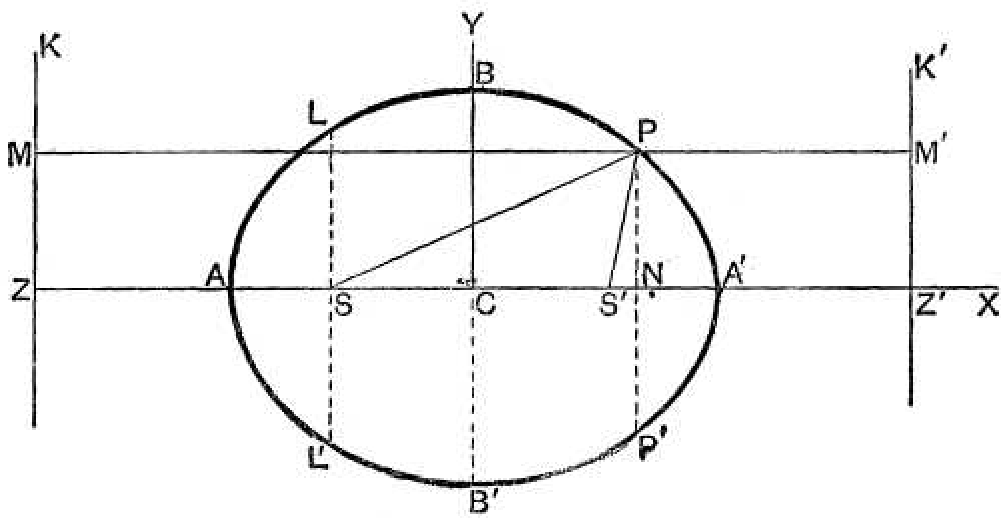
Chapter 12

THE ELLIPSE

247. THE ellipse is a conic section in which the eccentricity e is less than unity.

To find the equation to an ellipse.

Let ZK be the directrix, S the focus, and let SZ be perpendicular to the directrix.



There will be a point A on SZ, such that

$$SA = e \cdot AZ \qquad (1).$$

Since $e < 1$, there will be another point A', on ZS produced, such that

$$SA' = e \cdot A'Z \qquad (2).$$

Let the length AA' be called $2a$, and let C be the middle point of AA'. Adding (1) and (2), we have

$$2a = AA' = e\,(AZ + A'Z) = 2\,.\,e\,.\,CZ,$$

i.e.
$$CZ = \frac{a}{e} \text{ .... } (3).$$

Subtracting (1) from (2), we have

$$e\,(A'Z - AZ) = SA' - SA = (SC + CA') - (CA - CS),$$

i.e.
$$e\,.\,AA' = 2CS,$$

and hence
$$CS = a\,.\,e \text{ .... } (4).$$

Let C be the origin, CA' the axis of x, and a line through C perpendicular to AA' the axis of y.

Let P be any point on the curve, whose coordinates are x and y, and let PM be the perpendicular upon the directrix, and PN the perpendicular upon AA'.

The focus S is the point $(-ae, 0)$.

The relation $SP^2 = e^2\,.\,PM^2 = e^2\,.\,ZN^2$ then gives

$$(x + ae)^2 + y^2 = e^2\left(x + \frac{a}{e}\right)^2, \text{ (Art. 20)},$$

i.e
$$x^2\,(1 - e^2) + y^2 = a^2\,(1 - e^2),$$

i.e.
$$\frac{x^2}{a^2} + \frac{y^2}{a^2\,(1 - e^2)} = 1 \text{ .... } (5).$$

If in this equation we put $x = 0$, we have

$$y = \pm\, a\,\sqrt{1 - e^2},$$

shewing that the curve meets the axis of y in two points, B and B', lying on opposite sides of C, such that

$$B'C = CB = a\,\sqrt{1 - e^2}, \ \text{i.e. } CB^2 = CA^2 - CS^2.$$

Let the length CB be called b, so that

$$b = a\,\sqrt{1 - e^2}.$$

The equation (5) then becomes

$$\mathbf{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1} \text{ .... } (6).$$

248. The equation (6) of the previous article may be written

$$\frac{y^2}{b^2}=1-\frac{x^2}{a^2}=\frac{a^2-x^2}{a^2}=\frac{(a+x)(a-x)}{a^2},$$

i.e.
$$\frac{PN^2}{b^2}=\frac{AN \cdot NA'}{a^2},$$

i.e.
$$PN^2 : AN \cdot NA' :: BC^2 : AC^2.$$

Def. The points A and A' are called the vertices of the curve, AA' is called the major axis, and BB' the minor axis. Also C is called the centre.

249. Since S is the point $(-ae, 0)$, the equation to the ellipse referred to S as origin is (Art. 128),

$$\frac{(x-ae)^2}{a^2}+\frac{y^2}{b^2}=1.$$

The equation referred to A as origin, and AX and a perpendicular line as axes, is

$$\frac{(x-a)^2}{a^2}+\frac{y^2}{b^2}=1,$$

i.e.
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{2x}{a}=0.$$

Similarly, the equation referred to ZX and ZK as axes is, since $CZ=-\frac{a}{e}$,

$$\frac{\left(x-\frac{a}{e}\right)^2}{a^2}+\frac{y^2}{b^2}=1.$$

The equation to the ellipse, whose focus and directrix are any given point and line, and whose eccentricity is known, is easily written down.

For example, if the focus be the point $(-2, 3)$, the directrix be the line $2x+3y+4=0$, and the eccentricity be $\frac{4}{5}$, the required equation is

$$(x+2)^2+(y-3)^2=(\tfrac{4}{5})^2\frac{(2x+3y+4)^2}{2^2+3^2},$$

i.e.
$$261x^2+181y^2-192xy+1044x-2334y+3969=0.$$

Generally, the equation to the ellipse, whose focus is the point (f, g), whose directrix is $Ax+By+C=0$, and whose eccentricity is e, is

$$(x-f)^2+(y-g)^2=e^2\frac{(Ax+By+C)^2}{A^2+B^2}.$$

250. *There exist a second focus and a second directrix for the curve.*

On the positive side of the origin take a point S', which is such that $SC=CS'=ae$, and another point Z', such that

$$ZC=CZ'=\frac{a}{e}.$$

Draw $Z'K'$ perpendicular to ZZ', and PM' perpendicular to $Z'K'$.

The equation (5) of Art. 247 may be written in the form

$$x^2-2aex+a^2e^2+y^2=e^2x^2-2aex+a^2,$$

i.e. $$(x-ae)^2+y^2=e^2\left(\frac{a}{e}-x\right)^2,$$

i.e. $$S'P^2=e^2 . PM'^2.$$

Hence any point P of the curve is such that its distance from S' is e times its distance from $Z'K'$, so that we should have obtained the same curve, if we had started with S' as focus, $Z'K'$ as directrix, and the same eccentricity.

251. *The sum of the focal distances of any point on the curve is equal to the major axis.*

For (Fig. Art. 247) we have

$$SP=e . PM, \text{ and } S'P=e . PM'.$$

Hence

$$SP+S'P=e\,(PM+PM')=e . MM'$$
$$=e . ZZ'=2e . CZ=2a \text{ (Art. 247.)}$$
$$=\text{the major axis.}$$

Also $\mathbf{SP}=e . PM=e . NZ=e . CZ+e . CN=\mathbf{a+ex'}$,

and $\mathbf{S'P}=e . PM'=e . NZ'=e . CZ'-e . CN=\mathbf{a-ex'}$,

where x' is the abscissa of P referred to the centre.

252. Mechanical construction for an ellipse.

By the preceding article we can get a simple mechanical method of constructing an ellipse.

Take a piece of thread, whose length is the major axis of the required ellipse, and fasten its ends at the points S and S' which are to be the foci.

Let the point of a pencil move on the paper, the point being always in contact with the string and keeping the two portions of the string between it and the fixed ends always tight. If the end of the pencil be moved about on the paper, so as to satisfy these conditions, it will trace out the curve on the paper. For the end of the pencil will be always in such a position that the sum of its distances from S and S' will be constant.

In practice, it is easier to fasten two drawing pins at S and S', and to have an endless piece of string whose total length is equal to the sum of SS' and AA'. This string must be passed round the two pins at S and S' and then be kept stretched by the pencil as before. By this second arrangement it will be found that the portions of the curve near A and A' can be more easily described than in the first method.

253. *Latus-rectum of the ellipse.*

Let LSL' be the double ordinate of the curve which passes through the focus S. By the definition of the curve, the semi-latus-rectum SL

$= e$ times the distance of L from the directrix

$= e . SZ = e\,(CZ - CS) = e . CZ - e . CS$

$= a - ae^2$ (by equations (3) and (4) of Art. 247)

$= \dfrac{b^2}{a}$. (Art. 247.)

254. *To trace the curve*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \ldots\ldots\ldots \quad \ldots\ldots\ldots\ldots\ldots (1).$$

The equation may be written in either of the forms

$$y = \pm b\sqrt{1 - \frac{x^2}{a^2}} \quad \text{..................(2)},$$

or

$$x = \pm a\sqrt{1 - \frac{y^2}{b^2}} \quad \text{..................(3)}.$$

From (2), it follows that if $x^2 > a^2$, *i.e.* if $x > a$ or $< -a$, then y is impossible. There is therefore no part of the curve to the right of A' or to the left of A.

From (3), it follows, similarly, that, if $y > b$ or $< -b$, x is impossible, and hence that there is no part of the curve above B or below B'.

If x lie between $-a$ and $+a$, the equation (2) gives two equal and opposite values for y, so that the curve is symmetrical with respect to the axis of x.

If y lie between $-b$ and $+b$, the equation (3) gives two equal and opposite values for x, so that the curve is symmetrical with respect to the axis of y.

If a number of values in succession be given to x, and the corresponding values of y be determined, we shall obtain a series of points which will all be found to lie on a curve of the shape given in the figure of Art. 247.

255. *The quantity* $\frac{x'^2}{a^2} + \frac{y'^2}{b^2} - 1$ *is negative, zero, or positive, according as the point* (x', y') *lies within, upon, or without the ellipse.*

Let Q be the point (x', y'), and let the ordinate QN through Q meet the curve in P, so that, by equation (6) of Art. 247,

$$\frac{PN^2}{b^2} = 1 - \frac{x'^2}{a^2}.$$

If Q be within the curve, then y', *i.e.* QN, is $< PN$, so that

$$\frac{y'^2}{b^2} < \frac{PN^2}{b^2}, \ \textit{i.e.} \ < 1 - \frac{x'^2}{a^2}.$$

Hence, in this case,

$$\frac{x'^2}{a^2} + \frac{y'^2}{b^2} < 1,$$

i.e.

$$\frac{x'^2}{a^2} + \frac{y'^2}{b^2} - 1 \text{ is negative.}$$

Similarly, if Q' be without the curve, $y' > PN$, and then $\frac{x'^2}{a^2} + \frac{y'^2}{b^2} - 1$ is positive.

256. *To find the length of a radius vector from the centre drawn in a given direction.*

The equation (6) of Art. 247 when transferred to polar coordinates becomes

$$\frac{r^2 \cos^2\theta}{a^2} + \frac{r^2 \sin^2\theta}{b^2} = 1,$$

giving

$$r^2 = \frac{a^2 b^2}{b^2 \cos^2\theta + a^2 \sin^2\theta}.$$

We thus have the value of the radius vector drawn at any inclination θ to the axis.

Since $r^2 = \frac{a^2b^2}{b^2 + (a^2 - b^2)\sin^2\theta}$, we see that the greatest value of r is when $\theta = 0$, and then it is equal to a.

Similarly, $\theta = 90°$ gives the least value of r, viz. b.

Also, for each value of θ, we have two equal and opposite values of r, so that any line through the centre meets the curve in two points equidistant from it.

257. Auxiliary circle. Def. The circle which is described on the major axis, AA', of an ellipse as diameter, is called the auxiliary circle of the ellipse.

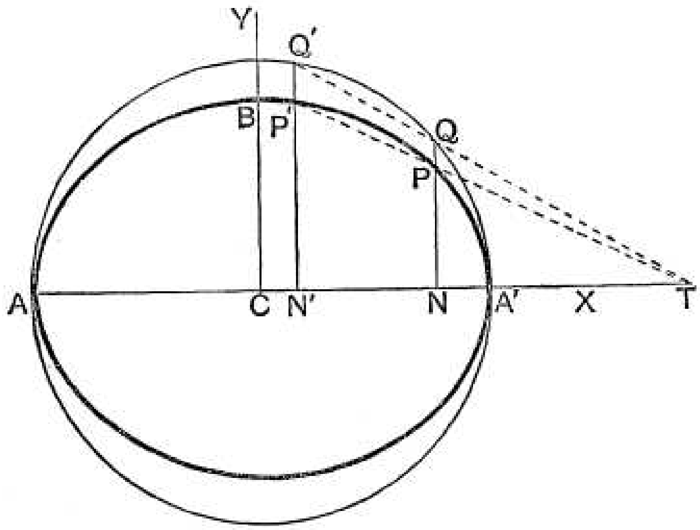
Let NP be any ordinate of the ellipse, and let it be produced to meet the auxiliary circle in Q.

Since the angle AQA' is a right angle, being the angle in a semicircle, we have, by Euc. VI. 8, $QN^2 = AN \cdot NA'$.

Hence Art. 248 gives

$$PN^2 : QN^2 :: BC^2 : AC^2,$$

so that $$\frac{\mathbf{PN}}{\mathbf{QN}} = \frac{\mathbf{BC}}{\mathbf{AC}} = \frac{\mathbf{b}}{\mathbf{a}}.$$



The point Q in which the ordinate NP meets the auxiliary circle is called the corresponding point to P.

The ordinates of any point on the ellipse and the corresponding point on the auxiliary circle are therefore to one another in the ratio $b : a$, *i.e.* in the ratio of the semi-minor to the semi-major axis of the ellipse.

The ellipse might therefore have been defined as follows:

Take a circle and from each point of it draw perpendiculars upon a diameter; the locus of the points dividing these perpendiculars in a given ratio is an ellipse, of which the given circle is the auxiliary circle.

258. Eccentric Angle. Def. The eccentric angle of any point P on the ellipse is the angle NCQ made with the major axis by the straight line CQ joining the centre C to the point Q on the auxiliary circle which corresponds to the point P.

This angle is generally called ϕ.

We have $$CN = CQ \,.\, \cos\phi = a\cos\phi,$$
and $$NQ = CQ\sin\phi = a\sin\phi.$$

Hence, by the last article,

$$NP = \frac{b}{a}\,.\,NQ = b\sin\phi.$$

The coordinates of any point P on the ellipse are therefore $a\cos\phi$ and $b\sin\phi$.

Since P is known when ϕ is given, it is often called "the point ϕ."

259. *To obtain the equation of the straight line joining two points on the ellipse whose eccentric angles are given.*

Let the eccentric angles of the two points, P and P', be ϕ and ϕ', so that the points have as coordinates

$$(a\cos\phi,\ b\sin\phi)\ \text{ and }\ (a\cos\phi',\ b\sin\phi').$$

The equation of the straight line joining them is

$$y - b\sin\phi = \frac{b\sin\phi' - b\sin\phi}{a\cos\phi' - a\cos\phi}(x - a\cos\phi)$$

$$= \frac{b}{a}\cdot\frac{2\cos\frac{1}{2}(\phi+\phi')\sin\frac{1}{2}(\phi'-\phi)}{2\sin\frac{1}{2}(\phi+\phi')\sin\frac{1}{2}(\phi-\phi')}(x - a\cos\phi)$$

$$= -\frac{b}{a}\cdot\frac{\cos\frac{1}{2}(\phi+\phi')}{\sin\frac{1}{2}(\phi'+\phi)}(x - a\cos\phi),$$

i.e.

$$\frac{x}{a}\cos\frac{\phi+\phi'}{2} + \frac{y}{b}\sin\frac{\phi+\phi'}{2} = \cos\phi\cos\frac{\phi+\phi'}{2} + \sin\phi\sin\frac{\phi+\phi'}{2}$$

$$= \cos\left[\phi - \frac{\phi+\phi'}{2}\right] = \cos\frac{\phi-\phi'}{2}\ \ldots\ldots\ldots\ldots(1).$$

This is the required equation.

Cor. The points on the auxiliary circle, corresponding to P and P', have as coordinates $(a\cos\phi,\ a\sin\phi)$ and $(a\cos\phi',\ a\sin\phi')$.

The equation to the line joining them is therefore (Art. 178)

$$\frac{x}{a}\cos\frac{\phi+\phi'}{2} + \frac{y}{a}\sin\frac{\phi+\phi'}{2} = \cos\frac{\phi-\phi'}{2}.$$

This straight line and (1) clearly make the same intercept on the major axis.

Hence the straight line joining any two points on an ellipse, and the straight line joining the corresponding points on the auxiliary circle, meet the major axis in the same point.

EXERCISES XXXII

1. Find the equation to the ellipses, whose centres are the origin, whose axes are the axes of coordinates, and which pass through (α) the points (2, 2), and (3, 1),
and (β) the points (1, 4) and (−6, 1).

Find the equation of the ellipse referred to its centre

2. whose latus rectum is 5 and whose eccentricity is $\frac{2}{3}$,

3. whose minor axis is equal to the distance between the foci and whose latus rectum is 10,

4. whose foci are the points (4, 0) and (−4, 0) and whose eccentricity is $\frac{1}{3}$.

5. Find the latus rectum, the eccentricity, and the coordinates of the foci, of the ellipses

(1) $x^2+3y^2=a^2$, (2) $5x^2+4y^2=1$, and (3) $9x^2+5y^2-30y=0$.

6. Find the eccentricity of an ellipse, if its latus rectum be equal to one half its minor axis.

7. Find the equation to the ellipse, whose focus is the point (−1, 1), whose directrix is the straight line $x-y+3=0$, and whose eccentricity is $\frac{1}{2}$.

8. Is the point (4, −3) within or without the ellipse

$$5x^2+7y^2=11^2?$$

9. Find the lengths of, and the equations to, the focal radii drawn to the point $(4\sqrt{3}, 5)$ of the ellipse

$$25x^2+16y^2=1600.$$

10. Prove that the sum of the squares of the reciprocals of two perpendicular diameters of an ellipse is constant.

11. Find the inclination to the major axis of the diameter of the ellipse the square of whose length is (1) the arithmetical mean, (2) the geometrical mean, and (3) the harmonical mean, between the squares on the major and minor axes.

12. Find the locus of the middle points of chords of an ellipse which are drawn through the positive end of the minor axis.

13. Prove that the locus of the intersection of AP with the straight line through A' perpendicular to $A'P$ is a straight line which is perpendicular to the major axis.

14. Q is the point on the auxiliary circle corresponding to P on the ellipse; PLM is drawn parallel to CQ to meet the axes in L and M; prove that $PL=b$ and $PM=a$.

15. Prove that the area of the triangle formed by three points on an ellipse, whose eccentric angles are θ, ϕ, and ψ, is

$$\tfrac{1}{2}ab\sin\frac{\phi-\psi}{2}\sin\frac{\psi-\theta}{2}\sin\frac{\theta-\phi}{2}.$$

Prove also that its area is to the area of the triangle formed by the corresponding points on the auxiliary circle as $b : a$, and hence that its area is a maximum when the latter triangle is equilateral, *i.e.* when

$$\phi-\theta=\psi-\phi=\frac{2\pi}{3}.$$

16. Any point P of an ellipse is joined to the extremities of the major axis; prove that the portion of a directrix intercepted by them subtends a right angle at the corresponding focus.

17. Shew that the perpendiculars from the centre upon all chords, which join the ends of perpendicular diameters, are of constant length.

18. If α, β, γ, and δ be the eccentric angles of the four points of intersection of the ellipse and any circle, prove that

$\alpha+\beta+\gamma+\delta$ is an even multiple of π radians.

[See *Trigonometry*, Part II, Art. 31.]

19. The tangent at any point P of a circle meets the tangent at a fixed point A in T, and T is joined to B, the other end of the diameter through A; prove that the locus of the intersection of AP and BT is an ellipse whose eccentricity is $\frac{1}{\sqrt{2}}$.

20. From any point P on the ellipse, PN is drawn perpendicular to the axis and produced to Q, so that NQ equals PS, where S is a focus; prove that the locus of Q is the two straight lines $y\pm ex+a=0$.

21. Given the base of a triangle and the sum of its sides, prove that the locus of the centre of its incircle is an ellipse.

