

7. Adjoint and Inverse of a Matrix

Exercise 7.1

1 A. Question

Find the adjoint of each of the following Matrices.

$$\begin{bmatrix} -3 & 5 \\ 2 & 4 \end{bmatrix}$$

Verify that $(\text{adj } A) A = |A| I = A (\text{adj } A)$ for the above matrices.

Answer

$$A = \begin{bmatrix} -3 & 5 \\ 2 & 4 \end{bmatrix}$$

Cofactors of A are

$$C_{11} = 4$$

$$C_{12} = -2$$

$$C_{21} = -5$$

$$C_{22} = -3$$

$$\text{Since, } \text{adj } A = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}^T$$

$$(\text{adj } A) = \begin{bmatrix} 4 & -2 \\ -5 & -3 \end{bmatrix}^T$$

$$= \begin{bmatrix} 4 & -5 \\ -2 & -3 \end{bmatrix}$$

$$\text{Now, } (\text{adj } A)A = \begin{bmatrix} 4 & -5 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} -3 & 5 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} -12 - 10 & 20 - 20 \\ 6 - 6 & -10 - 12 \end{bmatrix}$$

$$(\text{adj } A)A = \begin{bmatrix} -22 & 0 \\ 0 & -22 \end{bmatrix}$$

$$\text{And, } |A|.I = \begin{vmatrix} -3 & 5 \\ 2 & 4 \end{vmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = (-22) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -22 & 0 \\ 0 & -22 \end{bmatrix}$$

$$\text{Also, } A(\text{adj } A) = \begin{bmatrix} -3 & 5 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 4 & -5 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} -12 - 10 & 20 - 20 \\ 6 - 6 & -10 - 12 \end{bmatrix}$$

$$A(\text{adj } A) = \begin{bmatrix} -22 & 0 \\ 0 & -22 \end{bmatrix}$$

Hence, $(\text{adj } A)A = |A|.I = A.(\text{adj } A)$

1 B. Question

Find the adjoint of each of the following Matrices.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Verify that $(\text{adj } A) A = |A| I = A (\text{adj } A)$ for the above matrices.

Answer

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Cofactors of A are

$$C_{11} = d$$

$$C_{12} = -c$$

$$C_{21} = -b$$

$$C_{22} = a$$

$$\text{Since, } \text{adj } A = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}^T$$

$$(\text{adj } A) = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}^T$$

$$= \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\text{Now, } (\text{adj } A)A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ad - bc & bd - bd \\ -ac + ac & -bc + ad \end{bmatrix}$$

$$(\text{adj } A)A = \begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix}$$

$$\text{And, } |A|.I = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = (ad - bc) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix}$$

$$\text{Also, } A(\text{adj } A) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix}$$

$$\text{Hence, } (\text{adj } A)A = |A|.I = A.(\text{adj } A)$$

1 C. Question

Find the adjoint of each of the following Matrices.

$$\begin{bmatrix} \cos a & \sin a \\ \sin a & \cos a \end{bmatrix}$$

Verify that $(\text{adj } A)A = |A|.I = A.(\text{adj } A)$ for the above matrices.

Answer

$$A = \begin{bmatrix} \cos \alpha & \sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix}$$

Cofactors of A are

$$C_{11} = \cos \alpha$$

$$C_{12} = -\sin \alpha$$

$$C_{21} = -\sin \alpha$$

$$C_{22} = \cos \alpha$$

$$\text{Since, } \text{adj } A = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}^T$$

$$(\text{adj } A) = \begin{bmatrix} \cos \alpha & -\sin \alpha \\ -\sin \alpha & \cos \alpha \end{bmatrix}^T$$

$$= \begin{bmatrix} \cos \alpha & -\sin \alpha \\ -\sin \alpha & \cos \alpha \end{bmatrix}$$

$$\begin{aligned}\text{Now, } (\text{adj } A)A &= \begin{bmatrix} \cos\alpha & -\sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \\ &= \begin{bmatrix} -\sin^2\alpha + \cos^2\alpha & \cos\alpha \sin\alpha - \sin\alpha \cos\alpha \\ -\cos\alpha \sin\alpha + \sin\alpha \cos\alpha & -\sin^2\alpha + \cos^2\alpha \end{bmatrix}\end{aligned}$$

$$(\text{adj } A)A = \begin{bmatrix} \cos 2\alpha & 0 \\ 0 & \cos 2\alpha \end{bmatrix}$$

$$\text{And, } |A|.I = \begin{vmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & \cos\alpha \end{vmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= (\cos^2\alpha - \sin^2\alpha) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\alpha - \sin^2\alpha & 0 \\ 0 & \cos^2\alpha - \sin^2\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos 2\alpha & 0 \\ 0 & \cos 2\alpha \end{bmatrix}$$

$$\text{Also, } A(\text{adj } A) = \begin{bmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\alpha & -\sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} = \begin{bmatrix} \cos^2\alpha - \sin^2\alpha & 0 \\ 0 & \cos^2\alpha - \sin^2\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos 2\alpha & 0 \\ 0 & \cos 2\alpha \end{bmatrix}$$

$$\text{Hence, } (\text{adj } A)A = |A|.I = A.(\text{adj } A)$$

1 D. Question

Find the adjoint of each of the following Matrices.

$$\begin{bmatrix} 1 & \tan \alpha / 2 \\ -\tan \alpha / 2 & 1 \end{bmatrix}$$

Verify that $(\text{adj } A)A = |A|.I = A(\text{adj } A)$ for the above matrices.

Answer

$$A = \begin{bmatrix} 1 & \tan \frac{\alpha}{2} \\ -\tan \frac{\alpha}{2} & 1 \end{bmatrix}$$

Cofactors of A are

$$C_{11} = 1$$

$$C_{12} = \tan \frac{\alpha}{2}$$

$$C_{21} = -\tan \frac{\alpha}{2}$$

$$C_{22} = 1$$

$$\text{Since, } \text{adj } A = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}^T$$

$$(\text{adj } A) = \begin{bmatrix} 1 & \tan \frac{\alpha}{2} \\ -\tan \frac{\alpha}{2} & 1 \end{bmatrix}^T$$

$$= \begin{bmatrix} 1 & -\tan \frac{\alpha}{2} \\ \tan \frac{\alpha}{2} & 1 \end{bmatrix}$$

$$\text{Now, } (\text{adj } A)A = \begin{bmatrix} 1 & -\tan\frac{\alpha}{2} \\ \tan\frac{\alpha}{2} & 1 \end{bmatrix} \begin{bmatrix} 1 & \tan\frac{\alpha}{2} \\ -\tan\frac{\alpha}{2} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 + \tan^2\frac{\alpha}{2} & \tan\frac{\alpha}{2} - \tan\frac{\alpha}{2} \\ \tan\frac{\alpha}{2} - \tan\frac{\alpha}{2} & 1 + \tan^2\frac{\alpha}{2} \end{bmatrix}$$

$$(\text{adj } A)A = \begin{bmatrix} 1 + \tan^2\frac{\alpha}{2} & 0 \\ 0 & 1 + \tan^2\frac{\alpha}{2} \end{bmatrix}$$

$$\text{And, } |A|.I = \begin{vmatrix} 1 & \tan\frac{\alpha}{2} \\ -\tan\frac{\alpha}{2} & 1 \end{vmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = (1 + \tan^2\frac{\alpha}{2}) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 + \tan^2\frac{\alpha}{2} & 0 \\ 0 & 1 + \tan^2\frac{\alpha}{2} \end{bmatrix}$$

$$\text{Also, } A(\text{adj } A) = \begin{bmatrix} 1 & \tan\frac{\alpha}{2} \\ -\tan\frac{\alpha}{2} & 1 \end{bmatrix} \begin{bmatrix} 1 & -\tan\frac{\alpha}{2} \\ \tan\frac{\alpha}{2} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 + \tan^2\frac{\alpha}{2} & \tan\frac{\alpha}{2} - \tan\frac{\alpha}{2} \\ \tan\frac{\alpha}{2} - \tan\frac{\alpha}{2} & 1 + \tan^2\frac{\alpha}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 + \tan^2\frac{\alpha}{2} & 0 \\ 0 & 1 + \tan^2\frac{\alpha}{2} \end{bmatrix}$$

$$\text{Hence, } (\text{adj } A)A = |A|.I = A.(\text{adj } A)$$

2 A. Question

Find the adjoint of each of the following Matrices and Verify that $(\text{adj } A) A = |A| I = A (\text{adj } A)$

$$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

Verify that $(\text{adj } A) A = |A| I = A (\text{adj } A)$ for the above matrices.

Answer

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

Cofactors of A are:

$$C_{11} = -3 \quad C_{21} = 2 \quad C_{31} = 2$$

$$C_{12} = 2 \quad C_{22} = -3 \quad C_{32} = 2$$

$$C_{13} = 2 \quad C_{23} = 2 \quad C_{33} = -3$$

$$\text{adj } A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$$

$$= \begin{bmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{bmatrix}$$

$$\begin{aligned}
 \text{Now, } (\text{adj } A).A &= \begin{bmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \\
 &= \begin{bmatrix} -3+4+4 & -6+2+4 & -6+4+2 \\ 2-3+4 & 4-3+4 & 4-6+2 \\ 2+4-6 & 4+2-6 & 4+4-3 \end{bmatrix} \\
 &= \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}
 \end{aligned}$$

$$\begin{aligned}
 \text{Also, } |A|.I &= \begin{vmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{vmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = (-3+4+4) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
 &= \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}
 \end{aligned}$$

$$\begin{aligned}
 \text{Then, } A.(\text{adj } A) &= \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{bmatrix} \\
 &= \begin{bmatrix} -3+4+4 & -6+2+4 & -6+4+2 \\ 2-3+4 & 4-3+4 & 4-6+2 \\ 2+4-6 & 4+2-6 & 4+4-3 \end{bmatrix} \\
 &= \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}
 \end{aligned}$$

Since, $(\text{adj } A).A = |A|.I = A(\text{adj } A)$

2 B. Question

Find the adjoint of each of the following Matrices and Verify that $(\text{adj } A) A = |A| I = A (\text{adj } A)$

$$\begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$

Verify that $(\text{adj } A) A = |A| I = A (\text{adj } A)$ for the above matrices.

Answer

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$

Cofactors of A

$$C_{11} = 2 \quad C_{21} = 3 \quad C_{31} = -13$$

$$C_{12} = -3 \quad C_{22} = 6 \quad C_{32} = 9$$

$$C_{13} = 5 \quad C_{23} = -3 \quad C_{33} = -1$$

$$\begin{aligned}
 \text{adj } A &= \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T \\
 &= \begin{bmatrix} 2 & -3 & 5 \\ 3 & 6 & -3 \\ -13 & 9 & -1 \end{bmatrix}^T
 \end{aligned}$$

$$\text{adj } A = \begin{bmatrix} 2 & 3 & -13 \\ -3 & 6 & 9 \\ 5 & -3 & -1 \end{bmatrix}$$

$$\begin{aligned} \text{Now, } (\text{adj } A).A &= \begin{bmatrix} 2 & 3 & -13 \\ -3 & 6 & 9 \\ 5 & -3 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 1 \\ -1 & 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 2+6+13 & 4+9-13 & 10+3-13 \\ -3+12-9 & -6+18+9 & -15+6+9 \\ 5-6+1 & 10-9-1 & 25-3-1 \end{bmatrix} \\ &= \begin{bmatrix} 21 & 0 & 0 \\ 0 & 21 & 0 \\ 0 & 0 & 21 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \text{Also, } |A|.I &= \begin{vmatrix} 1 & 2 & 5 \\ 2 & 3 & 1 \\ -1 & 1 & 1 \end{vmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = [1(3-1) - 2(2+1) + 5(2+3)] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= (21) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 21 & 0 & 0 \\ 0 & 21 & 0 \\ 0 & 0 & 21 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \text{Then, } A.(\text{adj } A) &= \begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 1 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & -13 \\ -3 & 6 & 9 \\ 5 & -3 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 2-6+25 & 3+12-15 & -13+18-5 \\ 4-9+5 & 6+18-3 & -26+27-1 \\ -2-3+5 & -3+6-3 & 13+9-1 \end{bmatrix} \\ &= \begin{bmatrix} 21 & 0 & 0 \\ 0 & 21 & 0 \\ 0 & 0 & 21 \end{bmatrix} \end{aligned}$$

Hence, $(\text{adj } A).A = |A|.I = A(\text{adj } A)$

2 C. Question

Find the adjoint of each of the following Matrices and Verify that $(\text{adj } A) A = |A| I = A (\text{adj } A)$

$$\begin{bmatrix} 2 & -1 & 3 \\ 4 & 2 & 5 \\ 0 & 4 & -1 \end{bmatrix}$$

Verify that $(\text{adj } A) A = |A| I = A (\text{adj } A)$ for the above matrices.

Answer

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 4 & 2 & 5 \\ 0 & 4 & -1 \end{bmatrix}$$

Cofactors of A

$$C_{11} = -22 \quad C_{21} = 11 \quad C_{31} = -11$$

$$C_{12} = 4 \quad C_{22} = -2 \quad C_{32} = 2$$

$$C_{13} = 16 \quad C_{23} = -8 \quad C_{33} = 8$$

$$\text{adj } A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$$

$$= \begin{bmatrix} -22 & 4 & 16 \\ 11 & -2 & -8 \\ -11 & 2 & 8 \end{bmatrix}^T$$

$$\text{adj } A = \begin{bmatrix} -22 & 11 & -11 \\ 4 & -2 & 2 \\ 16 & -8 & 8 \end{bmatrix}$$

$$\text{Now, } (\text{adj } A).A = \begin{bmatrix} -22 & 11 & -11 \\ 4 & -2 & 2 \\ 16 & -8 & 8 \end{bmatrix} \begin{bmatrix} 2 & -1 & 3 \\ 4 & 2 & 5 \\ 0 & 4 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -44 + 44 + 0 & 22 + 22 - 44 & -66 + 55 + 11 \\ 8 - 8 + 0 & -4 - 4 + 8 & 12 - 10 - 2 \\ 32 - 32 + 0 & -16 - 16 + 32 & 48 - 40 - 8 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{Also, } |A|.I = \begin{vmatrix} 2 & -1 & 3 \\ 4 & 2 & 5 \\ 0 & 4 & -1 \end{vmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= [2(-2 - 20) + 1(-4 - 0) + 3(16 - 0)] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= (-44 - 4 + 48) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{Then, } A.(\text{adj } A) = \begin{bmatrix} 2 & -1 & 3 \\ 4 & 2 & 5 \\ 0 & 4 & -1 \end{bmatrix} \begin{bmatrix} -22 & 11 & -11 \\ 4 & -2 & 2 \\ 16 & -8 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} -44 - 4 + 48 & 22 + 2 - 24 & -22 - 2 + 24 \\ -88 + 8 + 80 & 44 - 4 - 40 & -44 + 4 + 40 \\ 0 + 16 - 16 & 0 - 8 + 8 & 0 + 8 - 8 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{Hence, } (\text{adj } A).A = |A|.I = A(\text{adj } A)$$

2 D. Question

Find the adjoint of each of the following Matrices and Verify that $(\text{adj } A) A = |A| I = A (\text{adj } A)$

$$\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 1 & 1 & 3 \end{bmatrix}$$

Verify that $(\text{adj } A) A = |A| I = A (\text{adj } A)$ for the above matrices.

