{\lfloor 4}-\dfrac{x^6}{\lfloor 6}+\ldots+(-1)^r\dfrac{x^{2r}}{\lfloor 2r}+\ldots$ 18. $\dfrac{x}{1-x}+\log_e(1-x)$.

24. ·69314718; 1·09861229; 1·60943792; $a=-\log_e\left(1-\dfrac{1}{10}\right)=\cdot105360516$;

$$b=-\log_e\left(1-\frac{4}{100}\right)=\cdot040821995;\quad c=\log_e\left(1+\frac{1}{80}\right)=\cdot012422520.$$

XVIII. a. Page 202.

1. £1146. 14*s*. 10*d*. 2. £720. 3. 14·2 years.
4. £6768. 7*s*. 10½*d*. 5. 9·6 years. 8. £496. 19*s*. 4¾*d*.
9. A little less than 7 years. 10. £119. 16*s*. 4¾*d*.

XVIII. b. Page 207.

1. 6 per cent. 2. £3137. 2*s*. 2⅔*d*. 3. £110.
4. 3 per cent. 5. 28$\frac{4}{7}$ years. 6. £1275. 7. £926. 2*s*.
8. £6755. 13*s*. 9. £183. 18*s*. 10. 3⅓ per cent. 11. £616. 9*s*. 1½*d*.
13. £1308. 12*s*. 4½*d*. 15. £4200.

XIX. a. Pages 213, 214.

8. a^3+2b^3 is the greater. **12.** $x^3>$ or $<x^2+x+2$, according as $x>$ or <2.
14. The greatest value of x is 1. **15.** 4; 8.
22. $4^4 . 5^5$; when $x=3$. **23.** 9, when $x=1$.

XIX. b. Pages 218, 219.

10. $\dfrac{3^3 . 5^5}{2^8} a^8$; $\sqrt{\dfrac{3}{5}} . \sqrt[3]{\dfrac{2}{5}}$.

XX. Page 228.

1. $-\dfrac{10}{7}$; $-\dfrac{9}{4}$. **2.** 9; $\dfrac{1}{9}$. **3.** $\dfrac{1}{2}$; $\dfrac{5}{3}$.
4. $-\dfrac{15}{8}$; 6. **5.** 1; 0. **6.** 0; -30. **7.** $-\dfrac{3}{2}$.
8. $\log a - \log b$. **9.** 2. **10.** me^{ma}. **11.** $\dfrac{1}{2\sqrt{a}}$.
12. $\dfrac{1}{3}$. **13.** -1. **14.** $\dfrac{\sqrt{2a}}{a\sqrt{3}+1}$. **15.** $\sqrt{a}$.
16. 0. **17.** $\dfrac{3}{2}$. **18.** $e^{\frac{2}{a}}$.

XXI. a. Pages 241, 242.

1. Convergent. **2.** Convergent. **3.** Convergent.
4. $x<1$, or $x=1$, convergent; $x>1$, divergent.
5. Same result as Ex. 4. **6.** Convergent. **7.** Divergent.
8. $x<1$, convergent; $x>1$, or $x=1$, divergent.
9. Divergent except when $p>2$.
10. $x<1$, or $x=1$, convergent; $x>1$, divergent.
11. If $x<1$, convergent; $x>1$, or $x=1$, divergent.
12. Same result as Ex. 11. **13.** Divergent, except when $p>1$.
14. $x<1$, or $x=1$, convergent; $x>1$, divergent.
15. Convergent. **16.** Divergent.
17. (1) Divergent. (2) Convergent.
18. (1) Divergent. (2) Convergent.

XXI. b. Page 252.

1. $x<1$, or $x=1$, convergent; $x>1$, divergent.
2. Same result as Ex. 1. **3.** Same result as Ex. 1.
4. $x<\dfrac{1}{e}$, or $x=\dfrac{1}{e}$, convergent; $x>\dfrac{1}{e}$, divergent.
5. $x<e$, convergent; $x>e$, or $x=e$, divergent.

6. $x<1$, convergent; $x>1$, or $x=1$, divergent.

7. Divergent.

8. $x<\frac{1}{e}$, convergent; $x>\frac{1}{e}$, or $x=\frac{1}{e}$, divergent.

9. $x<1$, convergent; $x>1$, divergent. If $x=1$ and if $\gamma-\alpha-\beta$ is positive, convergent; if $\gamma-\alpha-\beta$ is negative, or zero, divergent.

10. $x<1$, convergent; $x>1$, or $x=1$, divergent. The results hold for all values of q, positive or negative.

11. a negative, or zero, convergent; a positive, divergent.

XXII. a. Page 256.

1. $\frac{1}{3}n(4n^2-1)$.

2. $\frac{1}{4}n(n+1)(n+2)(n+3)$.

3. $\frac{1}{12}n(n+1)(n+2)(3n+5)$.

4. $n^2(2n^2-1)$.

5. $\frac{1}{30}n(n+1)(2n+1)(3n^2+3n-1)$.

6. $p^3=q^2$.

7. $b^3=27a^2d,\ c^3=27ad^2$.

8. $ad=bf,\ 4a^2c-b^2=8a^3f$.

13. $abc+2fgh-af^2-bg^2-ch^2=0$.

XXII. b. Page 260.

1. $1+3x+4x^2+7x^3$.

2. $1-7x-x^2-43x^3$.

3. $\frac{1}{2}+\frac{1}{4}x-\frac{3}{8}x^2+\frac{1}{16}x^3$.

4. $\frac{3}{2}+\frac{5}{4}x+\frac{11}{8}x^2+\frac{21}{16}x^3$.

5. $1-ax+a(a+1)x^2-(a^3+2a^2-1)x^3$.

6. $a=1,\ b=2$.

7. $a=1,\ b=-1,\ c=2$.

9. The next term is $+\cdot 0000000000003$.

11. $\dfrac{a^n}{(1-a)(1-a^2)(1-a^3)\ldots\ldots(1-a^n)}$.

XXIII. Pages 265, 266.

1. $\dfrac{4}{1-3x}-\dfrac{5}{1-2x}$.

2. $\dfrac{7}{3x-5}-\dfrac{5}{4x+3}$.

3. $\dfrac{4}{1-2x}-\dfrac{3}{1-x}$.

4. $\dfrac{2}{x-1}+\dfrac{3}{x-2}-\dfrac{4}{x-3}$.

5. $1+\dfrac{1}{x}-\dfrac{1}{5(x-1)}-\dfrac{8}{5(2x+3)}$

6. $\dfrac{1}{x-1}-\dfrac{1}{x+2}-\dfrac{3}{(x+2)^2}$.

7. $x-2+\dfrac{17}{16(x+1)}-\dfrac{11}{4(x+1)^2}-\dfrac{17}{16(x-3)}$.

8. $\dfrac{41x+3}{x^2+1}-\dfrac{15}{x+5}$.

9. $\dfrac{3x}{x^2+2x-5}-\dfrac{1}{x-3}$.

10. $\dfrac{5}{(x-1)^4}-\dfrac{7}{(x-1)^3}+\dfrac{1}{(x-1)^2}+\dfrac{3}{x-1}$.

11. $\dfrac{1}{x-1}-\dfrac{1}{x+1}+\dfrac{3}{(x+1)^2}-\dfrac{3}{(x+1)^3}+\dfrac{2}{(x+1)^4}$.

12. $\dfrac{4}{3(1+7x)}-\dfrac{1}{3(1+4x)}$; $\dfrac{(-1)^r}{3}(4 \,.\, 7^r-4^r)\,x^r$.

13. $\dfrac{11}{3(1-x)}-\dfrac{4}{3(2+x)}$; $\dfrac{1}{3}\left(11+\dfrac{(-1)^{r-1}}{2^{r-1}}\right)x^r$.

14. $1+\dfrac{7}{3(x+5)}-\dfrac{7}{3(x+2)}$; $(-1)^r\dfrac{7}{3}\left(\dfrac{1}{5^{r+1}}-\dfrac{1}{2^{r+1}}\right)x^r$.

15. $\dfrac{1}{1-x}-\dfrac{1}{1+x}-\dfrac{4}{1-2x}$; $\{1+(-1)^{r-1}-2^{r+2}\}\,x^r$.

16. $\dfrac{4}{3(1+2x)}-\dfrac{1}{3(1-x)}+\dfrac{3}{(1-x)^2}$; $\dfrac{1}{3}\{9r+8+(-1)^r\,2^{r+2}\}\,x^r$.

17. $\dfrac{1}{4(1-4x)}+\dfrac{11}{4(1-4x)^2}$; $4^{r-1}(12+11r)\,x^r$.

18. $\dfrac{2}{1+x}+\dfrac{3}{(1+x)^2}-\dfrac{6}{2+3x}$; $(-1)^r\left(3r+5-\dfrac{3^{r+1}}{2^r}\right)x^r$.

19. $\dfrac{3}{2(x-1)}+\dfrac{1-3x}{2(1+x^2)}$; r even, $\dfrac{1}{2}\{(-1)^{\frac{r}{2}}-3\}\,x^r$; r odd, $-\dfrac{3}{2}\{1+(-1)^{\frac{r-1}{2}}\}\,x^r$.