22. With a given point and line as focus and directrix, a series of ellipses are described; prove that the locus of the extremities of their minor axes is a parabola.

23. A line of fixed length $a+b$ moves so that its ends are always on two fixed perpendicular straight lines; prove that the locus of a point, which divides this line into portions of length a and b, is an ellipse.

24. Prove that the extremities of the latera recta of all ellipses, having a given major axis $2a$, lie on the parabola $x^2=-a(y-a)$.

ANSWERS

1. (α) $3x^2+5y^2=32$; (β) $3x^2+7y^2=115$.

2. $20x^2+36y^2=405$. **3.** $x^2+2y^2=100$. **4.** $8x^2+9y^2=1152$.

5. (1) $\frac{2a}{3}$; $\frac{1}{3}\sqrt{6}$; $\left(\pm\frac{a}{3}\sqrt{6},\ 0\right)$; (2) $\frac{4}{5}$; $\frac{1}{5}\sqrt{5}$; $(0, \pm\frac{1}{10}\sqrt{5})$;

(3) $\frac{10}{3}$; $\frac{2}{3}$; $(0, 5)$ and $(0, 1)$.

6. $\frac{\sqrt{3}}{2}$. **7.** $7x^2+2xy+7y^2+10x-10y+7=0$. **8.** Without.

9. $x+4\sqrt{3}y=24\sqrt{3}$; $11x-4\sqrt{3}y=24\sqrt{3}$; 7 and 13.

11. (1) $\tan^{-1}\frac{b}{a}$; (2) $\tan^{-1}\sqrt{\frac{b}{a}}$; (3) $45°$.

12. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{y}{b}$.

SOLUTIONS/HINTS

1. Let the equation be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

(α) Then $\frac{4}{a^2}+\frac{4}{b^2}=1$, and $\frac{9}{a^2}+\frac{1}{b^2}=1$,

whence $a^2=\frac{32}{3}$, and $b^2=\frac{32}{5}$.

(β) $\frac{1}{a^2}+\frac{16}{b^2}=1$, and $\frac{36}{a^2}+\frac{1}{b^2}=1$,

whence $a^2=\frac{115}{3}$, and $b^2=\frac{115}{7}$.

2. We have $\frac{b^2}{a}=\frac{5}{2}$, and $e=\frac{2}{3}$,

whence $a^2=\frac{81}{4}$, and $b^2=\frac{45}{4}$.

3. We have $b=ae$, and $\frac{b^2}{a}=5$.

$$\therefore\ a^2e^2=b^2=a^2(1-e^2);\quad \therefore\ e=\frac{1}{\sqrt{2}}.$$

$$\therefore\ a(1-e^2)=5;\quad \therefore\ a^2=100,\ b^2=50.$$

Hence the required equation is $x^2+2y^2=100$.

4. $ae=4$, and $e=\frac{1}{3}$; $\therefore\ a=12$,

and $b^2=a^2(1-e^2)=\frac{144.8}{9}=128$.

Hence the required equation is $8x^2 + 9y^2 = 1152$.

5. Let A and B be the lengths of the semi-major and semi-minor axes in each case.

(1) $A^2 = a^2,\ B^2 = \frac{a^2}{3}$; $\therefore$ the latus rectum $= \frac{2B^2}{A} = \frac{2}{3}a$.

$$e^2 = \frac{A^2 - B^2}{A^2} = \frac{2}{3}; \quad \therefore e = \tfrac{1}{3}\sqrt{6}.$$

$\therefore$ the coordinates of the foci are $(\pm \frac{1}{3}\sqrt{6}a,\ 0)$.

(2) $A^2 = \frac{1}{4},\ B^2 = \frac{1}{5}$; $\therefore$ the latus rectum $= \frac{2B^2}{A} = \frac{4}{5}$;

$$e^2 = \frac{A^2 - B^2}{A^2} = \frac{1}{5}; \quad \therefore e = \frac{1}{5}\sqrt{5}.$$

$\therefore$ the coordinates of the foci are $(0,\ \pm \frac{1}{10}\sqrt{5})$.

(3) The equation may be written $\frac{x^2}{5} + \frac{(y-3)^2}{9} = 1$.

$\therefore A^2 = 9,\ B^2 = 5$; $\therefore$ the latus rectum $= \frac{2B^2}{A} = \frac{10}{3}$.

$$e^2 = 1 - \frac{B^2}{A^2} = 1 - \frac{5}{9} = \frac{4}{9}; \quad \therefore e = \frac{2}{3}.$$

The coordinates of the foci are $(0,\ 3 \pm 2)$, *i.e.* $(0, 5)$ and $(0, 1)$.

6. We have $\frac{2b^2}{a} = b$. $\therefore \frac{b}{a} = \frac{1}{2}$; $\therefore e^2 = 1 - \frac{b^2}{a^2} = 1 - \frac{1}{4}$;

$$\therefore e = \frac{\sqrt{3}}{2}.$$

7. $(x+1)^2 + (y-1)^2 = \frac{1}{4}\left\{\frac{x - y + 3}{\sqrt{2}}\right\}^2$, etc.

8. $\frac{5 \cdot 4^2 + 7 \cdot 3^2}{11} - 1$ is positive; $\therefore$ the point lies without the ellipse. [Art. 255.]

9. Semi-major axis $= 10$; semi-minor axis $= 8$; eccentricity $= \frac{3}{5}$.

$\therefore$ the required lengths are $10 \pm \frac{3}{5} \times 5$, *i.e.* 7 and 13. [Art. 251.]

The coordinates of the foci are $(0, \pm 6)$; hence the required equations are $\dfrac{x}{4\sqrt{3}} = \dfrac{y \mp 6}{5 \mp 6}$, etc.

10. Let r_1 and r_2 be the lengths of the semi-diameters whose vectorial angles are θ and $\theta + \dfrac{\pi}{2}$; then

$$\frac{1}{r_1^2} + \frac{1}{r_2^2} = \frac{b^2\cos^2\theta + a^2\sin^2\theta}{a^2b^2} + \frac{b^2\sin^2\theta + a^2\cos^2\theta}{a^2b^2} \quad \text{[Art. 256]}$$
$$= \frac{1}{a^2} + \frac{1}{b^2}.$$

11. By Art. 256,

(1) $$\frac{a^2+b^2}{2} = \frac{a^2b^2}{b^2\cos^2\theta + a^2\sin^2\theta};$$

$\therefore (a^2+b^2)(b^2+a^2t^2) = 2a^2b^2(1+t^2)$, where $t = \tan\theta$,

whence $$t = \frac{b}{a}.$$

(2) $ab = \dfrac{a^2b^2}{b^2\cos^2\theta + a^2\sin^2\theta}$, whence $t = \sqrt{\dfrac{b}{a}}$.

(3) $$\frac{2a^2b^2}{a^2+b^2} = \frac{a^2b^2}{b^2\cos^2\theta + a^2\sin^2\theta},$$

whence $$t = 1. \quad \therefore \theta = 45^\circ.$$

12. Let $(a\cos\phi, b\sin\phi)$ be the other extremity of one of the chords. Then if (x, y) be a point on the locus,

$$2x = a\cos\phi, \text{ and } 2y = b + b\sin\phi.$$

Eliminating ϕ, we have

$$\left(\frac{2x}{a}\right)^2 + \left(\frac{2y-b}{b}\right)^2 = 1, \ \textit{i.e.}\ \frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{y}{b}.$$

13. If P be the point $(a\cos\phi, b\sin\phi)$ the equation of AP is

$$\frac{y}{b\sin\phi} = \frac{x-a}{a\cos\phi - a}, \text{ or } y = -\frac{b}{a}\cdot\cot\frac{\phi}{2}(x-a).$$

The "m" of $A'P = \dfrac{b \sin \phi}{a \cos \phi + a} = \dfrac{b}{a} \tan \dfrac{\phi}{2}$.

The equation of the line through A' perpendicular to $A'P$ is $y = -\dfrac{a}{b} . \cot \dfrac{\phi}{2} (x + a)$.

Eliminating ϕ, $\dfrac{x + a}{x - a} = \dfrac{b^2}{a^2}$. Hence, etc.

14. $PM = CN \sec \phi = a$; $PL = PN . \operatorname{cosec} \phi = b$.

15. (i) See Ex. II. 6. (ii) Put $b = a$.

16. The equations of AP and $A'P$ are

$$y = -\frac{b}{a} . \cot \frac{\phi}{2} (x - a), \text{ and}$$

$$y = \frac{b}{a} \tan \frac{\phi}{2} (x + a). \quad \text{(See No. 13.)}$$

If these cut a directrix $\left(x = \dfrac{a}{e}\right)$ in Y and Y',

$$XY = -b \cot \frac{\phi}{2} \left(\frac{1}{e} - 1\right), \text{ and } XY' = b \tan \frac{\phi}{2} \left(\frac{1}{e} + 1\right).$$

$$\therefore XY . XY' = -b^2 \left(1 - \frac{1}{e^2}\right) = \frac{a^2 (1 - e^2)^2}{e^2} = SX^2.$$

$\therefore Y\hat{S}Y'$ is a right angle.

17. The equation of the lines joining the origin to the common points of the ellipse and the line

$$x \cos \alpha + y \sin \alpha = p \text{ is} \quad \text{[Art. 122]}$$

$$p^2 (b^2 x^2 + a^2 y^2) = a^2 b^2 (x \cos \alpha + y \sin \alpha)^2.$$

If these are at right angles, then

$$p^2 (b^2 + a^2) = a^2 b^2. \quad \text{[Art. 111.]}$$

18. Let the equation to the circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0.$$

If this meets the ellipse in the point whose eccentric angle is θ, then it is satisfied by $x = a\cos\theta$, $y = b\sin\theta$, so that

$$a^2\cos^2\theta + b^2\sin^2\theta + 2ga\cos\theta + 2fb\sin\theta + c = 0. \quad \ldots(1)$$

Let $t \equiv \tan\frac{\theta}{2}$, so that $\sin\theta = \frac{2t}{1+t^2}$ and $\cos\theta = \frac{1-t^2}{1+t^2}$.

Then (1) becomes, on substitution,

$$a^2(1-t^2)^2 + 4b^2t^2 + 2ga(1-t^4) + 4fbt(1+t^2) + c(1+t^2)^2 = 0,$$

i.e. $$t^4(a^2 - 2ga + c) + 4fbt^3 + t^2(4b^2 - 2a^2 + 2c) + 4fbt + a^2 + 2ga + c = 0. \quad \ldots(2)$$

This is an equation with four roots.

Also $s_1 = \text{sum of the roots} = -\frac{4fb}{a^2 - 2ga + c}$,

$$s_2 = \text{sum taken two at a time} = \frac{4b^2 - 2a^2 + 2c}{a^2 - 2ga + c},$$

$$s_3 = \text{sum taken three at a time} = -\frac{4fb}{a^2 - 2ga + c},$$

and $$s_4 = \text{sum taken four at a time} = \frac{a^2 + 2ga + c}{a^2 - 2ga + c}.$$

$$\therefore \tan\frac{\theta_1 + \theta_2 + \theta_3 + \theta_4}{2} = \frac{s_1 - s_3}{1 - s_2 + s_4} = \frac{0}{4(a^2 - b^2)} = 0 = \tan n\pi.$$

$\therefore \theta_1 + \theta_2 + \theta_3 + \theta_4 = 2n\pi$ = an even multiple of two right angles.

19. Let $x^2 + y^2 = a^2$ be the equation of the circle and let the coordinates of A and B be $(\pm a, 0)$. Let P be the point $(a\cos\alpha, a\sin\alpha)$.

T is the intersection of the lines

$x = a$, and $x\cos\alpha + y\sin\alpha = a$.

$\therefore$ its coordinates are $\left(a,\ a\tan\frac{\alpha}{2}\right)$.

Hence the equation of TB is $x + a = 2y\cot\frac{\alpha}{2}$, and the equation of PA is $x - a = -y\tan\frac{\alpha}{2}$.

Eliminating a, $x^2 - a^2 = -2y^2$, or $x^2 + 2y^2 = a^2$, which is an ellipse whose eccentricity is $\frac{1}{\sqrt{2}}$.

20. If (x, y) be the coordinates of P,

$$y = NQ = -(PS) = -(a \pm ex). \quad \text{[Art. 251.]}$$

$$\therefore\ y + a \pm ex = 0.$$

21. Take the base BC for axis of x, and the perpendicular through O, its middle point for axis of y. Let P be the incentre and PN perpendicular to BC. Let the coordinates of P be (x, y).

Then $x = ON = (s - b) - \frac{a}{2}$, (if $c > b$), $= \frac{c - b}{2}$.

$$y^2 = \frac{\Delta^2}{s^2} = \lambda\{a^2 - (b - c)^2\}, \text{ where } \lambda \text{ is a constant,}$$

$$= \lambda(a^2 - 4x^2).$$

$\therefore\ y^2 + 4\lambda x^2 = \lambda a^2$, which is the equation of an ellipse.

22. Taking the figure of Art. 247, let S be the given point and ZK the given line, then

$$\frac{a}{e} - ae = SZ = \text{constant.} \quad \therefore \frac{b^2}{ae} = \text{constant} = \lambda.$$

Hence $y^2 = \lambda x$ is the equation to the locus of B referred to ZS and LSL' as axes. Hence etc.

23. If the line is inclined at θ to the axis of x, and (x, y) are the coordinates of the point whose locus is required, we have $x = a\cos\theta$, and $y = b\sin\theta$.

$$\therefore\ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \text{ which is an ellipse.}$$

24. In the figure of Art. 247, the coordinates of L are

$$x = -ae, \text{ and } y = \frac{b^2}{a} = a - ae^2.$$

$$\therefore\ a(y - a) = -a^2e^2 = -x^2.$$

So L' lies on the parabola $x^2 = a(y + a)$.

260. *To find the intersections of any straight line with the ellipse*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad\text{.......................}(1).$$

Let the equation of the straight line be

$$y = mx + c \quad\text{.......................}(2).$$

The coordinates of the points of intersection of (1) and (2) satisfy both equations and are therefore obtained by solving them as simultaneous equations.

Substituting for y in (1) from (2), the abscissae of the points of intersection are given by the equation

$$\frac{x^2}{a^2}+\frac{(mx+c)^2}{b^2}=1,$$

i.e. $$x^2(a^2m^2+b^2)+2a^2mcx+a^2(c^2-b^2)=0 \quad\text{.......}(3).$$

This is a quadratic equation and hence has two roots, real, coincident, or imaginary.

Also corresponding to each value of x we have from (2) one value of y.

The straight line therefore meets the curve in two points real, coincident, or imaginary.

The roots of the equation (3) are real, coincident, or imaginary according as

$(2a^2mc)^2-4(b^2+a^2m^2)\times a^2(c^2-b^2)$ is positive, zero, or negative,

i.e. according as $b^2(b^2+a^2m^2)-b^2c^2$ is positive, zero, or negative,

i.e. according as c^2 is $<=$ or $> a^2m^2+b^2$.

261. *To find the length of the chord intercepted by the ellipse on the straight line* $y=mx+c$.

As in Art. 204, we have

$$x_1+x_2=-\frac{2a^2mc}{a^2m^2+b^2}, \text{ and } x_1x_2=\frac{a^2(c^2-b^2)}{a^2m^2+b^2},$$

so that $$x_1-x_2=\frac{2ab\sqrt{a^2m^2+b^2-c^2}}{a^2m^2+b^2}.$$

The equation (4) then becomes

$$y - y' = -\frac{b^2}{a^2}\frac{x'}{y'}(x - x'),$$

i.e. $$\frac{xx'}{a^2} + \frac{yy'}{b^2} = \frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 1, \text{ by equation (2).}$$

The required equation is therefore

$$\mathbf{\frac{xx'}{a^2} + \frac{yy'}{b^2} = 1}.$$

Cor. The equation to the tangent is therefore obtained from the equation to the curve by the rule of Art. 152.

263. *To find the equation to a tangent in terms of the tangent of its inclination to the major axis.*

As in Art. 260, the straight line

$$y = mx + c \text{ ....(1)}$$

meets the ellipse in points whose abscissae are given by

$$x^2(b^2 + a^2m^2) + 2mca^2x + a^2(c^2 - b^2) = 0,$$

and, by the same article, the roots of this equation are coincident if

$$c = \sqrt{a^2m^2 + b^2}.$$

In this case the straight line (1) is a tangent, and it becomes

$$\mathbf{y = mx + \sqrt{a^2m^2 + b^2}} \text{ ....(2).}$$

This is the required equation.

Since the radical sign on the right-hand of (2) may have either + or − prefixed to it, we see that there are two tangents to the ellipse having the same m, *i.e.* there are two tangents parallel to any given direction.

The above form of the equation to the tangent may be deduced from the equation of Art. 262, as in the case of the parabola (Art. 206). It will be found that the point of contact is the point

$$\left(\frac{-a^2m}{\sqrt{a^2m^2+b^2}},\ \frac{b^2}{\sqrt{a^2m^2+b^2}}\right).$$

264. By a proof similar to that of the last article, it may be shewn that the straight line

$$x \cos \alpha + y \sin \alpha = p$$

touches the ellipse, if

$$\mathbf{p^2 = a^2 \cos^2 \alpha + b^2 \sin^2 \alpha.}$$

Similarly, it may be shewn that the straight line

$$lx + my = n$$

touches the ellipse, if $a^2l^2 + b^2m^2 = n^2$.

265. *Equation to the tangent at the point whose eccentric angle is ϕ.*

The coordinates of the point are $(a \cos \phi, b \sin \phi)$.

Substituting $x' = a \cos \phi$ and $y' = b \sin \phi$ in the equation of Art. 262, we have, as the required equation,

$$\frac{\mathbf{x}}{\mathbf{a}} \cos \phi + \frac{\mathbf{y}}{\mathbf{b}} \sin \phi = 1 \quad \text{..............} (1).$$

This equation may also be deduced from Art. 259.

For the equation of the tangent at the point "ϕ" is obtained by making $\phi' = \phi$ in the result of that article.

Ex. *Find the intersection of the tangents at the points ϕ and ϕ'.*

The equations to the tangents are

$$\frac{x}{a} \cos \phi + \frac{y}{b} \sin \phi - 1 = 0,$$

and

$$\frac{x}{a} \cos \phi' + \frac{y}{b} \sin \phi' - 1 = 0.$$

The required point is found by solving these equations.

We obtain

$$\frac{\frac{x}{a}}{\sin \phi - \sin \phi'} = \frac{\frac{y}{b}}{\cos \phi' - \cos \phi} = \frac{-1}{\sin \phi' \cos \phi - \cos \phi' \sin \phi} = \frac{1}{\sin (\phi - \phi')},$$

i.e.

$$\frac{x}{2a \cos \frac{\phi + \phi'}{2} \sin \frac{\phi - \phi'}{2}} = \frac{y}{2b \sin \frac{\phi + \phi'}{2} \sin \frac{\phi - \phi}{2}} = \frac{1}{2 \sin \frac{\phi - \phi'}{2} \cos \frac{\phi - \phi}{2}}.$$

Hence $x = a\dfrac{\cos\frac{1}{2}(\phi+\phi')}{\cos\frac{1}{2}(\phi-\phi')}$, and $y = b\dfrac{\sin\frac{1}{2}(\phi+\phi')}{\cos\frac{1}{2}(\phi-\phi')}$.

266. *Equation to the* **normal** *at the point* (x', y').

The required normal is the straight line which passes through the point (x', y') and is perpendicular to the tangent, *i.e.* to the straight line

$$y = -\frac{b^2x'}{a^2y'}x + \frac{b^2}{y'}.$$

Its equation is therefore

$$y - y' = m(x - x'),$$

where $m\left(-\dfrac{b^2x'}{a^2y'}\right) = -1$, *i.e.* $m = \dfrac{a^2y'}{b^2x'}$, (Art. 69).

The equation to the normal is therefore $y - y' = \dfrac{a^2y'}{b^2x'}(x - x')$,

i.e.
$$\mathbf{\frac{x - x'}{\frac{x'}{a^2}} = \frac{y - y'}{\frac{y'}{b^2}}}.$$

267. *Equation to the normal at the point whose eccentric angle is* ϕ.

The coordinates of the point are $a\cos\phi$ and $b\sin\phi$.