Answer

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 1 & 1 & 3 \end{bmatrix}$$

Cofactors of A

$$C_{11} = 3 \quad C_{21} = -1 \quad C_{31} = -1$$

$$C_{12} = -15 \quad C_{22} = 7 \quad C_{32} = -5$$

$$C_{13} = 4 \quad C_{23} = -2 \quad C_{33} = 2$$

$$\text{adj } A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$$

$$= \begin{bmatrix} 3 & -15 & 4 \\ -3 & 7 & -2 \\ 1 & -5 & 2 \end{bmatrix}^T$$

$$\text{adj } A = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 7 & -5 \\ 4 & -2 & 2 \end{bmatrix}$$

$$\text{Now, } (\text{adj } A) \cdot A = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 7 & -5 \\ 4 & -2 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 1 & 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 6 - 5 + 1 & 0 - 1 + 1 & -3 + 0 + 3 \\ -30 + 35 - 5 & 0 + 7 - 5 & 15 - 0 - 15 \\ 8 - 10 + 2 & 0 - 2 + 2 & -4 - 0 + 6 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\text{Also, } |A| \cdot I = \begin{vmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 1 & 1 & 3 \end{vmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= [2(3 - 0) + 0(15 - 0) - 1(5 - 1)] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= (6 - 4) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\text{Then, } A \cdot (\text{adj } A) = \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} 3 & -1 & 1 \\ -15 & 7 & -5 \\ 4 & -2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 6 + 0 - 4 & -2 + 0 + 2 & 2 - 0 - 2 \\ 15 - 15 + 0 & -5 + 7 + 0 & 5 - 5 + 0 \\ 3 - 15 + 12 & -1 + 7 - 6 & 1 - 5 + 6 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Hence, $(\text{adj } A) \cdot A = |A| \cdot I = A(\text{adj } A)$

3. Question

For the matrix $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & 0 \\ 18 & 2 & 10 \end{bmatrix}$, show that $A(\text{adj } A) = O$.

Answer

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & 0 \\ 18 & 2 & 10 \end{bmatrix}$$

Cofactors of A

$$C_{11} = 30 \quad C_{21} = 12 \quad C_{31} = -3$$

$$C_{12} = -20 \quad C_{22} = -8 \quad C_{32} = 2$$

$$C_{13} = -50 \quad C_{23} = -20 \quad C_{33} = 5$$

$$\text{adj } A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$$

$$= \begin{bmatrix} 30 & -20 & -50 \\ 12 & -8 & -20 \\ -3 & 2 & 5 \end{bmatrix}^T$$

$$\text{So, adj}(A) = \begin{bmatrix} 30 & 12 & -3 \\ -20 & -8 & 2 \\ -50 & -20 & 5 \end{bmatrix}$$

$$\text{Now, } A(\text{adj } A) = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & 0 \\ 18 & 2 & 10 \end{bmatrix} \begin{bmatrix} 30 & 12 & -3 \\ -20 & -8 & 2 \\ -50 & -20 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 30 + 20 - 50 & 12 + 8 - 20 & -3 - 2 + 5 \\ 60 - 60 + 0 & 24 - 24 + 0 & -6 + 6 + 0 \\ 540 - 40 - 500 & 216 - 16 - 200 & -54 + 4 + 50 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, $A(\text{adj } A) = O$

4. Question

If $A = \begin{bmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & 3 \end{bmatrix}$, show that $\text{adj } A = A$.

Answer

$$A = \begin{bmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & 3 \end{bmatrix}$$

Cofactors of A

$$C_{11} = -4 \quad C_{21} = -3 \quad C_{31} = -3$$

$$C_{12} = 1 \quad C_{22} = 0 \quad C_{32} = 1$$

$$C_{13} = 4 \quad C_{23} = 4 \quad C_{33} = 3$$

$$\text{adj } A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$$

$$= \begin{bmatrix} -4 & 1 & 4 \\ -3 & 0 & 4 \\ -3 & 1 & 3 \end{bmatrix}^T$$

$$\text{So, adj } A = \begin{bmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & 3 \end{bmatrix}$$

Hence, $\text{adj } A = A$

5. Question

$$\text{If } A = \begin{bmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}, \text{ show that } \text{adj } A = 3A^T.$$

Answer

$$A = \begin{bmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

Cofactors of A are:

$$C_{11} = -3 \quad C_{21} = 6 \quad C_{31} = 6$$

$$C_{12} = -6 \quad C_{22} = 3 \quad C_{32} = -6$$

$$C_{13} = -6 \quad C_{23} = -6 \quad C_{33} = 3$$

$$\text{adj } A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$$

$$= \begin{bmatrix} -3 & -6 & -6 \\ 6 & 3 & -6 \\ 6 & -6 & 3 \end{bmatrix}^T$$

$$\text{So, adj } A = \begin{bmatrix} -3 & 6 & 6 \\ -6 & 3 & -6 \\ -6 & -6 & 3 \end{bmatrix}$$

$$\text{Now, } 3A^T = 3 \begin{bmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 6 & 6 \\ -6 & 3 & -6 \\ -6 & -6 & 3 \end{bmatrix}$$

Hence, $\text{adj } A = 3.A^T$

6. Question

$$\text{Find } A (\text{adj } A) \text{ for the matrix } A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 2 & -1 \\ -4 & 5 & 2 \end{bmatrix}.$$

Answer

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 2 & -1 \\ -4 & 5 & 2 \end{bmatrix}$$

Cofactors of A are:

$$C_{11} = 9 \quad C_{21} = 19 \quad C_{31} = -4$$

$$C_{12} = 4 \quad C_{22} = 14 \quad C_{32} = 1$$

$$C_{13} = 8 \quad C_{23} = 3 \quad C_{33} = 2$$

$$\text{adj } A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$$

$$= \begin{bmatrix} 9 & 4 & 8 \\ 19 & 14 & 3 \\ -4 & 1 & 2 \end{bmatrix}^T$$

$$\text{So, adj } A = \begin{bmatrix} 9 & 19 & -4 \\ 4 & 14 & 1 \\ 8 & 3 & 2 \end{bmatrix}$$

$$\text{Now, } A \cdot \text{adj } A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 2 & -1 \\ -4 & 5 & 2 \end{bmatrix} \begin{bmatrix} 9 & 19 & -4 \\ 4 & 14 & 1 \\ 8 & 3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 9 - 8 + 24 & 19 - 28 + 9 & -4 - 2 + 6 \\ 0 + 8 - 8 & 0 + 28 - 3 & 0 + 2 - 2 \\ -36 + 20 + 16 & -76 + 70 + 6 & 16 + 5 + 4 \end{bmatrix}$$

$$= \begin{bmatrix} 25 & 0 & 0 \\ 0 & 25 & 0 \\ 0 & 0 & 25 \end{bmatrix}$$

Hence, $A \cdot \text{adj } A = 25 \cdot I_3$

7 A. Question

Find the inverse of each of the following matrices:

$$\begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$$

Answer

$$\text{Now, } |A| = \cos \theta (\cos \theta) + \sin \theta (\sin \theta)$$

$$= 1$$

Hence, A^{-1} exists.

Cofactors of A are

$$C_{11} = \cos \theta$$

$$C_{12} = \sin \theta$$

$$C_{21} = -\sin \theta$$

$$C_{22} = \cos \theta$$

$$\text{Since, adj } A = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}^T$$

$$(\text{adj } A) = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}^T$$

$$= \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

$$\text{Now, } A^{-1} = \frac{1}{|A|} \cdot \text{adj } A$$

$$A^{-1} = \frac{1}{1} \cdot \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

7 B. Question

Find the inverse of each of the following matrices:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Answer

$$\text{Now, } |A| = -1 \neq 0$$

Hence, A^{-1} exists.

Cofactors of A are

$$C_{11} = 0$$

$$C_{12} = -1$$

$$C_{21} = -1$$

$$C_{22} = 0$$

$$\text{Since, } \text{adj } A = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}^T$$

$$(\text{adj } A) = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}^T$$

$$= \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

$$\text{Now, } A^{-1} = \frac{1}{|A|} \cdot \text{adj } A$$

$$A^{-1} = -\frac{1}{1} \cdot \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

7 C. Question

Find the inverse of each of the following matrices:

$$\begin{bmatrix} a & b \\ c & \frac{1+bc}{a} \end{bmatrix}$$

Answer

$$\text{Now, } |A| = \frac{a+abc}{a} - bc = \frac{a+abc-abc}{a} = 1 \neq 0$$

Hence, A^{-1} exists.

Cofactors of A are

$$C_{11} = \frac{1+bc}{a}$$

$$C_{12} = -c$$

$$C_{21} = -b$$

$$C_{22} = a$$

$$\text{Since, adj } A = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}^T$$

$$(\text{adj } A) = \begin{bmatrix} \frac{1+bc}{a} & -c \\ -b & a \end{bmatrix}^T$$

$$= \begin{bmatrix} \frac{1+bc}{a} & -b \\ -c & a \end{bmatrix}$$

$$\text{Now, } A^{-1} = \frac{1}{|A|} \cdot \text{adj } A$$

$$A^{-1} = \frac{1}{1} \cdot \begin{bmatrix} \frac{1+bc}{a} & -b \\ -c & a \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \frac{1+bc}{a} & -b \\ -c & a \end{bmatrix}$$

7 D. Question

Find the inverse of each of the following matrices:

$$\begin{bmatrix} 2 & 5 \\ -3 & 1 \end{bmatrix}$$

Answer

$$\text{Now, } |A| = 2 + 15 = 17$$

Hence, A^{-1} exists.

Cofactors of A are

$$C_{11} = 1$$

$$C_{12} = 3$$

$$C_{21} = -5$$

$$C_{22} = 2$$

$$\text{Since, adj } A = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}^T$$

$$(\text{adj } A) = \begin{bmatrix} 1 & 3 \\ -5 & 2 \end{bmatrix}^T$$

$$= \begin{bmatrix} 0 & -5 \\ 3 & 2 \end{bmatrix}$$

$$\text{Now, } A^{-1} = \frac{1}{|A|} \cdot \text{adj } A$$

$$A^{-1} = \frac{1}{17} \cdot \begin{bmatrix} 0 & -5 \\ 3 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{17} \begin{bmatrix} 0 & -5 \\ 3 & 2 \end{bmatrix}$$

8 A. Question

Find the inverse of each of the following matrices.

$$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$$

Answer

$$|A| = 1 \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} - 2 \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} + 3 \begin{vmatrix} 2 & 3 \\ 3 & 1 \end{vmatrix}$$

$$= 1(6 - 1) - 2(4 - 3) + 3(2 - 9)$$

$$= 5 - 2 - 21$$

$$= -18$$

Hence, A^{-1} exists

Cofactors of A are:

$$C_{11} = 5 \quad C_{21} = -1 \quad C_{31} = -7$$

$$C_{12} = -1 \quad C_{22} = -7 \quad C_{32} = 5$$

$$C_{13} = -7 \quad C_{23} = 5 \quad C_{33} = -1$$

$$\text{adj } A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$$

$$= \begin{bmatrix} 5 & -1 & -7 \\ -1 & -7 & 5 \\ -7 & 5 & -1 \end{bmatrix}^T$$

$$\text{So, adj } A = \begin{bmatrix} 5 & -1 & -7 \\ -1 & -7 & 5 \\ -7 & 5 & -1 \end{bmatrix}$$

$$\text{Now, } A^{-1} = \frac{1}{|A|} \text{adj } A$$

$$\text{So, } A^{-1} = \frac{1}{(-18)} \begin{bmatrix} 5 & -1 & -7 \\ -1 & -7 & 5 \\ -7 & 5 & -1 \end{bmatrix}$$

$$\text{Hence, } A^{-1} = \begin{bmatrix} \frac{-5}{18} & \frac{1}{18} & \frac{7}{18} \\ \frac{1}{18} & \frac{7}{18} & \frac{-5}{18} \\ \frac{7}{18} & \frac{-5}{18} & \frac{1}{18} \end{bmatrix}$$

8 B. Question

Find the inverse of each of the following matrices.

$$\begin{bmatrix} 1 & 2 & 5 \\ 1 & -1 & -1 \\ 2 & 3 & -1 \end{bmatrix}$$

Answer

$$|A| = 1 \begin{vmatrix} -1 & -1 \\ 3 & -1 \end{vmatrix} - 2 \begin{vmatrix} 1 & -1 \\ 2 & -1 \end{vmatrix} + 5 \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix}$$

$$= 1(1 + 3) - 2(-1 + 2) + 5(3 + 2)$$

$$= 4 - 2 + 25$$

$$= 27$$

Hence, A^{-1} exists

Cofactors of A are:

$$C_{11} = 4 \quad C_{21} = 17 \quad C_{31} = 3$$

$$C_{12} = -1 \quad C_{22} = -11 \quad C_{32} = 6$$

$$C_{13} = 5 \quad C_{23} = 1 \quad C_{33} = -3$$

$$\text{adj } A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$$

$$= \begin{bmatrix} 4 & -1 & 5 \\ 17 & -11 & 1 \\ 3 & 6 & -3 \end{bmatrix}^T$$

$$\text{So, adj } A = \begin{bmatrix} 4 & 17 & 3 \\ -1 & -11 & 6 \\ 5 & 1 & -3 \end{bmatrix}$$

$$\text{Now, } A^{-1} = \frac{1}{|A|} \text{adj } A$$

$$\text{So, } A^{-1} = \frac{1}{(27)} \begin{bmatrix} 4 & 17 & 3 \\ -1 & -11 & 6 \\ 5 & 1 & -3 \end{bmatrix}$$

$$\text{Hence, } A^{-1} = \begin{bmatrix} \frac{4}{27} & \frac{17}{27} & \frac{3}{27} \\ \frac{-1}{27} & \frac{-11}{27} & \frac{6}{27} \\ \frac{5}{27} & \frac{1}{27} & \frac{-3}{27} \end{bmatrix} = \begin{bmatrix} \frac{4}{27} & \frac{17}{27} & \frac{1}{9} \\ \frac{-1}{27} & \frac{-11}{27} & \frac{2}{9} \\ \frac{5}{27} & \frac{1}{27} & \frac{-1}{9} \end{bmatrix}$$

8 C. Question

Find the inverse of each of the following matrices.

$$\begin{bmatrix} 2 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

Answer

$$|A| = 2 \begin{vmatrix} -1 & -1 \\ 1 & 2 \end{vmatrix} + 1 \begin{vmatrix} -1 & -1 \\ 1 & 2 \end{vmatrix} + 1 \begin{vmatrix} -1 & 2 \\ 1 & -1 \end{vmatrix}$$

$$= 2(4 - 1) + 1(-2 + 1) + 1(1 - 2)$$

$$= 6 - 2$$

$$= -4$$

Hence, A^{-1} exists

Cofactors of A are:

$$C_{11} = 3 \quad C_{21} = 1 \quad C_{31} = -1$$

$$C_{12} = +1 \quad C_{22} = 3 \quad C_{32} = 1$$

$$C_{13} = -1 \quad C_{23} = 1 \quad C_{33} = 3$$

$$\text{adj } A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$$

$$= \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}^T$$

$$\text{So, adj } A = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

$$\text{Now, } A^{-1} = \frac{1}{|A|} \text{adj } A$$

$$\text{So, } A^{-1} = \frac{1}{4} \cdot \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

$$\text{Hence, } A^{-1} = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} & \frac{-1}{4} \\ \frac{1}{4} & \frac{3}{4} & \frac{1}{4} \\ \frac{-1}{4} & \frac{1}{4} & \frac{3}{4} \end{bmatrix}$$

8 D. Question

Find the inverse of each of the following matrices.

$$\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$$

Answer

$$|A| = 2 \begin{vmatrix} 1 & 0 \\ 1 & 3 \end{vmatrix} - 0 \begin{vmatrix} 5 & 0 \\ 0 & 3 \end{vmatrix} - 1 \begin{vmatrix} 5 & 1 \\ 0 & 1 \end{vmatrix}$$

$$= 2(3 - 0) - 0 - 1(5)$$

$$= 6 - 5$$

$$= 1$$

Hence, A^{-1} exists

Cofactors of A are:

$$C_{11} = 3 \quad C_{21} = -1 \quad C_{31} = 1$$

$$C_{12} = -15 \quad C_{22} = 6 \quad C_{32} = -5$$

$$C_{13} = -5 \quad C_{23} = -2 \quad C_{33} = 2$$

$$\text{adj } A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$$

$$= \begin{bmatrix} 3 & -15 & 5 \\ -1 & 6 & -2 \\ 1 & -5 & 2 \end{bmatrix}^T$$

$$\text{So, adj } A = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$$

$$\text{Now, } A^{-1} = \frac{1}{|A|} \cdot \text{adj } A$$

$$\text{So, } A^{-1} = \frac{1}{1} \cdot \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$$

$$\text{Hence, } A^{-1} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$$

8 E. Question

Find the inverse of each of the following matrices.