20. $\dfrac{2}{(1-x)^3}-\dfrac{3}{(1-x)^2}+\dfrac{2}{1-x}$; $(r^2+1)\,x^r$.

21. $\left\{\dfrac{a^{r+2}}{(a-b)(a-c)}+\dfrac{b^{r+2}}{(b-c)(b-a)}+\dfrac{c^{r+2}}{(c-a)(c-b)}\right\}x^r$.

22. $-\dfrac{5}{(2-x)^2}-\dfrac{2}{2-x}+\dfrac{1}{(1-x)^2}+\dfrac{2}{1-x}$; $\left\{r+3-\dfrac{5r+9}{2^{r+2}}\right\}x^r$.

23. (1) $\dfrac{1}{x(1-x)}\left\{\dfrac{1}{1+x^{n+1}}-\dfrac{1}{1+x}\right\}$.

(2) $\dfrac{1}{(1-a)^2}\left\{\dfrac{1}{1+a^n x}-\dfrac{1}{1+a^{n+1}x}-\dfrac{1}{1+x}+\dfrac{1}{1+ax}\right\}$.

24. $\dfrac{1}{x(1-x)(1-x^2)}$.

25. $\dfrac{1}{(1-x)^2}\left\{\dfrac{x}{1-x}-\dfrac{x^2}{1-x^2}-\dfrac{x^{n+1}}{1-x^{n+1}}+\dfrac{x^{n+2}}{1-x^{n+2}}\right\}$.

XXIV. Page 272.

1. $\dfrac{1+3x}{(1-x)^2}$; $(4r+1)\,x^r$.

2. $\dfrac{2+x}{1+x-2x^2}$; $\{1+(-1)^r\,2^r\}\,x^r$.

3. $\dfrac{2-3x}{1-3x+2x^2}$; $(1+2^r)\,x^r$.

4. $\dfrac{7-20x}{1-2x-3x^2}$; $\left\{\dfrac{27}{4}(-1)^r+\dfrac{3^r}{4}\right\}x^r$.

5. $\dfrac{3-12x+11x^2}{1-6x+11x^2-6x^3}$; $(3^r+2^r+1)\,x^r$.

6. $3^{n-1}+2^{n-1}$; $\dfrac{1}{2}(3^n-1)+2^n-1$.

7. $(2\,.\,3^{n-1}-3\,.\,2^{n-1})\,x^{n-1}$; $\dfrac{2(1-3^n x^n)}{1-3x}-\dfrac{3(1-2^n x^n)}{1-2x}$.

8. $(4^{n-1}+3^{n-1})\,x^{n-1}$; $\dfrac{1-4^n x^n}{1-4x}+\dfrac{1-3^n x^n}{1-3x}$.

9. $(1+3^{n-1}-2^{n-1})\,x^{n-1}$; $\dfrac{1-x^n}{1-x}+\dfrac{1-3^nx^n}{1-3x}-\dfrac{1-2^nx^n}{1-2x}$.

10. $\dfrac{8}{5}(-1)^n+\dfrac{2^{2n-3}}{5}$; $\dfrac{4}{5}\{(-1)^n-1\}+\dfrac{1}{30}(2^{2n}-1)$.

11. $u_n-3u_{n-1}+3u_{n-2}-u_{n-3}=0$; $u_n-4u_{n-1}+6u_{n-2}-4u_{n-3}+u_{n-4}=0$.

12. $S_n=S_\infty-\Sigma$, where $\Sigma=$ sum to infinity beginning with $(n+1)^{\text{th}}$ term. This may easily be shewn to agree with the result in Art. 325.

13. $(2n+1)^2+\dfrac{2}{3}(2^{2n+1}+1)$.

XXV. a. Pages 277, 278.

1. $\dfrac{2}{1}, \dfrac{13}{6}, \dfrac{15}{7}, \dfrac{28}{13}, \dfrac{323}{150}, \dfrac{674}{313}$.

2. $\dfrac{1}{2}, \dfrac{2}{5}, \dfrac{7}{17}, \dfrac{9}{22}, \dfrac{43}{105}, \dfrac{95}{232}, \dfrac{613}{1497}$.

3. $\dfrac{3}{1}, \dfrac{10}{3}, \dfrac{13}{4}, \dfrac{36}{11}, \dfrac{85}{26}, \dfrac{121}{37}, \dfrac{1174}{359}$.

4. $1+\dfrac{1}{2+}\,\dfrac{1}{2+}\,\dfrac{1}{2+}\,\dfrac{1}{1+}\,\dfrac{1}{1+}\,\dfrac{1}{2+}\,\dfrac{1}{2}$; $\dfrac{17}{12}$.

5. $5+\dfrac{1}{4+}\,\dfrac{1}{3+}\,\dfrac{1}{2+}\,\dfrac{1}{1+}\,\dfrac{1}{3}$; $\dfrac{157}{30}$.

6. $\dfrac{1}{3+}\,\dfrac{1}{3+}\,\dfrac{1}{3+}\,\dfrac{1}{3+}\,\dfrac{1}{3+}\,\dfrac{1}{3+}\,\dfrac{1}{3}$; $\dfrac{33}{109}$.

7. $\dfrac{1}{3+}\,\dfrac{1}{5+}\,\dfrac{1}{1+}\,\dfrac{1}{1+}\,\dfrac{1}{3+}\,\dfrac{1}{2+}\,\dfrac{1}{1+}\,\dfrac{1}{5}$; $\dfrac{11}{35}$.

8. $\dfrac{1}{2+}\,\dfrac{1}{1+}\,\dfrac{1}{2+}\,\dfrac{1}{2+}\,\dfrac{1}{1+}\,\dfrac{1}{3}$; $\dfrac{7}{19}$.

9. $1+\dfrac{1}{7+}\,\dfrac{1}{5+}\,\dfrac{1}{6+}\,\dfrac{1}{1+}\,\dfrac{1}{3}$; $\dfrac{254}{223}$.

10. $\dfrac{1}{3+}\,\dfrac{1}{3+}\,\dfrac{1}{3+}\,\dfrac{1}{6+}\,\dfrac{1}{1+}\,\dfrac{1}{2+}\,\dfrac{1}{1+}\,\dfrac{1}{10}$; $\dfrac{63}{208}$.

11. $4+\dfrac{1}{3+}\,\dfrac{1}{6+}\,\dfrac{1}{3}$; $\dfrac{259}{60}$.

13. $\dfrac{1}{4}, \dfrac{7}{29}, \dfrac{8}{33}, \dfrac{39}{161}, \dfrac{47}{194}$.

16. $n-1+\dfrac{1}{(n+1)+}\,\dfrac{1}{(n-1)+}\,\dfrac{1}{n+1}$; and the first three convergents are

$$\frac{n-1}{1},\ \frac{n^2}{n+1},\ \frac{n^3-n^2+n-1}{n^2}.$$

XXV. b. Pages 281—283.

1. $\dfrac{1}{(203)^2}$ and $\dfrac{1}{2\,(1250)^2}$.

2. $\dfrac{151}{115}$.

4. $\dfrac{1}{a+}\,\dfrac{1}{(a+1)+}\,\dfrac{1}{(a+2)+}\,\dfrac{1}{a+3}$; $\dfrac{a^2+3a+3}{a^3+3a^2+4a+2}$.

XXVI. Pages 290, 291.

1. $x=711t+100$, $y=775t+109$; $x=100$, $y=109$.
2. $x=519t-73$, $y=455t-64$; $x=446$, $y=391$.
3. $x=393t+320$, $y=436t+355$; $x=320$, $y=355$.
4. Four.
5. Seven.
6. $\frac{5}{7}$, $\frac{4}{9}$.
7. $\frac{5}{12}$, $\frac{3}{8}$; $\frac{11}{12}$, $\frac{7}{8}$; or $\frac{1}{8}$, $\frac{1}{12}$; $\frac{5}{8}$, $\frac{7}{12}$.
8. £6. 13*s*.
9. $x=9$, $y=8$, $z=3$.
10. $x=5$, $y=6$, $z=7$.
11. $x=4$, $y=2$, $z=7$.
12. $x=2$, $y=9$, $z=7$.
13. $x=3, 7, 2, 6, 1$; $y=11, 4, 8, 1, 5$; $z=1, 1, 2, 2, 3$.
14. $x=1, 3, 2$; $y=5, 1, 3$; $z=2, 4, 3$.
15. $280t+93$.
16. 181, 412.
17. Denary 248, Septenary 503, Nonary 305.
18. $a=11, 10, 9, 8, 6, 4, 3$; $b=66, 30, 18, 12, 6, 3, 2$.
19. The 107th and 104th divisions, reckoning from either end.
20. 50, 41, 35 times, excluding the first time.
21. 425.
22. 899.
23. 1829 and 1363.

