PUZZLER

One of the most popular early bicycles was the penny–farthing, introduced in 1870. The bicycle was so named because the size relationship of its two wheels was about the same as the size relationship of the penny and the farthing, two English coins. When the rider looks down at the top of the front wheel, he sees it moving forward faster than he and the handlebars are moving. Yet the center of the wheel does not appear to be moving at all relative to the handlebars. How can different parts of the rolling wheel move at different linear speeds? *(© Steve Lovegrove/Tasmanian Photo Library)*

chapter 11

Rolling Motion and Angular Momentum

Chapter Outline

In the preceding chapter we learned how to treat a rigid body rotating about a fixed axis; in the present chapter, we move on to the more general case in which the axis of rotation is not fixed in space. We begin by describing such motion, which is called *rolling motion*. The central topic of this chapter is, however, angular momentum, a quantity that plays a key role in rotational dynamics. In analogy to the conservation of linear momentum, we find that the angular momentum of a rigid object is always conserved if no external torques act on the object. Like the law of conservation of linear momentum, the law of conservation of angular momentum is a fundamental law of physics, equally valid for relativistic and quantum systems.

11.1 ROLLING MOTION OF A RIGID OBJECT

7.1

In this section we treat the motion of a rigid object rotating about a moving axis. In general, such motion is very complex. However, we can simplify matters by restricting our discussion to a homogeneous rigid object having a high degree of symmetry, such as a cylinder, sphere, or hoop. Furthermore, we assume that the object undergoes rolling motion along a flat surface. We shall see that if an object such as a cylinder rolls without slipping on the surface (we call this *pure rolling motion*), a simple relationship exists between its rotational and translational motions.

Suppose a cylinder is rolling on a straight path. As Figure 11.1 shows, the center of mass moves in a straight line, but a point on the rim moves in a more complex path called a *cycloid*. This means that the axis of rotation remains parallel to its initial orientation in space. Consider a uniform cylinder of radius R rolling without slipping on a horizontal surface (Fig. 11.2). As the cylinder rotates through an angle θ, its center of mass moves a linear distance $s = R\theta$ (see Eq. 10.1a). Therefore, the linear speed of the center of mass for pure rolling motion is given by

$$v_{\text{CM}} = \frac{ds}{dt} = R\frac{d\theta}{dt} = R\omega \qquad \textbf{(11.1)}$$

where ω is the angular velocity of the cylinder. Equation 11.1 holds whenever a cylinder or sphere rolls without slipping and is the **condition for pure rolling**

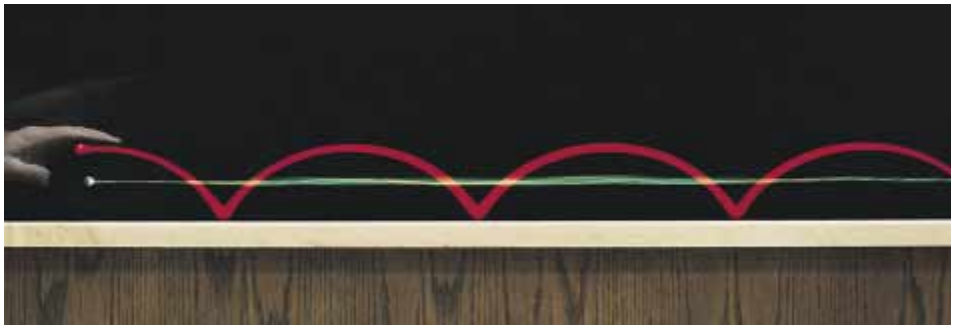


Figure 11.1 One light source at the center of a rolling cylinder and another at one point on the rim illustrate the different paths these two points take. The center moves in a straight line (green line), whereas the point on the rim moves in the path called a *cycloid* (red curve). *(Henry Leap and Jim Lehman)*