$$\begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix}$$

Answer

$$|A| = 0 \begin{vmatrix} -3 & 0 \\ -3 & 4 \end{vmatrix} - 1 \begin{vmatrix} 4 & 4 \\ 3 & 4 \end{vmatrix} - 1 \begin{vmatrix} 4 & -3 \\ 3 & -3 \end{vmatrix}$$

$$= 0 - 1(16 - 12) - 1(-12 + 9)$$

$$= -4 + 3$$

$$= -1$$

Hence, A^{-1} exists

Cofactors of A are:

$$C_{11} = 0 \quad C_{21} = -1 \quad C_{31} = 1$$

$$C_{12} = -4 \quad C_{22} = 3 \quad C_{32} = -4$$

$$C_{13} = -3 \quad C_{23} = 3 \quad C_{33} = -4$$

$$\text{adj } A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$$

$$= \begin{bmatrix} 0 & -4 & -3 \\ -1 & 3 & 3 \\ 1 & -4 & -4 \end{bmatrix}^T$$

$$\text{So, adj } A = \begin{bmatrix} 0 & -1 & 1 \\ -4 & 3 & -4 \\ -3 & 3 & -4 \end{bmatrix}$$

$$\text{Now, } A^{-1} = \frac{1}{|A|} \cdot \text{adj } A$$

$$\text{So, } A^{-1} = \frac{1}{-1} \cdot \begin{bmatrix} 0 & -1 & 1 \\ -4 & 3 & -4 \\ -3 & 3 & -4 \end{bmatrix}$$

$$\text{Hence, } A^{-1} = \begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix}$$

8 F. Question

Find the inverse of each of the following matrices.

$$\begin{bmatrix} 0 & 0 & -1 \\ 3 & 4 & 5 \\ -2 & -4 & -7 \end{bmatrix}$$

Answer

$$|A| = 0 \begin{vmatrix} 4 & 5 \\ -4 & -7 \end{vmatrix} - 0 \begin{vmatrix} 3 & 5 \\ -2 & -7 \end{vmatrix} - 1 \begin{vmatrix} 3 & 4 \\ -2 & -4 \end{vmatrix}$$

$$= 0 - 0 - 1(-12 + 8)$$

$$= 4$$

Hence, A^{-1} exists

Cofactors of A are:

$$C_{11} = -8 \quad C_{21} = 4 \quad C_{31} = 4$$

$$C_{12} = 11 \quad C_{22} = -2 \quad C_{32} = -3$$

$$C_{13} = -4 \quad C_{23} = 0 \quad C_{33} = 0$$

$$\text{adj } A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$$

$$= \begin{bmatrix} 8 & 11 & -4 \\ 4 & -2 & 0 \\ 4 & -3 & 0 \end{bmatrix}^T$$

$$\text{So, adj } A = \begin{bmatrix} 8 & 4 & 4 \\ 11 & -2 & -3 \\ -4 & 0 & 0 \end{bmatrix}$$

$$\text{Now, } A^{-1} = \frac{1}{|A|} \text{adj } A$$

$$\text{So, } A^{-1} = \frac{1}{4} \begin{bmatrix} 8 & 4 & 4 \\ 11 & -2 & -3 \\ -4 & 0 & 0 \end{bmatrix}$$

$$\text{Hence, } A^{-1} = \begin{bmatrix} 2 & 1 & 1 \\ \frac{11}{4} & \frac{-1}{2} & \frac{-3}{4} \\ -1 & 0 & 0 \end{bmatrix}$$

8 G. Question

Find the inverse of each of the following matrices.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos a & \sin a \\ 0 & \sin a & -\cos a \end{bmatrix}$$

Answer

$$|A| = 1 \begin{vmatrix} \cos \alpha & \sin \alpha \\ \sin \alpha & -\cos \alpha \end{vmatrix} - 0 + 0$$

$$= (-\cos^2 \alpha - \sin^2 \alpha)$$

$$= -1$$

Hence, A^{-1} exists

Cofactors of A are:

$$C_{11} = -1 \quad C_{21} = 0 \quad C_{31} = 0$$

$$C_{12} = 0 \quad C_{22} = -\cos \alpha \quad C_{32} = -\sin \alpha$$

$$C_{13} = 0 \quad C_{23} = -\sin \alpha \quad C_{33} = \cos \alpha$$

$$\text{adj } A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$$

$$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & -\cos \alpha & -\sin \alpha \\ 0 & -\sin \alpha & \cos \alpha \end{bmatrix}^T$$

$$\text{So, adj } A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -\cos \alpha & -\sin \alpha \\ 0 & -\sin \alpha & \cos \alpha \end{bmatrix}$$

$$\text{Now, } A^{-1} = \frac{1}{|A|} \text{adj } A$$

$$\text{So, } A^{-1} = \frac{1}{-1} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -\cos \alpha & -\sin \alpha \\ 0 & -\sin \alpha & \cos \alpha \end{bmatrix}$$

$$\text{Hence, } A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & \sin \alpha \\ 0 & \sin \alpha & -\cos \alpha \end{bmatrix}$$

9 A. Question

Find the inverse of each of the following matrices and verify that $A^{-1} A = I_3$.

$$\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$$

Answer

$$|A| = 1 \begin{vmatrix} 4 & 3 \\ 3 & 4 \end{vmatrix} - 3 \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} + 3 \begin{vmatrix} 1 & 4 \\ 1 & 3 \end{vmatrix}$$

$$= 1(16 - 9) - 3(4 - 3) + 3(3 - 4)$$

$$= 7 - 3 - 3$$

$$= 1$$

Hence, A^{-1} exists

Cofactors of A are:

$$C_{11} = 7 \quad C_{21} = -3 \quad C_{31} = -3$$

$$C_{12} = -1 \quad C_{22} = -1 \quad C_{32} = 0$$

$$C_{13} = -1 \quad C_{23} = 0 \quad C_{33} = 1$$

$$\text{adj } A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$$

$$= \begin{bmatrix} 7 & -1 & -1 \\ -3 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}^T$$

$$\text{So, adj } A = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$\text{Now, } A^{-1} = \frac{1}{1} \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$\text{Also, } A^{-1} \cdot A = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 7-3-3 & 21-12-9 & 21-9-12 \\ -1+1+0 & -3+4+0 & -3+3+0 \\ -1+0+1 & -3+0+3 & -3+0+4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{Hence, } A^{-1} \cdot A = I$$

9 B. Question

Find the inverse of each of the following matrices and verify that $A^{-1} A = I_3$.

$$\begin{bmatrix} 2 & 3 & 1 \\ 3 & 4 & 1 \\ 3 & 7 & 2 \end{bmatrix}$$

Answer

$$|A| = 2 \begin{vmatrix} 4 & 1 \\ 7 & 2 \end{vmatrix} - 3 \begin{vmatrix} 3 & 1 \\ 3 & 2 \end{vmatrix} + 1 \begin{vmatrix} 3 & 4 \\ 3 & 7 \end{vmatrix}$$

$$= 2(8 - 7) - 3(6 - 3) + 1(21 - 12)$$

$$= 2 - 9 + 9$$

$$= 2$$

Hence, A^{-1} exists

Cofactors of A are:

$$C_{11} = 1 \quad C_{21} = 1 \quad C_{31} = -1$$

$$C_{12} = -3 \quad C_{22} = 1 \quad C_{32} = 1$$

$$C_{13} = 9 \quad C_{23} = -5 \quad C_{33} = -1$$

$$\text{adj } A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$$

$$= \begin{bmatrix} 1 & -3 & 9 \\ 1 & 1 & -5 \\ -1 & 1 & -1 \end{bmatrix}^T$$

$$\text{So, adj } A = \begin{bmatrix} 1 & 1 & -1 \\ -3 & 1 & 1 \\ 9 & -5 & -1 \end{bmatrix}$$

$$\text{Now, } A^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 1 & -1 \\ -3 & 1 & 1 \\ 9 & -5 & -1 \end{bmatrix}$$

$$\text{Also, } A^{-1}.A = \frac{1}{2} \begin{bmatrix} 1 & 1 & -1 \\ -3 & 1 & 1 \\ 9 & -5 & -1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 1 \\ 3 & 4 & 1 \\ 3 & 7 & 2 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 2+3-3 & 3+4-7 & 1+1-2 \\ -6+3+3 & -9+4+7 & -3+1+2 \\ 18-15-3 & 27-20-7 & 9-5-2 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{Hence, } A^{-1}.A = I$$

10 A. Question

For the following pairs of matrices verify that $(AB)^{-1} = B^{-1}A^{-1}$:

$$A = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix} \text{ and } B = \begin{bmatrix} 4 & 6 \\ 3 & 2 \end{bmatrix}$$

Answer

$$A = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}, |A| = 1$$

$$\text{Then, adj } A = \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{\text{adj } A}{|A|} = \frac{1}{1} \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}$$

$$B = \begin{bmatrix} 4 & 6 \\ 3 & 2 \end{bmatrix}, |B| = -10$$

$$\text{Then, adj } B = \begin{bmatrix} 2 & -6 \\ -3 & 4 \end{bmatrix}$$

$$B^{-1} = -\frac{1}{10} \begin{bmatrix} 2 & -6 \\ -3 & 4 \end{bmatrix}$$

$$\text{Also, } A.B = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix} \begin{bmatrix} 4 & 6 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 12+6 & 18+4 \\ 28+15 & 42+10 \end{bmatrix}$$

$$AB = \begin{bmatrix} 18 & 22 \\ 43 & 52 \end{bmatrix}$$

$$|AB| = 936 - 946 = -10$$

$$\text{Adj}(AB) = \begin{bmatrix} 52 & -22 \\ -43 & 18 \end{bmatrix}$$

$$(AB)^{-1} = \frac{1}{-10} \begin{bmatrix} 52 & -22 \\ -43 & 18 \end{bmatrix} = \begin{bmatrix} -52 & 22 \\ 43 & -18 \end{bmatrix}$$

$$\begin{aligned}\text{Now } B^{-1}A^{-1} &= \frac{1}{-10} \begin{bmatrix} 2 & -6 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix} \\ &= \frac{1}{-10} \begin{bmatrix} 10 + 42 & -4 - 18 \\ -15 - 28 & 6 + 12 \end{bmatrix} \\ &= \frac{1}{10} \begin{bmatrix} -52 & 22 \\ 43 & -18 \end{bmatrix}\end{aligned}$$

$$\text{Hence, } (AB)^{-1} = B^{-1}A^{-1}$$

10 B. Question

For the following pairs of matrices verify that $(AB)^{-1} = B^{-1}A^{-1}$:

$$A = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} \text{ and } B = \begin{bmatrix} 4 & 5 \\ 3 & 4 \end{bmatrix}$$

Answer

$$|A| = 1$$

$$\text{Adj } A = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{\text{adj } A}{|A|} = \frac{1}{1} \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$$

$$B = \begin{bmatrix} 4 & 5 \\ 3 & 4 \end{bmatrix}$$

$$|B| = -1$$

$$B^{-1} = \frac{\text{adj } B}{|B|} = \frac{1}{-1} \begin{bmatrix} 4 & -5 \\ -3 & 4 \end{bmatrix}$$

$$\text{Also, } AB = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} 4 & 5 \\ 3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 14 \\ 29 & 37 \end{bmatrix}$$

$$|AB| = 407 - 406 = 1$$

$$\text{And, } \text{adj}(AB) = \begin{bmatrix} 37 & -14 \\ -29 & 11 \end{bmatrix}$$

$$(AB)^{-1} = \frac{\text{adj } AB}{|AB|}$$

$$= \begin{bmatrix} 37 & -14 \\ -29 & 11 \end{bmatrix}$$

$$\text{Now, } B^{-1}A^{-1} = \begin{bmatrix} 4 & -5 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 37 & -14 \\ -29 & 11 \end{bmatrix}$$

$$\text{Hence, } (AB)^{-1} = B^{-1}A^{-1}$$

11. Question

$$\text{Let } A = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix} \text{ and } B = \begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix}. \text{ Find } (AB)^{-1}.$$

Answer

$$A = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}$$

$$|A| = 15 - 14 = 1$$

$$\text{adj } A = \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{\text{adj}}{|A|} = \frac{1}{1} \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}$$

$$B = \begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix}$$

$$|B| = 54 - 56 = -2 \quad \text{adj } B = \begin{bmatrix} 9 & -7 \\ -8 & 6 \end{bmatrix}$$

$$B^{-1} = \frac{\text{adj}B}{|B|} = \frac{1}{-2} \begin{bmatrix} 9 & -7 \\ -8 & 6 \end{bmatrix}$$

$$\text{Now, } (AB)^{-1} = B^{-1}A^{-1}$$

$$= \frac{1}{-2} \begin{bmatrix} 9 & -7 \\ -8 & 6 \end{bmatrix} \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}$$

$$= \frac{1}{-2} \begin{bmatrix} 45 + 49 & -18 - 21 \\ -40 - 42 & 16 + 18 \end{bmatrix}$$

$$= \frac{1}{-2} \begin{bmatrix} 94 & -39 \\ -82 & 34 \end{bmatrix}$$

$$(AB)^{-1} = \begin{bmatrix} -47 & \frac{39}{2} \\ 41 & -17 \end{bmatrix}$$

12. Question

$$\text{Given } A = \begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix}, \text{ compute } A^{-1} \text{ and show that } 2A^{-1} = 9I - A.$$

Answer

$$A = \begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix}$$

$$|A| = 14 - 12 = 2 \quad \text{adj } A = \begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{2} \begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}$$

$$\text{To Show: } 2A^{-1} = 9I - A$$

$$\text{L.H.S } 2A^{-1} = 2 \cdot \frac{1}{2} \begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}$$

$$\text{R.H.S } 9I - A = \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}$$

$$\text{Hence, } 2A^{-1} = 9I - A$$

13. Question

$$\text{If } A = \begin{bmatrix} 4 & 5 \\ 2 & 1 \end{bmatrix}, \text{ then show that } A - 3I = 2(I + 3A^{-1}).$$

Answer

$$A = \begin{bmatrix} 4 & 5 \\ 2 & 1 \end{bmatrix}$$

$$|A| = 4 - 10 = -6 \quad \text{adj } A = \begin{bmatrix} 1 & -5 \\ -2 & 4 \end{bmatrix}$$

$$A^{-1} = \frac{1}{-6} \begin{bmatrix} 1 & -5 \\ -2 & 4 \end{bmatrix}$$

$$\text{To Show: } A - 3I = 2(I + 3A^{-1})$$

$$\text{LHS } A - 3I = \begin{bmatrix} 4 & 5 \\ 2 & 1 \end{bmatrix} - 3 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 5 \\ 2 & -2 \end{bmatrix}$$

$$\text{R.H.S } 2(I + 3A^{-1}) = 2I + 6A^{-1} = 2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 6 \cdot \frac{1}{-6} \begin{bmatrix} 1 & -5 \\ -2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} -1 & 5 \\ 2 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 5 \\ 2 & -4 \end{bmatrix}$$

$$\text{Hence, } A - 3I = 2(I + 3A^{-1})$$

14. Question

$$\text{Find the inverse of the matrix } A = \begin{bmatrix} a & b \\ c & \frac{1+bc}{a} \end{bmatrix} \text{ and show that } aA^{-1} = (a^2 + bc + 1)I - aA.$$

Answer

$$A = \begin{bmatrix} a & b \\ c & \frac{1+bc}{a} \end{bmatrix}$$

$$\text{Now, } |A| = \frac{a+abc}{a} - bc = \frac{a+abc-abc}{a} = 1 \neq 0$$

Hence, A^{-1} exists.