XXVII. a. Pages 294, 295.

1. $1+\frac{1}{1+}\,\frac{1}{2+}\cdots$; $\frac{26}{15}$.
2. $2+\frac{1}{4+}\cdots$; $\frac{2889}{1292}$.
3. $2+\frac{1}{2+}\,\frac{1}{4+}\cdots$; $\frac{485}{198}$
4. $2+\frac{1}{1+}\,\frac{1}{4+}\cdots$; $\frac{99}{35}$.
5. $3+\frac{1}{3+}\,\frac{1}{6+}\cdots$; $\frac{3970}{1197}$.
6. $3+\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{6+}\cdots$; $\frac{119}{33}$.
7. $3+\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{6+}\cdots$; $\frac{116}{31}$.
8. $4+\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{4+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{8+}\cdots$; $\frac{197}{42}$.
9. $3+\frac{1}{2+}\,\frac{1}{6+}\cdots$; $\frac{1351}{390}$.
10. $5+\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{10+}\cdots$; $\frac{198}{35}$.
11. $6+\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{12+}$; $\frac{161}{24}$.
12. $12+\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{5+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{24+}$; $\frac{253}{20}$.
13. $\frac{1}{4+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{8+}\cdots$; $\frac{12}{55}$.
14. $\frac{1}{5+}\,\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{10+}\cdots$; $\frac{47}{270}$.
15. $1+\frac{1}{10+}\,\frac{1}{2+}\cdots$; $\frac{5291}{4830}$.
16. $\frac{1}{1+}\,\frac{1}{3+}\,\frac{1}{1+}\,\frac{1}{16+}\,\frac{1}{1+}\,\frac{1}{3+}\,\frac{1}{2+}\,\frac{1}{3+}\,\frac{1}{1+}\,\frac{1}{16+}\cdots$; $\frac{280}{351}$.

17. $\dfrac{1}{(65)^2}$ and $\dfrac{1}{2\,(528)^2}$.

18. $\dfrac{1}{(191)^2}$ and $\dfrac{1}{2\,(240)^2}$.

19. $\dfrac{4030}{401}$.

20. $\dfrac{1677}{433}$.

21. $\dfrac{1}{2+}\,\dfrac{1}{2+}\,\dfrac{1}{2+}\dots$

22. $4+\dfrac{1}{1+}\,\dfrac{1}{1+}\,\dfrac{1}{1+}\,\dfrac{1}{4+}\dots$

23. $1+\dfrac{1}{2+}\,\dfrac{1}{3+}\,\dfrac{1}{1+}\dots$

24. $4+\dfrac{1}{3+}\,\dfrac{1}{3+}\dots$; $\dfrac{1}{1+}\,\dfrac{1}{2+}\,\dfrac{1}{3+}\,\dfrac{1}{3+}\,\dfrac{1}{3+}\dots$.

25. $\sqrt{10}$.

26. Positive root of $x^2+3x-3=0$.

27. Positive root of $3x^2-10x-4=0$.

28. $4\sqrt{2}$.

30. $\dfrac{1}{2}$.

XXVII. b. Pages 301, 302.

1. $a+\dfrac{1}{2a+}\,\dfrac{1}{2a+}\,\dfrac{1}{2a+}\dots$; $\dfrac{8a^4+8a^2+1}{8a^3+4a}$.

2. $a-1+\dfrac{1}{2+}\,\dfrac{1}{2\,(a-1)+}\,\dfrac{1}{2+}\,\dfrac{1}{2\,(a-1)+}\dots$; $\dfrac{8a^2-8a+1}{8a-4}$.

3. $a-1+\dfrac{1}{1+}\,\dfrac{1}{2\,(a-1)+}\,\dfrac{1}{1+}\,\dfrac{1}{2\,(a-1)+}\dots$; $\dfrac{2a^2-1}{2a}$.

4. $1+\dfrac{1}{2a+}\,\dfrac{1}{2+}\,\dfrac{1}{2a+}\,\dfrac{1}{2+}\dots$; $\dfrac{8a^2+8a+1}{8a^2+4a}$.

5. $a+\dfrac{1}{b+}\,\dfrac{1}{2a+}\,\dfrac{1}{b+}\,\dfrac{1}{2a+}\dots$; $\dfrac{2a^2b^2+4ab+1}{2ab^2+2b}$.

6. $a-1+\dfrac{1}{1+}\,\dfrac{1}{2\,(n-1)+}\,\dfrac{1}{1+}\,\dfrac{1}{2\,(a-1)+}\dots$; $\dfrac{2an-1}{2n}$.

7. $\dfrac{432a^5+180a^3+15a}{144a^4+36a^2+1}$.

XXVIII. Page 311.

1. $x=7$ or 1, $y=4$; $x=7$ or 5, $y=6$.

2. $x=2$, $y=1$.

3. $x=3$, $y=1$, 11; $x=7$, $y=9$, 19; $x=10$, $y=18$, 22.

4. $x=2, 3, 6, 11$; $y=12, 7, 4, 3$.

5. $x=3, 2$; $y=1, 4$.

6. $x=79, 27, 17, 13, 11, 9$; $y=157, 51, 29, 19, 13, 3$.

7. $x=15$, $y=4$.

8. $x=170$, $y=39$.

9. $x=32$, $y=5$.

10. $x=164$, $y=21$.

11. $x=4$, $y=1$.

12. $2x=(2+\sqrt{3})^n+(2-\sqrt{3})^n$; $2\sqrt{3}\,.\,y=(2+\sqrt{3})^n-(2-\sqrt{3})^n$; n being any integer.

13. $2x=(2+\sqrt{5})^n+(2-\sqrt{5})^n$; $2\sqrt{5}\,.\,y=(2+\sqrt{5})^n-(2-\sqrt{5})^n$; n being any even positive integer.

14. $2x=(4+\sqrt{17})^n+(4-\sqrt{17})^n$; $2\sqrt{17}\,.\,y=(4+\sqrt{17})^n-(4-\sqrt{17})^n$; n being any odd positive integer.

The form of the answers to 15—17, 19, 20 will vary according to the mode of factorising the two sides of the equation.

15. $x=m^2-3n^2,\ y=m^2-2mn.$ 16. $x=-m^2+2mn+n^2;\ y=m^2-n^2.$

17. $x=2mn,\ y=5m^2-n^2.$ 18. 53, 52; 19, 16; 13, 8; 11, 4.

19. $m^2-n^2;\ 2mn;\ m^2+n^2.$ 20. $m^2-n^2;\ 2mn+n^2.$

21. Hendriek, Anna; Claas, Catriin; Cornelius, Geertruij.

XXIX. a. Pages 321, 322.

1. $\frac{1}{4}n(n+1)(n+2)(n+3).$ 2. $\frac{1}{5}n(n+1)(n+2)(n+3)(n+4).$

3. $\frac{1}{12}(3n-2)(3n+1)(3n+4)(3n+7)+\frac{56}{12}=\frac{n}{4}(27n^3+90n^2+45n-50).$

4. $\frac{n}{4}(n+1)(n+6)(n+7).$ 5. $\frac{n}{4}(n+1)(n+8)(n+9).$

6. $\frac{n}{n+1};\ 1.$ 7. $\frac{n}{3n+1};\ \frac{1}{3}.$

8. $\frac{1}{12}-\frac{1}{4(2n+1)(2n+3)};\ \frac{1}{12}.$ 9. $\frac{1}{24}-\frac{1}{6(3n+1)(3n+4)};\ \frac{1}{24}.$

10. $\frac{5}{4}-\frac{2n+5}{2(n+1)(n+2)};\ \frac{5}{4}.$ 11. $\frac{1}{6}-\frac{1}{n+3}+\frac{2}{(n+3)(n+4)};\ \frac{1}{6}.$

12. $\frac{3}{4}-\frac{2}{n+2}+\frac{1}{2(n+1)(n+2)};\ \frac{3}{4}.$ 13. $\frac{n}{10}(n+1)(n+2)(n+3)(2n+3).$

14. $\frac{1}{4}n^2(n^2-1).$ 15. $\frac{n}{10}(n-1)(n+1)(n+2)(2n+1).$

16. $\frac{1}{15}(n+1)(n+2)(3n^3+36n^2+151n+240)-32.$

17. $\frac{(n-1)n(n+1)(n+2)}{6(2n+1)}.$ 18. $\frac{n(n+1)(n+2)}{3}-\frac{n}{n+1}.$

19. $\frac{n(n+3)}{2}+\frac{3}{2}-\frac{2}{n+2}-\frac{1}{(n+1)(n+2)}.$ 20. $n+1-\frac{1}{n+1}.$

XXIX. b. Pages 332, 333.

1. $3n^2+n;\ n(n+1)^2.$ 2. $5n^2+3n;\ \frac{1}{3}n(n+1)(5n+7).$

3. $n^2(n+1);\ \frac{1}{12}n(n+1)(n+2)(3n+1).$

4. $-4n^2(n-3);\ -n(n+1)(n^2-3n-2).$

5. $n(n+1)(n+2)(n+4);\ \frac{1}{20}n(n+1)(n+2)(n+3)(4n+21).$

6. $\frac{1+x^2}{(1-x)^3}.$ 7. $\frac{1-x+6x^2-2x^3}{(1-x)^3}.$ 8. $\frac{2-x+x^2}{(1-x)^3}.$

9. $\frac{1-x}{(1+x)^2}.$ 10. $\frac{1+11x+11x^2+x^3}{(1-x)^5}.$ 11. $\frac{9}{4}.$

12. $\frac{25}{54}.$ 13. $3\cdot 2^n+n+2;\ 6(2^n-1)+\frac{n(n+5)}{2}.$

14. $n^3-(n+1)^2$; $\frac{n}{12}(3n^3+2n^2-15n-26)$.