Hence, in the result of the last article putting

$$x' = a\cos\phi \text{ and } y' = b\sin\phi,$$

it becomes
$$\frac{x - a\cos\phi}{\frac{\cos\phi}{a}} = \frac{y - b\sin\phi}{\frac{\sin\phi}{b}},$$

i.e.
$$\frac{ax}{\cos\phi} - a^2 = \frac{by}{\sin\phi} - b^2.$$

The required normal is therefore

$$\mathbf{ax\sec\phi - by\,\mathrm{cosec}\,\phi = a^2 - b^2.}$$

***268.** *Equation to the normal in the form $y = mx + c$.*

The equation to the normal at (x', y') is, as in Art. 266,

$$y = \frac{a^2 y'}{b^2 x'} x - y' \left(\frac{a^2}{b^2} - 1 \right).$$

Let $\frac{a^2 y'}{b^2 x'} = m$, so that $\frac{x'}{a} = \frac{ay'}{b^2 m}$.

Hence, since (x', y') satisfies the relation $\frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 1$, we obtain

$$y' = \frac{b^2 m}{\sqrt{a^2 + b^2 m^2}}.$$

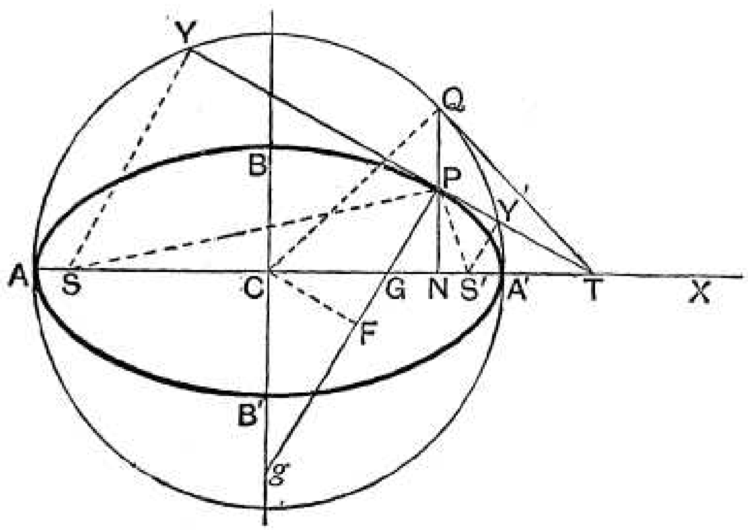
The equation to the normal is therefore

$$y = mx - \frac{(a^2 - b^2) m}{\sqrt{a^2 + b^2 m^2}}.$$

This is not as important an equation as the corresponding equation in the case of the parabola. (Art. 208.)

When it is desired to have the equation to the normal expressed in terms of one independent parameter it is generally better to use the equation of the previous article.

269. *To find the length of the subtangent and subnormal.*



Let the tangent and normal at P, the point (x', y'), meet the axis in T and G respectively, and let PN be the ordinate of P.

The equation to the tangent at P is (Art. 262)

$$\frac{xx'}{a^2}+\frac{yy'}{b^2}=1 \text{ .... } (1).$$

To find where the straight line meets the axis we put $y=0$ and have

$$x=\frac{a^2}{x'}, \quad i.e. \;\; CT=\frac{a^2}{CN},$$

i.e. $$CT \,.\, CN=a^2=CA^2 \text{ .... } (2).$$

Hence the subtangent NT

$$=CT-CN=\frac{a^2}{x'}-x'=\frac{a^2-x'^2}{x'}.$$

The equation to the normal is (Art. 266)

$$\frac{x-x'}{\frac{x'}{a^2}}=\frac{y-y'}{\frac{y'}{b^2}}.$$

To find where it meets the axis, we put $y=0$, and have

$$\frac{x-x'}{\frac{x'}{a^2}}=\frac{-y'}{\frac{y'}{b^2}}=-b^2,$$

i.e. $$CG=x=x'-\frac{b^2}{a^2}x'=\frac{a^2-b^2}{a^2}x'=e^2 \,.\, x'=e^2 \,.\, CN \text{...} (3).$$

Hence the subnormal NG

$$=CN-CG=(1-e^2)\,CN,$$

i.e. $$NG :: NC :: 1-e^2 : 1$$

$$:: b^2 : a^2. \qquad \text{(Art. 247.)}$$

Cor. If the tangent meet the minor axis in t and Pn be perpendicular to it, we may, similarly, prove that

$$Ct \,.\, Cn=b^2.$$

270. Some properties of the ellipse.

(α) *$SG=e\,.\,SP$, and the tangent and normal at P bisect the external and internal angles between the focal distances of* ***P.***

By Art. 269, we have $CG=e^2x'$.

Hence $\quad SG = SC + CG = ae + e^2x' = e \,.\, SP$, by Art. 251.

Also $\quad S'G = CS' - CG = e(a - ex') = e \,.\, S'P.$

Hence $\quad SG : S'G :: SP : S'P.$

Therefore, by Euc. VI, 3, PG bisects the angle SPS'.

It follows that the tangent bisects the exterior angle between SP and $S'P$.

(β) *If SY and $S'Y'$ be the perpendiculars from the foci upon the tangent at any point P of the ellipse, then Y and Y' lie on the auxiliary circle, and $SY \,.\, S'Y' = b^2$. Also CY and $S'P$ are parallel.*

The equation to any tangent is

$$x \cos \alpha + y \sin \alpha = p \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1),$$

where $\quad p = \sqrt{a^2 \cos^2 \alpha + b^2 \sin^2 \alpha}$ (Art. 264).

The perpendicular SY to (1) passes through the point $(-ae, 0)$ and its equation, by Art. 70, is therefore

$$(x + ae) \sin \alpha - y \cos \alpha = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2).$$

If Y be the point (h, k) then, since Y lies on both (1) and (2), we have

$$h \cos \alpha + k \sin \alpha = \sqrt{a^2 \cos^2 \alpha + b^2 \sin^2 \alpha},$$

and $\quad h \sin \alpha - k \cos \alpha = -ae \sin \alpha = -\sqrt{a^2 - b^2} \sin^2 \alpha.$

Squaring and adding these equations, we have $h^2 + k^2 = a^2$, so that Y lies on the auxiliary circle $x^2 + y^2 = a^2$.

Similarly it may be proved that Y' lies on this circle.

Again S is the point $(-ae, 0)$ and S' is $(ae, 0)$.

Hence, from (1),

$$SY = p + ae \cos \alpha, \text{ and } S'Y' = p - ae \cos \alpha. \quad \text{(Art. 75.)}$$

Thus
$$\begin{aligned} SY \,.\, S'Y' &= p^2 - a^2e^2 \cos^2 \alpha \\ &= a^2 \cos^2 \alpha + b^2 \sin^2 \alpha - (a^2 - b^2) \cos^2 \alpha \\ &= b^2. \end{aligned}$$

Also
$$CT = \frac{a^2}{CN},$$

and therefore
$$S'T = \frac{a^2}{CN} - ae = \frac{a(a - eCN)}{CN}.$$

$$\therefore \frac{CT}{S'T} = \frac{a}{a - e \,.\, CN} = \frac{CY}{S'P}.$$

Hence CY and $S'P$ are parallel. Similarly CY' and SP are parallel.

(γ) *If the normal at any point P meet the major and minor axes in G and g, and if CF be the perpendicular upon this normal, then $PF \cdot PG = b^2$ and $PF \cdot Pg = a^2$.*

The tangent at any point P (the point "ϕ") is

$$\frac{x}{a}\cos\phi + \frac{y}{b}\sin\phi = 1.$$

Hence PF = perpendicular from C upon this tangent

$$= \frac{1}{\sqrt{\dfrac{\cos^2\phi}{a^2} + \dfrac{\sin^2\phi}{b^2}}} = \frac{ab}{\sqrt{b^2\cos^2\phi + a^2\sin^2\phi}} \text{ .... (1).}$$

The normal at P is

$$\frac{ax}{\cos\phi} - \frac{by}{\sin\phi} = a^2 - b^2 \text{ .... (2).}$$

If we put $y = 0$, we have $CG = \dfrac{a^2 - b^2}{a}\cos\phi$.

$$\therefore PG^2 = \left(a\cos\phi - \frac{a^2 - b^2}{a}\cos\phi\right)^2 + b^2\sin^2\phi$$

$$= \frac{b^4}{a^2}\cos^2\phi + b^2\sin^2\phi,$$

i.e.
$$PG = \frac{b}{a}\sqrt{b^2\cos^2\phi + a^2\sin^2\phi}.$$

From this and (1), we have $PF \cdot PG = b^2$.

If we put $x = 0$ in (2), we see that g is the point

$$\left(0, -\frac{a^2 - b^2}{b}\sin\phi\right).$$

Hence
$$Pg^2 = a^2\cos^2\phi + \left(b\sin\phi + \frac{a^2 - b^2}{b}\sin\phi\right)^2,$$

so that
$$Pg = \frac{a}{b}\sqrt{b^2\cos^2\phi + a^2\sin^2\phi}.$$

From this result and (1) we therefore have

$$PF \cdot Pg = a^2.$$

271. *To find the locus of the point of intersection of tangents which meet at right angles.*

Any tangent to the ellipse is

$$y = mx + \sqrt{a^2m^2 + b^2},$$

and a perpendicular tangent is

$$y = -\frac{1}{m}x + \sqrt{a^2\left(-\frac{1}{m}\right)^2 + b^2}.$$

Hence, if (h, k) be their point of intersection, we have

$$k - mh = \sqrt{a^2m^2 + b^2} \quad \text{..................(1)},$$

and
$$mk + h = \sqrt{a^2 + b^2m^2} \quad \text{..................(2)}.$$

If between (1) and (2) we eliminate m, we shall have a relation between h and k. Squaring and adding these equations, we have

$$(k^2 + h^2)(1 + m^2) = (a^2 + b^2)(1 + m^2),$$

i.e.
$$h^2 + k^2 = a^2 + b^2.$$

Hence the locus of the point (h, k) is the circle

$$x^2 + y^2 = a^2 + b^2,$$

i.e. a circle, whose centre is the centre of the ellipse, and whose radius is the length of the line joining the ends of the major and minor axis. This circle is called the **Director Circle.**

EXAMPLES XXXIII

Find the equation to the tangent and normal

1. at the point $(1, \frac{4}{3})$ of the ellipse $4x^2 + 9y^2 = 20$,

2. at the point of the ellipse $5x^2 + 3y^2 = 137$ whose ordinate is 2,

3. at the ends of the latera recta of the ellipse $9x^2 + 16y^2 = 144$.

4. Prove that the straight line $y = x + \sqrt{\frac{7}{12}}$ touches the ellipse

$$3x^2 + 4y^2 = 1.$$

5. Find the equations to the tangents to the ellipse $4x^2 + 3y^2 = 5$ which are parallel to the straight line $y = 3x + 7$.

Find also the coordinates of the points of contact of the tangents which are inclined at 60° to the axis of x.

6. Find the equations to the tangents at the ends of the latera recta of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, and shew that they pass through the intersections of the axis and the directrices.

7. Find the points on the ellipse such that the tangents there are equally inclined to the axes. Prove also that the length of the perpendicular from the centre on either of these tangents is

$$\sqrt{\frac{a^2 + b^2}{2}}.$$

8. In an ellipse, referred to its centre, the length of the subtangent corresponding to the point $(3, \frac{12}{5})$ is $\frac{16}{3}$; prove that the eccentricity is $\frac{4}{5}$.

9. Prove that the sum of the squares of the perpendiculars on any tangent from two points on the minor axis, each distant $\sqrt{a^2-b^2}$ from the centre, is $2a^2$.

10. Find the equations to the normals at the ends of the latera recta, and prove that each passes through an end of the minor axis if $e^4+e^2=1$.

11. If any ordinate MP meet the tangent at L in Q, prove that MQ and SP are equal.

12. Two tangents to the ellipse intersect at right angles; prove that the sum of the squares of the chords which the auxiliary circle intercepts on them is constant, and equal to the square on the line joining the foci.

13. If P be a point on the ellipse, whose ordinate is y', prove that the angle between the tangent at P and the focal distance of P is $\tan^{-1}\dfrac{b^2}{aey'}$.

14. Shew that the angle between the tangents to the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ and the circle $x^2+y^2=ab$ at their points of intersection is $\tan^{-1}\dfrac{a-b}{\sqrt{ab}}$.

15. A circle, of radius r, is concentric with the ellipse; prove that the common tangent is inclined to the major axis at an angle $\tan^{-1}\sqrt{\dfrac{r^2-b^2}{a^2-r^2}}$ and find its length.

16. Prove that the common tangent of the ellipses

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{2x}{c} \text{ and } \frac{x^2}{b^2}+\frac{y^2}{a^2}+\frac{2x}{c}=0$$

subtends a right angle at the origin.

17. Prove that $PG \,.\, Pg = SP \,.\, S'P$, and $CG \,.\, CT = CS^2$.

18. The tangent at P meets the axes in T and t, and CY is the perpendicular on it from the centre; prove that (1) $Tt \,.\, PY = a^2-b^2$, and (2) the least value of Tt is $a+b$.

19. Prove that the perpendicular from the focus upon any tangent and the line joining the centre to the point of contact meet on the corresponding directrix.

20. Prove that the straight lines, joining each focus to the foot of the perpendicular from the other focus upon the tangent at any point P, meet on the normal at P and bisect it.

21. Prove that the circle on any focal distance as diameter touches the auxiliary circle.

22. Find the tangent of the angle between CP and the normal at P, and prove that its greatest value is $\frac{a^2-b^2}{2ab}$.

23. Prove that the straight line $lx+my=n$ is a normal to the ellipse, if $\frac{a^2}{l^2}+\frac{b^2}{m^2}=\frac{(a^2-b^2)^2}{n^2}$.

24. Find the locus of the point of intersection of the two straight lines $\frac{tx}{a}-\frac{y}{b}+t=0$ and $\frac{x}{a}+\frac{ty}{b}-1=0$.

Prove also that they meet at the point whose eccentric angle is $2\tan^{-1}t$.

25. Prove that the locus of the middle points of the portions of tangents included between the axes is the curve

$$\frac{a^2}{x^2}+\frac{b^2}{y^2}=4.$$

26. Any ordinate NP of an ellipse meets the auxiliary circle in Q; prove that the locus of the intersection of the normals at P and Q is the circle $\qquad x^2+y^2=(a+b)^2$.

27. The normal at P meets the axes in G and g; shew that the loci of the middle points of PG and Gg are respectively the ellipses

$$\frac{4x^2}{a^2(1+e^2)^2}+\frac{4y^2}{b^2}=1, \text{ and } a^2x^2+b^2y^2=\tfrac{1}{4}(a^2-b^2)^2.$$

28. Prove that the locus of the feet of the perpendicular drawn from the centre upon any tangent to the ellipse is

$$r^2=a^2\cos^2\theta+b^2\sin^2\theta. \quad [\textit{Use Art. } 264.]$$

29. If a number of ellipses be described, having the same major axis, but a variable minor axis, prove that the tangents at the ends of their latera recta pass through one or other of two fixed points.

30. The normal GP is produced to Q, so that $GQ=n\,.\,GP$.

Prove that the locus of Q is the ellipse $\frac{x^2}{a^2(n+e^2-ne^2)^2}+\frac{y^2}{n^2b^2}=1$.

31. If the straight line $y=mx+c$ meet the ellipse, prove that the equation to the circle, described on the line joining the points of intersection as diameter, is

$$(a^2m^2+b^2)(x^2+y^2)+2ma^2cx-2b^2cy+c^2(a^2+b^2)-a^2b^2(1+m^2)=0.$$

32. PM and PN are perpendiculars upon the axes from any point P on the ellipse. Prove that MN is always normal to a fixed concentric ellipse.

33. Prove that the sum of the eccentric angles of the extremities of a chord, which is drawn in a given direction, is constant, and equal to twice the eccentric angle of the point at which the tangent is parallel to the given direction.

34. A tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ meets the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=a+b$$

in the points P and Q; prove that the tangents at P and Q are at right angles.

ANSWERS

1. $16x^2-9y^2=36$.

2. $25x^2-144y^2=900$.

3. $65x^2-36y^2=441$.

4. $x^2-y^2=32$.

5. 6, 4, $(\pm\sqrt{13},\ 0)$, $2\frac{2}{3}$.

6. $3x^2-y^2=3a^2$.

7. $7y^2+24xy-24ax-6ay+15a^2=0$; $\left(-\frac{a}{3},\ a\right)$; $12x-9y+29a=0$.

8. $(5,\ -\frac{20}{3})$.

9. $24y-30x=\pm\sqrt{161}$.

14. $y=\pm x\pm\sqrt{a^2-b^2}$; $(a^2+b^2)\sqrt{\frac{2}{a^2-b^2}}$.

15. $9y=32x$.

16. $125x-48y=481$.

29. (1) $b^4x^2+a^4y^2=a^2b^2(b^2-a^2)$; (2) $x=a\cdot\frac{a^2-b^2}{a^2+b^2}$;

(3) $x^2(a^2+2b^2)-a^2y^2-2a^3ex+a^2(a^2-b^2)=0$.

SOLUTIONS/HINTS

1. (1) Substitute x for x^2 and $\frac{4}{3}y$ for y^2, [Art. 262].

(2) Put $x'=1$, $y'=\frac{4}{3}$, $a^2=5$, and $b^2=\frac{20}{9}$ in the equation of Art. 266.

2. If $y=2$, $5x^2=125$ and $\therefore\ x=\pm 5$.

(1) Substitute $\pm 5x$ for x^2 and $2y$ for y^2.

(2) Put $x'=\pm 5$, $y'=2$, $a^2=\frac{137}{5}$, and $b^2=\frac{137}{3}$ in the equation of Art. 266.

3. We have $a=4$ and $b=3$. $\therefore\ e^2=1-\frac{9}{16}=\frac{7}{16}$, so that $e=\frac{1}{4}\sqrt{7}$.

Hence the coordinates of the extremities of the latera recta are $(\pm\sqrt{7},\ \pm\frac{9}{4})$.

Substitute in the equations of Arts. 262, 266.

4. The common points are given by

$$3x^2 + 4\left(x + \sqrt{\tfrac{7}{12}}\right)^2 = 1, \text{ or } 21x^2 + 4\sqrt{21}\,.\,x + 4 = 0,$$

which has equal roots.

5. (1) The intersections of the ellipse with the line $y = 3x + c$ are given by

$$4x^2 + 3(3x + c)^2 = 5, \text{ or } 31x^2 + 18cx + 3c^2 - 5 = 0,$$

which has equal roots if

$$81c^2 = 31(3c^2 - 5), \text{ whence } c = \pm \tfrac{1}{2}\sqrt{\tfrac{155}{3}}.$$

(2) By Art. 265 the "m" of the tangent at the point ϕ

$$= -\frac{b}{a}\cot\phi = -\frac{2}{\sqrt{3}}\cot\phi,$$

and this is given to be $+\sqrt{3}$.

$$\therefore \cot\phi = -\tfrac{3}{2}. \quad \therefore \cos\phi = \pm\frac{3}{\sqrt{13}},\ \sin\phi = \mp\frac{2}{\sqrt{13}}.$$

$$\therefore \text{ the required points are } \left(\pm\frac{3\sqrt{65}}{26},\ \mp\frac{2}{39}\sqrt{195}\right).$$

6. Put $x = \pm ae$, $y = \pm\dfrac{b^2}{a}$ in the equation of Art. 262.

Hence the tangents at the end of the latera recta are

$$\pm ex \pm y = a.$$

These cut the axis of x where $x = \pm\dfrac{a}{e}$, *i.e.* at the points Z or Z' [Art. 247].