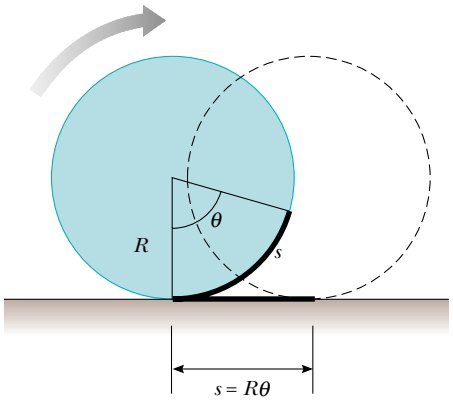


Figure 11.2 In pure rolling motion, as the cylinder rotates through an angle θ, its center of mass moves a linear distance $s = R\theta$.

motion. The magnitude of the linear acceleration of the center of mass for pure rolling motion is

$$a_{\text{CM}} = \frac{dv_{\text{CM}}}{dt} = R\frac{d\omega}{dt} = R\alpha \tag{11.2}$$

where α is the angular acceleration of the cylinder.

The linear velocities of the center of mass and of various points on and within the cylinder are illustrated in Figure 11.3. A short time after the moment shown in the drawing, the rim point labeled P will have rotated from the six o'clock position to, say, the seven o'clock position, the point Q will have rotated from the ten o'clock position to the eleven o'clock position, and so on. Note that the linear velocity of any point is in a direction perpendicular to the line from that point to the contact point P. At any instant, the part of the rim that is at point P is at rest relative to the surface because slipping does not occur.

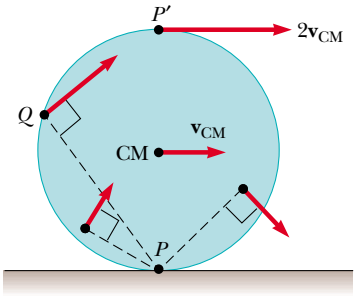


Figure 11.3 All points on a rolling object move in a direction perpendicular to an axis through the instantaneous point of contact P. In other words, all points rotate about P. The center of mass of the object moves with a velocity $\mathbf{v}_{\text{CM}}$, and the point P' moves with a velocity $2\mathbf{v}_{\text{CM}}$.

All points on the cylinder have the same angular speed. Therefore, because the distance from P' to P is twice the distance from P to the center of mass, P' has a speed $2v_{\text{CM}} = 2R\omega$. To see why this is so, let us model the rolling motion of the cylinder in Figure 11.4 as a combination of translational (linear) motion and rotational motion. For the pure translational motion shown in Figure 11.4a, imagine that the cylinder does not rotate, so that each point on it moves to the right with speed v_{CM}. For the pure rotational motion shown in Figure 11.4b, imagine that a rotation axis through the center of mass is stationary, so that each point on the cylinder has the same rotational speed ω. The combination of these two motions represents the rolling motion shown in Figure 11.4c. Note in Figure 11.4c that the top of the cylinder has linear speed $v_{\text{CM}} + R\omega = v_{\text{CM}} + v_{\text{CM}} = 2v_{\text{CM}}$, which is greater than the linear speed of any other point on the cylinder. As noted earlier, the center of mass moves with linear speed v_{CM} while the contact point between the surface and cylinder has a linear speed of zero.

7.2

We can express the total kinetic energy of the rolling cylinder as

$$K = \tfrac{1}{2}I_P\omega^2 \tag{11.3}$$

where I_P is the moment of inertia about a rotation axis through P. Applying the parallel-axis theorem, we can substitute $I_P = I_{\text{CM}} + MR^2$ into Equation 11.3 to obtain