15. $3^{n-1}+n$; $\frac{3^n+n^2+n-1}{2}$.

16. $2^{n+1}-n^2-2n$; $2^{n+2}-4-\frac{1}{6}n(n+1)(2n+7)$.

17. $3^n-1+\frac{1}{2}n(n+3)$; $\frac{1}{2}(3^{n+1}-3)+\frac{n(n+1)(n+5)}{6}-n$.

18. $\frac{1-x^n}{(1-x)^2}-\frac{nx^n}{1-x}$.

19. $\frac{1-x^n}{(1-x)^3}-\frac{nx^n}{(1-x)^2}-\frac{n(n+1)x^n}{2(1-x)}$.

20. $1-\frac{1}{n+1}\cdot\frac{1}{2^n}$.

21. $\frac{n-1}{n+2}\cdot\frac{4^{n+1}}{3}+\frac{2}{3}$.

22. $\frac{n(n+1)(3n^3+27n^2+58n+2)}{15}$.

23. $\frac{n(n+1)(12n^3+33n^2+37n+8)}{60}$.

24. $\frac{n(n+1)(9n^2+13n+8)}{12}$.

25. $\frac{1}{2}-\frac{1}{2}\cdot\frac{1}{1.3.5.7\ldots\ldots(2n+1)}$.

26. $1-\frac{2^{n+1}}{\lfloor n+2}$.

27. $(n^2-n+4)2^n-4$.

28. $(n-1)3^{n+1}+3$.

29. $\frac{1}{2}-\frac{1.3.5\ldots\ldots(2n+1)}{2.4.6\ldots\ldots(2n+2)}$.

30. $\frac{n}{n+1}\cdot 2^n$.

31. $\frac{1}{4}-\frac{1}{2(n+1)(n+2)}\cdot\frac{1}{3^n}$.

32. $\frac{1}{2}-\frac{n+1}{\lfloor n+2}$.

33. $1-\frac{n+4}{(n+1)(n+2)}\cdot\frac{1}{2^{n+1}}$.

XXIX. c. Pages 338—340.

1. $\frac{1}{2}(e^x-e^{-x})-x$.

2. $1+\frac{1-x}{x}\log(1-x)$.

3. $\frac{1}{4}(e^x-e^{-x}-ie^{ix}+ie^{-ix})$.

4. $\frac{1}{(r-2)\lfloor r-1}$.

5. $(1+x)e^x$.

6. $\frac{(p+q)^r}{\lfloor r}$.

7. 1.

8. $n(2n-1)$.

9. 0.

10. 4.

11. $\log_e 2-\frac{1}{2}$.

12. $3(e-1)$.

13. $e^x-\log(1+x)$.

14. (1) $\frac{n^7}{7}+\frac{n^6}{2}+\frac{n^5}{2}-\frac{n^3}{6}+\frac{n}{42}$. (2) $\frac{n^8}{8}+\frac{n^7}{2}+\frac{7n^6}{12}-\frac{7n^4}{24}+\frac{n^2}{12}$.

15. $15e$.

17. (1) $n+1$.

19. (1) $\frac{1}{2}\left(1-\frac{1}{1+n+n^2}\right)$; (2) $3-\frac{2+(-1)^n}{n+1}$.

20. $\frac{(1+x)^2}{2x^2}\log(1+x)-\frac{3x+2}{4x}$.

21. $n(n+1)2^{n-3}$.

22. (1) $\frac{1}{3}\left\{1+\frac{(-1)^{n+1}}{2^{n+1}+(-1)^{n+1}}\right\}$. (2) $\frac{1}{2}\left\{\frac{3}{2}+(-1)^{n-1}\frac{2n+3}{(n+1)(n+2)}\right\}$.

XXX. a. Pages 348, 349.

1. 3, 6, 15, 42. 2. 1617, 180, 1859. 6. 48.
7. 23. 33. 8987.

XXX. b. Pages 356—358.

20. $x=139t+61$, where t is an integer.

XXXI. a. Pages 367—369.

2. $1+\frac{1}{x-}\frac{1}{4+}\frac{1}{x}$. 18. 1; it can be shewn that $q_n=1+p_n$.

XXXII. a. Pages 376, 377.

1. (1) $\frac{1}{9}$; (2) $\frac{5}{36}$. 2. $\frac{8}{663}$. 3. $\frac{1}{56}$. 4. $\frac{3}{8}$.
5. 2 to 3. 6. $\frac{4}{270725}$. 8. 43 to 34. 9. 36 : 30 : 25.
10. $\frac{2197}{20825}$. 11. 952 to 715. 14. $\frac{1}{6}$. 15. $\frac{2}{7}$.
16. $\frac{11}{4165}$. 17. $\frac{n(n-1)}{(m+n)(m+n-1)}$.

XXXII. b. Pages 383, 384.

1. $\frac{5}{36}$. 2. $\frac{16}{5525}$. 3. $\frac{52}{77}$. 4. $\frac{16}{21}$. 5. $\frac{8}{15}$.
6. $\frac{72}{289}$. 7. (1) $\frac{2197}{20825}$; (2) $\frac{2816}{4165}$. 8. $\frac{4651}{7776}$. 9. $\frac{209}{343}$.
10. $\frac{1}{7}$. 11. $\frac{91}{216}$. 13. $\frac{10}{19}$. 14. $\frac{63}{256}$. 15. $\frac{1}{32}$.
16. $\frac{16}{37}, \frac{12}{37}, \frac{9}{37}$. 17. $\frac{22}{35}, \frac{13}{35}$. 18. $n-3$ to 2.
19. 13 to 5. 20. $\frac{45927}{50000}$.

XXXII. c. Pages 389, 390

1. $\frac{2133}{3125}$. 2. $\frac{5}{16}$. 3. $\frac{4}{9}$. 4. Florins. 5. $\frac{1}{2}$.
6. 17*s*. $2\frac{2}{5}d$. 7. $\frac{4}{63}$. 8. $\frac{7}{27}$. 9. 11 to 5. 10. $\frac{1}{8}$.

11. A £5; B £11. 12. $\frac{20}{27}$. 13. $4\frac{1}{6}$ shillings.

14. (1) $\frac{250}{7776}$; (2) $\frac{276}{7776}$. 15. $4d$. 16. $\frac{3}{4}$. 17. $M+\frac{1}{2}m$.

XXXII. d. Pages 399, 400.

1. $\frac{2}{5}$. 2. $\frac{1}{35}$. 3. $\frac{12}{17}$. 4. B $\frac{2}{5}$; C $\frac{4}{15}$.

5. $\frac{2}{n(n+1)}$. 6. $\frac{32}{41}$. 7. $\frac{11}{15}$. 8. $2s.$ $3d.$ 9. $\frac{1}{5}$.

10. $\frac{1}{3}$. 11. $\frac{40}{41}$. 12. $\frac{11}{50}$. 13. £1. 14. (1) $\frac{3}{5}$; (2) $\frac{7}{8}$.

15. £8. 17. $\frac{n-1}{mn-1}$, $\frac{n-1}{mn-rn-1}$. 18. $\frac{13}{14}$.

XXXII. e. Pages 405—408.

1. 7 to 5. 2. $\frac{1}{126}$. 3. $\frac{12393}{12500}$. 5. $\frac{275}{504}$.

6. $1 : \frac{5}{6} : \left(\frac{5}{6}\right)^2 : \left(\frac{5}{6}\right)^3 : \left(\frac{5}{6}\right)^4$. 7. $\frac{16}{21}$. 8. 6; each equal to $\frac{1}{6}$.

9. $\frac{13}{28}$. 10. $\frac{343}{1695}$. 11. 11 to 5. 13. A, $\frac{169}{324}$; B, $\frac{155}{324}$.

14. $\frac{1}{168}$, $\frac{1}{126}$. 16. $\frac{25}{216}$. 17. $\frac{149}{2401}$. 18. $\frac{33}{1000}$, $\frac{1}{60}$.

20. One guinea. 22. $\frac{140}{141}$. 23. $\frac{n(n+1)}{2}$ shillings. 26. 15 to 1.

28. $\frac{1}{4}$. 29. $\frac{1}{4}$. 30. $\frac{1265}{1286}$; £$\frac{5087}{5144}$. 31. $\left(\frac{a-b}{a}\right)^2$.

32. If $b>\frac{a}{2}$, the chance is $1-3\left(\frac{a-b}{a}\right)^2$;

If $b<\frac{a}{2}$, the chance is $\left(\frac{3b-a}{a}\right)^2$.