7. By Art. 265, we have $-\dfrac{b}{a}\cot\phi = \pm\tan 45° = \pm 1.$

$$\therefore \sin\phi = \pm\frac{b}{\sqrt{a^2 + b^2}},\ \cos\phi = \pm\frac{a}{\sqrt{a^2 + b^2}}.$$

$$\therefore \text{ the required points are } \left(\pm\frac{a^2}{\sqrt{a^2 + b^2}},\ \pm\frac{b^2}{\sqrt{a^2 + b^2}}\right).$$

$$p^2 = a^2\cos^2\alpha + b^2\sin^2\alpha \text{ [Art. 264]} = \frac{a^2 + b^2}{2}, \text{ if } \alpha = 45°.$$

8. $$\frac{a^2-x'^2}{x'}=NT=\frac{16}{3}, \quad \text{[Art. 269]}.$$

$$\therefore \; a^2-9=16. \qquad \therefore \; a=5.$$

Also $\frac{9}{a^2}+\frac{144}{25b^2}=1$, whence $b^2=9$ and $b=3$.

$$e^2=1-\frac{b^2}{a^2}=1-\frac{9}{25}=\frac{16}{25}. \qquad \therefore \; e=\frac{4}{5}.$$

9. Let $x\cos\alpha+y\sin\alpha=p$ be the equation of any tangent, so that $p^2=a^2\cos^2\alpha+b^2\sin^2\alpha$.

The required sum

$$=(p+\sqrt{a^2-b^2}\sin\alpha)^2+(p-\sqrt{a^2-b^2}\sin\alpha)^2$$
$$=2\{p^2+(a^2-b^2)\sin^2\alpha\}$$
$$=2\{a^2\cos^2\alpha+b^2\sin^2\alpha+a^2\sin^2\alpha-b^2\sin^2\alpha\}=2a^2.$$

10. Putting $x'=ae$, and $y'=\frac{b^2}{a}$ in the equation of Art 266, we obtain $(x-ae)=ey-ae(1-e^2)$.

This passes through $(0,-b)$ if

$$ae=be+ae(1-e^2), \text{ or } b=ae^2.$$

Hence $a^2e^4=b^2=a^2(1-e^2)$, whence $e^4+e^2=1$. Similarly for the other normals.

11. See No. 6. The equation of the tangent at L is
$y=a-ex$, *i.e.* $QM=a-e\,.\,CM=SP$. [Art. 251.]

12. Let YY' and ZZ' be the intercepts.

The equation to a tangent YY' to the ellipse is

$$y=mx+\sqrt{a^2m^2+b^2}.$$

Put $c^2=a^2m^2+b^2$ in the result of Art. 154.

$$\therefore \; YY'=\frac{2ae}{\sqrt{1+m^2}}.$$

Similarly $$ZZ'=\frac{2ae}{\sqrt{1+\frac{1}{m^2}}}=\frac{2aem}{\sqrt{1+m^2}}.$$

$$\therefore \; YY'^2+ZZ'^2=4a^2e^2.$$

13. Let θ be the required angle, since the tangent is equally inclined to SP and $S'P$ [Art. 270 (a)],

$$\sin a = \frac{SY}{SP} = \frac{S'Y'}{S'P}.$$

$$\therefore \sin^2 a = \frac{SY \cdot S'Y'}{SP \cdot S'P} = \frac{b^2}{a^2 - e^2x'^2}. \quad \text{[Art. 270 } (\beta).]$$

$$\therefore \tan^2 a = \frac{b^2}{e^2(a^2 - x'^2)} = \frac{b^4}{e^2(a^2b^2 - b^2x'^2)} = \frac{b^4}{a^2e^2y'^2}.$$

14. Solving, a common point is

$$x = \sqrt{\frac{a^2b}{a+b}}, \quad y = \sqrt{\frac{ab^2}{a+b}}.$$

Then "m" of the tangent to the ellipse at this point

$$= -\frac{b^2}{a^2}\sqrt{\frac{a^2b}{ab^2}} = -\frac{b^{\frac{3}{2}}}{a^{\frac{3}{2}}}. \quad \text{[Art. 262.]}$$

And the "m" of the tangent to the circle

$$= -\frac{a^{\frac{1}{2}}}{b^{\frac{1}{2}}}. \quad \text{[Art. 158.]}$$

The required angle

$$= \tan^{-1}\frac{\dfrac{a^{\frac{1}{2}}}{b^{\frac{1}{2}}} - \dfrac{b^{\frac{3}{2}}}{a^{\frac{3}{2}}}}{1 + \dfrac{b}{a}} = \tan^{-1}\frac{a-b}{\sqrt{ab}}.$$

15. The line $x\cos a + y\sin a = r$ (which is any tangent to the circle) will also touch the ellipse if

$$r^2 = a^2\cos^2 a + b^2\sin^2 a, \text{ whence } \tan^2 a = \frac{a^2 - r^2}{r^2 - b^2}.$$

The "m" of the tangent $= -\cot a = -\sqrt{\dfrac{r^2 - b^2}{a^2 - r^2}}.$

If the normal to the ellipse at the point of contact of this tangent meet the axis in G, the length of the common tangent

$$= CG \sin \alpha = e^2 CN . \sin \alpha = e^2 . \frac{a^2}{CT} \sin \alpha = e^2 . \frac{a^2 \cos \alpha}{CY} . \sin \alpha$$

[Art. 269]

$$= e^2 . \frac{a^2 \cos \alpha}{r} . \sin \alpha = \frac{a^2 e^2 . \sqrt{a^2 - r^2} . \sqrt{r^2 - b^2}}{r . (a^2 - b^2)}$$

$$= \frac{\sqrt{a^2 - r^2} . \sqrt{r^2 - b^2}}{r}.$$

16. The equations may be written

$$\left(\frac{x}{a} - \frac{a}{c}\right)^2 + \left(\frac{y^2}{b^2} - \frac{a^2}{c^2}\right) = 0,$$

and $$\left(\frac{x}{b} + \frac{b}{c}\right)^2 + \left(\frac{y^2}{a^2} - \frac{b^2}{c^2}\right) = 0.$$

Hence the line $y = \frac{ab}{c}$ is a common tangent, since it cuts each in coincident points; also the points of contact are

$$\left.\begin{array}{l} x' = \frac{a^2}{c} \\ y' = \frac{ab}{c} \end{array}\right\}, \text{ and } \left.\begin{array}{l} x'' = -\frac{b^2}{c} \\ y'' = \frac{ab}{c} \end{array}\right\}. \quad \therefore \ x'x'' + y'y'' = 0,$$

i.e. the points of contact subtend a right angle at the origin.

17. $PG . Pg = b^2 \cos^2 \phi + a^2 \sin^2 \phi$. [Art. 270 ($\gamma$).]

$SP . S'P = (a + e \cos \phi)(a - e \cos \phi)$ [Art. 251]

$$= a^2 - a^2 e^2 \cos^2 \phi = a^2 - (a^2 - b^2) \cos^2 \phi = b^2 \cos^2 \phi + a^2 \sin^2 \phi.$$

$CG = e^2 . CN$, and $CT = \frac{a^2}{CN}$. [Art. 269.]

$\therefore \ CG . CT = a^2 e^2 = CS^2$.

18. In the equation $\frac{x}{a} . \cos \phi + \frac{y}{b} \sin \phi = 1$, put $x = 0$ and $y = 0$, successively; $\therefore \ CT = a \sec \phi$, $Ct = b \operatorname{cosec} \phi$,

and $$Tt = \sqrt{a^2 \sec^2 \phi + b^2 \operatorname{cosec}^2 \phi}.$$

$PY =$ the perpendicular from C upon the normal

$$ax \sec \phi - by \operatorname{cosec} \phi = a^2 - b^2;$$

$$\therefore\ PY = \frac{a^2 - b^2}{\sqrt{a^2 \sec^2\phi + b^2 \operatorname{cosec}^2\phi}}. \quad \therefore\ Tt\,.\,PY = a^2 - b^2.$$

$$Tt^2 = a^2\sec^2\phi + b^2\operatorname{cosec}^2\phi$$
$$= (a^2\tan^2\phi + b^2\cot^2\phi - 2ab) + a^2 + b^2 + 2ab$$
$$= (a+b)^2 + (a\tan\phi - b\cot\phi)^2.$$

Hence the minimum value of Tt is $a+b$, when

$$a\tan\phi - b\cot\phi = 0,\ \textit{i.e.}\text{ when }\tan\phi = \sqrt{\frac{b}{a}}.$$

19. If P be the point $(a\cos\phi,\ b\sin\phi)$, the equations of CP and SY are

$$xb\sin\phi - ya\cos\phi = 0,\ \text{ and }\ a\sin\phi\,(x - ae) - yb\cos\phi = 0.$$

Solving, we have $x = \frac{a}{e}$, *i.e.* their point of intersection lies on the corresponding directrix.

20. In the figure of Art. 269, if SP and $S'Y'$ meet in W, SY' is a median of the triangle $SS'W$, and therefore it bisects PG which is parallel to $S'W$. Similarly $S'Y$ bisects PG.

21. In the figure of Art. 269, the circle on $S'P$ as diameter clearly passes through Y', and since CY' is parallel to SP it bisects $S'P$. Hence the line joining the centres of the two circles passes through their point of intersection, *i.e.* the two circles touch.

22. The "m" of $CP = \frac{b}{a}\tan\phi$. The "m" of the normal $= \frac{a}{b}\tan\phi$.

$$\therefore\ \text{the required tangent} = \frac{-\frac{b}{a}\tan\phi + \frac{a}{b}\tan\phi}{1 + \tan^2\phi}$$

$$= \frac{a^2 - b^2}{ab}\,.\sin\phi\,.\cos\phi = \frac{a^2 - b^2}{2ab}\,.\sin 2\phi,$$

so that its greatest value $= \frac{a^2 - b^2}{2ab}$.

23. The lines

$$lx + my = m \text{ and } ax\sec\phi - by\operatorname{cosec}\phi = a^2 - b^2$$

are coincident if $\frac{l\cos\phi}{a} = -\frac{m\sin\phi}{b} = \frac{n}{a^2-b^2}$.

Eliminating ϕ, we have $\frac{a^2}{l^2} + \frac{b^2}{m^2} = \frac{(a^2-b^2)^2}{n^2}$.

24. We have $1 + \frac{x}{a} = \frac{y}{b}\cdot\frac{1}{t}$, and $1 - \frac{x}{a} = \frac{y}{b}\cdot t$.

$\therefore 1 - \frac{x^2}{a^2} = \frac{y^2}{b^2}$. $\therefore \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, which is an ellipse.

Let $t \equiv \tan\frac{\theta}{2}$. Solving the two given equations, we have $\frac{x}{a} = \frac{1-t^2}{1+t^2} = \cos\theta$, and $\frac{y}{b} = \frac{2t}{1+t^2} = \sin\theta$.

Hence the required eccentric angle is θ, *i.e.* $2\tan^{-1} t$.

25. The intercepts on the axes of the tangent at ϕ are $a\sec\phi$, $b\operatorname{cosec}\phi$. Hence if (x, y) be the point whose locus is required, $2x = a\sec\phi$, $2y = a\operatorname{cosec}\phi$.

Eliminating ϕ, $\frac{a^2}{x^2} + \frac{b^2}{y^2} = 4$.

26. The equations of the two normals are

$$ax\sec\phi - by\operatorname{cosec}\phi = a^2 - b^2, \text{ and } y = x\tan\phi.$$

The latter gives $\sec\phi = \frac{\sqrt{x^2+y^2}}{x}$, $\operatorname{cosec}\phi = \frac{\sqrt{x^2+y^2}}{y}$.

Substituting in the former, we have

$$(a-b)\sqrt{x^2+y^2} = a^2 - b^2. \quad \therefore x^2 + y^2 = (a+b)^2.$$

27. Let (x, y) be the coordinates of the middle point of PG. $\therefore 2x = CG + CN = (1+e^2)a\cos\phi$, and $2y = b\sin\phi$.

Eliminating ϕ, $\frac{4x^2}{a^2(1+e^2)^2} + \frac{4y^2}{b^2} = 1$.

Putting $x = 0$, and $y = 0$ successively in the equation

$$ax\sec\phi - by\operatorname{cosec}\phi = a^2 - b^2, \text{ we have}$$

$$CG = \frac{a^2-b^2}{a}\cdot\cos\phi, \text{ and } Cg = -\frac{a^2-b^2}{b}\cdot\sin\phi.$$

Hence, if (x, y) are the coordinates of the middle point of Gg, $2x = \frac{a^2-b^2}{a} . \cos\phi$, $2y = -\frac{a^2-b^2}{b} . \sin\phi$.

Eliminating ϕ, we have $a^2x^2 + b^2y^2 = \frac{1}{4}(a^2-b^2)^2$.

28. See Art. 264. (p, a) are the polar coordinates of the point whose locus is required.

29. See No. 6. The tangents are the four lines $\pm ex \pm y = a$, which pass through one or other of the points $(0, \pm a)$.

30. If (x, y) are the coordinates of Q, and QM be perpendicular to the major axis,

$$x = CM = CG + GM = e^2 . CN + nGN$$
$$= e^2 . CN + n(CN - CG) = a\cos\phi\,(n + e^2 - ne^2);$$

and
$$y = QM = n . PN = n . b\sin\phi.$$

Eliminating ϕ, $\frac{x^2}{a^2(n+e^2-ne^2)^2} + \frac{y^2}{n^2b^2} = 1.$

31. If x_1, x_2 be the roots of equation (3) in Art. 260,

$$x_1 + x_2 = -\frac{2a^2mc}{a^2m^2+b^2}, \text{ and } x_1x_2 = \frac{a^2(c^2-b^2)}{a^2m^2+b^2}.$$

Also
$$y_1 + y_2 = m(x_1+x_2) + 2c,$$

and
$$y_1y_2 = m^2x_1x_2 + cm(x_1+x_2) + c^2.$$

The equation of the required circle is

$$(x-x_1)(x-x_2) + (y-y_1)(y-y_2) = 0,$$

or

$$x^2 + y^2 - (x_1+x_2)(x+my-cm) - 2cy + x_1x_2(1+m^2) + c^2 = 0.$$

Substituting for $x_1 + x_2$ and x_1x_2 we obtain

$$(a^2m^2+b^2)(x^2+y^2) + 2ma^2cx - 2b^2cy$$
$$+ c^2(a^2+b^2) - a^2b^2(1+m^2) = 0.$$

32. If P be the point "ϕ," the equation of MN is

$$\frac{x}{a\cos\phi} + \frac{y}{b\sin\phi} = 1, \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

and A and B can be found so that this will be identical with

$$Ax \sec\phi - By \operatorname{cosec}\phi = A^2 - B^2, \quad \text{.........(2)}$$

which is a normal to the ellipse $\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1$.

On comparing (1) and (2) we easily have

$$\frac{A}{-b} = \frac{B}{a} = \frac{ab}{a^2 - b^2}.$$

33. From the equation of Art. 259, we have

$$-\frac{b}{a} \cdot \cot\frac{(\phi + \phi')}{2} = \text{cons.} \quad \therefore \ \phi + \phi' = \text{cons.}$$

If the chord be parallel to the tangent at α, then

$$-\frac{b}{a} \cdot \cot\frac{\phi + \phi'}{2} = -\frac{b}{a} \cdot \cot\alpha. \quad \therefore \ \phi + \phi' = 2\alpha.$$

34. Any chord of the second ellipse, viz.

$$\frac{x}{\sqrt{a(a+b)}} \cos\tfrac{1}{2}(\alpha + \beta) + \frac{y}{\sqrt{b(a+b)}} \sin\tfrac{1}{2}(\alpha+\beta) = \cos\tfrac{1}{2}(\alpha - \beta),$$

will touch the first ellipse if

$$\frac{a}{a+b} \cos^2\tfrac{1}{2}(\alpha+\beta) + \frac{b}{a+b} \cdot \sin^2\tfrac{1}{2}(\alpha+\beta) = \cos^2\tfrac{1}{2}(\alpha-\beta)$$

[Art. 264].

$$\therefore \ a\{1 + \cos(\alpha+\beta)\} + b\{1 - \cos(\alpha+\beta)\} = (a+b)\{1 + \cos(\alpha-\beta)\};$$

whence $\frac{\cos(\alpha+\beta)}{\cos(\alpha-\beta)} = \frac{a+b}{a-b}$; $\therefore \ \frac{\cos\alpha\cos\beta}{\sin\alpha\sin\beta} = -\frac{a}{b}$;

and this is the condition that the tangents at P and Q are at right angles, since their "m's" are

$$-\sqrt{\frac{b}{a}} \cot\alpha; \ -\sqrt{\frac{b}{a}} \cot\beta.$$

272. *To prove that through any given point* (x_1, y_1) *there pass, in general, two tangents to an ellipse.*

The equation to any tangent is (by Art. 263)

$$y = mx + \sqrt{a^2m^2 + b^2} \quad \text{..................(1).}$$

If this pass through the fixed point (x_1, y_1), we have

$$y_1 - mx_1 = \sqrt{a^2m^2 + b^2},$$

i.e. $$y_1^2 - 2mx_1y_1 + m^2x_1^2 = a^2m^2 + b^2,$$

i.e. $$m^2(x_1^2 - a^2) - 2mx_1y_1 + (y_1^2 - b^2) = 0 \quad \text{.........(2).}$$

For any given values of x_1 and y_1 this equation is in general a quadratic equation and gives two values of m (real or imaginary).

Corresponding to each value of m we have, by substituting in (1), a different tangent.

The roots of (2) are real and different, if

$$(-2x_1y_1)^2 - 4(x_1^2 - a^2)(y_1^2 - b^2) \text{ be positive,}$$

i.e. if $$b^2x_1^2 + a^2y_1^2 - a^2b^2 \text{ be positive,}$$

i.e. if $$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1 \text{ be positive,}$$

i.e. if the point (x_1, y_1) be outside the curve.

The roots are equal, if

$$b^2x_1^2 + a^2y_1^2 - a^2b^2$$

be zero, *i.e.* if the point (x_1, y_1) lie on the curve.

The roots are imaginary, if

$$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1$$

be negative, *i.e.* if the point (x_1, y_1) lie within the curve (Art. 255).

273. *Equation to the chord of contact of tangents drawn from a point* (x_1, y_1).

The equation to the tangent at any point Q, whose coordinates are x' and y', is

$$\frac{xx'}{a^2} + \frac{yy'}{b^2} = 1.$$

Also the tangent at the point R, whose coordinates are x'' and y'', is

$$\frac{xx''}{a^2} + \frac{yy''}{b^2} = 1.$$

If these tangents meet at the point T, whose coordinates are x_1 and y_1, we have

$$\frac{x_1x'}{a^2} + \frac{y_1y'}{b^2} = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

and

$$\frac{x_1x''}{a^2} + \frac{y_1y''}{b^2} = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

The equation to QR is then

$$\mathbf{\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1} \quad \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

For, since (1) is true, the point (x', y') lies on (3).

Also, since (2) is true, the point (x'', y'') lies on (3).

Hence (3) must be the equation to the straight line joining (x', y') and (x'', y''), *i.e.* it must be the equation to QR the required chord of contact of tangents from (x_1, y_1).

274. *To find the equation of the polar of the point* (x_1, y_1) *with respect to the ellipse*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \qquad \text{[Art. 162.]}$$

Let Q and R be the points in which any chord drawn through the point (x_1, y_1) meets the ellipse [Fig. Art. 214].

Let the tangents at Q and R meet in the point whose coordinates are (h, k).

We require the locus of (h, k).

Since QR is the chord of contact of tangents from (h, k), its equation (Art. 273) is

$$\frac{xh}{a^2} + \frac{yk}{b^2} = 1.$$

Since this straight line passes through the point (x_1, y_1), we have

$$\frac{hx_1}{a^2} + \frac{ky_1}{b^2} = 1 \text{ .... (1).}$$

Since the relation (1) is true, it follows that the point (h, k) lies on the straight line

$$\frac{\mathbf{xx_1}}{\mathbf{a^2}} + \frac{\mathbf{yy_1}}{\mathbf{b^2}} = \mathbf{1} \text{ ....(2).}$$

Hence (2) is the equation to the polar of the point (x_1, y_1).

Cor. The polar of the focus (ae, o) is

$$\frac{x \cdot ae}{a^2} = 1, \text{ i.e. } x = \frac{a}{e},$$

i.e. the corresponding directrix.

275. When the point (x_1, y_1) lies outside the ellipse, the equation to its polar is the same as the equation of the chord of contact of tangents from it.

When (x_1, y_1) is on the ellipse, its polar is the same as the tangent at it.

As in Art. 215 the polar of (x_1, y_1) might have been defined as the chord of contact of the tangents, real or imaginary, drawn from it.

276. By a proof similar to that of Art. 217 it can be shewn that *If the polar of P pass through T, then the polar of T passes through P.*

277. *To find the coordinates of the pole of any given line*

$$Ax + By + C = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Let (x_1, y_1) be its pole. Then (1) must be the same as the polar of (x_1, y_1), *i.e.*

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Comparing (1) and (2), as in Art. 218, the required pole is easily seen to be

$$\left(-\frac{Aa^2}{C}, -\frac{Bb^2}{C}\right).$$

278. *To find the equation to the pair of tangents that can be drawn to the ellipse from the point* (x_1, y_1).

Let (h, k) be any point on either of the tangents that can be drawn to the ellipse.