$$K = \tfrac{1}{2}I_{\text{CM}}\omega^2 + \tfrac{1}{2}MR^2\omega^2$$

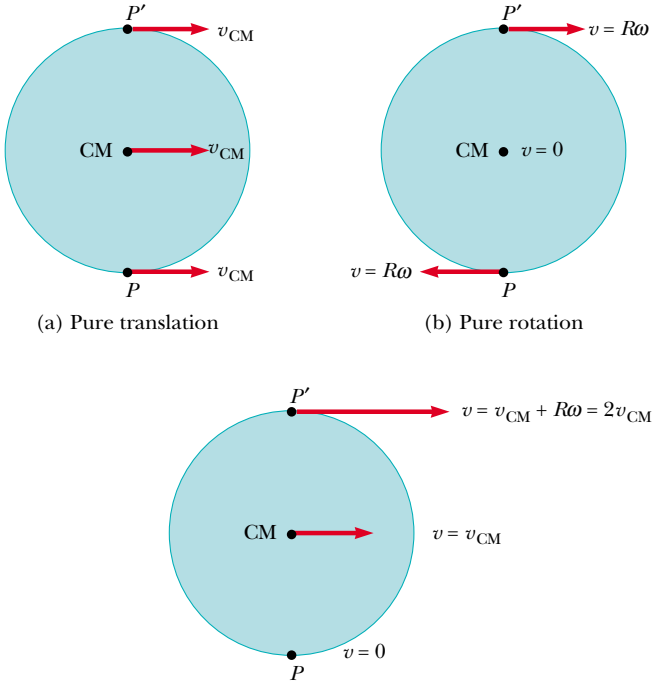


Figure 11.4 The motion of a rolling object can be modeled as a combination of pure translation and pure rotation.

or, because $v_{CM} = R\omega$,

Total kinetic energy of a rolling body

$$K = \tfrac{1}{2}I_{CM}\omega^2 + \tfrac{1}{2}Mv_{CM}^2 \qquad (11.4)$$

The term $\frac{1}{2}I_{CM}\omega^2$ represents the rotational kinetic energy of the cylinder about its center of mass, and the term $\frac{1}{2}Mv_{CM}^2$ represents the kinetic energy the cylinder would have if it were just translating through space without rotating. Thus, we can say that the **total kinetic energy of a rolling object is the sum of the rotational kinetic energy about the center of mass and the translational kinetic energy of the center of mass.**

We can use energy methods to treat a class of problems concerning the rolling motion of a sphere down a rough incline. (The analysis that follows also applies to the rolling motion of a cylinder or hoop.) We assume that the sphere in Figure 11.5 rolls without slipping and is released from rest at the top of the incline. Note that accelerated rolling motion is possible only if a frictional force is present between the sphere and the incline to produce a net torque about the center of mass. Despite the presence of friction, no loss of mechanical energy occurs because the contact point is at rest relative to the surface at any instant. On the other hand, if the sphere were to slip, mechanical energy would be lost as motion progressed.

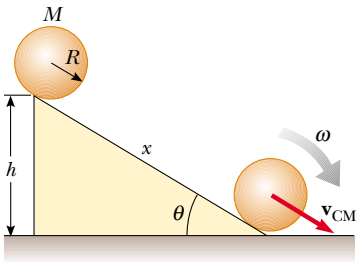


Figure 11.5 A sphere rolling down an incline. Mechanical energy is conserved if no slipping occurs.

Using the fact that $v_{CM} = R\omega$ for pure rolling motion, we can express Equation 11.4 as

$$K = \tfrac{1}{2}I_{CM}\left(\frac{v_{CM}}{R}\right)^2 + \tfrac{1}{2}Mv_{CM}^2$$

$$K = \tfrac{1}{2}\left(\frac{I_{CM}}{R^2} + M\right)v_{CM}^2 \qquad (11.5)$$

By the time the sphere reaches the bottom of the incline, work equal to Mgh has been done on it by the gravitational field, where h is the height of the incline. Because the sphere starts from rest at the top, its kinetic energy at the bottom, given by Equation 11.5, must equal this work done. Therefore, the speed of the center of mass at the bottom can be obtained by equating these two quantities:

$$\tfrac{1}{2}\left(\frac{I_{CM}}{R^2} + M\right) v_{CM}^2 = Mgh$$

$$v_{CM} = \left(\frac{2gh}{1 + I_{CM}/MR^2}\right)^{1/2} \tag{11.6}$$

Quick Quiz 11.1

Imagine that you slide your textbook across a gymnasium floor with a certain initial speed. It quickly stops moving because of friction between it and the floor. Yet, if you were to start a basketball rolling with the same initial speed, it would probably keep rolling from one end of the gym to the other. Why does a basketball roll so far? Doesn't friction affect its motion?