XXXIII. a. Pages 419, 420, 421.

1. 7. 2. 0. 3. 1. 4. $abc+2fgh-af^2-bg^2-ch^2$.

5. $1+x^2+y^2+z^2$. 6. xy. 7. 0. 8. $4abc$. 9. 0.

10. 3. 11. $3abc-a^3-b^3-c^3=0$. 13. (1) $x=a$, or b; (2) $x=4$.

20. $\begin{vmatrix} b^2+c^2 & ab & ac \\ ba & c^2+a^2 & bc \\ ca & cb & a^2+b^2 \end{vmatrix}$. 22. $\lambda^3(\lambda^2+a^2+b^2+c^2)^3$.

26. The determinant is equal to $\begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix} \times \begin{vmatrix} 1 & -2x & x^2 \\ 1 & -2y & y^2 \\ 1 & -2z & z^2 \end{vmatrix}$.

27. $\begin{vmatrix} u & w' & v' \\ w' & v & u' \\ v' & u' & w \end{vmatrix} = 0.$

28. $\begin{vmatrix} u & w' & v' \\ w' & v & u' \\ v' & u' & w \end{vmatrix} \div \begin{vmatrix} u & w' & v' & a \\ w' & v & u' & b \\ v' & u' & w & c \\ a & b & c & 0 \end{vmatrix}$.

XXXIII. b. Pages 427, 428.

1. 1.

2. 0; add first and second rows, third and fourth rows.

3. $(a+3)(a-1)^3$.

4. $a^2+b^2+c^2-2bc-2ca-2ab$.

5. 6; from the first column subtract three times the third, from the second subtract twice the third, and from the fourth subtract four times the third.

6. $abcd\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)$.

7. $-(x+y+z)(y+z-x)(z+x-y)(x+y-z)$.

8. $(ax-by+cz)^2$.

9. a^4.

12. $x=\frac{(k-b)(k-c)}{(a-b)(a-c)}$; &c.

13. $x=\frac{k(k-b)(k-c)}{a(a-b)(a-c)}$; &c.

14. $x=\frac{(k-b)(k-c)(k-d)}{(a-b)(a-c)(a-d)}$; &c.

XXXIV. a. Pages 439, 440.

1. -102.

2. $3a+b=27$.

3. x^3-2x^2+x+1; $-15x+11$.

4. $a=3$.

5. $x^{-4}+5x^{-5}+18x^{-6}+54x^{-7}$; $147x^{-4}-356x^{-5}+90x^{-6}+432x^{-7}$.

6. $(b-c)(c-a)(a-b)(a+b+c)$.

7. $-(b-c)(c-a)(a-b)(b+c)(c+a)(a+b)$.

8. $24abc$.

9. $(b+c)(c+a)(a+b)$.

10. $(b-c)(c-a)(a-b)(a^2+b^2+c^2+bc+ca+ab)$.

11. $3abc(b+c)(c+a)(a+b)$.

12. $12abc(a+b+c)$.

13. $80abc(a^2+b^2+c^2)$.

14. $3(b-c)(c-a)(a-b)(x-a)(x-b)(x-c)$.

28. $\frac{x}{(x-a)(x-b)(x-c)}$.

29. 2.

30. $\frac{(p-x)(q-x)}{(a+x)(b+x)(c+x)}$.

31. -1.

32. $a+b+c+d$.

XXXIV. b. Pages 442, 443.

5. 0.

7. $A=ax+by+ay$, $B=bx-ay$.

28. $(a^2+bc)(b^2+ca)(c^2+ab)$.

XXXIV. c. Pages 449, 450.

1. $x^3+xy^2+ay^2=0$. 2. $x+a=0$. 3. $x^2+y^2=a^2$.
4. $y^2=a(x-3a)$. 5. $a^6-a^3=1$. 6. $x^2+y^2=2a^2$.
7. $b^4c^4+c^4a^4+a^4b^4=a^2b^2c^2d^2$. 8. $y^2-4ax=k^2(x+a)^2$.
9. $a^4-4ac^3+3b^4=0$. 10. $a^4-2a^2b^2-b^4+2c^4=0$.
11. $\dfrac{a}{1+a}+\dfrac{b}{1+b}+\dfrac{c}{1+c}+\dfrac{d}{1+d}=1$. 12. $5a^2b^3=6c^5$.
13. $ab=1+c$. 14. $a^3+b^3+c^3+abc=0$.
15. $(a+b)^{\frac{2}{3}}-(a-b)^{\frac{2}{3}}=4c^{\frac{2}{3}}$. 16. $a^2+b^2+c^2\pm 2abc=1$.
17. $abc=(4-a-b-c)^2$. 18. $a^2-4abc+ac^3+4b^3-b^2c^2=0$.
20. $c^2(a+b-1)^2-c(a+b-1)(a^2-2ab+b^2-a-b)+ab=0$.
22. $$\frac{1}{(a-b)cr+(a-c)bq}+\frac{1}{(b-c)ap+(b-a)cr}+\frac{1}{(c-a)bq+(c-b)ap}=\frac{1}{bcqr+carp+abpq}.$$
23. $$\begin{vmatrix} ab'-a'b & ac'-a'c & ad'-a'd \\ ac'-a'c & ad'-a'd+bc'-b'c & bd'-b'd \\ ad'-a'd & bd'-b'd & cd'-c'd \end{vmatrix}=0.$$

XXXV. a. Pages 456, 457.

1. $6x^4-13x^3-12x^2+39x-18=0$. 2. $x^6+2x^5-11x^4-12x^3+36x^2=0$.
3. $x^6-5x^5-8x^4+40x^3+16x^2-80x=0$.
4. $x^4-2(a^2+b^2)x^2+(a^2-b^2)^2$. 5. 1, 3, 5, 7.
6. $\frac{3}{2}, -\frac{3}{2}, -4$. 7. $\frac{1}{2}, \frac{1}{2}, -6$. 8. $6, 2, \frac{2}{3}$.
9. $-\frac{3}{2}, -2, 4$. 10. $-\frac{3}{2}, -\frac{3}{4}, \frac{1}{3}$. 11. $\pm\sqrt{3}, \frac{3}{4}, -\frac{1}{2}$.
12. $\frac{8}{9}, -\frac{2}{3}, \frac{1}{2}$. 13. $\frac{1}{4}, \frac{1}{2}, \frac{3}{4}$. 14. $\frac{4}{3}, \frac{3}{2}, 1\pm\sqrt{2}$.
15. $-4, -1, 2, 5$. 16. $\frac{8}{9}, \frac{4}{3}, 2, 3$. 17. $-\frac{4}{3}, -\frac{3}{2}, -\frac{5}{3}$.
18. (1) $\dfrac{q^2-2pr}{r^2}$; (2) $\dfrac{p^2-2q}{r^2}$. 19. (1) $-6q$; (2) $\dfrac{q}{r}$.
20. $-2q, -3r$. 21. $2q^2$.

XXXV. b. Pages 460, 461.

1. $3, -\frac{2}{3}, \dfrac{1\pm\sqrt{-3}}{2}$. 2. $-\frac{3}{2}, -\frac{1}{3}, 2\pm\sqrt{3}$.
3. $-1\pm\sqrt{2}, -1\pm\sqrt{-1}$. 4. $\pm\sqrt{-1}, -2\pm\sqrt{-1}$.
5. $-1, \pm\sqrt{3}, 1\pm 2\sqrt{-1}$. 6. $x^4-2x^2+25=0$.
7. $x^4-8x^2+36=0$. 8. $x^4+16=0$.

9. $x^4-10x^2+1=0$. 10. $x^4-10x^3-19x^2+480x-1392=0$.
11. $x^4-6x^3+18x^2-26x+21=0$. 12. $x^8-16x^6+88x^4+192x^2+144=0$.
13. One positive, one negative, two imaginary. [Compare Art. 554.]
15. One positive, one negative, at least four imaginary. [Compare Art. 554.]
16. Six. 17. (1) $pq=r$; (2) $p^3r=q^3$. 20. q^2-2pr.
21. $pq-r$. 22. $\frac{pq}{r}-3$. 23. $pq-3r$.
24. $pr-4s$. 25. $p^4-4p^2q+2q^2+4pr-4s$.

XXXV. c. Pages 470, 471.

1. $x^4-6x^3+15x^2-12x+1$. 2. $x^4-37x^2-123x-110$.
3. $2x^4+8x^3-x^2-8x-20$. 4. x^4-24x^2-1.
5. $16axh(x^6+7x^4h^2+7x^2h^4+h^6)+2bh(5x^4+10x^2h^2+h^4)+2ch$.
10. 2, 2, -1, -3. 11. 1, 1, 1, 3. 12. 3, 3, 3, 2, 2.
13. -2, $\frac{1\pm\sqrt{-3}}{2}$, $\frac{1\pm\sqrt{-3}}{2}$. 14. $\frac{1}{2}$, $\frac{1}{2}$, $\frac{1}{2}$, -2.
15. 1, 1, 1, -1, -1, 2. 16. $\pm\sqrt{3}$, $\pm\sqrt{3}$, $1\pm\sqrt{-1}$.
17. a, a, $-a$, b. 18. $\pm\sqrt{\frac{3}{2}}$, $\frac{1\pm\sqrt{-7}}{2}$; $\pm\sqrt{\frac{3}{2}}$, $\frac{1\pm\sqrt{-23}}{4}$.
19. 0, 1, $-\frac{3}{2}$, $-\frac{5}{2}$; 0, 1, $-\frac{3}{2}$, $-\frac{5}{3}$. 20. $n^nr^{n-2}=4p^n(n-2)^{n-2}$.
22. (1) -2; (2) -1. 27. 5. 28. 99, 795.