The equation of the straight line joining (h, k) to (x_1, y_1) is

$$y - y_1 = \frac{k - y_1}{h - x_1}(x - x_1),$$

i.e.

$$y = \frac{k - y_1}{h - x_1}x + \frac{hy_1 - kx_1}{h - x_1}.$$

If this straight line touch the ellipse, it must be of the form

$$y = mx + \sqrt{a^2m^2 + b^2}. \qquad \text{(Art. 263.)}$$

Hence

$$m = \frac{k - y_1}{h - x_1}, \quad \text{and} \quad \left(\frac{hy_1 - kx_1}{h - x_1}\right)^2 = a^2m^2 + b^2.$$

Hence

$$\left(\frac{hy_1 - kx_1}{h - x_1}\right)^2 = a^2\left(\frac{k - y_1}{h - x_1}\right)^2 + b^2.$$

But this is the condition that the point (h, k) may lie on the locus

$$(xy_1 - x_1y)^2 = a^2(y - y_1)^2 + b^2(x - x_1)^2 \quad \ldots\ldots (1).$$

This equation is therefore the equation to the required tangents.

It would be found that (1) is equivalent to

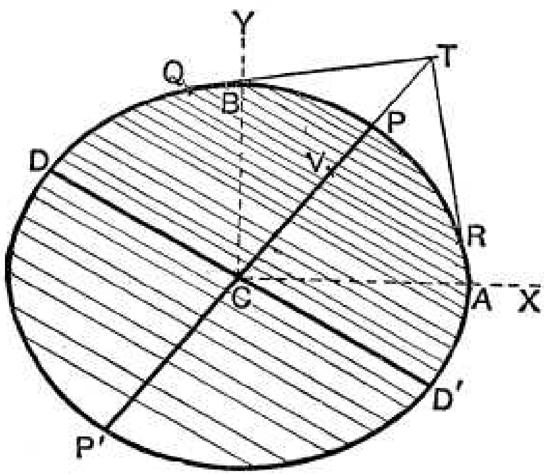
$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{x_1^{\,2}}{a^2}+\frac{y_1^{\,2}}{b^2}-1\right)=\left(\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1\right)^2.$$

279. *To find the locus of the middle points of parallel chords of the ellipse.*

Let the chords make with the axis an angle whose tangent is m, so that the equation to any one of them, QR, is

$$y=mx+c \quad \text{.......................}(1),$$

where c is different for the different chords.



This straight line meets the ellipse in points whose abscissae are given by the equation

$$\frac{x^2}{a^2}+\frac{(mx+c)^2}{b^2}=1,$$

i.e. $$x^2(a^2m^2+b^2)+2a^2mcx+a^2(c^2-b^2)=0 \quad \text{......}(2).$$

Let the roots of this equation, *i.e.* the abscissae of Q and R, be x_1 and x_2, and let V, the middle point of QR, be the point (h, k).

Then, by Arts. 22 and 1, we have

$$h=\frac{x_1+x_2}{2}=-\frac{a^2mc}{a^2m^2+b^2} \quad \text{..............}(3).$$

Also V lies on the straight line (1), so that

$$k = mh + c \qquad \text{.......................(4).}$$

If between (3) and (4) we eliminate c, we have

$$h = -\frac{a^2m\,(k - mh)}{a^2m^2 + b^2},$$

i.e.

$$b^2h = -a^2mk \qquad \text{....................(5).}$$

Hence the point (h, k) always lies on the straight line

$$y = -\frac{b^2}{a^2m}x \qquad \text{....................(6).}$$

The required locus is therefore the straight line

$$y = m_1x, \text{ where } m_1 = -\frac{b^2}{a^2m},$$

i.e.

$$mm_1 = -\frac{b^2}{a^2} \qquad \text{....................(7).}$$

280. *Equation to the chord whose middle point is* (h, k).

The required equation is (1) of the foregoing article, where m and c are given by equations (4) and (5), so that

$$m = -\frac{b^2h}{a^2k}, \text{ and } c = \frac{a^2k^2 + b^2h^2}{a^2k}.$$

The required equation is therefore

$$y = -\frac{b^2h}{a^2k}x + \frac{a^2k^2 + b^2h^2}{a^2k},$$

i.e.

$$\frac{k}{b^2}(y - k) + \frac{h}{a^2}(x - h) = 0.$$

It is therefore parallel to the polar of (h, k).

281. Diameter. Def. The locus of the middle points of parallel chords of an ellipse is called a diameter, and the chords are called its double ordinates.

By equation (6) of Art. 279 we see that any diameter passes through the centre C.

Also, by equation (7), we see that the diameter $y = m_1x$ bisects all chords parallel to the diameter $y = mx$, if

$$mm_1 = -\frac{b^2}{a^2} \qquad \text{....................(1).}$$

But the symmetry of the result (1) shows that, in this case, the diameter $y = mx$ bisects all chords parallel to the diameter $y = m_1x$.

Such a pair of diameters are called Conjugate Diameters. Hence

Conjugate Diameters. Def. Two diameters are said to be conjugate when each bisects all chords parallel to the other.

Two diameters $y = mx$ and $y = m_1x$ are therefore conjugate, if

$$\mathbf{mm_1 = -\frac{b^2}{a^2}}.$$

282. *The tangent at the extremity of any diameter is parallel to the chords which it bisects.*

In the Figure of Art. 279 let (x', y') be the point P on the ellipse, the tangent at which is parallel to the chord QR, whose equation is

$$y = mx + c \qquad\text{.......................(1).}$$

The tangent at the point (x', y') is

$$\frac{xx'}{a^2} + \frac{yy'}{b^2} = 1 \qquad\text{.................... (2).}$$

Since (1) and (2) are parallel, we have

$$m = -\frac{b^2x'}{a^2y'},$$

i.e. the point (x', y') lies on the straight line

$$y = -\frac{b^2}{a^2m}x.$$

But, by Art. 279, this is the diameter which bisects QR and all chords which are parallel to it.

Cor. It follows that two conjugate diameters CP and CD are such that each is parallel to the tangent at the extremity of the other. Hence, given either of these, we have a geometrical construction for the other.

283. *The tangents at the ends of any chord meet on the diameter which bisects the chord.*

Let the equation to the chord QR (Art. 279) be

$$y = mx + c \quad\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Let T be the point of intersection of the tangents at Q and R, and let its coordinates be x_1 and y_1.

Since QR is the chord of contact of tangents from T, its equation is, by Art. 273,

$$\frac{xh}{a^2} + \frac{yk}{b^2} = 1 \quad\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

The equations (1) and (2) therefore represent the same straight line, so that

$$m = -\frac{b^2h}{a^2k},$$

i.e. (h, k) lies on the straight line

$$y = -\frac{b^2}{a^2m}x,$$

which, by Art. 279, is the equation to the diameter bisecting the chord QR. Hence T lies on the straight line CP.

284. *If the eccentric angles of the ends, P and D, of a pair of conjugate diameters be ϕ and ϕ', then ϕ and ϕ' differ by a right angle.*

Since P is the point $(a\cos\phi,\ b\sin\phi)$, the equation to CP is

$$y = x\,.\,\frac{b}{a}\tan\phi \quad\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

So the equation to CD is

$$y = x\,.\,\frac{b}{a}\tan\phi' \quad\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

These diameters are (Art. 281) conjugate if

$$\frac{b^2}{a^2}\tan\phi\tan\phi' = -\frac{b^2}{a^2},$$

i.e. if $\qquad \tan\phi = -\cot\phi' = \tan(\phi' \pm 90^\circ),$

i.e. if $\qquad \phi - \phi' = \pm 90^\circ.$

Cor. 1. The points on the auxiliary circle corresponding to P and D subtend a right angle at the centre.

For if p and d be these points then, by Art. 258, we have

$$\angle pCA' = \phi \text{ and } \angle dCA' = \phi'.$$

Hence

$$\angle pCd = \angle dCA' - \angle pCA' = \phi - \phi' = 90°.$$

Cor. 2. In the figure of Art. 286 if P be the point ϕ, then D is the point $\phi + 90°$ and D' is the point $\phi - 90°$.

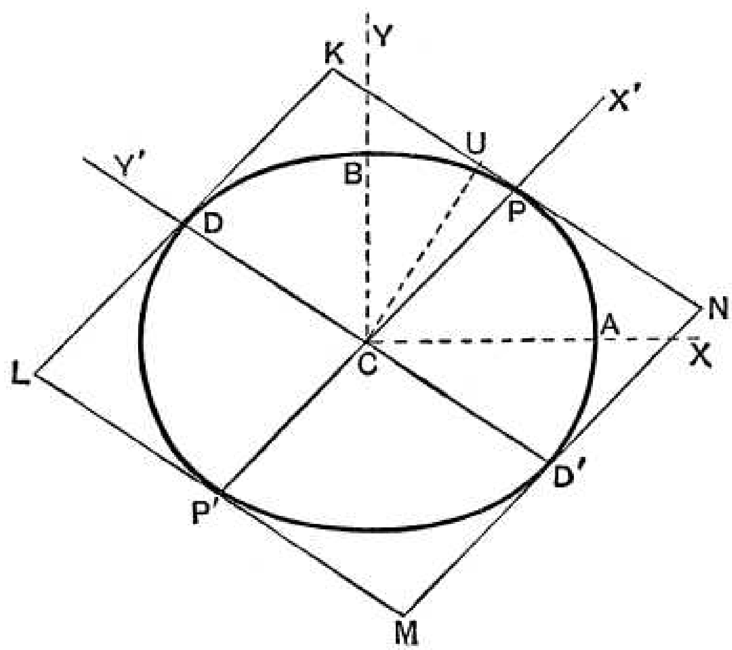
285. From the previous article it follows that if P be the point $(a \cos \phi, b \sin \phi)$, then D is the point

$$\{a \cos(90° + \phi), \quad b \sin(90° + \phi)\} \text{ i.e. } (-a \sin \phi, b \cos \phi).$$

Hence, if PN and DM be the ordinates of P and D, we have

$$\frac{NP}{b} = -\frac{CM}{a}, \text{ and } \frac{CN}{a} = \frac{MD}{b}.$$

286. *If PCP' and DCD' be a pair of conjugate diameters, then* (1) *$CP^2 + CD^2$ is constant, and* (2) *the area of the parallelogram formed by the tangents at the ends of these diameters is constant.*



Let P be the point ϕ, so that its coordinates are $a\cos\phi$ and $b\sin\phi$. Then D is the point $90^\circ+\phi$, so that its coordinates are

$$a\cos(90^\circ+\phi) \text{ and } b\sin(90^\circ+\phi),$$

i.e.

$$-a\sin\phi \text{ and } b\cos\phi.$$

(1) We therefore have

$$CP^2 = a^2\cos^2\phi + b^2\sin^2\phi,$$

and

$$CD^2 = a^2\sin^2\phi + b^2\cos^2\phi.$$

Hence $\quad CP^2 + CD^2 = a^2 + b^2$

= the sum of the squares of the semi-axes of the ellipse.

(2) Let $KLMN$ be the parallelogram formed by the tangents at P, D, P', and D'.

By Euc. I. 36, we have

$$\text{area } KLMN = 4 \,.\, \text{area } CPKD$$
$$= 4 \,.\, CU \,.\, PK = 4\, CU \,.\, CD,$$

where CU is the perpendicular from C upon the tangent at P.

Now the equation to the tangent at P is

$$\frac{x}{a}\cos\phi + \frac{y}{b}\sin\phi - 1 = 0,$$

so that (Art. 75) we have

$$CU = \frac{1}{\sqrt{\dfrac{\cos^2\phi}{a^2} + \dfrac{\sin^2\phi}{b^2}}} = \frac{ab}{\sqrt{a^2\sin^2\phi + b^2\cos^2\phi}} = \frac{ab}{CD}.$$

Hence $\quad CU \,.\, CD = ab.$

Thus the area of the parallelogram $KLMN = 4ab$,

which is equal to the rectangle formed by the tangents at the ends of the major and minor axes.

287. *The product of the focal distances of a point P is equal to the square on the semidiameter parallel to the tangent at P.*

If P be the point ϕ, then, by Art. 251, we have

$$SP = a + ae\cos\phi, \text{ and } S'P = a - ae\cos\phi.$$

Hence $$SP \cdot S'P = a^2 - a^2e^2 \cos^2 \phi$$
$$= a^2 - (a^2 - b^2) \cos^2 \phi$$
$$= a^2 \sin^2 \phi + b^2 \cos^2 \phi$$
$$= CD^2.$$

288. Ex. *If P and D be the ends of conjugate diameters, find the locus of*

(1) *the middle point of PD,*

(2) *the intersection of the tangents at P and D,*

and (3) *the foot of the perpendicular from the centre upon PD.*

P is the point $(a \cos \phi,\ b \sin \phi)$ and D is $(-a \sin \phi,\ b \cos \phi)$.

(1) If (x, y) be the middle point of PD, we have

$$x = \frac{a \cos \phi - a \sin \phi}{2}, \text{ and } y = \frac{b \sin \phi + b \cos \phi}{2}.$$

If we eliminate ϕ we shall get the required locus. We obtain

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \tfrac{1}{4} [(\cos \phi - \sin \phi)^2 + (\sin \phi + \cos \phi)^2] = \tfrac{1}{2}.$$

The locus is therefore a concentric and similar ellipse.

[N.B. Two ellipses are similar if the ratios of their axes are the same, so that they have the same eccentricity.]

(2) The tangents are

$$\frac{x}{a} \cos \phi + \frac{y}{b} \sin \phi = 1,$$

and $$-\frac{x}{a} \sin \phi + \frac{y}{b} \cos \phi = 1.$$

Both of these equations hold at the intersection of the tangents. If we eliminate ϕ we shall have the equation of the locus of their intersections.

By squaring and adding, we have

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2,$$

so that the locus is another similar and concentric ellipse.

(3) By Art. 259, on putting $\phi' = 90° + \phi$, the equation to PD is

$$\frac{x}{a} \cos (45° + \phi) + \frac{y}{b} \sin (45° + \phi) = \cos 45°.$$

Let the length of the perpendicular from the centre be p and let it make an angle ω with the axis. Then this line must be equivalent to

$$x \cos \omega + y \sin \omega = p.$$

Comparing the equations, we have

$$\cos(45^\circ+\phi)=\frac{a\cos\omega\cos 45^\circ}{p}, \text{ and } \sin(45^\circ+\phi)=\frac{b\sin\omega\cos 45^\circ}{p}.$$

Hence, by squaring and adding, $2p^2=a^2\cos^2\omega+b^2\sin^2\omega$, *i.e.* the locus required is the curve

$$2r^2=a^2\cos^2\theta+b^2\sin^2\theta,\ \ i.e.\ \ 2(x^2+y^2)^2=a^2x^2+b^2y^2.$$

289. *Equiconjugate diameters.* Let P and D be extremities of equiconjugate diameters, so that $CP^2=CD^2$.

If the eccentric angle of P be ϕ, we then have

$$a^2\cos^2\phi+b^2\sin^2\phi=a^2\sin^2\phi+b^2\cos^2\phi,$$

giving $$\tan^2\phi=1,$$

i.e. $$\phi=45^\circ, \text{ or } 135^\circ.$$

The equation to CP is then

$$y=x\,.\frac{b}{a}\tan\phi,$$

i.e. $$y=\pm\frac{b}{a}x \text{ ....(1)},$$

and that to CD is $$y=-x\frac{b}{a}\cot\phi,$$

i.e. $$y=\mp\frac{b}{a}x \text{ .... (2)}.$$

If a rectangle be formed whose sides are the tangents at A, A', B, and B' the lines (1) and (2) are easily seen to be its diagonals.

The directions of the equiconjugates are therefore along the diagonals of the circumscribing rectangle.

The length of each equiconjugate is, by Art. 286,

$$\sqrt{\frac{a^2+b^2}{2}}.$$

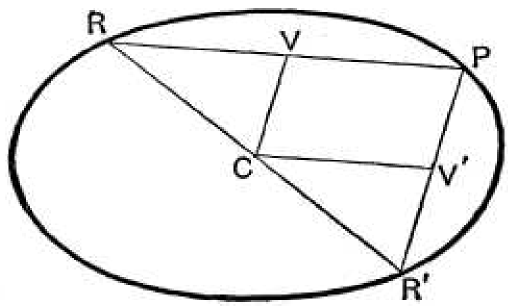
290. Supplemental chords. Def. The chords joining any point P on an ellipse to the extremities, R and R', of any diameter of the ellipse are called supplemental chords.

Supplemental chords are parallel to conjugate diameters.

Let P be the point whose eccentric angle is ϕ, and R and R' the points whose eccentric angles are ϕ_1 and $180^\circ + \phi_1$.

The equations to PR and PR' are then (Art. 259)

$$\frac{x}{a}\cos\frac{\phi+\phi_1}{2}+\frac{y}{b}\sin\frac{\phi+\phi_1}{2}=\cos\frac{\phi-\phi_1}{2} \ldots\ldots (1),$$



and

$$\frac{x}{a}\cos\frac{\phi+180^\circ+\phi_1}{2}+\frac{y}{b}\sin\frac{\phi+180^\circ+\phi_1}{2}=\cos\frac{\phi-180^\circ-\phi_1}{2},$$

i.e.
$$-\frac{x}{a}\sin\frac{\phi+\phi_1}{2}+\frac{y}{b}\cos\frac{\phi+\phi_1}{2}=\sin\frac{\phi-\phi_1}{2} \ldots (2).$$

The "m" of the straight line (1) $=-\frac{b}{a}\cot\frac{\phi+\phi_1}{2}$.

The "m" of the line (2) $=\frac{b}{a}\tan\frac{\phi+\phi_1}{2}$.

The product of these "m's" $=-\frac{b^2}{a^2}$, so that, by Art. 281, the lines PR and PR' are parallel to conjugate diameters.

This proposition may also be easily proved geometrically.

For let V and V' be the middle points of PR and PR'.

Since V and C are respectively the middle points of RP and RR', the line CV is parallel to PR'. Similarly CV' is parallel to PR.

Since CV bisects PR it bisects all chords parallel to PR, *i.e.* all chords parallel to CV'. So CV' bisects all chords parallel to CV.

Hence CV and CV' are in the direction of conjugate diameters and therefore PR' and PR, being parallel to CV and CV' respectively, are parallel to conjugate diameters.

291. *To find the equation to an ellipse referred to a pair of conjugate diameters.*

Let the conjugate semi-diameters be CP and CD (Fig. Art. 286), whose lengths are a' and b' respectively.

If we transform the equation to the ellipse, referred to its principal axes, to CP and CD as axes of coordinates, then, since the origin is unaltered, it becomes, by Art. 134, of the form

$$Ax^2 + 2Hxy + By^2 = 1 \text{ .... } (1).$$

Now the point P, $(a', 0)$, lies on (1), so that

$$Aa'^2 = 1 \text{ .... } (2).$$

So since Q, the point $(0, b')$, lies on (1), we have

$$Bb'^2 = 1.$$

Hence $$A = \frac{1}{a'^2}, \text{ and } B = \frac{1}{b'^2}.$$

Also, since CP bisects all chords parallel to CD, therefore for each value of x we have two equal and opposite values of y. This cannot be unless $H = 0$.

The equation then becomes

$$\frac{x^2}{a'^2} + \frac{y^2}{b'^2} = 1.$$

Cor. If the axes be the equiconjugate diameters, the equation is $x^2 + y^2 = a'^2$. The equation is thus the same in form as the equation to a circle. In the case of the ellipse however the axes are oblique.

292. It will be noted that the equation to the ellipse, when referred to a pair of conjugate diameters, is of the same form as it is when referred to its principal axes. The latter are merely a particular case of a pair of conjugate diameters.

Just as in Art. 262, it may be shewn that the equation to the tangent at the point (x', y') is

$$\frac{xx'}{a'^2} + \frac{yy'}{b'^2} = 1.$$

Similarly for the equation to the polar.

Ex. *If QVQ' be a double ordinate of the diameter CP, and if the tangent at Q meet CP in T, then $CV.CT = CP^2$.*

If Q be the point (x', y'), the tangent at it is

$$\frac{xx'}{a'^2} + \frac{yy'}{b'^2} = 1.$$

Putting $y = 0$, we have $\qquad x = \dfrac{a'^2}{x'}$,

i.e.
$$CT = \frac{a'^2}{x'} = \frac{CP^2}{CV},$$

i.e.
$$CV.CT = CP^2.$$

EXAMPLES XXXIV

1. In the ellipse $\frac{x^2}{36} + \frac{y^2}{9} = 1$, find the equation to the chord which passes through the point (2, 1) and is bisected at that point.