Cofactors of A are

$$C_{11} = \frac{1+bc}{a} \quad C_{12} = -c$$

$$C_{21} = -b \quad C_{22} = a$$

$$\text{Since, } \text{adj } A = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}^T$$

$$(\text{adj } A) = \begin{bmatrix} \frac{1+bc}{a} & -c \\ -b & a \end{bmatrix}^T$$

$$= \begin{bmatrix} \frac{1+bc}{a} & -b \\ -c & a \end{bmatrix}$$

$$\text{Now, } A^{-1} = \frac{1}{|A|} \text{adj } A$$

$$A^{-1} = \frac{1}{1} \cdot \begin{bmatrix} \frac{1+bc}{a} & -b \\ -c & a \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \frac{1+bc}{a} & -b \\ -c & a \end{bmatrix}$$

To show. $aA^{-1} = (a^2 + bc + 1)I - aA$.

$$\text{LHS } aA^{-1} = a \begin{bmatrix} \frac{1+bc}{a} & -b \\ -c & a \end{bmatrix}$$

$$= \begin{bmatrix} 1+bc & -ab \\ -ac & a^2 \end{bmatrix}$$

$$\text{RHS } (a^2 + bc + 1)I - aA = \begin{bmatrix} a^2 + bc + 1 & 0 \\ 0 & a^2 + bc + 1 \end{bmatrix} - \begin{bmatrix} a^2 & ab \\ ac & 1+bc \end{bmatrix}$$

$$= \begin{bmatrix} 1+bc & -ab \\ -ac & a^2 \end{bmatrix}$$

Hence, LHS = RHS

15. Question

$$\text{Given } A = \begin{bmatrix} 5 & 0 & 4 \\ 2 & 3 & 2 \\ 1 & 2 & 1 \end{bmatrix}, B^{-1} = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}. \text{ Compute } (AB)^{-1}.$$

Answer

$$A = \begin{bmatrix} 5 & 0 & 4 \\ 2 & 3 & 2 \\ 1 & 2 & 1 \end{bmatrix} \text{ and } B^{-1} = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$$

$$\text{Here, } (AB)^{-1} = B^{-1} A^{-1}$$

$$|A| = -5 + 4 = -1$$

Cofactors of A are:

$$C_{11} = -1 \quad C_{21} = 8 \quad C_{31} = -12$$

$$C_{12} = 0 \quad C_{22} = 1 \quad C_{32} = -2$$

$$C_{13} = 1 \quad C_{23} = -10 \quad C_{33} = 15$$

$$\text{adj } A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$$

$$= \begin{bmatrix} -1 & 0 & 1 \\ 8 & 1 & -10 \\ -12 & -2 & 15 \end{bmatrix}^T$$

$$\text{So, adj } A = \begin{bmatrix} -1 & 8 & -12 \\ 0 & 1 & -2 \\ 1 & -10 & 15 \end{bmatrix}$$

$$\text{Now, } A^{-1} = \frac{1}{-1} \begin{bmatrix} -1 & 8 & -12 \\ 0 & 1 & -2 \\ 1 & -10 & 15 \end{bmatrix}$$

$$(AB)^{-1} = B^{-1} A^{-1}$$

$$= \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & -8 & 12 \\ 0 & -1 & 2 \\ -1 & 10 & -15 \end{bmatrix}$$

$$= \begin{bmatrix} 1 + 0 - 3 & -8 - 3 + 30 & 12 + 6 - 45 \\ 1 + 0 - 3 & -8 - 4 + 30 & 12 + 8 - 45 \\ 1 + 0 - 4 & -8 - 3 + 40 & 12 + 6 - 60 \end{bmatrix}$$

$$\text{Hence, } = \begin{bmatrix} -2 & 19 & -27 \\ -2 & 18 & -25 \\ -3 & 29 & 42 \end{bmatrix}$$

16 A. Question

$$\text{Let } F(\alpha) = \begin{bmatrix} \cos \alpha & -\sin \alpha & 0 \\ \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } G(\beta) = \begin{bmatrix} \cos \beta & 0 & \sin \beta \\ 0 & 1 & 0 \\ -\sin \beta & 0 & \cos \beta \end{bmatrix}. \text{ Show that}$$

$$[F(\alpha)]^{-1} = F(-\alpha)$$

Answer

$$F(\alpha) = \begin{bmatrix} \cos \alpha & -\sin \alpha & 0 \\ \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$|F(\alpha)| = \cos^2 \alpha + \sin^2 \alpha = 1$$

Cofactors of A are:

$$C_{11} = \cos \alpha \quad C_{21} = \sin \alpha \quad C_{31} = 0$$

$$C_{12} = -\sin \alpha \quad C_{22} = \cos \alpha \quad C_{32} = 0$$

$$C_{13} = 0 \quad C_{23} = -1 \quad C_{33} = 1$$

$$\text{adj } F(\alpha) = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$$

$$= \begin{bmatrix} \cos \alpha & -\sin \alpha & 0 \\ \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}^T$$

$$\text{So, adj } F(\alpha) = \begin{bmatrix} \cos \alpha & \sin \alpha & 0 \\ -\sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \dots\dots (i)$$

$$\text{Now, } [F(\alpha)]^{-1} = \frac{1}{1} \begin{bmatrix} \cos \alpha & \sin \alpha & 0 \\ -\sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{And, } F(-\alpha) = \begin{bmatrix} \cos(-\alpha) & \sin(-\alpha) & 0 \\ \sin(-\alpha) & \cos(-\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix} \dots\dots (ii)$$

$$= \begin{bmatrix} \cos \alpha & -\sin \alpha & 0 \\ \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{Hence, } [F(\alpha)]^{-1} = F(-\alpha)$$

16 B. Question

$$\text{Let } F(\alpha) = \begin{bmatrix} \cos \alpha & -\sin \alpha & 0 \\ \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } G(\beta) = \begin{bmatrix} \cos \beta & 0 & \sin \beta \\ 0 & 1 & 0 \\ -\sin \beta & 0 & \cos \beta \end{bmatrix}. \text{ Show that}$$

$$[G(\beta)]^{-1} = G(-\beta)$$

Answer

$$G(\beta) = \begin{bmatrix} \cos \beta & 0 & \sin \beta \\ 0 & 1 & 0 \\ -\sin \beta & 0 & \cos \beta \end{bmatrix}$$

$$|G(\beta)| = \cos^2 \beta + \sin^2 \beta = 1$$

Cofactors of A are:

$$C_{11} = \cos \beta \quad C_{21} = \sin \alpha \quad C_{31} = \sin \beta$$

$$C_{12} = 0 \quad C_{22} = 1 \quad C_{32} = 0$$

$$C_{13} = \sin \beta \quad C_{23} = 0 \quad C_{33} = \cos \beta$$

$$\text{Adj } G(\beta) = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$$

$$= \begin{bmatrix} \cos \beta & 0 & \sin \beta \\ 0 & 1 & 0 \\ -\sin \beta & 0 & \cos \beta \end{bmatrix}^T$$

$$\text{So, adj } G(\beta) = \begin{bmatrix} \cos \beta & 0 & -\sin \beta \\ 0 & 1 & 0 \\ \sin \beta & 0 & \cos \beta \end{bmatrix} \dots\dots (i)$$

$$\text{Now, } [G(\beta)]^{-1} = \frac{1}{1} \begin{bmatrix} \cos \beta & 0 & -\sin \beta \\ 0 & 1 & 0 \\ \sin \beta & 0 & \cos \beta \end{bmatrix}$$

$$\text{And, } G(-\beta) = \begin{bmatrix} \cos(-\beta) & 0 & \sin(-\beta) \\ 0 & 1 & 0 \\ \sin(-\beta) & 0 & \cos(-\beta) \end{bmatrix}$$

$$= \begin{bmatrix} \cos \beta & 0 & -\sin \beta \\ 0 & 1 & 0 \\ \sin \beta & 0 & \cos \beta \end{bmatrix}$$

$$\text{Hence, } [G(\beta)]^{-1} = G(-\beta)$$

16 C. Question

$$\text{Let } F(\alpha) = \begin{bmatrix} \cos \alpha & -\sin \alpha & 0 \\ \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } G(\beta) = \begin{bmatrix} \cos \beta & 0 & \sin \beta \\ 0 & 1 & 0 \\ -\sin \beta & 0 & \cos \beta \end{bmatrix}. \text{ Show that}$$

$$[F(\alpha)G(\beta)]^{-1} = G(-\beta)F(-\alpha).$$

Answer

We have to show that

$$[F(\alpha)G(\beta)]^{-1} = G(-\beta)F(-\alpha)$$

We have already shown that

$$[G(\beta)]^{-1} = G(-\beta)$$

$$[F(\alpha)]^{-1} = F(-\alpha)$$

$$\text{And LHS} = [F(\alpha)G(\beta)]^{-1}$$

$$= [G(\beta)]^{-1} [F(\alpha)]^{-1}$$

$$= G(-\beta) F(-\alpha)$$

$$\text{Hence} = \text{RHS}$$

17. Question

If $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$, verify that $A^2 - 4A + I = O$, where $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Hence, find A^{-1} .

Answer

$$A^2 = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 + 3 & 6 + 6 \\ 2 + 2 & 3 + 4 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix}$$

$$4A = 4 \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 12 \\ 4 & 8 \end{bmatrix}$$

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\text{Now, } A^2 - 4A + I = \begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix} - \begin{bmatrix} 8 & 12 \\ 4 & 8 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 7 - 8 + 1 & 12 - 12 + 0 \\ 4 - 4 + 0 & 7 - 8 + 1 \end{bmatrix}$$

$$\text{Hence, } = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\text{Now, } A^2 - 4A + I = 0$$

$$A.A - 4A = -I$$

Multiply by A^{-1} both sides

$$A.A(A^{-1}) - 4A A^{-1} = -I A^{-1}$$

$$AI - 4I = -A^{-1}$$

$$A^{-1} = 4I - A = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$$

18. Question

Show that $A = \begin{bmatrix} -8 & 5 \\ 2 & 4 \end{bmatrix}$ satisfies the equation $A^2 + 4A - 42I = O$. Hence, find A^{-1} .

Answer

$$A = \begin{bmatrix} -8 & 5 \\ 2 & 4 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} -8 & 5 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} -8 & 5 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 64 + 10 & -40 + 20 \\ -16 + 8 & 10 + 16 \end{bmatrix}$$

$$= \begin{bmatrix} 74 & -20 \\ -8 & 26 \end{bmatrix}$$

$$4A = 4 \begin{bmatrix} -8 & 5 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} -32 & 20 \\ 8 & 16 \end{bmatrix}$$

$$42I = 42 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 42 & 0 \\ 0 & 42 \end{bmatrix}$$

Now,

$$A^2 + 4A - 42I = \begin{bmatrix} 74 & -20 \\ -8 & 26 \end{bmatrix} + \begin{bmatrix} -32 & 20 \\ 8 & 16 \end{bmatrix} - \begin{bmatrix} 42 & 0 \\ 0 & 42 \end{bmatrix}$$

$$= \begin{bmatrix} 74 - 32 & -20 + 20 \\ -8 + 8 & 26 - 42 \end{bmatrix}$$

$$\text{Hence, } = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\text{Now, } A^2 + 4A - 42I = 0$$

$$= A^{-1} \cdot A \cdot A + 4 A^{-1} \cdot A - 42 A^{-1} \cdot I = 0$$

$$= IA + 4I - 42A^{-1} = 0$$

$$= 42A^{-1} = A + 4I$$

$$= A^{-1} = \frac{1}{42}[A + 4I]$$

$$= \frac{1}{42} \left[\begin{bmatrix} -8 & 5 \\ 2 & 4 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \right]$$

$$A^{-1} = \frac{1}{42} \begin{bmatrix} -4 & 5 \\ 2 & 8 \end{bmatrix}$$

19. Question

If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, show that $A^2 - 5A + 7I = O$. Hence, find A^{-1} .

Answer

$$A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 9-1 & 3+2 \\ -3-2 & -1+4 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix}$$

$$\text{Now, } A^2 - 5A + 7I = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - 5 \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} + 7 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8-15+7 & 5-5+0 \\ -5+5+0 & 3-10+7 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\text{So, } A^2 - 5A + 7I = 0$$

Multiply by A^{-1} both sides

$$= A \cdot A \cdot A^{-1} - 5A \cdot A^{-1} + 7I \cdot A^{-1} = 0$$

$$= A - 5I + 7A^{-1} = 0$$

$$= A^{-1} = \frac{1}{7}[5I - A]$$

$$= A^{-1} = \frac{1}{7} \cdot \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$$

$$= A^{-1} = \frac{1}{7} \cdot \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$$

20. Question

If $A = \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}$ find x and y such $A^2 - xA + yI = O$. Hence, evaluate A^{-1} .

Answer

$$A = \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 16 + 6 & 12 + 15 \\ 8 + 10 & 6 + 25 \end{bmatrix}$$

$$= \begin{bmatrix} 22 & 27 \\ 18 & 31 \end{bmatrix}$$

$$\text{Now, } A^2 - xA + yI = \begin{bmatrix} 22 & 27 \\ 18 & 31 \end{bmatrix} - x \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix} + y \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= 22 - 4x + y = 0 \text{ or } 4x - y = 22$$

$$= 18 - 2x = 0 \text{ or } x = 9$$

$$= y = 14$$

$$\text{So, } A^2 - 5A + 7I = 0$$

Multiply by A^{-1} both sides

$$= A.A.A^{-1} - 9A.A^{-1} + 14I.A^{-1} = 0$$

$$= A - 9I + 14A^{-1} = 0$$

$$= A^{-1} = \frac{1}{14}[9I - A]$$

$$= A^{-1} = \frac{1}{14} \cdot \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}$$

$$= A^{-1} = \frac{1}{7} \cdot \begin{bmatrix} 5 & -3 \\ -2 & 4 \end{bmatrix}$$

21. Question

If $A = \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix}$, find the value of λ so that $A^2 = \lambda A - 2I$. Hence, find A^{-1} .

Answer

$$A = \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix} = \begin{bmatrix} 9 - 8 & -6 + 4 \\ 12 - 8 & -8 + 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -2 \\ 4 & -4 \end{bmatrix}$$

$$\text{Now, } A^2 = \lambda A - 2I$$

$$= \lambda A = A^2 + 2I$$

$$= \begin{bmatrix} 1 & -2 \\ 4 & -4 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix}$$

$$= \lambda \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 3\lambda & -2\lambda \\ 4\lambda & -2\lambda \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix}$$

$$= 3\lambda = 3 \text{ or } \lambda = 1$$

$$\text{So, } A^2 = A - 2I$$

Multiply by A^{-1} both sides

$$= A.A. A^{-1} = A. A^{-1} - 2I. A^{-1} = 0$$

$$= 2A^{-1} = I - A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ -4 & 3 \end{bmatrix}$$

$$\text{Hence, } A^{-1} = \frac{1}{2} \cdot \begin{bmatrix} -2 & 2 \\ -4 & 3 \end{bmatrix}$$

22. Question

Show that $A = \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix}$ satisfies the equation $x^2 - 3A - 7I = 0$. Thus, find A^{-1} .