XXXV. d. Pages 478, 479.

1. $y^3-24y^2+9y-24=0$. 2. $y^4-5y^3+3y^2-9y+27=0$.
3. 1, 1, -2, $-\frac{1}{2}$. 4. $3\pm2\sqrt{2}$, $2\pm\sqrt{3}$.
5. 1, $\frac{1\pm\sqrt{-3}}{2}$, $\frac{3\pm\sqrt{5}}{2}$. 6. 2, 2, $\frac{1}{2}$, $\frac{1}{2}$, $\frac{1}{2}(1\pm\sqrt{-3})$.
7. 4, 2, $\frac{4}{3}$. 8. 6, 3, 2. 10. $\frac{1}{4}$, 1, $-\frac{1}{2}$, $-\frac{1}{5}$.
11. $y^3-2y+1=0$. 12. $y^4-4y^2+1=0$. 13. $y^5-7y^3+12y^2-7y=0$.
14. $y^6-60y^4-320y^3-717y^2-773y-42=0$.
15. $y^3-\frac{9y^2}{2}+\frac{13y}{2}-\frac{15}{4}=0$. 16. $y^5+11y^4+42y^3+57y^2-13y-60=0$.
17. $y^3-8y^2+19y-15=0$. 18. $y^4+3y^3+4y^2+3y+1=0$.
19. $y^3+33y^2+12y+8=0$. 20. $ry^3+kqy^2+k^3=0$.
21. $y^3-q^2y^2-2qr^2y-r^4=0$. 22. $ry^3-qy^2-1=0$.
23. $ry^3+q(1-r)y^2+(1-r)^3=0$. 24. $y^3-2qy^2+q^2y+r^2=0$.
25. $y^3+3ry^2+(q^3+3r^2)y+r^3=0$.
26. $r^3y^3+3r^3y^2+(3r^2+q^3)ry+r(r^2+2q^3)=0$. 28. ±1, ±2, 5.

XXXV. e. Pages 488, 489.

1. $5, \frac{-5 \pm \sqrt{-3}}{2}$. 2. $10, -5 \pm 7\sqrt{-3}$. 3. $4, -2 \pm 5\sqrt{-3}$.

4. $-6, 3 \pm 4\sqrt{-3}$. 5. $-\frac{1}{4}, \frac{2 \pm \sqrt{-3}}{7}$. 6. 11, 11, 7.

7. $-\frac{1}{2}, -\frac{1 \pm \sqrt{-3}}{2}$. 9. $4, -1, -\frac{1}{2}(3 \pm \sqrt{-31})$.

10. $4, -2, -1 \pm \sqrt{-1}$. 11. $\pm 1, -4 \pm \sqrt{6}$. 12. $1, 2, -2, -3$.

13. $1 \pm \sqrt{2}, -1 \pm \sqrt{-1}$. 14. $1, -3, 2 \pm \sqrt{5}$.

15. $2, 2, \frac{1}{2}, \frac{1}{2}$. 16. $1, 4 \pm \sqrt{15}, -\frac{3 \pm \sqrt{5}}{2}$.

17. $-4, -4, -4, 3$. 18. $q^3 + 8r^2 = 0$; $\frac{3}{2}, \frac{-3 \pm 3\sqrt{5}}{4}$.

22. $-2 \pm \sqrt{6}, \pm\sqrt{2}, 2 \pm \sqrt{2}$. 23. $s^3y^4 + qs(1-s)^2y^2 + r(1-s)^3y + (1-s)^4 = 0$.

25. $2 \pm \sqrt{3}$. 26. $\frac{5 \pm \sqrt{13}}{2}$.

28. $x^4 - 8x^3 + 21x^2 - 20x + 5 = (x^2 - 5x + 5)(x^2 - 3x + 1)$; on putting $x = 4 - y$, the expressions $x^2 - 5x + 5$ and $x^2 - 3x + 1$ become $y^2 - 3y + 1$ and $y^2 - 5y + 5$ respectively, so that we merely reproduce the original equation.

MISCELLANEOUS EXAMPLES. Pages 490—524.

2. 6, 8. 3. Eight.

4. (1) $1 \pm \sqrt{5}$; $1 \pm 2\sqrt{5}$.
   (2) $x = 1, y = 3, z = -5$; or $x = -1, y = -3$; $z = 5$.

6. (1) $1, -\frac{a+2b}{2a+b}$. (2) 3. 7. First term 1; common difference $\frac{1}{3}$.

8. $p^2 - q$; $-p(p^2 - 3q)$; $(p^2 - q)(p^2 - 3q)$.

9. $\frac{1}{2}(ab + a^{-1}b^{-1})$. 10. $\frac{7}{13}$. 13. A, 7 minutes; B, 8 minutes.

14. $a^4 + b^4 + c^4 = b^2c^2 + c^2a^2 + a^2b^2$.

15. $x^2 = y^2 = \frac{d}{a+b+c}$; or $\frac{x}{c-a} = \frac{y}{a-b} = k$;
   where $k^2a(a^2 + b^2 + c^2 - bc - ca - ab) = d$.

16. One mile per hour.

17. (1) $(b+c)(c+a)(a+b)$. (2) $\sqrt{\frac{5-4x}{2}} + \sqrt{\frac{2x-3}{2}}$. 18. $\frac{35}{9}$; 2268.

19. (1) $\frac{21 \pm \sqrt{105}}{14}$.
   (2) $x = y = \pm\sqrt{ab}$; $\frac{x}{2a+b} = \frac{y}{-(3a+2b)} = \pm\sqrt{\frac{ab}{b^2 + ab - a^2}}$.

22. $1et5$; nine. 23. $\frac{1}{2}\{(1 + 2 + 3 + \ldots + n)^2 - (1^2 + 2^2 + 3^2 + \ldots + n^2)\}$.

24. Wages 15*s*.; loaf 6*d*.

25. 6, 10, 14, 18.

26. (1) 1, $\frac{c(a-b)}{a(b-c)}$. (2) $\frac{ab(c+d)-cd(a+b)}{ab-cd}$.

28. $88\frac{8}{9}$ miles.

29. $x=3k$, $y=4k$, $z=5k$; where $k^3=1$, so that $k=1$, ω, or ω^2.

30. 480.

31. Either 33 half-crowns, 19 shillings, 8 fourpenny pieces;
or 37 half-crowns, 6 shillings, 17 fourpenny pieces.

32. $a=6$, $b=7$.

33. 40 minutes.

35. $1+x+\frac{1}{2}x^2-\frac{1}{2}x^3-\frac{13}{8}x^4$.

37. $\frac{-1\pm\sqrt{-3}}{2}$, or $\frac{1\pm\sqrt{21}}{2}$. [$x^4-x-5(x^2+x+1)=0$.]

38. $a=8$; $\frac{x-4}{x-5}$.

40. The first term.

41. 13, 9.

42. $\frac{1+4b^2c^2+9c^2a^2+a^2b^2}{a^2+b^2+c^2}$.

43. (1) 3, -2, $\frac{-1\pm\sqrt{-39}}{2}$. [Add x^2+4 to each side.]

(2) $x=1,\ -\frac{1}{2},\ -1,\ 0,\ 0;$

$x=1,\ -\frac{1}{2},\ 0,\ -1,\ 0;$

$z=1,\ -\frac{1}{2},\ 0,\ 0,\ -1.$

47. 5780.

48. 150 persons changed their mind; at first the minority was 250, the majority 350.

50. 936 men.

51. (1) 0, $\frac{2^m-1}{2^m+1}a$. (2) $\frac{ad-bc}{a-b-c+d}$.

[Put $(a-c)(b-d)=\{(x-c)-(x-a)\}\{(x-d)-(x-b)\}$; then square.]

53. 6, $-\frac{161}{30}$.

55. $m=\frac{2b\sqrt{a}}{\sqrt{a}+\sqrt{b}}$, $n=\frac{2a\sqrt{b}}{\sqrt{a}+\sqrt{b}}$.

58. (1) 1. (2) $\pm 4\sqrt{2}$ [putting $x^2-16=y^4$, we find $y^4-16-4y(y^2-4)=0$.]

60. $\frac{(a-c)p}{b-c}$ males; $\frac{(b-a)p}{b-c}$ females.

63. 0, $a+b$, $\frac{a^2+b^2}{a+b}$.