2. Find, with respect to the ellipse $4x^2 + 7y^2 = 8$,

(1) the polar of the point $(-\frac{1}{2}, 1)$, and

(2) the pole of the straight line $12x + 7y + 16 = 0$.

3. Tangents are drawn from the point (3, 2) to the ellipse $x^2 + 4y^2 = 9$. Find the equation to their chord of contact and the equation of the straight line joining (3, 2) to the middle point of this chord of contact.

4. Write down the equation of the pair of tangents drawn to the ellipse $3x^2 + 2y^2 = 5$ from the point (1, 2), and prove that the angle between them is $\tan^{-1}\frac{12\sqrt{5}}{5}$.

5. In the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, write down the equations to the diameters which are conjugate to the diameters whose equations are

$$x - y = 0, \quad x + y = 0, \quad y = \frac{a}{b}x, \quad \text{and} \quad y = \frac{b}{a}x.$$

6. Shew that the diameters whose equations are $y + 3x = 0$ and $4y - x = 0$, are conjugate diameters of the ellipse $3x^2 + 4y^2 = 5$.

7. If the product of the perpendiculars from the foci upon the polar of P be constant and equal to c^2, prove that the locus of P is the ellipse $b^4x^2(c^2 + a^2e^2) + c^2a^4y^2 = a^4b^4$.

8. Shew that the four lines which join the foci to two points P and Q on an ellipse all touch a circle whose centre is the pole of PQ.

9. If the pole of the normal at P lie on the normal at Q, then shew that the pole of the normal at Q lies on the normal at P.

10. CK is the perpendicular from the centre on the polar of any point P, and PM is the perpendicular from P on the same polar and is produced to meet the major axis in L. Shew that (1) $CK . PL = b^2$, and (2) the product of the perpendiculars from the foci on the polar $= CK . LM$.

What do these theorems become when P is on the ellipse?

11. In the previous question, if PN be the ordinate of P and the polar meet the axis in T, shew that $CL = e^2 . CN$ and $CT . CN = a^2$.

12. If tangents TP and TQ be drawn from a point T, whose coordinates are h and k, prove that the area of the triangle TPQ is

$$ab\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}-1\right)^{\frac{3}{2}} \div \left(\frac{h^2}{a^2}+\frac{k^2}{b^2}\right),$$

and that the area of the quadrilateral $CPTQ$ is

$$ab\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}-1\right)^{\frac{1}{2}}.$$

13. Tangents are drawn to the ellipse from the point

$$\left(\frac{a^2}{\sqrt{a^2-b^2}},\ \sqrt{a^2+b^2}\right);$$

prove that they intercept on the ordinate through the nearer focus a distance equal to the major axis.

14. Prove that the angle between the tangents that can be drawn from any point (x_1, y_1) to the ellipse is

$$\tan^{-1}\frac{2ab\sqrt{\dfrac{{x_1}^2}{a^2}+\dfrac{{y_1}^2}{b^2}-1}}{{x_1}^2+{y_1}^2-a^2-b^2}.$$

15. If T be the point (x_1, y_1), shew that the equation to the straight lines joining it to the foci, S and S', is

$$(x_1y-xy_1)^2-a^2e^2(y-y_1)^2=0.$$

Prove that the bisector of the angle between these lines also bisects the angle between the tangents TP and TQ that can be drawn from T, and hence that

$$\angle STP = \angle S'TQ.$$

16. If two tangents to an ellipse and one of its foci be given, prove that the locus of its centre is a straight line.

17. Prove that the straight lines joining the centre to the intersections of the straight line $y = mx + \sqrt{\dfrac{a^2m^2+b^2}{2}}$ with the ellipse are conjugate diameters.

18. Any tangent to an ellipse meets the director circle in p and d; prove that Cp and Cd are in the directions of conjugate diameters of the ellipse.

19. If CP be conjugate to the normal at Q, prove that CQ is conjugate to the normal at P.

20. If a fixed straight line parallel to either axis meet a pair of conjugate diameters in the points K and L, shew that the circle described on KL as diameter passes through two fixed points on the other axis.

21. Prove that a chord which joins the ends of a pair of conjugate diameters of an ellipse always touches a similar ellipse.

22. The eccentric angles of two points P and Q on the ellipse are ϕ_1 and ϕ_2; prove that the area of the parallelogram formed by the tangents at the ends of the diameters through P and Q is

$$4ab \operatorname{cosec}(\phi_1 - \phi_2),$$

and hence that it is least when P and Q are at the end of conjugate diameters.

23. A pair of conjugate diameters is produced to meet the directrix; shew that the orthocentre of the triangle so formed is at the focus.

24. If the tangent at any point P meet in the points L and L'

(1) two parallel tangents, or (2) two conjugate diameters,

prove that in each case the rectangle $LP \cdot PL'$ is equal to the square on the semidiameter which is parallel to the tangent at P.

25. A point is such that the perpendicular from the centre on its polar with respect to the ellipse is constant and equal to c; shew that its locus is the ellipse

$$\frac{x^2}{a^4} + \frac{y^2}{b^4} = \frac{1}{c^2}.$$

26. Tangents are drawn from any point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ to the circle $x^2 + y^2 = r^2$; prove that the chords of contact are tangents to the ellipse $a^2x^2 + b^2y^2 = r^4$.

If $\frac{1}{r^2} = \frac{1}{a^2} + \frac{1}{b^2}$, prove that the lines joining the centre to the points of contact with the circle are conjugate diameters of the second ellipse.

27. CP and CD are conjugate diameters of the ellipse; prove that the locus of the orthocentre of the triangle CPD is the curve

$$2(b^2y^2 + a^2x^2)^3 = (a^2 - b^2)^2 (b^2y^2 - a^2x^2)^2.$$

28. If circles be described on two semi-conjugate diameters of the ellipse as diameters, prove that the locus of their second points of intersection is the curve $2(x^2 + y^2)^2 = a^2x^2 + b^2y^2$.

ANSWERS

1. $x+2y=4$. **2.** $2x-7y+8=0$; $(-\frac{3}{2}, -\frac{1}{2})$.

3. $3x+8y=9$; $2x=3y$.

4. $9x^2-24xy-4y^2+30x+40y-55=0$.

5. $a^2y+b^2x=0$; $a^2y-b^2x=0$; $a^3y+b^3x=0$; $ay+bx=0$.

SOLUTIONS/HINTS

1. By Art. 280, the required equation is

$$\frac{1}{9}(y-1)+\frac{2}{36}(x-2)=0, \text{ or } x+2y=4.$$

2. The required equation is [Art. 274]

$$4x(-\tfrac{1}{2})+7y(1)=8, \text{ or } 7y-2x=8.$$

The required point is [Art. 277]

$$(-\tfrac{12}{16}\cdot 2, -\tfrac{7}{16}\cdot\tfrac{8}{7}), \text{ or } (-\tfrac{3}{2}, -\tfrac{1}{2}).$$

3. $3x+8y=9$. [Art. 274.] $2x=3y$. [Art. 283.]

4. Substitute, in equation (1) of Art. 278,

$$x_1=1, \; y_1=2, \; a^2=\tfrac{5}{3}, \; b^2=\tfrac{5}{2}.$$

The lines are parallel to $9x^2-24xy-4y^2=0$, and the angle between them is [Art. 110]

$$\tan^{-1}\frac{2\sqrt{144+36}}{5}=\tan^{-1}\frac{12\sqrt{5}}{5}.$$

5 and 6. See Art. 281.

7. The polar of (x_1, y_1) is $b^2xx_1+a^2yy_1=a^2b^2$.

$\therefore$ the product of the perpendiculars from $(\pm ae, 0)$ upon it

$$=\frac{(a^2b^2-b^2aex_1)(a^2b^2+b^2aex_1)}{x_1^2b^4+y_1^2a^4}.$$

Equating this to c^2, the required locus is

$$b^4x^2(c^2+a^2e^2)+c^2a^4y^2=a^4b^4.$$

8. Let T be the pole of the chord PQ, and let SQ, $S'P$ intersect in O. Then since TP bisects the angle SPS' externally and TS bisects the angle PSQ, $\therefore$ T is an excentre of the triangle SPO. Similarly it is an excentre of the triangle $S'QO$. The perpendiculars from T' upon the four straight lines SP, SQ, $S'P$, $S'Q$ are therefore equal.

9. Let the eccentric angles of P and Q be θ and ϕ. If (x_1, y_1) be the pole of the normal at P, viz.

$$\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2,$$

then as in Art. 277 we have

$$x_1 = \frac{a^3}{a^2-b^2}\frac{1}{\cos\theta} \text{ and } y_1 = -\frac{b^3}{a^2-b^2}\frac{1}{\sin\theta}.$$

This point lies on the normal at Q if

$$\frac{a}{\cos\phi}\left(\frac{a^3}{a^2-b^2}\frac{1}{\cos\theta}\right) - \frac{b}{\sin\phi}\left(\frac{-b^3}{a^2-b^2}\frac{1}{\sin\theta}\right) = a^2 - b^2,$$

i.e. if $$a^4 \sec\theta \sec\phi + b^4 \operatorname{cosec}\theta \operatorname{cosec}\phi = (a^2-b^2)^2.$$

This, being symmetrical in θ and ϕ, is also the condition that the pole of the normal at Q lies on the normal at P.

10. Let (h, k) be the coordinates of P. Then CK is the perpendicular from $(0, 0)$ upon the line $b^2hx + a^2ky = a^2b^2$

and $$\therefore = \frac{a^2b^2}{\sqrt{b^4h^2 + a^4k^2}}.$$

The equation of the line PL is $b^2h(y-k) = a^2k(x-h)$.

Putting $y = 0$, the coordinates of L are $\left(h - \frac{b^2}{a^2}h, 0\right)$;

$$\therefore PL = \sqrt{\frac{b^4}{a^4}h^2 + k^2} = \frac{\sqrt{b^4h^2 + a^4k^2}}{a^2}; \ \therefore PL \, . \, CK = b^2.$$

Again, $$PM = \frac{b^2h^2 + a^2k^2 - a^2b^2}{\sqrt{b^4h^2 + a^4k^2}};$$

$$\therefore LM = PL - PM = \frac{b^2(a^2 - e^2h^2)}{\sqrt{b^4h^2 + a^4k^2}};$$

$$\therefore CK \, . \, LM = \frac{a^2b^4(a^2 - e^2h^2)}{b^4h^2 + a^4k^2}.$$

Also the product of the perpendiculars from ($\pm ae$, 0) upon the polar of $P = \dfrac{a^4b^4 - a^2b^4e^2h^2}{b^4h^2 + a^4k^2} = CK\,.\,LM.$

When P is on the ellipse each of these theorems becomes

$$PF\,.\,PG = b^2 \quad [\text{Art. } 270\ (\gamma)].$$

11. By the previous example,

$$CL = h - \frac{b^2}{a^2}h = e^2h = e^2CN. \text{ Also } CT = \frac{a^2}{h};\ \therefore\ CT\,.\,CN = a^2.$$

12. By Art. 261, the chord intercepted by the ellipse on the line $b^2hx + a^2ky = a^2b^2$ is

$$\frac{2\sqrt{a^4k^2 + b^4h^2}\,.\,\sqrt{b^2h^2 + a^2k^2 - a^2b^2}}{b^2h^2 + a^2k^2};$$

also the perpendicular TM on it from T

$$= \frac{b^2h^2 + a^2k^2 - a^2b^2}{\sqrt{a^4k^2 + b^4h^2}}.$$

$$\therefore\ \triangle TPQ = \frac{(b^2h^2 + a^2k^2 - a^2b^2)^{\frac{3}{2}}}{b^2h^2 + a^2k^2}.$$

Also $$CK + TM = \frac{b^2h^2 + a^2k^2}{\sqrt{b^4h^2 + a^4k^2}}.$$

$\therefore$ Area of the quadrilateral $CPTQ$

$$= \tfrac{1}{2}\,(CK + TM)\,.\,PQ = \sqrt{b^2h^2 + a^2k^2 - a^2b^2}.$$

13. Put $\dfrac{a}{e} = x_1$, $y_1 = \sqrt{a^2 + b^2} = c$, and $x = ae$ in the equation of Art. 278, and we have

$$\left(aec - y\frac{a}{e}\right)^2 = a^2\,(y - c)^2 + b^2\left(ae - \frac{a}{e}\right)^2.$$

$$\therefore\ y^2\frac{a^2}{e^2} + a^2e^2c^2 - 2a^2cy = a^2y^2 - 2a^2cy + a^2c^2 + \frac{a^2b^2}{e^2}(1 - e^2)^2.$$

$$\therefore\ a^2y^2\,(1 - e^2) = a^2c^2e^2\,(1 - e^2) + a^2b^2\,(1 - e^2)^2.$$

$$\therefore\ y^2 = e^2c^2 + b^2(1 - e^2) = e^2a^2 + e^2b^2 + b^2 - e^2b^2 = a^2.$$

$\therefore$ Difference of roots $= 2a$.

14. The terms of the second degree in the equation of Art. 278 are $x^2(y_1^2-b^2)-2x_1y_1xy+y^2(x_1^2-a^2)$.

$\therefore$ By Art. 110, the required angle

$$=\tan^{-1}\frac{2\sqrt{x_1^2y_1^2-(y_1^2-b^2)(x_1^2-a^2)}}{x_1^2+y_1^2-a^2-b^2}$$

$$=\tan^{-1}\frac{2\sqrt{b^2x_1^2+a^2y_1^2-a^2b^2}}{x_1^2+y_1^2-a^2-b^2}.$$

15. The equations of the required lines are $\dfrac{x\pm ae}{x_1\pm ae}=\dfrac{y}{y_1}$,

i.e. $x_1y-xy_1=\pm ae(y-y_1)$; or, expressed in one equation,

$$(x_1y-xy_1)^2-a^2e^2(y-y_1)^2=0.$$

The lines through the origin parallel to the bisectors of the angles between these are [Art. 112]

$$\frac{x^2-y^2}{y_1^2-x_1^2+a^2-b^2}=-\frac{xy}{x_1y_1},$$

which are also the bisectors of the angles between the lines

$$x^2(y_1^2-b^2)-2x_1y_1xy+y^2(x_1^2-a^2)=0,$$

which by Art. 278 are the lines through the centre parallel to the tangents drawn from (x_1, y_1).

16. If TP and TQ be the tangents the other focus lies on a line TS' through T (lying between TP and TQ) such that $Q\hat{T}S'=P\hat{T}S$. Hence the centre will lie on a line which bisects the distance between S and TS' and is parallel to it.

17. The equation of the required lines is [Art. 122]

$$\frac{(b^2x^2+a^2y^2)(a^2m^2+b^2)}{2}=a^2b^2(y-mx)^2,$$

or $$x^2(a^2b^2m^2-b^4)-4a^2b^2mxy+y^2(a^2b^2-a^4m^2)=0;$$

$$\therefore\ m_1m_2=\frac{a^2b^2m^2-b^4}{a^2b^2-a^4m^2}=-\frac{b^2}{a^2}.$$

Hence, by Art. 281, these lines are conjugate diameters.

18. The lines joining the origin to the common points of the circle $x^2+y^2=a^2+b^2$ and the line

$$x\cos\alpha+y\sin\alpha=\sqrt{a^2\cos^2\alpha+b^2\sin^2\alpha}$$

are [Art. 122]

$$(x^2+y^2)(a^2\cos^2\alpha+b^2\sin^2\alpha)=(a^2+b^2)(x\cos\alpha+y\sin\alpha)^2,$$

or $\quad b^2\cos 2\alpha x^2+2xy\cos\alpha\sin\alpha(a^2+b^2)-a^2\cos 2\alpha y^2=0,$

whence $$m_1m_2=-\frac{b^2}{a^2}.$$

19. Let P and Q be the points ϕ and θ. Since CP is conjugate to the normal at Q,

$$\frac{b}{a}\tan\phi\,.\,\frac{a}{b}\,.\tan\theta=-\frac{b^2}{a^2}.\quad\therefore\ \tan\phi\,.\tan\theta=-\frac{b^2}{a^2},$$

which is, by symmetry, also the condition that CQ should be conjugate to the normal at P.

20. Let $lb^2x^2+2mxy-la^2y^2=0$ be the equation of the conjugate diameters. The points in which they meet the fixed line $y=c$ are given by $lb^2x^2+2cmx-la^2c^2=0$.

$$\therefore\ x_1+x_2=-\frac{2cm}{lb^2},\ \text{and}\ x_1x_2=-\frac{a^2c^2}{b^2}.$$

The equation of the circle on KL as diameter is, by Art. 145, $\quad (x-x_1)(x-x_2)+(y-c)^2=0,$

or $$lb^2x^2+2cmx-la^2c^2+lb^2(y-c)^2=0.$$

This cuts $x=0$ in the fixed points given by $(y-c)^2=\dfrac{a^2c^2}{b^2}$.

21. The equation of the chord joining the points "α" and "β," if $\alpha-\beta=\frac{\pi}{2}$, is, [Art. 259],

$$\frac{x}{a}\,.\cos\tfrac{1}{2}(\alpha+\beta)+\frac{y}{b}\sin\tfrac{1}{2}(\alpha+\beta)=\frac{1}{\sqrt{2}},$$

which is a tangent to the similar ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=\dfrac{1}{2}$.

22. Let PCP', QCQ' be the diameters, and $TRT'R'$ the parallelogram formed by the tangents.

The coordinates of T are $\left[a\dfrac{\cos\frac{1}{2}(\phi_1+\phi_2)}{\cos\frac{1}{2}(\phi_1-\phi_2)},\ b\dfrac{\sin\frac{1}{2}(\phi_1+\phi_2)}{\cos\frac{1}{2}(\phi_1-\phi_2)}\right]$.

Hence, by Ex. 12, the area of $CPTQ$

$$=\frac{\{a^2b^2\cos^2\frac{1}{2}(\phi_1+\phi_2)+a^2b^2\sin^2\frac{1}{2}(\phi_1+\phi_2)-a^2b^2\cos^2\frac{1}{2}(\phi_1-\phi_2)\}^{\frac{1}{2}}}{\cos\frac{1}{2}(\phi_1-\phi_2)}$$

$$=ab\tan\tfrac{1}{2}(\phi_1-\phi_2);$$

and (changing ϕ_1 into $\pi+\phi_1$) the area of

$$CPRQ'=ab\cot\tfrac{1}{2}(\phi_1-\phi_2).$$

Hence the area of $TRT'R'$

$$=2ab\{\tan\tfrac{1}{2}(\phi_1-\phi_2)+\cot\tfrac{1}{2}(\phi_1-\phi_2)\}=4ab\operatorname{cosec}(\phi_1-\phi_2).$$

This is least when $\operatorname{cosec}(\phi_1-\phi_2)=1$, *i.e.* when $\phi_1-\phi_2=\dfrac{\pi}{2}$.

23. Let $y=\dfrac{b}{a}\tan\phi\,.\,x$, ...(i) and $y=-\dfrac{b}{a}\cot\phi\,.\,x$ ...(ii) be the equations of the conjugate diameters.

The point where (i) cuts the directrix

$$x=\frac{a}{e}\text{ is }\left(\frac{a}{e},\ \frac{b}{e}\,.\tan\phi\right).$$

The equation of the line through this point perpendicular to (ii) is

$$a\sin\phi\left(x-\frac{a}{e}\right)-b\cos\phi\left(y-\frac{b}{e}\tan\phi\right)=0.$$

This line cuts the axis of x where $x=\dfrac{a}{e}-\dfrac{b^2}{ae}=ae.$

24. Let $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ be the equation to the ellipse referred to CP and its conjugate diameter as axes.

The equations to two parallel tangents are

$$\frac{xx'}{a^2}+\frac{yy'}{b^2}=\pm 1.$$

Putting $x=a$, $LP=\dfrac{b^2}{y'}\left\{1-\dfrac{x'}{a}\right\}$, and $L'P=\dfrac{b^2}{y'}\left\{1+\dfrac{x'}{a}\right\}$.

$$\therefore\ LP\,.\,PL'=\frac{b^4}{y'^2}\left\{1-\frac{x'^2}{a^2}\right\}=b^2.$$

Putting $x=a$ in the equations of any two conjugate diameters, viz. $y=\frac{b}{a}\tan\phi\,.\,x$ and $y=-\frac{b}{a}\cot\phi\,.\,x$, we have

$$LP=b\tan\phi \text{ and } L'P=b\cot\phi. \quad \therefore\ LP\,.\,PL'=b^2.$$

25. See Ex. 10. $CK=\frac{a^2b^2}{\sqrt{a^4k^2+b^4h^2}}=c.$ Hence etc.

26. The polar of $(a\cos\phi,\, b\sin\phi)$ with respect to the circle $x^2+y^2=r^2$ is $ax\cos\phi+by\sin\phi=r^2$, or

$$\frac{x}{r^2/a}\cos\phi+\frac{y}{r^2/b}\sin\phi=1,$$

which is a tangent to the ellipse $a^2x^2+b^2y^2=r^4$.