Answer

$$A = \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 25 - 3 & 15 - 6 \\ -5 + 2 & -3 + 4 \end{bmatrix}$$

$$= \begin{bmatrix} 22 & 9 \\ -3 & 1 \end{bmatrix}$$

$$\text{Now, } A^2 - 3A - 7I = 0$$

$$= \begin{bmatrix} 22 & 9 \\ -3 & 1 \end{bmatrix} - 3 \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix} - 7 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 22 - 15 - 7 & 9 - 9 - 0 \\ -3 + 3 - 0 & 1 + 6 - 7 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\text{So, } A^2 - 3A - 7I = 0$$

Multiply by A^{-1} both sides

$$= A.A. A^{-1} - 3A. A^{-1} - 7I. A^{-1} = 0$$

$$= A - 3I - 7A^{-1} = 0$$

$$= 7A^{-1} = A - 3I$$

$$= A^{-1} = \frac{1}{7} \cdot \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix} - 3 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\text{Hence, } A^{-1} = \frac{1}{7} \cdot \begin{bmatrix} 2 & 3 \\ -1 & -5 \end{bmatrix}$$

23. Question

Show that $A = \begin{bmatrix} 6 & 5 \\ 7 & 6 \end{bmatrix}$ satisfies the equation $x^2 - 12x + 1 = 0$. Thus, find A^{-1}

Answer

$$A = \begin{bmatrix} 6 & 5 \\ 7 & 6 \end{bmatrix}$$

We have $A^2 - 12A + I = 0$

$$\begin{aligned} A^2 &= \begin{bmatrix} 6 & 5 \\ 7 & 6 \end{bmatrix} \begin{bmatrix} 6 & 5 \\ 7 & 6 \end{bmatrix} = \begin{bmatrix} 36 + 35 & 30 + 30 \\ 42 + 42 & 35 + 36 \end{bmatrix} \\ &= \begin{bmatrix} 71 & 60 \\ 84 & 71 \end{bmatrix} \end{aligned}$$

Now, $A^2 - 12A + I = 0$

$$\begin{aligned} &= \begin{bmatrix} 71 & 60 \\ 84 & 71 \end{bmatrix} - 12 \begin{bmatrix} 6 & 5 \\ 7 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 71 - 72 + 1 & 60 - 60 + 0 \\ 84 - 82 + 0 & 71 - 72 + 1 \end{bmatrix} \end{aligned}$$

$$\text{Hence, } = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Also, $A^2 - 12A + I = 0$

$$= A - 12I + A^{-1} = 0$$

$$= A^{-1} = 12I - A$$

$$= 12 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 6 & 5 \\ 7 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 12 - 6 & 0 - 5 \\ 0 - 7 & 12 - 6 \end{bmatrix}$$

$$\text{Hence, } A^{-1} = \begin{bmatrix} 6 & -5 \\ -7 & 6 \end{bmatrix}$$

24. Question

For the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$. Show that $A^3 - 6A^2 + 5A + 11I_3 = O$. Hence, find A^{-1} .

Answer

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$$

$$A^3 = A^2 \cdot A$$

$$A^2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 + 1 + 2 & 1 + 2 - 1 & 1 - 3 + 3 \\ 1 + 2 - 6 & 1 + 4 + 3 & 1 - 6 - 9 \\ 2 - 1 + 6 & 2 - 2 - 3 & 2 + 3 + 9 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix}$$

$$\begin{aligned}
 A^2.A &= \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} \\
 &= \begin{bmatrix} 4+2+2 & 4+4-1 & 4-6+3 \\ -3+8-28 & -3+16+14 & -3-24-42 \\ 7-3+28 & 7-6-14 & 7+9+42 \end{bmatrix} \\
 &= \begin{bmatrix} 8 & 7 & 1 \\ -23 & 27 & -69 \\ 32 & -13 & 58 \end{bmatrix}
 \end{aligned}$$

$$\text{Now, } A^3 - 6A^2 + 5A + 11I$$

$$\begin{aligned}
 &\begin{bmatrix} 8 & 7 & 1 \\ -23 & 27 & -69 \\ 32 & -13 & 58 \end{bmatrix} - 6 \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix} + 5 \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} + 11 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
 &= \begin{bmatrix} 8-24 & 7-12 & 1-6 \\ -23+18 & 27-48 & -69+84 \\ 32-42 & -13+18 & 58-84 \end{bmatrix} + \begin{bmatrix} 5+11 & 5+0 & 5+0 \\ 5+0 & 10+11 & -15+0 \\ 10+0 & -5+0 & 15+11 \end{bmatrix} \\
 &= \begin{bmatrix} -16 & -5 & -5 \\ -5 & -21 & 15 \\ -10 & 5 & 26 \end{bmatrix} + \begin{bmatrix} 16 & 5 & 5 \\ 5 & 21 & -15 \\ 10 & -5 & 26 \end{bmatrix} \\
 &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
 \end{aligned}$$

$$\text{Thus, } A^3 - 6A^2 + 5A + 11I$$

$$\text{Now, } (AAA) A^{-1} - 6(A.A) A^{-1} + 5.A A^{-1} + 11I.A^{-1} = 0$$

$$AA(A^{-1}A) - 6A(A^{-1}A) + 5(A^{-1}A) = -1(A^{-1}I)$$

$$A^2 - 6A + 5I = 11 A^{-1}$$

$$= A^{-1} = -\frac{1}{11}(A^2 - 6A + 5I)$$

Now,

$$A^2 - 6A + 5I$$

$$\begin{aligned}
 &= \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix} - 6 \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} + 5 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
 &= \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix} - \begin{bmatrix} 6 & 6 & 6 \\ 6 & 12 & -18 \\ 12 & -6 & 18 \end{bmatrix} + \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} \\
 &= \begin{bmatrix} 9 & 2 & 1 \\ -3 & 13 & -14 \\ 7 & -3 & 19 \end{bmatrix} - \begin{bmatrix} 6 & 6 & 6 \\ 6 & 12 & -18 \\ 12 & -6 & 18 \end{bmatrix} \\
 &= \begin{bmatrix} 3 & -4 & -5 \\ -9 & 1 & 4 \\ -5 & 3 & 1 \end{bmatrix}
 \end{aligned}$$

$$\text{Hence, } A^{-1} = -\frac{1}{11} \begin{bmatrix} 3 & -4 & -5 \\ -9 & 1 & 4 \\ -5 & 3 & 1 \end{bmatrix}$$

25. Question

Show that the matrix, $A = \begin{bmatrix} 1 & 0 & -1 \\ -2 & -1 & 2 \\ 3 & 4 & 1 \end{bmatrix}$ satisfies the equation, $A^3 - A^2 - 3A - I_3 = O$. Hence, find A^{-1} .

Answer

$$A = \begin{bmatrix} 1 & 0 & -2 \\ -2 & -1 & 2 \\ 3 & 4 & 1 \end{bmatrix}$$

$$A^3 = A^2.A$$

$$A^2 = \begin{bmatrix} 1 & 0 & -2 \\ -2 & -1 & 2 \\ 3 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \\ -2 & -1 & 2 \\ 3 & 4 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0-6 & 0+0-8 & -2+0-2 \\ -2+2+6 & 0+1+8 & 4-2+2 \\ 3-8+3 & 0-4+4 & -6+8+1 \end{bmatrix} = \begin{bmatrix} -5 & -8 & -4 \\ 6 & 9 & 4 \\ -2 & 0 & 3 \end{bmatrix}$$

$$A^2.A = \begin{bmatrix} -5 & -8 & -4 \\ 6 & 9 & 4 \\ -2 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \\ -2 & -1 & 2 \\ 3 & 4 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -5+16-12 & 0-8+16 & 10-16-4 \\ 6-18+12 & 0-9+16 & -12+18+4 \\ -2-0+9 & 0-0-12 & 4+0+3 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -8 & -10 \\ 0 & 7 & 10 \\ 7 & 12 & 7 \end{bmatrix}$$

$$\text{Now, } A^3 - A^2 - 3A - I$$

$$\begin{bmatrix} -1 & -8 & -10 \\ 0 & 7 & 10 \\ 7 & 12 & 7 \end{bmatrix} - \begin{bmatrix} -5 & -8 & -4 \\ 6 & 9 & 4 \\ -2 & 0 & 3 \end{bmatrix} - 3 \begin{bmatrix} 1 & 0 & -2 \\ -2 & -1 & 2 \\ 3 & 4 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1+5 & -8+8 & -10+4 \\ 0-6 & 7-9 & 10-4 \\ 7+2 & 12-0 & 7-3 \end{bmatrix} + \begin{bmatrix} -3-1 & -0-0 & 6-0 \\ 6-0 & +3-1 & -6-0 \\ -9-0 & -12+0 & -3-1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 & -6 \\ -6 & -2 & 6 \\ 9 & 12 & 4 \end{bmatrix} + \begin{bmatrix} -4 & 0 & 6 \\ 6 & 2 & -6 \\ -9 & -12 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{Thus, } A^3 - A^2 - 3A - I$$

$$\text{Now, } (AAA) A^{-1} - (A.A) A^{-1} - 3.A A^{-1} - I.A^{-1} = 0$$

$$AA(A^{-1}A) - A(A^{-1}A) - 3(A^{-1}A) = -1(A^{-1}I)$$

$$A^2 - A - 3A - I = 0$$

$$= A^{-1} = (A^2 - A - 3I)$$

Now,

$$(A^2 - A - 3I) = \begin{bmatrix} -1 & -8 & -10 \\ 0 & 7 & 10 \\ 7 & 12 & 7 \end{bmatrix} - \begin{bmatrix} 1 & 0 & -2 \\ -2 & -1 & 2 \\ 3 & 4 & 1 \end{bmatrix} - 3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned}
&= \begin{bmatrix} -5 & -8 & -4 \\ 6 & 9 & 4 \\ -2 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 0 & -2 \\ -2 & -1 & 2 \\ 3 & 4 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \\
&= \begin{bmatrix} -5-1-3 & -8-0-0 & -4+2-0 \\ 6+2-0 & 7+1-3 & 4-2-0 \\ -2-3-0 & 0-4-0 & 3-1-3 \end{bmatrix} \\
&= \begin{bmatrix} -9 & -8 & -2 \\ 8 & 7 & 2 \\ -5 & -4 & -1 \end{bmatrix}
\end{aligned}$$

$$\text{Hence, } A^{-1} = \begin{bmatrix} -9 & -8 & -2 \\ 8 & 7 & 2 \\ -5 & -4 & -1 \end{bmatrix}$$

26. Question

$$\text{If } A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}. \text{ Verify that } A^3 - 6A^2 + 9A - 4I = O \text{ and hence find } A^{-1}.$$

Answer

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$A^3 = A^2 \cdot A$$

$$\begin{aligned}
A^2 &= \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \\
&= \begin{bmatrix} 4+1+1 & -2-2-2 & 2+1+2 \\ -2-2-2 & 1+2+1 & -1-2-2 \\ 2+1+2 & -1-2-2 & 1+1+4 \end{bmatrix} = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
A^2 \cdot A &= \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \\
&= \begin{bmatrix} 12+5+5 & -6-10-5 & 6+5+10 \\ -10-6-5 & 5+12+5 & -5-6-10 \\ 10+5+6 & -5-10-6 & 5+5+12 \end{bmatrix} \\
&= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}
\end{aligned}$$

$$\text{Now, } A^3 - 6A^2 + 9A - 4I$$

$$\begin{aligned}
&\begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - 6 \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} + 9 \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - 4 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 22-36+18-4 & -21+30-9-0 & 21-30+9-0 \\ -21+30-9-0 & 22-36+18-4 & -21+30-9-0 \\ 21-30+9-0 & -21+30-9-0 & 22-36+18-4 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\end{aligned}$$

$$\text{Thus, } A^3 - 6A^2 + 9A - 4I$$

$$\text{Now, } (AAA) A^{-1} - 6(A.A) A^{-1} + 9.A A^{-1} - 4I.A^{-1} = 0$$

$$A^2 - 6A + 9I = 4A^{-1}$$

$$= A^{-1} = \frac{1}{4}(A^2 - 6A + 9I)$$

Now,

$$(A^2 - 6A + 9I) = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} - 6 \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + 9 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} - \begin{bmatrix} 12 & -6 & 6 \\ -6 & 12 & -6 \\ 6 & -6 & 12 \end{bmatrix} + \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 6-12+9 & -5+6+0 & 5-6+0 \\ -5+6+0 & 6-12+9 & -5+6+0 \\ 5-6+0 & -5-6+0 & 6-12+3 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

$$\text{Hence, } A^{-1} = \frac{1}{4} \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

27. Question

$$\text{If } A = \frac{1}{9} \begin{bmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{bmatrix}, \text{ prove that } A^{-1} = A^T.$$

Answer

$$A = \frac{1}{9} \begin{bmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{bmatrix} A^T = \frac{1}{9} \begin{bmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{bmatrix}$$

$$|A| = \frac{1}{9} [-8(16 + 56) - 1(16 - 7) + 4(-32 - 4)]$$

$$= -81$$

Cofactors of A are:

$$C_{11} = 72 \quad C_{21} = -36 \quad C_{31} = -9$$

$$C_{12} = -9 \quad C_{22} = -36 \quad C_{32} = 72$$

$$C_{13} = -36 \quad C_{23} = -63 \quad C_{33} = -36$$

$$\text{adj } A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$$

$$= \begin{bmatrix} 72 & -9 & -36 \\ -36 & -36 & -63 \\ -9 & 72 & -36 \end{bmatrix}^T$$

$$\text{So, adj } A = \begin{bmatrix} 72 & -36 & -9 \\ -9 & -36 & 72 \\ -36 & -63 & -36 \end{bmatrix}$$

$$\text{Now, } A^{-1} = \frac{1}{-81} \begin{bmatrix} 72 & -36 & -9 \\ -9 & -36 & 72 \\ -36 & -63 & -36 \end{bmatrix}$$

$$\text{Hence, } A^{-1} = \frac{1}{9} \begin{bmatrix} -8 & 4 & 1 \\ 1 & 4 & -8 \\ 4 & 7 & 4 \end{bmatrix} = A^T$$

28. Question

$$\text{If } A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}, \text{ show that } A^{-1} = A^3.$$

Answer

$$A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$$

$$|A| = 3 + 6 - 8 = 1$$

Cofactors of A are:

$$C_{11} = 1 \quad C_{21} = -1 \quad C_{31} = 0$$

$$C_{12} = -2 \quad C_{22} = 3 \quad C_{32} = -4$$

$$C_{13} = -2 \quad C_{23} = 3 \quad C_{33} = -3$$

$$\text{adj } A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$$

$$= \begin{bmatrix} 1 & -2 & -2 \\ -1 & 3 & 3 \\ 0 & -4 & -3 \end{bmatrix}^T$$

$$\text{So, adj } A = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$$

$$\text{Now, } A^{-1} = \frac{1}{1} \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$$

$$\text{Also, } A^2 = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 9-6+0 & -9+9-4 & 12-12+4 \\ 6-6+0 & -6+9-4 & 8-12+4 \\ 0-2+0 & 0+3-1 & 0-4+1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -4 & 4 \\ 0 & -1 & 0 \\ -2 & 2 & -3 \end{bmatrix}$$

$$A^3 = A^2 \cdot A = \begin{bmatrix} 3 & -4 & 4 \\ 0 & -1 & 0 \\ -2 & 2 & -3 \end{bmatrix} \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$$

Hence, $A^{-1} = A^3$

29. Question

If $A = \begin{bmatrix} -1 & 2 & 0 \\ -1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$, Show that $A^2 = A^{-1}$.

Answer

$$|A| = -1 \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} - 2 \begin{vmatrix} -1 & 1 \\ 0 & 0 \end{vmatrix} + 0$$

$$|A| = -1(0 - 1) - 2(0) + 0$$

$$= 1 - 0 + 0$$

$$|A| = 1$$

$$A = \begin{bmatrix} -1 & 2 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$A^2 = A.A = \begin{bmatrix} -1 & 2 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 2 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 - 2 + 0 & -2 + 2 + 0 & 0 + 2 + 0 \\ 1 - 1 + 1 & -2 + 1 + 1 & -1 + 1 - 0 \\ 0 - 1 + 0 & 0 + 1 - 0 & 0 + 1 - 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 2 \\ 0 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$

Cofactors of A are:

$$C_{11} = -1 \quad C_{21} = 0 \quad C_{31} = 2$$

$$C_{12} = 0 \quad C_{22} = 0 \quad C_{32} = 1$$

$$C_{13} = -1 \quad C_{23} = 1 \quad C_{33} = 1$$

$$\text{adj } A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$$

$$= \begin{bmatrix} -1 & 0 & -1 \\ 0 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix}^T$$

$$\text{So, adj } A = \begin{bmatrix} -1 & 0 & 2 \\ 0 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$

$$\text{Now, } A^{-1} = \frac{1}{1} \begin{bmatrix} -1 & 0 & 2 \\ 0 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$

$$\text{Hence, } A^{-1} = \begin{bmatrix} -1 & 0 & 2 \\ 0 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix} = A^2$$

30. Question

Solve the matrix equation $\begin{bmatrix} 5 & 4 \\ 1 & 1 \end{bmatrix} X = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}$, where X is a 2x2 matrix.