64. Common difference of the A. P. is $\frac{b-a}{n-1}$; common difference of the A.P. which is the reciprocal of the H.P. is $\frac{a-b}{ab(n-1)}$. [The r^{th} term is $\frac{a(n-r)+b(r-1)}{n-1}$; the $(n-r+1)^{\text{th}}$ term is $\frac{ab(n-1)}{a(n-r)+b(r-1)}$.]

68. 19.

69. £78.

70. 0, $\frac{1\pm\sqrt{-3}}{2}$, $\frac{-1\pm\sqrt{-3}}{2}$.

[$(a+b)^3-a^3-b^3=3ab(a+b)$, and $(a-b)^3-a^3+b^3=-3ab(a-b)$.]

72. (1) $x=\pm\frac{\log 3}{\log 6}=\pm\cdot 614$. (2) $x=\pm\frac{2(1-2\log 2)}{1-\log 2}=\pm 1\cdot 139$.

73. 7, 2. 74. 8 hours.

79. (1) $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}=0$, or $\frac{a+b+c}{abc}$. (2) $x=y=z=1$.

80. $a=3$, $b=1$. 81. [Put $x-a=u$ and $y-b=v$.] 82. $x=3$. 84. 126.

85. Sums invested were £7700 and £3500: the fortune of each was £1400.

86. 503 in scale seven. 91. 25 miles from London.

95. $x=5,\ 1,\ \frac{15\pm 6\sqrt{-1}}{29}$; $y=3,\ \frac{3}{5},\ \frac{25\pm 10\sqrt{-1}}{29}$. 96. $\sqrt{\frac{5}{3}}$. 98. $\frac{1}{2c}$.

100. Generating function is $\frac{1+4x}{1-x-2x^2}$; sum $=\frac{2(1-2^nx^n)}{1-2x}-\frac{1-(-1)^nx^n}{1+x}$.

n^{th} term $=\{2^n+(-1)^n\}\,x^{n-1}$.

107. a^2+b-c^2-d. 108. 12 persons, £14. 18*s*.

109. (1) $x=a$, $y=b$, $z=c$. (2) $x=3$, or 1; $y=1$, or 3.

111. $1+\frac{1}{1+}\,\frac{1}{12+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{9}$; $x=948$, $y=492$. 113. £12. 15*s*.

117. (1) $x=a$, $y=b$; $x=a$, $y=2a$; $x=2b$, $y=b$.
(2) $x=3$ or 1, $y=2$, $z=1$ or 3;
$x=\frac{-1\pm\sqrt{29}}{2}$, $y=-3$; $z=\frac{-1\mp\sqrt{29}}{2}$.

120. (1) $1-\frac{1}{(n+1)^2}$.
(2) $\frac{a(x^n-1)}{x-1}+\frac{b}{(x-1)^3}\{x^{n+2}+x^{n+1}-(n+1)^2x^2+(2n^2+2n-1)x-n^2\}$.

121. $\frac{1}{3}$. 122. (1) $\frac{-5\pm\sqrt{-11}}{6}$ or $\frac{-3\pm\sqrt{5}}{2}$.
(2) $x=0$, $y=0$, $z=0$; $x=\pm 2$, $y=\pm 1$, $z=\pm 3$.

124. $\frac{13x-23}{3(x^2-3x-1)}-\frac{10x-1}{3(x^2+x+1)}$; $-\frac{r+4}{2^{r+1}}$.

125. $l=1$; scale of relation is $1-x-2x^2$; general term is $\{2^{n-3}+(-1)^{n-1}\}\,x^{n-1}$.

127. (1) $x=-6,\ 2$; $y=9,\ -3$. (2) $x=\frac{1}{a}$; $y=\frac{1}{b}$.

128. (1) $\frac{a^2}{2}$. (2) $\frac{2\sqrt{3}}{9}$. 129. 12, 16; or 48, 4.

130. (1) $x=\pm 7$.
(2) $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}=\pm\frac{k}{2abc}$, where $k^2=2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4$.

133. 11, $r-1$. 134. 384 sq. yds. 136. $a=\pm 2$, $b=3$, $c=\pm 2$.

137. (1) $x=\pm\frac{7}{\sqrt{2}}$, $y=\pm\frac{9}{\sqrt{2}}$. (2) $\pm\frac{1}{\sqrt{2}}$; $\pm\sqrt{\frac{13}{10}}$.

138. £3. 2*s*. at the first sale and £2. 12*s*. at the second sale.

139. (1) $\frac{1}{6}n(n+1)(2n+1)$. (2) $\frac{1}{60}n(n+1)(n+2)(3n^2+6n+1)$.

(3) $\frac{1}{6}n(n+1)(4n-1)$.

141. (1) $x=1$ or $\frac{14}{5}$; $y=3$ or $\frac{15}{7}$.

(2) x, y, z may have the permutations of the values 3, 5, 7.

142. $y^3+qy^2-q^2y-q^3-8r=0$.

143. (1) $\frac{x(x^n-1)}{(x-1)^2}-\frac{n}{x-1}$. (2) $\frac{3+14x-157x^2}{1+5x-50x^2-8x^3}$.

(3) $2^{n+1}+\frac{1}{2}n(n+7)-2$. **144.** $2(b^3-d^3)=3(b^2-c^2)(b-a)$.

145. -2, -2, -2, $\frac{2}{3}$.

146.

A walks in successive days	1, 3, 5, 7, 9,	11, 13, 15, 17, 19, 21, 23,	miles,
B walks	12, 13,	14, 15, 16, 17, 18, 19, 20,	

so that B overtakes A in 2 days and passes him on the third day; A subsequently gains on B and overtakes him on B's 9^{th} day.

147. $\frac{\sqrt{6}-1}{5}$. **148.** $-(a+b+c)$, $-(a+\omega b+\omega^2 c)$, $-(a+\omega^2 b+\omega c)$.

150. n^{th} term is $\frac{n(a^n-b^n)}{a-b}x^{n-1}$; Sum $=A-B$,

where $A=\frac{a(1-na^n x^n)}{1-ax}+\frac{a^2x(1-a^{n-1}x^{n-1})}{(1-ax)^2}$, and B denotes a corresponding function of B.

151. $qy^3-2p^2y^2-5pqy-2p^3-q^2=0$.

153. (1) -7, $\frac{7\pm3\sqrt{-3}}{2}$. (2) ±1, ±3, 4. **154.** 3 days.

156. (1) $\frac{1}{2}$, $-\frac{1}{12}$, $\frac{5\pm\sqrt{-39}}{24}$. $[(12x-1)(12x-2)(12x-3)(12x-4)=120.]$

(2) $1\pm\sqrt{19}$. $\left[\frac{92}{585}=\frac{1}{5}+\frac{1}{9}-\frac{2}{13}.\right]$

157. 22 years nearly. **161.** 44 hours.

162. (1) $x^2=y^2=\frac{-7\pm\sqrt{217}}{4}$; $x=\pm1$, ±2; $y=\mp2$, ∓1; $x=-y=\pm\sqrt{3}$

(2) $x=k(b^4+c^4-a^2b^2-a^2c^2)$, &c., where $2k^2(a^6+b^6+c^6-3a^2b^2c^2)=1$.
[It is easy to shew that $a^2x+b^2y+c^2z=0$, and

$$a^2y+b^2z+c^2x=x^3+y^3+z^3-3xyz=a^2z+b^2x+c^2y.]$$

163. $2(a+b+c)x=(bc+ca+ab)\pm\sqrt{(bc+ca+ab)^2-4abc(a+b+c)}$.
[Equation reduces to $(a+b+c)x^2-(bc+ca+ab)x+abc=0$.]

164. (1) $\frac{1}{12}n(n+1)(n+2)(3n+13)$. (2) $2e-5$.

166. (1) $x=\frac{51+30\sqrt{2}}{8}a$, $y=\frac{17a}{8}$. [Eliminate x.]

(2) x, y, z are the permutations of the quantities $2, \frac{1+\sqrt{-3}}{2}, \frac{1-\sqrt{-3}}{2}$.

167. $(x+y+z)^2=3k^2$. 168. 2. 169. $x^3+y^3+z^3-3xyz$.

170. He walks $3\frac{3}{4}$ miles, drives $7\frac{1}{2}$ miles, rides 10 miles per hour. $AB=37\frac{1}{2}$, $BC=30$, $CA=15$ miles.

172. (1) $x=13$ or 10, $y=10$ or 13.

(2) $x=\frac{d(a-b)}{d-c}$; $y=\frac{c(a-b)}{d-c}$; $z=\frac{b(d-c)}{a-b}$; $u=\frac{a(d-c)}{a-b}$.

174. £3200. 176. $ry^3+3ry^2+(3r-p^3)y+r^3=0$.

177. $p=(ac\pm bd)(eg\pm fh)+(bc\mp ad)(fg\mp eh)$;
$q=(bc\mp ad)(eg\pm fh)-(ac\pm bd)(fg\mp eh)$.