The equation of these lines is [Art. 122]

$$(x^2+y^2)\,r^2=(ax\cos\phi+by\sin\phi)^2$$

or $$x^2(a^2\cos^2\phi-r^2)+\ \dots\ +y^2(b^2\sin^2\phi-r^2)=0$$

which are conjugate diameters of the second ellipse if

$$\frac{a^2\cos^2\phi-r^2}{b^2\sin^2\phi-r^2}=-\frac{r^4}{b^2}\div\frac{r^4}{a^2}=-\frac{a^2}{b^2},\ \textit{i.e.}\text{ if }\frac{1}{r^2}=\frac{1}{a^2}+\frac{1}{b^2}.$$

27. See Art. 288. The m's of CP and CD are

$$\frac{b}{a}\tan\phi \text{ and } -\frac{b}{a}\cot\phi.$$

Hence the equations of the lines through P and D perpendicular respectively to CD and CP are

$$a\sin\phi\,(x-a\cos\phi)-b\cos\phi\,(y-b\sin\phi)=0,$$

and $$a\cos\phi\,(x+a\sin\phi)+b\sin\phi\,(y-b\cos\phi)=0,$$

or $$ax\sin\phi-by\cos\phi=(a^2-b^2)\sin\phi\cos\phi,$$

and $$ax\cos\phi+by\sin\phi=-(a^2-b^2)\sin\phi\cos\phi.$$

By adding, $$\frac{\sin\phi}{by-ax}=\frac{\cos\phi}{ax+by}=\frac{1}{\sqrt{2(a^2x^2+b^2y^2)}};$$

whence, substituting for $\cos\phi$ and $\sin\phi$,

$$2(b^2y^2+a^2x^2)^3=(a^2-b^2)^2(b^2y^2-a^2x^2)^2.$$

28. This is the same as Art. 288 (3).

293. *To prove that, in general, four normals can be drawn from any point to an ellipse, and that the sum of the eccentric angles of their feet is equal to an odd multiple of two right angles.*

The normal at any point, whose eccentric angle is ϕ, is

$$\frac{ax}{\cos\phi} - \frac{by}{\sin\phi} = a^2 - b^2 = a^2e^2.$$

If this normal pass through the point (h, k), we have

$$\frac{ah}{\cos\phi} - \frac{bk}{\sin\phi} = a^2e^2 \quad \text{.................} \quad (1).$$

For a given point (h, k) this equation gives the eccentric angles of the feet of the normals which pass through (h, k).

Let $\tan\frac{\phi}{2} = t$, so that

$$\cos\phi = \frac{1 - \tan^2\frac{\phi}{2}}{1 + \tan^2\frac{\phi}{2}} = \frac{1-t^2}{1+t^2}, \text{ and } \sin\phi = \frac{2\tan\frac{\phi}{2}}{1+\tan^2\frac{\phi}{2}} = \frac{2t}{1+t^2}.$$

Substituting these values in (1), we have

$$ah\frac{1+t^2}{1-t^2} - bk\frac{1+t^2}{2t} = a^2e^2,$$

i.e. $$bkt^4 + 2t^3(ah + a^2e^2) + 2t(ah - a^2e^2) - bk = 0 \ \ldots \ (2).$$

Let t_1, t_2, t_3, and t_4 be the roots of this equation, so that, by Art. 2,

$$t_1 + t_2 + t_3 + t_4 = -2\frac{ah + a^2e^2}{bk} \quad \text{...........} \quad (3),$$

$$t_1t_2 + t_1t_3 + t_1t_4 + t_2t_3 + t_2t_4 + t_3t_4 = 0 \quad \text{.........} \quad (4),$$

$$t_2t_3t_4 + t_3t_4t_1 + t_4t_1t_2 + t_1t_2t_3 = -2\frac{ah - a^2e^2}{bk} \quad \text{......} \quad (5),$$

and $$t_1t_2t_3t_4 = -1 \quad \text{......................} \quad (6).$$

Hence (*Trigonometry*, Art. 125), we have

$$\tan\left(\frac{\phi_1}{2}+\frac{\phi_2}{2}+\frac{\phi_3}{2}+\frac{\phi_4}{2}\right)=\frac{s_1-s_3}{1-s_2+s_4}=\frac{s_1-s_3}{0}=\infty .$$

$$\therefore\ \frac{\phi_1+\phi_2+\phi_3+\phi_4}{2}=n\pi+\frac{\pi}{2},$$

and hence $\phi_1+\phi_2+\phi_3+\phi_4=(2n+1)\pi$

= an odd multiple of two right angles.

294. We shall conclude the chapter with some examples of loci connected with the ellipse.

Ex. 1. *Find the locus of the intersection of tangents at the ends of chords of an ellipse, which are of constant length* $2c$.

Let QR be any such chord, and let the tangents at Q and R meet in a point P, whose coordinates are (h, k).

Since QR is the polar of P, its equation is

$$\frac{xh}{a^2}+\frac{yk}{b^2}=1 \quad\text{...............................(1).}$$

The abscissæ of the points in which this straight line meets the ellipse are given by

$$\left(1-\frac{xh}{a^2}\right)^2=\frac{k^2}{b^2}\left(1-\frac{x^2}{a^2}\right),$$

i.e.

$$\frac{x^2}{a^2}\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}\right)-\frac{2xh}{a^2}+1-\frac{k^2}{b^2}=0.$$

If x_1 and x_2 be the roots of this equation, *i.e.* the abscissæ of Q and R, we have

$$x_1+x_2=\frac{2a^2b^2h}{b^2h^2+a^2k^2},\quad\text{and}\quad x_1x_2=\frac{a^4(b^2-k^2)}{b^2h^2+a^2k^2}.$$

$$\therefore (x_1-x_2)^2=(x_1+x_2)^2-4x_1x_2=\frac{4a^4[b^2h^2+a^2k^2-a^2b^2]k^2}{(b^2h^2+a^2k^2)^2} \quad\text{...(2).}$$

If y_1 and y_2 be the ordinates of Q and R, we have from (1)

$$\frac{x_1h}{a^2}+\frac{y_1k}{b^2}=1,$$

and

$$\frac{x_2h}{a^2}+\frac{y_2k}{b^2}=1,$$

so that, by subtraction,

$$y_2-y_1=-\frac{b^2h}{a^2k}(x_2-x_1).$$

The condition of the question therefore gives

$$4c^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 = \left(1 + \frac{b^4h^2}{a^4k^2}\right)(x_2 - x_1)^2$$

$$= \frac{4(a^4k^2 + b^4h^2)(b^2h^2 + a^2k^2 - a^2b^2)}{(b^2h^2 + a^2k^2)^2}, \text{ by (2).}$$

Hence the point (h, k) always lies on the curve

$$c^2\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right) = \left(\frac{a^2y^2}{b^2} + \frac{b^2x^2}{a^2}\right)\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right),$$

which is therefore the locus of P.

Ex. 2. *Find the locus* (1) *of the middle points, and* (2) *of the poles, of normal chords of the ellipse.*

The chord, whose middle point is (h, k), is parallel to the polar of (h, k), and is therefore

$$(x - h)\frac{h}{a^2} + (y - k)\frac{k}{b^2} = 0 \quad \text{.......................(1).}$$

If this be a normal, it must be the same as

$$ax \sec\theta - by \operatorname{cosec}\theta = a^2 - b^2 \quad \text{.....................(2).}$$

We therefore have

$$\frac{a\sec\theta}{\frac{h}{a^2}} = \frac{-b\operatorname{cosec}\theta}{\frac{k}{b^2}} = \frac{a^2 - b^2}{\frac{h^2}{a^2} + \frac{k^2}{b^2}},$$

so that

$$\cos\theta = \frac{a^3}{h(a^2 - b^2)}\left(\frac{h^2}{a^2} + \frac{k^2}{b^2}\right),$$

and

$$\sin\theta = -\frac{b^3}{k(a^2 - b^2)}\left(\frac{h^2}{a^2} + \frac{k^2}{b^2}\right).$$

Hence, by the elimination of θ,

$$\left(\frac{a^6}{h^2} + \frac{b^6}{k^2}\right)\left(\frac{h^2}{a^2} + \frac{k^2}{b^2}\right)^2 = (a^2 - b^2)^2.$$

The equation to the required locus is therefore

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^2\left(\frac{a^6}{x^2} + \frac{b^6}{y^2}\right) = (a^2 - b^2)^2.$$

Again, if (x_1, y_1) be the pole of the normal chord (2), the latter equation must be equivalent to the equation

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1 \quad \text{.............................(3).}$$

Comparing (2) and (3), we have

$$\frac{a^3\sec\theta}{x_1} = -\frac{b^3\operatorname{cosec}\theta}{y_1} = a^2 - b^2,$$

so that

$$1 = \cos^2\theta + \sin^2\theta = \left(\frac{a^6}{x_1^{\,2}} + \frac{b^6}{y_1^{\,2}}\right)\frac{1}{(a^2 - b^2)^2},$$

and hence the required locus is

$$\frac{a^6}{x^2}+\frac{b^6}{y^2}=(a^2-b^2)^2.$$

Ex. 3. *Chords of the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ *always touch the concentric and coaxal ellipse* $\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}=1$; *find the locus of their poles.*

Any tangent to the second ellipse is

$$y=mx+\sqrt{\alpha^2m^2+\beta^2} \quad \text{.........................(1).}$$

Let the tangents at the points where it meets the first ellipse meet in (h, k). Then (1) must be the same as the polar of (h, k) with respect to the first ellipse, *i.e.* it is the same as

$$\frac{xh}{a^2}+\frac{yk}{b^2}-1=0 \quad \text{..........................(2).}$$

Since (1) and (2) coincide, we have

$$\frac{m}{\frac{h}{a^2}}=\frac{-1}{\frac{k}{b^2}}=\frac{\sqrt{\alpha^2m^2+\beta^2}}{-1}.$$

Hence $\qquad m=-\frac{b^2}{a^2}\frac{h}{k},\;$ and $\;\sqrt{\alpha^2m^2+\beta^2}=\frac{b^2}{k}.$

Eliminating m, we have

$$\alpha^2\frac{b^4}{a^4}\frac{h^2}{k^2}+\beta^2=\frac{b^4}{k^2},$$

i.e. the point (h, k) lies on the ellipse

$$\frac{\alpha^2}{a^4}x^2+\frac{\beta^2}{b^4}y^2=1,$$

i.e. on a concentric and coaxal ellipse whose semi-axes are $\frac{a^2}{\alpha}$ and $\frac{b^2}{\beta}$ respectively.

EXAMPLES XXXV

The tangents drawn from a point P to the ellipse make angles θ_1 and θ_2 with the major axis; find the locus of P when

1. $\theta_1+\theta_2$ is constant $(=2\alpha)$. [*Compare Ex.* 1, *Art.* 235.]
2. $\tan\theta_1+\tan\theta_2$ is constant $(=c)$.
3. $\tan\theta_1-\tan\theta_2$ is constant $(=d)$.
4. $\tan^2\theta_1+\tan^2\theta_2$ is constant $(=\lambda)$.

Find the locus of the intersection of tangents

5. which meet at a given angle α.

6. if the sum of the eccentric angles of their points of contact be equal to a constant angle 2α.

7. if the difference of these eccentric angles be 120°.

8. if the lines joining the points of contact to the centre be perpendicular.

9. if the sum of the ordinates of the points of contact be equal to b.

Find the locus of the middle points of chords of an ellipse

10. whose distance from the centre is the constant length c.

11. which subtend a right angle at the centre.

12. which pass through the given point (h, k).

13. whose length is constant $(=2c)$.

14. whose poles are on the auxiliary circle.

15. the tangents at the ends of which intersect at right angles.

16. Prove that the locus of the intersection of normals at the ends of conjugate diameters is the curve

$$2(a^2x^2+b^2y^2)^3=(a^2-b^2)^2(a^2x^2-b^2y^2)^2.$$

17. Prove that the locus of the intersection of normals at the ends of chords, parallel to the tangent at the point whose eccentric angle is α, is the conic

$$2(ax\sin\alpha+by\cos\alpha)(ax\cos\alpha+by\sin\alpha)=(a^2-b^2)^2\sin 2\alpha\cos^2 2\alpha.$$

If the chords be parallel to an equiconjugate diameter, the locus is a diameter perpendicular to the other equiconjugate.

18. A parallelogram circumscribes the ellipse and two of its opposite angular points lie on the straight lines $x^2=h^2$; prove that the locus of the other two is the conic

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}\left(1-\frac{a^2}{h^2}\right)=1.$$

19. Circles of constant radius c are drawn to pass through the ends of a variable diameter of the ellipse. Prove that the locus of their centres is the curve

$$(x^2+y^2)(a^2x^2+b^2y^2+a^2b^2)=c^2(a^2x^2+b^2y^2).$$

20. The polar of a point P with respect to an ellipse touches a fixed circle, whose centre is on the major axis and which passes through the centre of the ellipse. Shew that the locus of P is a parabola, whose latus rectum is a third proportional to the diameter of the circle and the latus rectum of the ellipse.

21. Prove that the locus of the pole, with respect to the ellipse, of any tangent to the auxiliary circle is the curve $\dfrac{x^2}{a^4}+\dfrac{y^2}{b^4}=\dfrac{1}{a^2}$.

22. Shew that the locus of the pole, with respect to the auxiliary circle, of a tangent to the ellipse is a similar concentric ellipse, whose major axis is at right angles to that of the original ellipse.

23. Chords of the ellipse touch the parabola $ay^2 = -2b^2x$; prove that the locus of their poles is the parabola $ay^2 = 2b^2x$.

24. Prove that the sum of the angles that the four normals drawn from any point to an ellipse make with the axis is equal to the sum of the angles that the two tangents from the same point make with the axis.

[Use the equation of Art. 268.]

25. Triangles are formed by pairs of tangents drawn from any point on the ellipse

$$a^2x^2 + b^2y^2 = (a^2+b^2)^2 \text{ to the ellipse } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

and their chord of contact. Prove that the orthocentre of each such triangle lies on the ellipse.

26. An ellipse is rotated through a right angle in its own plane about its centre, which is fixed; prove that the locus of the point of intersection of a tangent to the ellipse in its original position with the tangent at the same point of the curve in its new position is

$$(x^2+y^2)(x^2+y^2-a^2-b^2) = 2(a^2-b^2)xy.$$

27. If Y and Z be the feet of the perpendiculars from the foci upon the tangent at any point P of an ellipse, prove that the tangents at Y and Z to the auxiliary circle meet on the ordinate of P and that the locus of their point of intersection is another ellipse.

28. Prove that the directrices of the two parabolas that can be drawn to have their foci at any given point P of the ellipse and to pass through its foci meet at an angle which is equal to twice the eccentric angle of P.

29. Chords at right angles are drawn through any point P of the ellipse, and the line joining their extremities meets the normal in the point Q. Prove that Q is the same for all such chords, its coordinates being $\dfrac{a^3e^2\cos\alpha}{a^2+b^2}$ and $\dfrac{-a^2be^2\sin\alpha}{a^2+b^2}$.

Prove also that the major axis is the bisector of the angle PCQ, and that the locus of Q for different positions of P is the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \left(\frac{a^2-b^2}{a^2+b^2}\right)^2.$$

ANSWERS

1. $x^2 - 2xy \cot 2\alpha - y^2 = a^2 - b^2$.

2. $cx^2 - 2xy = ca^2$.

3. $d^2(x^2 - a^2)^2 = 4(b^2x^2 + a^2y^2 - a^2b^2)$.

4. $\lambda(x^2 - a^2)^2 = 2(x^2y^2 + b^2x^2 + a^2y^2 - a^2b^2)$.

5. $(x^2 + y^2 - a^2 - b^2)^2 = 4\cot^2\alpha\,(b^2x^2 + a^2y^2 - a^2b^2)$.

6. $ay = bx \tan\alpha$.

7. $b^2x^2 + a^2y^2 = 4a^2b^2$.

8. $b^4x^2 + a^4y^2 = a^2b^2(a^2 + b^2)$.

9. $b^2x^2 + a^2y^2 = 2a^2by$.

10. $(b^2x^2 + a^2y^2)^2 = c^2(b^4x^2 + a^4y^2)$.

11. $(a^2 + b^2)(b^2x^2 + a^2y^2)^2 = a^2b^2(b^4x^2 + a^4y^2)$.

12. $b^2x(x - h) + a^2y(y - k) = 0$.

13. $c^2a^2b^2(b^2x^2 + a^2y^2) + (b^2x^2 + a^2y^2 - 1)(b^4x^2 + a^4y^2) = 0$.

14. $(b^2x^2 + a^2y^2)^2 = a^2b^4(x^2 + y^2)$.

15. $a^4b^4(x^2 + y^2) = (a^2 + b^2)(b^2x^2 + a^2y^2)^2$.

29. If the chords be PK and PK', let the equation to KK' be $y = mx + c$; transform the origin to P and, by means of Art. 122, find the condition that the angle KPK' is a right angle; substitute for c in the equation to KK', and find the point of intersection of KK' and the normal at P. See also Art. 404.

SOLUTIONS/HINTS

In Exs. 1 to 5, $\tan\theta_1$ and $\tan\theta_2$ are the roots of the equation $m^2(x^2 - a^2) - 2mxy + (y^2 - b^2) = 0$. [Art. 272].

1. $$\tan 2\alpha = \tan(\theta_1 + \theta_2) = \frac{\tan\theta_1 + \tan\theta_2}{1 - \tan\theta_1\tan\theta_2} = \frac{2xy}{x^2 - y^2 - a^2 + b^2}.$$

2. $$c = \tan\theta_1 + \tan\theta_2 = \frac{2xy}{x^2 - a^2}.$$

3. $d^2 = (\tan\theta_1 - \tan\theta_2)^2 = (\tan\theta_1 + \tan\theta_2)^2 - 4\tan\theta_1\tan\theta_2$.

$\therefore d^2(x^2 - a^2)^2 = 4\{x^2y^2 - (x^2 - a^2)(y^2 - b^2)\} = 4(b^2x^2 + a^2y^2 - a^2b^2)$.