Answer

$$\text{Let } A = \begin{bmatrix} 5 & 4 \\ 1 & 1 \end{bmatrix} \text{ } B = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}$$

$$\text{So, } AX = B$$

$$\text{Or, } X = A^{-1}B$$

$$|A| = 1$$

Cofactors of A are

$$C_{11} = 1 \quad C_{12} = -1$$

$$C_{21} = -4 \quad C_{22} = 5$$

$$\text{Since, } \text{adj } A = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}^T$$

$$(\text{adj } A) = \begin{bmatrix} 1 & -1 \\ -4 & 5 \end{bmatrix}^T$$

$$= \begin{bmatrix} 1 & -4 \\ -1 & 5 \end{bmatrix}$$

$$\text{Now, } A^{-1} = \frac{1}{|A|} \text{adj } A$$

$$A^{-1} = \frac{1}{1} \begin{bmatrix} 1 & -4 \\ -1 & 5 \end{bmatrix}$$

$$\text{So, } X = \begin{bmatrix} 1 & -4 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}$$

$$\text{Hence, } X = \begin{bmatrix} -3 & -14 \\ 4 & 17 \end{bmatrix}$$

31. Question

Find the matrix X satisfying the matrix equation: $X \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 14 & 7 \\ 7 & 7 \end{bmatrix}$.

Answer

$$\text{Let } A = \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix} \text{ } B = \begin{bmatrix} 14 & 7 \\ 7 & 7 \end{bmatrix}$$

$$\text{So, } AX = B$$

$$\text{Or, } X = A^{-1}B$$

$$|A| = -7$$

Cofactors of A are

$$C_{11} = -2 \quad C_{12} = 1$$

$$C_{21} = -3 \quad C_{22} = 5$$

$$\text{Since, } \text{adj } A = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}^T$$

$$(\text{adj } A) = \begin{bmatrix} -2 & 1 \\ -3 & 5 \end{bmatrix}^T$$

$$= \begin{bmatrix} -2 & -3 \\ 1 & 5 \end{bmatrix}$$

$$\text{Now, } A^{-1} = \frac{1}{|A|} \text{adj } A$$

$$A^{-1} = \frac{1}{7} \begin{bmatrix} 2 & 3 \\ -1 & -5 \end{bmatrix}$$

$$\text{So, } X = \frac{1}{7} \begin{bmatrix} 2 & 3 \\ -1 & -5 \end{bmatrix} \begin{bmatrix} 14 & 7 \\ 7 & 7 \end{bmatrix}$$

$$\text{Hence, } X = \frac{1}{7} \begin{bmatrix} 28 + 21 & 14 + 21 \\ -14 - 35 & -7 - 35 \end{bmatrix}$$

$$X = \begin{bmatrix} 7 & 5 \\ -7 & -6 \end{bmatrix}$$

32. Question

$$\text{Find the matrix } X \text{ for which: } \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix} X \begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 0 & 4 \end{bmatrix}.$$

Answer

$$\text{Let } A = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix} B = \begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix} C = \begin{bmatrix} 2 & -1 \\ 0 & 4 \end{bmatrix}$$

Then The given equations becomes as

$$AXB = C$$

$$= X = A^{-1}CB^{-1}$$

$$|A| = 35 - 14 = 21$$

$$|B| = -1 + 2 = 1$$

$$A^{-1} = \frac{\text{adj}(A)}{|A|} = \frac{1}{21} \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}$$

$$B^{-1} = \frac{\text{adj}(B)}{|B|} = \frac{1}{1} \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$$

$$= X = A^{-1}CB^{-1} = \frac{1}{21} \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$$

$$= \frac{1}{21} \begin{bmatrix} 10 + 0 & -5 - 8 \\ -14 + 0 & 7 + 12 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$$

$$= \frac{1}{21} \begin{bmatrix} 10 & -13 \\ -14 & 19 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$$

$$= \frac{1}{21} \begin{bmatrix} 10 - 26 & -10 + 13 \\ -14 + 38 & 14 - 19 \end{bmatrix}$$

$$\text{Hence, } X = \frac{1}{21} \begin{bmatrix} -16 & 3 \\ 24 & -5 \end{bmatrix}$$

33. Question

$$\text{Find the matrix } X \text{ satisfying the equation: } \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} X \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Answer

$$\text{Let } A = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} B = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix} C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Then The given equations becomes as

$$AXB = I$$

$$= X = A^{-1}B^{-1}$$

$$|A| = 6 - 5 = 1$$

$$|B| = 10 - 9 = 1$$

$$A^{-1} = \frac{\text{adj}(A)}{|A|} = \frac{1}{1} \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$$

$$B^{-1} = \frac{\text{adj}(B)}{|B|} = \frac{1}{1} \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}$$

$$= X = A^{-1}B^{-1} = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 6 + 3 & -9 - 5 \\ -10 - 6 & 15 + 10 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & -14 \\ -16 & 25 \end{bmatrix}$$

$$\text{Hence, } X = \begin{bmatrix} 9 & -14 \\ -16 & 25 \end{bmatrix}$$

34. Question

$$\text{If } A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}, \text{ find } A^{-1} \text{ and prove that } A^2 - 4A - 5I = O.$$

Answer

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 + 4 + 4 & 2 + 2 + 4 & 2 + 4 + 2 \\ 2 + 2 + 4 & 4 + 1 + 4 & 4 + 2 + 2 \\ 2 + 4 + 2 & 4 + 2 + 2 & 4 + 4 + 1 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix}$$

$$A^2 - 4A + 5I = 0$$

$$= \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix} - 4 \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} + 5 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 9 - 4 - 5 & 8 - 8 - 0 & 8 - 8 - 0 \\ 8 - 8 - 0 & 9 - 4 - 5 & 8 - 8 - 0 \\ 8 - 8 - 0 & 8 - 8 - 0 & 9 - 4 - 5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{Also, } A^2 - 4A - 5I = 0$$

$$\text{Now, } 6(A.A) A^{-1} - 4.A A^{-1} - 5I.A^{-1} = 0$$

$$= A - 4I - 5A^{-1} = 0$$

$$= A^{-1} = \frac{1}{5}(A - 4I)$$

$$= \frac{1}{5} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} - 4 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \frac{1}{5} \begin{bmatrix} 1-4 & 2-0 & 2-0 \\ 2-0 & 1-4 & 2-0 \\ 2-0 & 2-0 & 1-4 \end{bmatrix}$$

$$\text{Hence, } A^{-1} = \frac{1}{5} \begin{bmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{bmatrix}$$

35. Question

If A is a square matrix of order n, prove that $|A \text{ adj } A| = |A|^n$.

Answer

$$|A \text{ adj } A| = |A|^n$$

$$\text{LHS } |A \text{ adj } A|$$

$$|A|.|\text{adj } A|$$

$$|A|.|A|^{n-1}$$

$$|A|^{n-1+1}$$

$$|A|^n = \text{RHS}$$

Hence, LHS = RHS

36. Question

$$\text{If } A^{-1} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}, \text{ find } (AB)^{-1}.$$

Answer

$$A^{-1} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix} B = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

$$|B| = 1(3 - 0) - 2(-1 - 0) - 2(2 - 0)$$

$$= 3 + 2 - 4$$

$$|B| = 1$$

$$\text{Now, } B^{-1} = \frac{1}{|B|} \text{adj } B$$

Cofactors of B are:

$$C_{11} = -3 \quad C_{21} = 2 \quad C_{31} = 6$$

$$C_{12} = 1 \quad C_{22} = 1 \quad C_{32} = 2$$

$$C_{13} = 2 \quad C_{23} = 2 \quad C_{33} = 5$$

$$\text{adj } B = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$$

$$= \begin{bmatrix} 3 & 1 & 2 \\ 2 & 1 & 2 \\ 6 & 2 & 5 \end{bmatrix}^T$$

$$\text{So, adj } B = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$$

$$\text{Now, } B^{-1} = \frac{1}{1} \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$$

$$(AB)^{-1} = B^{-1} A^{-1}$$

$$= \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix} \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 9 - 30 + 30 & -3 + 12 - 12 & 3 - 10 + 12 \\ 3 - 15 + 10 & -1 + 6 - 4 & 1 - 5 + 4 \\ 6 - 30 + 25 & -2 + 12 - 10 & 2 - 10 + 10 \end{bmatrix}$$

$$\text{Hence, } = \begin{bmatrix} 9 & -3 & 5 \\ -2 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

37. Question

$$\text{If } A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & 4 \\ -2 & 2 & 1 \end{bmatrix}, \text{ find } (A^T)^{-1}.$$

Answer

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & 4 \\ -2 & 2 & 1 \end{bmatrix}$$

$$\text{Let } B = A^T = \begin{bmatrix} 1 & 0 & -2 \\ -2 & -1 & 2 \\ 3 & 4 & 1 \end{bmatrix}$$

$$|B| = \begin{vmatrix} 1 & 0 & -2 \\ -2 & -1 & 2 \\ 3 & 4 & 1 \end{vmatrix}$$

$$= (-1 - 8) - 0 - 2(-8 + 3) = -9 + 10 = 1$$

Cofactors of B are:

$$C_{11} = -9 \quad C_{21} = 8 \quad C_{31} = -5$$

$$C_{12} = -8 \quad C_{22} = 7 \quad C_{32} = -4$$

$$C_{13} = -2 \quad C_{23} = 2 \quad C_{33} = -1$$

$$\text{adj } B = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$$

$$= \begin{bmatrix} -9 & 8 & -5 \\ -8 & 7 & 4 \\ -2 & 2 & -1 \end{bmatrix}^T$$

$$\text{So, adj } B = \begin{bmatrix} -9 & -8 & -2 \\ 8 & 7 & 2 \\ -5 & 4 & -1 \end{bmatrix}$$

$$\text{Now, } B^{-1} = \frac{1}{1} \begin{bmatrix} -9 & -8 & -2 \\ 8 & 7 & 2 \\ -5 & 4 & -1 \end{bmatrix}$$

$$\text{Hence, } (A^T)^{-1} = \begin{bmatrix} -9 & -8 & -2 \\ 8 & 7 & 2 \\ -5 & 4 & -1 \end{bmatrix}$$

38. Question

$$\text{Find the adjoint of the matrix } A = \begin{bmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} \text{ and hence show that } A(\text{adj } A) = |A| I_3.$$

Answer

$$A = \begin{bmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

$$|A| = \begin{vmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{vmatrix}$$

$$= -1(1 - 4) + 2(2 + 4) - 2(-4 - 2)$$

$$= 3 + 12 + 12$$

$$|A| = 27$$

Cofactors of A

$$C_{11} = -3 \quad C_{21} = -6 \quad C_{31} = 6$$

$$C_{12} = -6 \quad C_{22} = 3 \quad C_{32} = -6$$

$$C_{13} = -6 \quad C_{23} = -6 \quad C_{33} = 3$$

$$\text{adj } A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$$

$$= \begin{bmatrix} -3 & -6 & -6 \\ 6 & 3 & -6 \\ 6 & -6 & 3 \end{bmatrix}^T$$

$$\text{So, adj } A = \begin{bmatrix} -3 & 6 & 6 \\ -6 & 3 & -6 \\ -6 & 6 & 3 \end{bmatrix}$$

$$A(\text{adj } A) = \begin{bmatrix} -1 & -2 & -2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} -3 & 6 & 6 \\ -6 & 3 & -6 \\ -6 & 6 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 27 & 0 & 0 \\ 0 & 27 & 0 \\ 0 & 0 & 27 \end{bmatrix}$$

$$A(\text{adj } A) = 27 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{Hence, } A(\text{adj } A) = |A|I$$

39. Question

$$\text{If } A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \text{ find } A^{-1} \text{ and show that } A^{-1} = \frac{1}{2}(A^2 - 3I).$$

Answer

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} |A| = 0 - 1(0 - 1) + 1(1 - 0) = 0 + 1 + 1 = 2$$

Cofactors of A are:

$$C_{11} = -1 \quad C_{21} = 1 \quad C_{31} = 1$$

$$C_{12} = 1 \quad C_{22} = -1 \quad C_{32} = 1$$

$$C_{13} = 1 \quad C_{23} = 1 \quad C_{33} = -1$$

$$\text{adj } A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$$

$$= \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}^T$$

$$\text{So, adj } A = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

$$\text{Now, } A^{-1} = \frac{1}{2} \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

$$A^2 - 3I = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} - 3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0+1+1 & 0+0+1 & 0+1+0 \\ 0++0+1 & 1+0+1 & 1+0+0 \\ 0+1+0 & 1+0+0 & 1+1+0 \end{bmatrix} - 3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

$$\text{Hence, } A^{-1} = \frac{1}{2}(A^2 - 3I)$$

Exercise 7.2

1. Question

Find the inverse of each of the following matrices by using elementary row transformations:

$$\begin{bmatrix} 7 & 1 \\ 4 & -3 \end{bmatrix}$$

Answer

Given:- 2 x 2 square matrix

Tip:- Algorithm to find Inverse of a square matrix of 'n' order by elementary row transformation

(i) Obtain the square matrix, say A

(ii) Write $A = I_n A$

(iii) Perform a sequence of elementary row operation successively on A on the LHS and pre-factor I_n on the RHS till we obtain the result

$$I_n = BA$$

(iv) Write $A^{-1} = B$

Now,

We have,

$$A = I_2 A$$

Where I_2 is 2 x 2 elementary matrix

$$\Rightarrow \begin{bmatrix} 7 & 1 \\ 4 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$$

Applying $r_1 \rightarrow \frac{1}{7}r_1$

$$\Rightarrow \begin{bmatrix} 1 & \frac{1}{7} \\ 4 & -3 \end{bmatrix} = \begin{bmatrix} \frac{1}{7} & 0 \\ 0 & 1 \end{bmatrix} A$$

Applying $r_2 \rightarrow r_2 - 4r_1$

$$\Rightarrow \begin{bmatrix} 1 & \frac{1}{7} \\ 0 & -\frac{25}{7} \end{bmatrix} = \begin{bmatrix} \frac{1}{7} & 0 \\ -\frac{4}{7} & 1 \end{bmatrix} A$$

Applying $r_2 \rightarrow -\frac{7}{25}r_2$

$$\Rightarrow \begin{bmatrix} 1 & \frac{1}{7} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{7} & 0 \\ \frac{4}{25} & -\frac{7}{25} \end{bmatrix} A$$

Applying $r_1 \rightarrow r_1 - \frac{1}{7}r_2$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{21}{175} & \frac{1}{25} \\ \frac{4}{25} & -\frac{7}{25} \end{bmatrix} A$$

Hence, it is of the form

$$I = BA$$

So, as we know that

$$I = A^{-1}A$$

Therefore

$$A^{-1} = B$$

$$\Rightarrow A^{-1} = \begin{bmatrix} \frac{21}{175} & \frac{1}{25} \\ \frac{4}{25} & -\frac{7}{25} \end{bmatrix} \text{ inverse of } A$$

2. Question

Find the inverse of each of the following matrices by using elementary row transformations:

$$\begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$$

Answer

Given:- 2 x 2 square matrix

Tip:- Algorithm to find Inverse of a square matrix of 'n' order by elementary row transformation

(i) Obtain the square matrix, say A

(ii) Write $A = I_n A$

(iii) Perform a sequence of elementary row operation successively on A on the LHS and pre-factor I_n on the RHS till we obtain the result

$$I_n = BA$$

(iv) Write $A^{-1} = B$

Now,

We have,

$$A = I_2 A$$

Where I_2 is 2 x 2 elementary matrix

$$\Rightarrow \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$$

Applying $r_1 \rightarrow \frac{1}{5}r_1$

$$\Rightarrow \begin{bmatrix} 1 & \frac{2}{5} \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & 0 \\ 0 & 1 \end{bmatrix} A$$

Applying $r_2 \rightarrow r_2 - 2r_1$

$$\Rightarrow \begin{bmatrix} 1 & \frac{2}{5} \\ 0 & \frac{1}{5} \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & 0 \\ -\frac{2}{5} & 1 \end{bmatrix} A$$

Applying $r_2 \rightarrow 5r_2$

$$\Rightarrow \begin{bmatrix} 1 & \frac{2}{5} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & 0 \\ -2 & 5 \end{bmatrix} A$$