178. $x=6, -5$; $\frac{13\pm\sqrt{-47}}{2}$; $\frac{-14\pm\sqrt{-74}}{2}$;

$y=5, -6$; $\frac{-13\pm\sqrt{-47}}{2}$; $\frac{14\pm\sqrt{-74}}{2}$.

[Put $x-y=u$ and $xy=v$, then $u^2+2v=61$, $u(61+v)=91$.]

182. 8987. 183. $y^3-by^2-acy-c^2=0$. $-1, -2, -\frac{1}{2}, \frac{3\pm\sqrt{5}}{2}$.

186. (1) x, y, z are the permutations of the quantities $1, \frac{1+\sqrt{-3}}{2}, \frac{1-\sqrt{-3}}{2}$.

(2) $x=\pm\frac{a(b^2+c^2)}{2bc}$, &c.

187. Conservatives; English 286, Scotch 19, Irish 35, Welsh 11.
Liberals; English 173, Scotch 41, Irish 68, Welsh 19.

191. (1) 7, 9, -3. (2) $2\pm\sqrt{-3}$, $-2\pm\sqrt{-1}$.

192. $2a_n=a+b+\frac{a-b}{3^n}$; $2b_n=a+b-\frac{a-b}{3^n}$. 201. $\frac{\lfloor m+n-2}{\lfloor m-1 \, \lfloor n-1}$.

202. 54, -26, $14\pm 840\sqrt{-1}$. 204. $\frac{n}{n+1}$, $\frac{4^{n+1}+4(-1)^{n+1}}{4^{n+1}-(-1)^{n+1}}$.

206. $\frac{3rm^3+nm^2q-3n^3}{rm^3+nm^2q+n^3}$. 207. 81 years nearly.

209. 7 Poles, 14 Turks, 15 Greeks, 24 Germans, 20 Italians.

210. $\frac{1}{2}-\frac{x}{4}-\frac{1+x^2}{2x}\log(1+x)$.

212. (1) $\frac{1}{4}n(n+1)(n+2)(n+3)$; (2) $\frac{4+3x+x^2}{(1+x)^3}$; (3) 23.

213. $-\frac{11}{97}$. 215. $x=a+\sqrt{\frac{(a^2+\beta)(a^2+\gamma)}{a^2+a}}$, &c. 217. 420.

223. (1) $x=y=\frac{1}{3}(\pm 15\pm\sqrt{-3})$, $z=\frac{1}{3}(\pm 15\mp 2\sqrt{-3})$;
or $x=4,\ 6,\ -4,\ -6$;
$y=6,\ 4,\ -6,\ -4$;
$z=5,\ 5,\ -5,\ -5$.

(2) $\dfrac{x-a}{a(b-c)}=\dfrac{y-b}{b(c-a)}=\dfrac{z-c}{c(a-b)}=\lambda$,
where $(b-c)(c-a)(a-b)\lambda=a^2+b^2+c^2-bc-ca-ab$.

226. 12 calves, 15 pigs, 20 sheep. **229.** $\text{Lim}\left\{n\left(\dfrac{u_n}{u_{n+1}}-1\right)\right\}=\dfrac{3}{2}$; convergent.

230. Scale of relation is $1-12x+32x^2$; n^{th} term $=\frac{1}{2}\{4^{n-1}+8^{n-1}\}$;
$S_n=\dfrac{2^{2n-1}}{3}+\dfrac{2^{3n-1}}{7}-\dfrac{5}{21}$.

231. $\dfrac{11}{243}$. **232.** $2x=\pm\sqrt{a^2-b^2+c^2}\pm\sqrt{a^2+b^2-c^2}$, &c.

233. $a^3+b^3+c^3=a^2(b+c)+b^2(c+a)+c^2(a+b)$.

235. (1) $(1-x)^4S=1+4x+x^2-(n+1)^3x^n+(3n^3+6n^2-4)x^{n+1}$
$-(3n^3+3n^2-3n+1)x^{n+2}+n^3x^{n+3}$.

(2) $\dfrac{1}{8}-\dfrac{1}{(n+1)^2(n+2)^3}$.

236. $1+a^3x^4+a^5x^8+a^8x^{12}+a^9x^{16}+a^{12}x^{20}+a^{14}x^{24}+a^{17}x^{28}+a^{17}x^{32}+a^{20}x^{36}$.

237. 3 hours 51 min. **240.** 2 or $-\dfrac{1}{2}$. **242.** -140.

244. 3, 4, 5, 6. **246.** $a^3(c^3-3d^3)^2=(ab^2+2d^3)(ab^2-d^3)^2$.

247. 2, 6, 1, 3. **248.** $\dfrac{6}{13}$.

249. (1) $2^{n+1}-2-\dfrac{1}{6}n(n+1)(2n+1)$.

(2) $\dfrac{2^{n+1}}{(n+1)(n+3)}-\dfrac{2}{3}$.

(3) $\dfrac{1-x^{n+1}}{1-x}+\dfrac{2(1-2^nx^n)}{1-4x^2}$ when n is even; $\dfrac{1-x^{n+1}}{1-x}+\dfrac{2(1-2^{n+1}x^{n+1})}{1-4x^2}$
when n is odd.

250. (1) $x=y=z=0$ or $\dfrac{a}{3}$. If however $x^2+y^2+z^2+yz+zx+xy=0$, then $x+y+z=-a$, and the solution is indeterminate.

(2) $\dfrac{x}{a(-a+b+c)}=\dfrac{y}{b(a-b+c)}=\dfrac{z}{c(a+b-c)}$

$$=\frac{1}{\pm\sqrt{(-a+b+c)(a-b+c)(a+b-c)}}.$$

253. $-(Ax+By+Cz)(-Ax+By+Cz)(Ax-By+Cz)(Ax+By-Cz)$ where $A=\sqrt{a(b-c)}$, &c.

256. (1) $x=1,\ \omega,\ \omega^2;$
$y=1,\ \omega^2,\ \omega;$
$z=-(a+b),\ -(a\omega+b\omega^2),\ -(a\omega^2+b\omega).$

(2) $\left.\begin{matrix} x=3, \text{ or } 7 \\ y=7, \text{ or } 3 \end{matrix}\right\}\ \left.\begin{matrix} z=6, \text{ or } -4 \\ u=4, \text{ or } -6 \end{matrix}\right\}.$

257. To at least $3r-2$ places. 258. Tea, 2*s.* 6*d.*; Coffee, 1*s.* 8*d.*

262. $2q^2-6pr+24s.$ 263. 11 turkeys, 9 geese, 3 ducks.

266. (1) x, y, z have the permutations of the values

$$a,\ \frac{1}{2}a\left(b-1+\sqrt{b^2-2b-3}\right),\ \frac{1}{2}a\left(b-1-\sqrt{b^2-2b-3}\right).$$

(2) $x=y=z=1;\quad x=\dfrac{a+b+c}{a-b-c};$ &c. 267. 0.

268. 16 Clergymen of average age 45 years;
24 Doctors of average age 35 years;
20 Lawyers of average age 30 years.

269. $(a_0a_2-a_1^2)(a_2a_4-a_3^2)=(a_1a_3-a_2^2)^2;$
or $a_0a_2a_4+2a_1a_2a_3-a_0a_3^2-a_1^2a_4-a_2^3=0.$

270. $x=\pm\dfrac{a^2}{\sqrt{a^2+b^2+c^2}}$, &c. $u=\pm\dfrac{bc}{\sqrt{a^2+b^2+c^2}}$, &c 273. $c^{-\frac{1}{2}}.$

274. (1) $\left(1-\dfrac{2}{x}\right)\log(1-x)-2.$ (2) $\dfrac{1}{a-1}\left\{1-\dfrac{\lfloor n+1}{(a+1)(a+2)\dots(a+n)}\right\}.$

275. (1) $x=\dfrac{3}{2},\ \dfrac{3}{2},\ 2;$

$y=-1,\ -\dfrac{4}{3},\ -1;$

$z=1,\ \dfrac{3}{4},\ \dfrac{3}{4}.$

(2) $x=\pm4,\ y=\pm5,\ u=\pm2,\ v=\pm1.$

$x=\pm\dfrac{5}{3}\sqrt{\dfrac{2}{3}},\ y=\mp2\sqrt{\dfrac{2}{3}},\ u=\mp\dfrac{1}{3}\sqrt{\dfrac{2}{3}},\ v=\pm\sqrt{\dfrac{2}{3}}.$

276. $a^2+b^2+c^2+d^2+\lambda.$ 277. $-p_1^3+3p_1p_2-3p_3.$

279. A, 6 birds; B, 4 birds. 281. 2.

287. $a,\ -5a,\ -5a.$ 289. $x_1=-\dfrac{(b_1-a_1)(b_1-a_2)\dots(b_1-a_n)}{(b_1-b_2)(b_1-b_3)\dots(b_1-b_n)}$, &c.

291. A worked 45 days; B, 24 days; C, 10 days.

294. $(b^2+c^2-a^2)(a^2-b^2+c^2)(a^2+b^2-c^2).$

300. Walked 3 miles, worked 4 hours a day;
or walked 4 miles, worked 3 hours a day.

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