4. $\lambda = \tan^2\theta_1 + \tan^2\theta_2 = (\tan\theta_1 + \tan\theta_2)^2 - 2\tan\theta_1\tan\theta_2$.

$$\therefore \lambda(x^2 - a^2)^2 = 2\{2x^2y^2 - (x^2 - a^2)(y^2 - b^2)\} = 2(x^2y^2 + b^2x^2 + a^2y^2 - a^2b^2).$$

5. $\tan^2\alpha = \tan^2(\theta_1 - \theta_2) = \dfrac{(\tan\theta_1 - \tan\theta_2)^2}{(1 + \tan\theta_1 \tan\theta_2)^2}$

$$= \frac{4\,(b^2x^2 + a^2y^2 - a^2b^2)}{(x^2 + y^2 - a^2 - b^2)^2}.$$

6. See Art. 265, Ex.

$$x = \frac{a\cos\frac{1}{2}(\phi_1 + \phi_2)}{\cos\frac{1}{2}(\phi_1 - \phi_2)} = \frac{a\cos\alpha}{\cos\frac{1}{2}(\phi_1 - \phi_2)},$$

$$y = \frac{b\sin\frac{1}{2}(\phi_1 + \phi_2)}{\cos\frac{1}{2}(\phi_1 - \phi_2)} = \frac{b\sin\alpha}{\cos\frac{1}{2}(\phi_1 - \phi_2)}.$$

$$\therefore\ ay = bx\tan\alpha.$$

7. See Art. 265, Ex. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = \dfrac{1}{\cos^2\frac{1}{2}(\phi_1 - \phi_2)} = 4.$

8. The equation of the lines joining the centre to the common points of the ellipse and

$$\frac{xx'}{a^2} + \frac{yy'}{b^2} = 1 \text{ is [Art. 122]}$$

$$\left\{\frac{xx'}{a^2} + \frac{yy'}{b^2}\right\}^2 = \frac{x^2}{a^2} + \frac{y^2}{b^2}.$$

Since these are at right angles, the locus of (x', y') is

$$\frac{x^2}{a^4} + \frac{y^2}{b^4} = \frac{1}{a^2} + \frac{1}{b^2}.$$

9. We have to eliminate α and β from

$$x = \frac{a\cos\frac{1}{2}(\alpha + \beta)}{\cos\frac{1}{2}(\alpha - \beta)}, \ \ldots\text{(i)} \qquad y = \frac{b\sin\frac{1}{2}(\alpha + \beta)}{\cos\frac{1}{2}(\alpha - \beta)}, \ \ldots\text{(ii)}$$

and $\qquad b\sin\alpha + b\sin\beta = b,$

i.e. $\qquad 2\sin\frac{1}{2}(\alpha + \beta)\cos\frac{1}{2}(\alpha - \beta) = 1. \quad \ldots\ldots\ldots\text{(iii)}$

Multiply (i) and (ii) by (iii);

$$\therefore\ x = 2a\sin\tfrac{1}{2}(\alpha + \beta)\cos\tfrac{1}{2}(\alpha + \beta); \quad \therefore\ \frac{x}{a} = \sin(\alpha + \beta);$$

and $\qquad y = 2b\sin^2\frac{1}{2}(\alpha + \beta). \qquad \therefore\ \dfrac{b - y}{b} = \cos(\alpha + \beta).$

$$\therefore \frac{(y-b)^2}{b^2} + \frac{x^2}{a^2} = 1, \text{ or } \frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{2y}{b}.$$

10. Taking the equation of Art. 280, viz.

$$b^2hx + a^2ky = b^2h^2 + a^2k^2,$$

we have $$c^2(b^4h^2 + a^4k^2) = (b^2h^2 + a^2k^2)^2,$$

so that the locus of (h, k) is

$$c^2(b^4x^2 + a^4y^2) = (b^2x^2 + a^2y^2)^2.$$

11. Taking the equation of Art. 280, the equation of the lines joining the centre to the common points of this line and the ellipse is [Art. 122]

$$a^2b^2(b^2hx + a^2ky)^2 = (b^2h^2 + a^2k^2)^2(b^2x^2 + a^2y^2).$$

If these are at right angles, then

$$a^2b^2(b^4h^2 + a^4k^2) = (b^2h^2 + a^2k^2)^2(a^2 + b^2). \quad \text{[Art. 111.]}$$

Hence the equation to the locus.

12. Interchanging x and h, and y and k in the equation of Art. 280, we have

$$\frac{y}{b^2}(y-k) + \frac{x}{a^2}(x-h) = 0.$$

13. From Arts. 261 and 280, we have

$$c = \frac{ab\sqrt{b^4h^2 + a^4k^2} \cdot \sqrt{a^2b^4h^2 + b^2a^4k^2 - (b^2h^2 + a^2k^2)^2}}{a^2b^2(b^2h^2 + a^2k^2)}.$$

$$\therefore c^2a^2b^2(b^2h^2 + a^2k^2)^2$$
$$= (b^4h^2 + a^4k^2)(b^2h^2 + a^2k^2)(a^2b^2 - b^2h^2 - a^2k^2),$$

or $$c^2a^2b^2(b^2h^2 + a^2k^2) + (b^4h^2 + a^4k^2)(b^2h^2 + a^2k^2 - a^2b^2) = 0.$$

Aliter. If θ be the inclination to the axis of x of the chord whose middle point is (h, k) then the point

$$(h + c\cos\theta,\ k + c\sin\theta),$$

and also the point $(h - c\cos\theta,\ k - c\sin\theta)$, lies on the ellipse.

$$\therefore b^2(h + c\cos\theta)^2 + a^2(k + c\sin\theta)^2 = a^2b^2,$$

and $$b^2(h - c\cos\theta)^2 + a^2(k - c\sin\theta)^2 = a^2b^2.$$

Hence, by adding and subtracting, we have

$$b^2h^2 + a^2k^2 - a^2b^2 + c^2(b^2\cos^2\theta + a^2\sin^2\theta) = 0, \quad \ldots(1)$$

and

$$b^2h\cos\theta + a^2k\sin\theta = 0. \quad \ldots\ldots\ldots\ldots(2)$$

(2) gives $$\frac{\cos\theta}{-a^2k} = \frac{\sin\theta}{b^2h} = \frac{1}{\sqrt{a^4k^2 + b^4h^2}}.$$

Substitute these values in (1).

14. The pole of the line $\frac{xh}{a^2} + \frac{yk}{b^2} = \frac{h^2}{a^2} + \frac{k^2}{b^2}$ is

$$\left[h\Big/\left(\frac{h^2}{a^2} + \frac{k^2}{b^2}\right),\ k\Big/\left(\frac{h^2}{a^2} + \frac{k^2}{b^2}\right)\right].$$

$$\therefore\ h^2 + k^2 = a^2\left(\frac{h^2}{a^2} + \frac{k^2}{b^2}\right)^2.$$

15. The same point must lie on the director circle.

$$\therefore\ h^2 + k^2 = (a^2 + b^2)\left(\frac{h^2}{a^2} + \frac{k^2}{b^2}\right)^2.$$

16. We have to eliminate ϕ between

$$ax\sec\phi - by\operatorname{cosec}\phi = a^2 - b^2, \quad \ldots\ldots\ldots\ldots(1)$$

and

$$-ax\operatorname{cosec}\phi - by\sec\phi = a^2 - b^2. \quad \ldots\ldots\ldots\ldots(2)$$

Multiply (1) by by and (2) by ax, and we have, on addition,

$$-(b^2y^2 + a^2x^2)\operatorname{cosec}\phi = (a^2 - b^2)(ax + by).$$

Multiply (1) by ax and (2) by by and we have, on subtraction,

$$(b^2y^2 + a^2x^2)\sec\phi = (a^2 - b^2)(ax - by).$$

$$\therefore\ \frac{1}{(ax + by)^2} + \frac{1}{(ax - by)^2} = \frac{(a^2 - b^2)^2}{(b^2y^2 + a^2x^2)^2}.$$

$$\therefore\ 2(b^2y^2 + a^2x^2)^3 = (a^2 - b^2)^2(a^2x^2 - b^2y^2)^2.$$

17. Since the sum of the eccentric angles of the ends of a chord $= 2\alpha$ (Ex. XXXIII. 33), we have to eliminate θ between

$$ax\sec(\alpha + \theta) - by\operatorname{cosec}(\alpha + \theta) = (a^2 - b^2),$$

and

$$ax\sec(\alpha - \theta) - by\operatorname{cosec}(\alpha - \theta) = (a^2 - b^2).$$

These equations become

$$\cos\theta(ax\sin\alpha - by\cos\alpha) + \sin\theta(ax\cos\alpha + by\sin\alpha)$$
$$= \tfrac{1}{2}(a^2 - b^2)\sin(2\alpha + 2\theta),$$

and $\cos\theta\,(ax\sin\alpha - by\cos\alpha) - \sin\theta\,(ax\cos\alpha + by\sin\alpha)$

$$= \tfrac{1}{2}(a^2 - b^2)\sin(2\alpha - 2\theta).$$

Adding,

$$2\cos\theta\,(ax\sin\alpha - by\cos\alpha) = (a^2 - b^2)\sin 2\alpha\cos 2\theta. \ldots(1)$$

Subtracting,

$$2\sin\theta\,(ax\cos\alpha + by\sin\alpha) = (a^2 - b^2)\cos 2\alpha\sin 2\theta,$$

whence $\cos\theta\,.\,\cos 2\alpha\,(a^2 - b^2) = ax\cos\alpha + by\sin\alpha.$

Substitute in (1) for $\cos\theta$;

$$\therefore\ 2(ax\cos\alpha + by\sin\alpha)(ax\sin\alpha - by\cos\alpha)\cos 2\alpha$$

$$= \sin 2\alpha\,\{2(ax\cos\alpha + by\sin\alpha)^2 - (a^2 - b^2)^2\cos^2 2\alpha\};$$

$$\therefore\ 2(ax\cos\alpha + by\sin\alpha)\{(ax\cos\alpha + by\sin\alpha)\sin 2\alpha$$

$$- (ax\sin\alpha - by\cos\alpha)\cos 2\alpha\} = (a^2 - b^2)^2\sin 2\alpha\cos^2 2\alpha;$$

$$\therefore\ 2(ax\cos\alpha + by\sin\alpha)(ax\sin\alpha + by\cos\alpha)$$

$$= (a^2 - b^2)^2\sin 2\alpha\cos^2 2\alpha.$$

If $\alpha = 45°$, this becomes $ax + by = 0$ and this is a diameter perpendicular to the other equiconjugate

$$ay - bx = 0.$$

18. Let α, β, $\pi + \alpha$, $\pi + \beta$ be the eccentric angles of the points of contact. Then if (x, y) be a point on the locus

$$h\cos\tfrac{1}{2}(\alpha - \beta) = a\cos\tfrac{1}{2}(\alpha + \beta), \ \ldots\ldots\ldots(\text{i})$$

$$x\sin\tfrac{1}{2}(\alpha - \beta) = a\sin\tfrac{1}{2}(\alpha + \beta), \ \ldots\ldots\ldots(\text{ii})$$

and

$$y\sin\tfrac{1}{2}(\alpha - \beta) = -b\cos\tfrac{1}{2}(\alpha + \beta). \ \ldots\ldots\ldots(\text{iii})$$

From (i) and (iii), $\cot\tfrac{1}{2}(\alpha - \beta) = -\dfrac{ay}{bh}$,

and from (ii) and (iii),

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \operatorname{cosec}^2\tfrac{1}{2}(\alpha - \beta) = 1 + \cot^2\frac{\alpha - \beta}{2} = 1 + \frac{a^2y^2}{b^2h^2}.$$

$$\therefore\ \frac{x^2}{a^2} + \frac{y^2}{b^2}\left(1 - \frac{a^2}{h^2}\right) = 1.$$

19. Let $(a\cos\phi,\ b\sin\phi)$ be one extremity of the diameter. Then the centres lie on the perpendicular diameter, viz. $\tan\phi = -\dfrac{ax}{by}$. $\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$

Also if (x, y) be the coordinates of a centre,

$$(x - a\cos\phi)^2 + (y - b\sin\phi)^2 = c^2.$$

Substituting for $\sin\phi$ and $\cos\phi$, from (1), we have

$$(x^2 + y^2 - c^2)(a^2x^2 + b^2y^2) + a^2b^2y^2 + a^2b^2x^2 = 0;$$

$$\therefore (x^2 + y^2)(a^2x^2 + b^2y^2 + a^2b^2) = c^2(a^2x^2 + b^2y^2).$$

20. Let $(x-c)^2 + y^2 = c^2$ be the equation of the circle. The equation of a tangent is $(x-c)\cos\theta + y\sin\theta = c$.

If this is identical with $b^2hx + a^2yk = a^2b^2$, we must have

$$\frac{\cos\theta}{b^2h} = \frac{\sin\theta}{a^2k} = \frac{c(1+\cos\theta)}{a^2b^2}.$$

Whence $\cos\theta = \dfrac{b^2ch}{b^2(a^2 - ch)}$, and $\sin\theta = \dfrac{a^2ck}{b^2(a^2 - ch)}$.

$$\therefore c^2(b^4h^2 + a^4k^2) = b^4(a^2 - ch)^2.$$

Hence the equation to the locus of (h, k) is

$$y^2 = \frac{b^4}{c^2} - \frac{2b^4}{a^2c}\cdot x,$$

which is a parabola; and, if l be its latus rectum,

$$2cl = 4\frac{b^4}{a^2} = \left(2\frac{b^2}{a}\right)^2.$$

21. The lines

$$x\cos\theta + y\sin\theta = a, \text{ and } b^2hx + a^2yk = a^2b^2,$$

are identical if $\dfrac{\cos\theta}{b^2h} = \dfrac{\sin\theta}{a^2k} = \dfrac{1}{ab^2}$.

Hence the locus of (h, k) is given by $b^4x^2 + a^4y^2 = a^2b^4$.

22. The lines

$$bx\cos\phi + ay\sin\phi = ab, \text{ and } xh + yk = a^2,$$

are identical if $\dfrac{b\cos\phi}{h} = \dfrac{a\sin\phi}{k} = \dfrac{b}{a}$.

$\therefore a^2h^2 + b^2k^2 = a^4$; which is a concentric ellipse, with its major axis along the axis of y and ratio of axes $= \dfrac{a}{b}$.

23. The lines

$$b^2hx + a^2ky = a^2b^2, \text{ and } my - m^2x = -\frac{b^2}{2a},$$

are identical if $\quad \dfrac{m^2}{b^2h} = -\dfrac{m}{a^2k} = \dfrac{1}{2a^3}$;

eliminating m, we have, for the locus of (h, k), $ay^2 = 2b^2x$.

24. From equation (2) of Art. 272 if $\tan\theta_1$ and $\tan\theta_2$ be the roots,

$$\tan\theta_1 + \tan\theta_2 = \frac{2x_1y_1}{x_1^2 - a^2}, \text{ and } \tan\theta_1\tan\theta_2 = \frac{y_1^2 - b^2}{x_1^2 - a^2}.$$

$$\therefore \tan(\theta_1 + \theta_2) = \frac{2x_1y_1}{x_1^2 - y_1^2 - a^2 + b^2}.$$

The equation of Art. 268 when rationalized is

$$m^4b^2x^2 - 2m^3b^2xy + m^2\{a^2x^2 + b^2y^2 - (a^2 - b^2)^2\} - 2a^2xym + a^2y^2 = 0.$$

If m_1, m_2, m_3, m_4 be the roots,

$$\text{the sum of the angles} = \tan^{-1}\frac{\Sigma m_1 - \Sigma m_1m_2m_3}{1 - \Sigma m_1m_2 + m_1m_2m_3m_4}$$

$$= \tan^{-1}\frac{2b^2xy - 2a^2xy}{b^2x^2 - \{a^2x^2 + b^2y^2 - (a^2 - b^2)^2\} + a^2y^2}$$

$$= \tan^{-1}\frac{2xy}{x^2 - y^2 - a^2 + b^2} = \theta_1 + \theta_2,$$

if the point (x, y) coincide with the point (x_1, y_1).

25. Let α and β be the eccentric angles of the points of contact. Then since the pole of the chord of contact, viz.

$$\left\{a\frac{\cos\frac{1}{2}(\alpha + \beta)}{\cos\frac{1}{2}(\alpha - \beta)},\ b\frac{\sin\frac{1}{2}(\alpha + \beta)}{\cos\frac{1}{2}(\alpha - \beta)}\right\}$$

lies on the first ellipse,

$$\therefore a^4\cos^2\tfrac{1}{2}(\alpha + \beta) + b^4\sin^2\tfrac{1}{2}(\alpha + \beta) = (a^2 + b^2)^2\cos^2\tfrac{1}{2}(\alpha - \beta), \quad \ldots\text{(i)}$$

$$\therefore a^4\{1 + \cos(\alpha + \beta)\} + b^4\{1 - \cos(\alpha + \beta)\} = (a^2 + b^2)^2\{1 + \cos(\alpha - \beta)\},$$

or $$a^2\sin\alpha\sin\beta + b^2\cos\alpha\cos\beta + \frac{a^2b^2}{a^2 + b^2} = 0. \quad \ldots\ldots\text{(ii)}$$

The orthocentre is the intersection of the lines

$$a\sin\alpha\,(x - a\cos\beta) - b\cos\alpha\,(y - b\sin\beta) = 0,$$

and $$a\sin\beta\,(x - a\cos\alpha) - b\cos\beta\,(y - b\sin\alpha) = 0.$$

Solving, $\quad ax\sin(\alpha-\beta)$

$= a^2(\sin\alpha\cos^2\beta - \sin\beta\cos^2\alpha) - b^2\cos\alpha\cos\beta(\sin\beta - \sin\alpha)$

$= a^2(\sin\alpha\cos^2\beta - \sin\beta\cos^2\alpha) + \left(a^2\sin\alpha\sin\beta + \dfrac{a^2b^2}{a^2+b^2}\right)(\sin\beta - \sin\alpha)$ by (ii),

$= a^2\sin\alpha - a^2\sin\beta + \dfrac{a^2b^2}{a^2+b^2}(\sin\beta - \sin\alpha)$,

$= \dfrac{a^4}{a^2+b^2}\cdot(\sin\alpha - \sin\beta)$.

$$\therefore\ x = \frac{a^3}{a^2+b^2}\cdot\frac{\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}};\ \text{similarly } y = \frac{b^3}{a^2+b^2}\cdot\frac{\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}.$$

Now (i) is the condition that this point should lie on the second ellipse.

26. Let $x\cos\alpha + y\sin\alpha = \sqrt{a^2\cos^2\alpha + b^2\sin^2\alpha}$ ...(1) be the equation to any tangent.

Its second position is at right angles to its former position, and is at the same distance from the origin.

Hence its equation is

$$x\cos(90^\circ+\alpha) + y\sin(90^\circ+\alpha) = \sqrt{a^2\cos^2\alpha + b^2\sin^2\alpha},$$

i.e. $\quad y\cos\alpha - x\sin\alpha = \sqrt{a^2\cos^2\alpha + b^2\sin^2\alpha}$. ....(2)

Subtracting (2) from (1), $\sin\alpha(x+y) = \cos\alpha(y-x)$.

Substituting for $\sin\alpha$ and $\cos\alpha$ in (1), we have

$$x(y+x) + y(y-x) = \sqrt{a^2(x+y)^2 + b^2(y-x)^2},$$

i.e. $\quad (x^2+y^2)^2 = (x^2+y^2)(a^2+b^2) + 2xy(a^2-b^2)$,

or $\quad (x^2+y^2)(x^2+y^2-a^2-b^2) = 2xy(a^2-b^2)$.

27. The second part is the same as No. 22.

Since, in the solution of that question,

$$\frac{b\cos\phi}{h} = \frac{b}{a},\ \therefore\ h = a\cos\phi.$$

28. Let $x\cos\alpha + y\sin\alpha = p$ be the equation to a directrix, and ϕ the eccentric angle of P. Now the perpendicular from S on the directrix $= SP$.

$$\therefore\ p - ae\cos\alpha = a - ae\cos\phi.$$

Similarly, $$p + ae\cos\alpha = a + ae\cos\phi.$$

Subtracting, $$\cos\alpha = \cos\phi,$$

so that the two values of α are ϕ and $-\phi$; hence the angle between the directrices $= 2\phi$.

29. Removing the origin to the point $(a\cos\alpha,\ b\sin\alpha)$, the equation to the ellipse becomes

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = -\left\{\frac{2x\cos\alpha}{a} + \frac{2y\sin\alpha}{b}\right\} \quad \text{.........(i)}$$

and the equation to the normal at "α" becomes

$$ax\sec\alpha = by\,\text{cosec}\,\alpha. \quad \text{..................(ii)}$$

The equation to the lines joining the origin to the common points of $lx + my = 1$ and the ellipse (i) is [Art. 122]

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = -2\,(lx + my)\left(\frac{x\cos\alpha}{a} + \frac{y\sin\alpha}{b}\right).$$

These are at right angles if

$$\frac{1}{a^2} + \frac{1}{b^2} = -\left\{\frac{2l\cos\alpha}{a} + \frac{2m\sin\alpha}{b}\right\}. \quad \text{............(iii)}$$

Solving (ii) with $lx + my = 1$, we obtain

$$x = \frac{b\cos\alpha}{lb\cos\alpha + ma\sin\alpha} = -\frac{2ab^2\cos\alpha}{a^2 + b^2},$$

and $$y = \frac{a\sin\alpha}{lb\cos\alpha + ma\sin\alpha} = -\frac{2a^2b\sin\alpha}{a^2 + b^2}, \text{ by (iii).}$$

The coordinates of this point referred to the original axes are $$x = a\cos\alpha - \frac{2ab^2\cos\alpha}{a^2 + b^2} = \frac{a^3e^2\cos\alpha}{a^2 + b^2},$$

and $$y = b\sin\alpha - \frac{2a^2b\sin\alpha}{a^2 + b^2} = -\frac{a^2be^2\sin\alpha}{a^2 + b^2}.$$

Eliminating α we obtain the required locus.

Also the "m" of $CP = \frac{b}{a}\tan\alpha$, and of $CQ = -\frac{b}{a}\tan\alpha$.