$\Rightarrow$

Applying $r_1 \rightarrow r_1 - \frac{2}{5}r_2$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix} A$$

Hence, it is of the form

$$I = BA$$

So, as we know that

$$I = A^{-1}A$$

Therefore

$$A^{-1} = B$$

$$\Rightarrow A^{-1} = \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix} \text{ inverse of } A$$

3. Question

Find the inverse of each of the following matrices by using elementary row transformations:

$$\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$$

Answer

Given:- 2 x 2 square matrix

Tip:- Algorithm to find Inverse of a square matrix of 'n' order by elementary row transformation

(i) Obtain the square matrix, say A

(ii) Write $A = I_n A$

(iii) Perform a sequence of elementary row operation successively on A on the LHS and pre-factor I_n on the RHS till we obtain the result

$$I_n = BA$$

(iv) Write $A^{-1} = B$

Now,

We have,

$$A = I_2 A$$

Where I_2 is 2 x 2 elementary matrix

$$\Rightarrow \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$$

Applying $r_2 \rightarrow r_2 - 2r_1$

$$\Rightarrow \begin{bmatrix} 1 & 2 \\ 0 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} A$$

Applying $r_2 \rightarrow -\frac{1}{5}r_2$

$$\Rightarrow \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{2}{5} & -\frac{1}{5} \end{bmatrix} A$$

Applying $r_1 \rightarrow r_1 - 2r_2$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & -\frac{1}{5} \end{bmatrix} A$$

Hence, it is of the form

$$I = BA$$

So, as we know that

$$I = A^{-1}A$$

Therefore

$$A^{-1} = B$$

$$\Rightarrow A^{-1} = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & -\frac{1}{5} \end{bmatrix} \text{ inverse of } A$$

4. Question

Find the inverse of each of the following matrices by using elementary row transformations:

$$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$$

Answer

Given:- 2 x 2 square matrix

Tip:- Algorithm to find Inverse of a square matrix of 'n' order by elementary row transformation

(i) Obtain the square matrix, say A

(ii) Write $A = I_n A$

(iii) Perform a sequence of elementary row operation successively on A on the LHS and pre-factor I_n on the RHS till we obtain the result

$$I_n = BA$$

(iv) Write $A^{-1} = B$

Now,

We have,

$$A = I_2 A$$

Where I_2 is 2 x 2 elementary matrix

$$\Rightarrow \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$$

Applying $r_1 \rightarrow \frac{1}{2}r_1$

$$\Rightarrow \begin{bmatrix} 1 & \frac{5}{2} \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} A$$

Applying $r_2 \rightarrow r_2 - r_1$

$$\Rightarrow \begin{bmatrix} 1 & \frac{5}{2} \\ 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ -\frac{1}{2} & 1 \end{bmatrix} A$$

Applying $r_2 \rightarrow 2r_2$

$$\Rightarrow \begin{bmatrix} 1 & \frac{5}{2} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ -1 & 2 \end{bmatrix} A$$

Applying $r_1 \rightarrow r_1 - \frac{5}{2}r_2$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} A$$

Hence, it is of the form

$$I = BA$$

So, as we know that

$$I = A^{-1}A$$

Therefore

$$A^{-1} = B$$

$$\Rightarrow A^{-1} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} \text{ inverse of } A$$

5. Question

Find the inverse of each of the following matrices by using elementary row transformations:

$$\begin{bmatrix} 3 & 10 \\ 2 & 7 \end{bmatrix}$$

Answer

Given:- 2 x 2 square matrix

Tip:- Algorithm to find Inverse of a square matrix of 'n' order by elementary row transformation

(i) Obtain the square matrix, say A

(ii) Write $A = I_n A$

(iii) Perform a sequence of elementary row operation successively on A on the LHS and pre-factor I_n on the RHS till we obtain the result

$$I_n = BA$$

(iv) Write $A^{-1} = B$

Now,

We have,

$$A = I_2 A$$

Where I_2 is 2 x 2 elementary matrix

$$\Rightarrow \begin{bmatrix} 3 & 10 \\ 2 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$$

Applying $r_1 \rightarrow \frac{1}{3}r_1$

$$\Rightarrow \begin{bmatrix} 1 & \frac{10}{3} \\ 2 & 7 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix} A$$

Applying $r_2 \rightarrow r_2 - 2r_1$

$$\Rightarrow \begin{bmatrix} 1 & \frac{10}{3} \\ 0 & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 \\ -\frac{2}{3} & 1 \end{bmatrix} A$$

Applying $r_2 \rightarrow 3r_2$

$$\Rightarrow \begin{bmatrix} 1 & \frac{10}{3} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 \\ -2 & 3 \end{bmatrix} A$$

Applying $r_1 \rightarrow r_1 - \frac{10}{3}r_2$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & -10 \\ -2 & 3 \end{bmatrix} A$$

Hence, it is of the form

$$I = BA$$

So, as we know that

$$I = A^{-1}A$$

Therefore

$$A^{-1} = B$$

$$\Rightarrow A^{-1} = \begin{bmatrix} 7 & -10 \\ -2 & 3 \end{bmatrix} \text{ inverse of } A$$

6. Question

Find the inverse of each of the following matrices by using elementary row transformations:

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$$

Answer

Given:- 3 x 3 square matrix

Tip:- Algorithm to find Inverse of a square matrix of 'n' order by elementary row transformation

(i) Obtain the square matrix, say A

(ii) Write $A = I_n A$

(iii) Perform a sequence of elementary row operation successively on A on the LHS and pre-factor I_n on the RHS till we obtain the result

$$I_n = BA$$

(iv) Write $A^{-1} = B$

Now,

We have,

$$A = I_3 A$$

Where I_3 is 3 x 3 elementary matrix

$$\Rightarrow \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $r_1 \leftrightarrow r_2$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $r_3 \rightarrow r_3 - 3r_1$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -5 & -8 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -3 & 1 \end{bmatrix} A$$

Applying $r_1 \rightarrow r_1 - 2r_2$ and $r_3 \rightarrow r_3 + 5r_2$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ 5 & -3 & 1 \end{bmatrix} A$$

Applying $r_3 \rightarrow \frac{1}{2}r_3$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 0 \\ \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{bmatrix} A$$

Applying $r_1 \rightarrow r_1 + r_3$ and $r_2 \rightarrow r_2 - 2r_3$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -4 & 3 & 1 \\ \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{bmatrix} A$$

Hence, it is of the form

$$I = BA$$

So, as we know that

$$I = A^{-1}A$$

Therefore

$$A^{-1} = B$$

$$\Rightarrow A^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -4 & 3 & 1 \\ \frac{5}{2} & -\frac{3}{2} & \frac{1}{2} \end{bmatrix} \text{ inverse of } A$$

7. Question

Find the inverse of each of the following matrices by using elementary row transformations:

$$\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$$

Answer

Given:- 3 x 3 square matrix

Tip:- Algorithm to find Inverse of a square matrix of 'n' order by elementary row transformation

(i) Obtain the square matrix, say A

(ii) Write $A = I_n A$

(iii) Perform a sequence of elementary row operation successively on A on the LHS and pre-factor I_n on the RHS till we obtain the result

$$I_n = BA$$

(iv) Write $A^{-1} = B$

Now,

We have,

$$A = I_3 A$$

Where I_3 is 3×3 elementary matrix

$$\Rightarrow \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $r_1 \rightarrow \frac{1}{2}r_1$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $r_2 \rightarrow r_2 - 5r_1$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{5}{2} \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -\frac{5}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $r_3 \rightarrow r_3 - r_2$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{5}{2} \\ 0 & 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -\frac{5}{2} & 1 & 0 \\ \frac{1}{2} & -1 & 1 \end{bmatrix} A$$

Applying $r_3 \rightarrow 2r_3$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{5}{2} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -\frac{5}{2} & 1 & 0 \\ 5 & -2 & 2 \end{bmatrix} A$$

Applying $r_1 \rightarrow r_1 + \frac{1}{2}r_3$ and $r_2 \rightarrow r_2 - \frac{5}{2}r_3$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix} A$$

Hence, it is of the form

$$I = BA$$

So, as we know that

$$I = A^{-1}A$$

Therefore

$$A^{-1} = B$$

$$\Rightarrow A^{-1} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix} \text{ inverse of } A$$

8. Question

Find the inverse of each of the following matrices by using elementary row transformations:

$$\begin{bmatrix} 2 & 3 & 1 \\ 2 & 4 & 1 \\ 3 & 7 & 2 \end{bmatrix}$$

Answer

Given:- 3 x 3 square matrix

Tip:- Algorithm to find Inverse of a square matrix of 'n' order by elementary row transformation

(i) Obtain the square matrix, say A

(ii) Write $A = I_n A$

(iii) Perform a sequence of elementary row operation successively on A on the LHS and pre-factor I_n on the RHS till we obtain the result

$$I_n = BA$$

(iv) Write $A^{-1} = B$

Now,

We have,

$$A = I_3 A$$

Where I_3 is 3 x 3 elementary matrix

$$\Rightarrow \begin{bmatrix} 2 & 3 & 1 \\ 2 & 4 & 1 \\ 3 & 7 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $r_1 \rightarrow \frac{1}{2}r_1$

$$\Rightarrow \begin{bmatrix} 1 & \frac{3}{2} & \frac{1}{2} \\ 2 & 4 & 1 \\ 3 & 7 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $r_2 \rightarrow r_2 - 2r_1$ and $r_3 \rightarrow r_3 - 3r_1$

$$\Rightarrow \begin{bmatrix} 1 & \frac{3}{2} & \frac{1}{2} \\ 0 & 1 & 0 \\ 0 & \frac{5}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -1 & 1 & 0 \\ -\frac{3}{2} & 0 & 1 \end{bmatrix} A$$

Applying $r_1 \rightarrow r_1 - \frac{3}{2}r_2$ and $r_3 \rightarrow r_3 - \frac{5}{2}r_2$

$$\Rightarrow \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 2 & -\frac{3}{2} & 0 \\ -1 & 1 & 0 \\ 1 & -\frac{5}{2} & 1 \end{bmatrix} A$$

Applying $r_3 \rightarrow 2r_3$

$$\Rightarrow \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -\frac{3}{2} & 0 \\ -1 & 1 & 0 \\ 2 & -5 & 2 \end{bmatrix} A$$

Applying $r_1 \rightarrow r_1 - \frac{1}{2}r_3$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 0 \\ 2 & -5 & 2 \end{bmatrix} A$$

Hence, it is of the form

$$I = BA$$

So, as we know that

$$I = A^{-1}A$$

Therefore

$$A^{-1} = B$$

$$\Rightarrow A^{-1} = \begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 0 \\ 2 & -5 & 2 \end{bmatrix} \text{ inverse of } A$$

9. Question

Find the inverse of each of the following matrices by using elementary row transformations:

$$\begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$$

Answer

Given:- 3 x 3 square matrix

Tip:- Algorithm to find Inverse of a square matrix of 'n' order by elementary row transformation

(i) Obtain the square matrix, say A

(ii) Write $A = I_n A$

(iii) Perform a sequence of elementary row operation successively on A on the LHS and pre-factor I_n on the RHS till we obtain the result

$$I_n = BA$$

(iv) Write $A^{-1} = B$

Now,

We have,

$$A = I_3 A$$

Where I_3 is 3 x 3 elementary matrix

$$\Rightarrow \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $r_1 \rightarrow \frac{1}{3}r_1$

$$\Rightarrow \begin{bmatrix} 1 & -1 & \frac{4}{3} \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $r_2 \rightarrow r_2 - 2r_1$

$$\Rightarrow \begin{bmatrix} 1 & -1 & \frac{4}{3} \\ 0 & -1 & \frac{4}{3} \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ -\frac{2}{3} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $r_2 \rightarrow -r_2$

$$\Rightarrow \begin{bmatrix} 1 & -1 & \frac{4}{3} \\ 0 & 1 & -\frac{4}{3} \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ \frac{2}{3} & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $r_1 \rightarrow r_1 + r_2$ and $r_3 \rightarrow r_3 + r_2$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{4}{3} \\ 0 & 0 & -\frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & -1 & 0 \\ \frac{2}{3} & -1 & 0 \\ \frac{2}{3} & -1 & 1 \end{bmatrix} A$$

Applying $r_3 \rightarrow -3r_3$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{4}{3} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & -1 & 0 \\ \frac{2}{3} & -1 & 0 \\ -2 & 3 & -3 \end{bmatrix} A$$

Applying $r_2 \rightarrow r_2 + \frac{4}{3}r_3$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix} A$$

Hence, it is of the form

$$I = BA$$

So, as we know that

$$I = A^{-1}A$$

Therefore

$$A^{-1} = B$$

$$\Rightarrow A^{-1} = \begin{bmatrix} \frac{1}{3} & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix} \text{ inverse of } A$$

10. Question

Find the inverse of each of the following matrices by using elementary row transformations:

$$\begin{bmatrix} 2 & -1 & 4 \\ 4 & 0 & 2 \\ 3 & -2 & 7 \end{bmatrix}$$

Answer

Given:- 3 x 3 square matrix

Tip:- Algorithm to find Inverse of a square matrix of 'n' order by elementary row transformation

(i) Obtain the square matrix, say A

(ii) Write $A = I_n A$

(iii) Perform a sequence of elementary row operation successively on A on the LHS and pre-factor I_n on the RHS till we obtain the result

$$I_n = BA$$

(iv) Write $A^{-1} = B$

Now,

We have,

$$A = I_3 A$$

Where I_3 is 3 x 3 elementary matrix

$$\Rightarrow \begin{bmatrix} 1 & 2 & 0 \\ 2 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $r_2 \rightarrow r_2 - 2r_1$ and $r_3 \rightarrow r_3 - r_1$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & -1 \\ 0 & -3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} A$$

Applying $r_2 \rightarrow -r_2$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & -3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix} A$$

Applying $r_1 \rightarrow r_1 - 2r_2$ and $r_3 \rightarrow r_3 + 3r_2$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 6 \end{bmatrix} = \begin{bmatrix} -3 & 2 & 0 \\ 2 & -1 & 0 \\ 5 & -3 & 1 \end{bmatrix} A$$

Applying $r_3 \rightarrow \frac{1}{6}r_3$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 2 & 0 \\ 2 & -1 & 0 \\ \frac{5}{6} & -\frac{1}{2} & \frac{1}{6} \end{bmatrix} A$$

Applying $r_1 \rightarrow r_1 + 2r_3$ and $r_2 \rightarrow r_2 - r_3$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{4}{3} & 1 & \frac{1}{3} \\ \frac{7}{6} & -\frac{1}{2} & -\frac{1}{6} \\ \frac{5}{6} & -\frac{1}{2} & \frac{1}{6} \end{bmatrix} A$$

Hence , it is of the form

$$I = BA$$

So, as we know that

$$I = A^{-1}A$$

Therefore

$$A^{-1} = B$$

$$\Rightarrow A^{-1} = \begin{bmatrix} -\frac{4}{3} & 1 & \frac{1}{3} \\ \frac{7}{6} & -\frac{1}{2} & -\frac{1}{6} \\ \frac{5}{6} & -\frac{1}{2} & \frac{1}{6} \end{bmatrix} \text{ inverse of A}$$

11. Question

Find the inverse of each of the following matrices by using elementary row transformations:

$$\begin{bmatrix} 2 & -1 & 3 \\ 1 & 2 & 4 \\ 3 & 1 & 1 \end{bmatrix}$$

Answer

Given:- 3 x 3 square matrix

Tip:- Algorithm to find Inverse of a square matrix of 'n' order by elementary row transformation

(i) Obtain the square matrix, say A

(ii) Write $A = I_n A$

(iii) Perform a sequence of elementary row operation successively on A on the LHS and pre-factor I_n on the RHS till we obtain the result

$$I_n = BA$$

(iv) Write $A^{-1} = B$

Now,

We have,

$$A = I_3 A$$

Where I_3 is 3 x 3 elementary matrix

$$\Rightarrow \begin{bmatrix} 2 & -1 & 3 \\ 1 & 2 & 4 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $r_1 \rightarrow \frac{1}{2}r_1$

$$\Rightarrow \begin{bmatrix} 1 & -\frac{1}{2} & \frac{3}{2} \\ 1 & 2 & 4 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $r_2 \rightarrow r_2 - r_1$ and $r_3 \rightarrow r_3 - 3r_1$

$$\Rightarrow \begin{bmatrix} 1 & -\frac{1}{2} & \frac{3}{2} \\ 0 & \frac{5}{2} & \frac{5}{2} \\ 0 & \frac{5}{2} & -\frac{7}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ -\frac{3}{2} & 0 & 1 \end{bmatrix} A$$

Applying $r_2 \rightarrow \frac{2}{5}r_2$

$$\Rightarrow \begin{bmatrix} 1 & -\frac{1}{2} & \frac{3}{2} \\ 0 & 1 & 1 \\ 0 & \frac{5}{2} & -\frac{7}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -\frac{1}{5} & \frac{2}{5} & 0 \\ -\frac{3}{2} & 0 & 1 \end{bmatrix} A$$

Applying $r_1 \rightarrow r_1 + \frac{1}{2}r_2$ and $r_3 \rightarrow r_3 - \frac{5}{2}r_2$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & -6 \end{bmatrix} = \begin{bmatrix} \frac{2}{5} & \frac{1}{5} & 0 \\ -\frac{1}{5} & \frac{2}{5} & 0 \\ -1 & -1 & 1 \end{bmatrix} A$$

Applying $r_3 \rightarrow -\frac{1}{6}r_3$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{2}{5} & \frac{1}{5} & 0 \\ -\frac{1}{5} & \frac{2}{5} & 0 \\ \frac{1}{6} & \frac{1}{6} & -\frac{1}{6} \end{bmatrix} A$$

Applying $r_2 \rightarrow r_2 - r_3$ and $r_1 \rightarrow r_1 - 2r_3$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{15} & \frac{-2}{15} & \frac{-1}{3} \\ -\frac{11}{30} & \frac{7}{30} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & -\frac{1}{6} \end{bmatrix} A$$

Hence , it is of the form

$$I = BA$$

So, as we know that

$$I = A^{-1}A$$

Therefore

$$A^{-1} = B$$

$$A^{-1} = \begin{bmatrix} \frac{1}{15} & \frac{-2}{15} & \frac{-1}{3} \\ -\frac{11}{30} & \frac{7}{30} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & -\frac{1}{6} \end{bmatrix} \text{ inverse of } A$$

12. Question

Find the inverse of each of the following matrices by using elementary row transformations:

$$\begin{bmatrix} 1 & 1 & 2 \\ 3 & 1 & 1 \\ 2 & 3 & 1 \end{bmatrix}$$

Answer

Given:- 3 x 3 square matrix

Tip:- Algorithm to find Inverse of a square matrix of 'n' order by elementary row transformation

(i) Obtain the square matrix, say A

(ii) Write $A = I_n A$

(iii) Perform a sequence of elementary row operation successively on A on the LHS and pre-factor I_n on the RHS till we obtain the result

$$I_n = BA$$

(iv) Write $A^{-1} = B$

Now,

We have,

$$A = I_3 A$$

Where I_3 is 3×3 elementary matrix

$$\Rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 3 & 1 & 1 \\ 2 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $r_2 \rightarrow r_2 - 3r_1$ and $r_3 \rightarrow r_3 - 2r_1$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 0 & -2 & -5 \\ 0 & 1 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} A$$

Applying $r_2 \rightarrow \frac{-1}{2}r_2$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & \frac{5}{2} \\ 0 & 1 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{3}{2} & -\frac{1}{2} & 0 \\ -2 & 0 & 1 \end{bmatrix} A$$

Applying $r_1 \rightarrow r_1 - r_2$ and $r_3 \rightarrow r_3 - r_2$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{5}{2} \\ 0 & 0 & -\frac{11}{2} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & 0 \\ \frac{3}{2} & -\frac{1}{2} & 0 \\ -\frac{7}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} A$$

Applying $r_3 \rightarrow -\frac{2}{11}r_3$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{5}{2} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & 0 \\ \frac{3}{2} & -\frac{1}{2} & 0 \\ \frac{7}{11} & -\frac{1}{11} & -\frac{2}{11} \end{bmatrix} A$$

Applying $r_1 \rightarrow r_1 + \frac{1}{2}r_3$ and $r_2 \rightarrow r_2 - \frac{5}{2}r_3$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{2}{11} & \frac{5}{11} & -\frac{1}{11} \\ -\frac{1}{11} & -\frac{3}{11} & \frac{5}{11} \\ \frac{7}{11} & -\frac{1}{11} & -\frac{2}{11} \end{bmatrix} A$$

Hence, it is of the form

$$I = BA$$

So, as we know that

$$I = A^{-1}A$$

Therefore

$$A^{-1} = B$$

$$\Rightarrow A^{-1} = \begin{bmatrix} -\frac{2}{11} & \frac{5}{11} & -\frac{1}{11} \\ -\frac{1}{11} & -\frac{3}{11} & \frac{5}{11} \\ \frac{7}{11} & -\frac{1}{11} & -\frac{2}{11} \end{bmatrix} \text{ inverse of } A$$

13. Question

Find the inverse of each of the following matrices by using elementary row transformations:

$$\begin{bmatrix} 2 & -1 & 4 \\ 4 & 0 & 2 \\ 3 & -2 & 7 \end{bmatrix}$$

Answer

Given:- 3 x 3 square matrix

Tip:- Algorithm to find Inverse of a square matrix of 'n' order by elementary row transformation

(i) Obtain the square matrix, say A

(ii) Write $A = I_n A$

(iii) Perform a sequence of elementary row operation successively on A on the LHS and pre-factor I_n on the RHS till we obtain the result

$$I_n = BA$$

(iv) Write $A^{-1} = B$

Now,

We have,

$$A = I_3 A$$

Where I_3 is 3 x 3 elementary matrix

$$\Rightarrow \begin{bmatrix} 2 & -1 & 4 \\ 4 & 0 & 2 \\ 3 & -2 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $r_1 \rightarrow \frac{1}{2}r_1$

$$\Rightarrow \begin{bmatrix} 1 & -\frac{1}{2} & 2 \\ 4 & 0 & 2 \\ 3 & -2 & 7 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $r_2 \rightarrow r_2 - 4r_1$ and $r_3 \rightarrow r_3 - 3r_1$

$$\Rightarrow \begin{bmatrix} 1 & -\frac{1}{2} & 2 \\ 0 & 2 & -6 \\ 0 & -\frac{1}{2} & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -2 & 1 & 0 \\ -\frac{3}{2} & 0 & 1 \end{bmatrix} A$$

Applying $r_2 \rightarrow \frac{1}{2}r_2$

$$\Rightarrow \begin{bmatrix} 1 & -\frac{1}{2} & 2 \\ 0 & 1 & -3 \\ 0 & -\frac{1}{2} & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -1 & \frac{1}{2} & 0 \\ -\frac{3}{2} & 0 & 1 \end{bmatrix} A$$

Applying $r_1 \rightarrow r_1 + \frac{1}{2}r_2$ and $r_3 \rightarrow r_3 - \frac{5}{2}r_2$

$$\Rightarrow \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & -3 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{4} & 0 \\ -1 & \frac{1}{2} & 0 \\ -2 & \frac{1}{4} & 1 \end{bmatrix} A$$

Applying $r_3 \rightarrow -2r_3$

$$\Rightarrow \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{4} & 0 \\ -1 & \frac{1}{2} & 0 \\ 4 & -\frac{1}{2} & -2 \end{bmatrix} AA$$

Applying $r_1 \rightarrow r_1 - \frac{1}{2}r_3$ and $r_2 \rightarrow r_2 + 3r_3$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & \frac{1}{2} & 1 \\ 11 & -1 & -6 \\ 4 & -\frac{1}{2} & -2 \end{bmatrix} A$$

Hence , it is of the form

$$I = BA$$

So, as we know that

$$I = A^{-1}A$$

Therefore

$$A^{-1} = B$$

$$\Rightarrow A^{-1} = \begin{bmatrix} -2 & \frac{1}{2} & 1 \\ 11 & -1 & -6 \\ 4 & -\frac{1}{2} & -2 \end{bmatrix} \text{ inverse of } A$$

14. Question

Find the inverse of each of the following matrices by using elementary row transformations:

$$\begin{bmatrix} 3 & 0 & -1 \\ 2 & 3 & 0 \\ 0 & 4 & 1 \end{bmatrix}$$

Answer

Given:- 3 x 3 square matrix

Tip:- Algorithm to find Inverse of a square matrix of 'n' order by elementary row transformation

(i) Obtain the square matrix, say A

(ii) Write $A = I_n A$

(iii) Perform a sequence of elementary row operation successively on A on the LHS and pre-factor I_n on the RHS till we obtain the result

$$I_n = BA$$

(iv) Write $A^{-1} = B$

Now,

We have,

$$A = I_3 A$$

Where I_3 is 3 x 3 elementary matrix

$$\Rightarrow \begin{bmatrix} 3 & 0 & -1 \\ 2 & 3 & 0 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $r_1 \rightarrow \frac{1}{3}r_1$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -\frac{1}{3} \\ 2 & 3 & 0 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $r_2 \rightarrow r_2 - 2r_1$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -\frac{1}{3} \\ 0 & 3 & \frac{2}{3} \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ -\frac{2}{3} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $r_2 \rightarrow \frac{1}{3}r_2$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -\frac{1}{3} \\ 0 & 1 & \frac{2}{9} \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ -\frac{2}{9} & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $r_3 \rightarrow r_3 - 4r_2$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -\frac{1}{3} \\ 0 & 1 & \frac{2}{9} \\ 0 & 0 & \frac{1}{9} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ -\frac{2}{9} & \frac{1}{3} & 0 \\ \frac{8}{9} & -\frac{4}{3} & 1 \end{bmatrix} A$$

Applying $r_3 \rightarrow 9r_3$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -\frac{1}{3} \\ 0 & 1 & \frac{2}{9} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ -\frac{2}{9} & \frac{1}{3} & 0 \\ 8 & -12 & 9 \end{bmatrix} A$$

Applying $r_1 \rightarrow r_1 + \frac{1}{3}r_3$ and $r_2 \rightarrow r_2 - \frac{2}{9}r_3$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -4 & 3 \\ -2 & 3 & -2 \\ 8 & -12 & 9 \end{bmatrix} A$$

Hence , it is of the form

$$I = BA$$

So, as we know that

$$I = A^{-1}A$$

Therefore

$$A^{-1} = B$$

$$\Rightarrow A^{-1} = \begin{bmatrix} 3 & -4 & 3 \\ -2 & 3 & -2 \\ 8 & -12 & 9 \end{bmatrix} \text{ inverse of } A$$

15. Question

Find the inverse of each of the following matrices by using elementary row transformations:

$$\begin{bmatrix} 1 & 3 & -2 \\ -3 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$$

Answer

Given:- 3 x 3 square matrix

Tip:- Algorithm to find Inverse of a square matrix of 'n' order by elementary row transformation

(i) Obtain the square matrix, say A

(ii) Write $A = I_n A$

(iii) Perform a sequence of elementary row operation successively on A on the LHS and pre-factor I_n on the RHS till we obtain the result

$$I_n = BA$$

(iv) Write $A^{-1} = B$

Now,

We have,

$$A = I_3 A$$

Where I_3 is 3 x 3 elementary matrix

$$\Rightarrow \begin{bmatrix} 1 & 3 & -2 \\ -3 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $r_2 \rightarrow r_2 + 3r_1$ and $r_3 \rightarrow r_3 - 2r_1$

$$\Rightarrow \begin{bmatrix} 1 & 3 & -2 \\ 0 & 9 & -5 \\ 0 & -5 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} A$$

Applying $r_2 \rightarrow \frac{-1}{2}r_2$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & \frac{5}{2} \\ 0 & 1 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{3}{2} & -\frac{1}{2} & 0 \\ -2 & 0 & 1 \end{bmatrix} A$$

Applying $r_1 \rightarrow r_1 - 3r_2$ and $r_3 \rightarrow r_3 + 5r_2$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -\frac{1}{3} \\ 0 & 1 & -\frac{5}{9} \\ 0 & 0 & \frac{11}{9} \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{3} & 0 \\ \frac{1}{3} & \frac{1}{9} & 0 \\ -\frac{1}{3} & \frac{5}{9} & 1 \end{bmatrix} A$$

Applying $r_1 \rightarrow \frac{9}{11}r_1$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -\frac{1}{3} \\ 0 & 1 & -\frac{5}{9} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{3} & 0 \\ \frac{1}{3} & \frac{1}{9} & 0 \\ -\frac{3}{11} & \frac{5}{11} & \frac{9}{11} \end{bmatrix} A$$

Applying $r_1 \rightarrow r_1 + \frac{1}{3}r_3$ and $r_2 \rightarrow r_2 + \frac{5}{9}r_3$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{11} & -\frac{2}{11} & \frac{3}{11} \\ \frac{2}{11} & \frac{4}{11} & \frac{5}{11} \\ -\frac{3}{11} & \frac{5}{11} & \frac{9}{11} \end{bmatrix} A$$

Hence , it is of the form

$$I = BA$$

So, as we know that

$$I = A^{-1}A$$

Therefore

$$A^{-1} = B$$

$$\Rightarrow A^{-1} = \begin{bmatrix} -\frac{1}{11} & -\frac{2}{11} & \frac{3}{11} \\ \frac{2}{11} & \frac{4}{11} & \frac{5}{11} \\ -\frac{3}{11} & \frac{5}{11} & \frac{9}{11} \end{bmatrix} \text{ inverse of } A$$

16. Question

Find the inverse of each of the following matrices by using elementary row transformations:

$$\begin{bmatrix} -1 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$$

Answer

Given:- 3 x 3 square matrix

Tip:- Algorithm to find Inverse of a square matrix of 'n' order by elementary row transformation

(i) Obtain the square matrix, say A

(ii) Write $A = I_n A$

(iii) Perform a sequence of elementary row operation successively on A on the LHS and pre-factor I_n on the RHS till we obtain the result

$$I_n = BA$$

(iv) Write $A^{-1} = B$

Now,

We have,

$$A = I_3 A$$

Where I_3 is 3 x 3 elementary matrix

$$\Rightarrow \begin{bmatrix} -1 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $r_1 \rightarrow -1r_1$

$$\Rightarrow \begin{bmatrix} 1 & -1 & -2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $r_2 \rightarrow r_2 - r_1$ and $r_3 \rightarrow r_3 - 3r_1$

$$\Rightarrow \begin{bmatrix} 1 & -1 & -2 \\ 0 & 3 & 5 \\ 0 & 4 & 7 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} A$$

Applying $r_2 \rightarrow \frac{1}{3}r_2$

$$\Rightarrow \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & \frac{5}{3} \\ 0 & 4 & 7 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 \\ 3 & 0 & 1 \end{bmatrix} A$$

Applying $r_1 \rightarrow r_1 + r_2$ and $r_3 \rightarrow r_3 - 4r_2$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -\frac{1}{3} \\ 0 & 1 & \frac{5}{3} \\ 0 & 0 & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} -\frac{2}{3} & \frac{1}{3} & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{5}{3} & -\frac{4}{3} & 1 \end{bmatrix} A$$

Applying $r_3 \rightarrow 3r_3$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -\frac{1}{3} \\ 0 & 1 & \frac{5}{3} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{2}{3} & \frac{1}{3} & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 \\ 5 & -4 & 3 \end{bmatrix} A$$

Applying $r_1 \rightarrow r_1 + \frac{1}{3}r_3$ and $r_2 \rightarrow r_2 - \frac{5}{3}r_3$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ -8 & 7 & -5 \\ 5 & -4 & 3 \end{bmatrix} A$$

Hence , it is of the form

$$I = BA$$

So, as we know that

$$I = A^{-1}A$$

Therefore

$$A^{-1} = B$$

$$\Rightarrow A^{-1} = \begin{bmatrix} 1 & -1 & 1 \\ -8 & 7 & -5 \\ 5 & -4 & 3 \end{bmatrix} \text{ inverse of } A$$