Example 11.1 Sphere Rolling Down an Incline

For the solid sphere shown in Figure 11.5, calculate the linear speed of the center of mass at the bottom of the incline and the magnitude of the linear acceleration of the center of mass.

Solution The sphere starts from the top of the incline with potential energy $U_g = Mgh$ and kinetic energy $K = 0$. As we have seen before, if it fell vertically from that height, it would have a linear speed of $\sqrt{2gh}$ at the moment before it hit the floor. After rolling down the incline, the linear speed of the center of mass must be less than this value because some of the initial potential energy is diverted into rotational kinetic energy rather than all being converted into translational kinetic energy. For a uniform solid sphere, $I_{CM} = \frac{2}{5}MR^2$ (see Table 10.2), and therefore Equation 11.6 gives

$$v_{CM} = \left(\frac{2gh}{1 + \dfrac{2/5MR^2}{MR^2}}\right)^{1/2} = \left(\frac{10}{7}gh\right)^{1/2}$$

which is less than $\sqrt{2gh}$.

To calculate the linear acceleration of the center of mass, we note that the vertical displacement is related to the displacement x along the incline through the relationship $h = x \sin\theta$. Hence, after squaring both sides, we can express the equation above as

$$v_{CM}^2 = \tfrac{10}{7} gx \sin\theta$$

Comparing this with the expression from kinematics, $v_{CM}^2 = 2a_{CM}x$ (see Eq. 2.12), we see that the acceleration of the center of mass is

$$a_{CM} = \tfrac{5}{7} g \sin\theta$$

These results are quite interesting in that both the speed and the acceleration of the center of mass are *independent* of the mass and the radius of the sphere! That is, **all homogeneous solid spheres experience the same speed and acceleration on a given incline.**

If we repeated the calculations for a hollow sphere, a solid cylinder, or a hoop, we would obtain similar results in which only the factor in front of $g \sin\theta$ would differ. The constant factors that appear in the expressions for v_{CM} and a_{CM} depend only on the moment of inertia about the center of mass for the specific body. In all cases, the acceleration of the center of mass is *less* than $g \sin\theta$, the value the acceleration would have if the incline were frictionless and no rolling occurred.

Example 11.2 Another Look at the Rolling Sphere

In this example, let us use dynamic methods to verify the results of Example 11.1. The free-body diagram for the sphere is illustrated in Figure 11.6.

Solution Newton's second law applied to the center of mass gives

$$(1) \qquad \Sigma F_x = Mg \sin\theta - f = Ma_{CM}$$

$$\Sigma F_y = n - Mg\cos\theta = 0$$

where x is measured along the slanted surface of the incline.

Now let us write an expression for the torque acting on the sphere. A convenient axis to choose is the one that passes

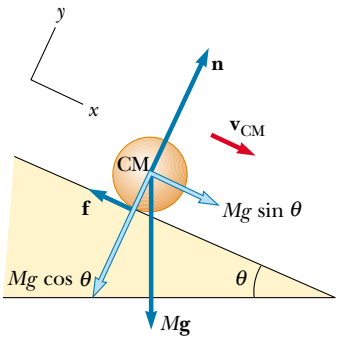


Figure 11.6 Free-body diagram for a solid sphere rolling down an incline.

through the center of the sphere and is perpendicular to the plane of the figure.[1] Because **n** and $M\mathbf{g}$ go through the center of mass, they have zero moment arm about this axis and thus do not contribute to the torque. However, the force of static friction produces a torque about this axis equal to fR in the clockwise direction; therefore, because τ is also in the clockwise direction,

$$\tau_{CM} = fR = I_{CM}\alpha$$

Because $I_{CM} = \frac{2}{5}MR^2$ and $\alpha = a_{CM}/R$, we obtain

$$(2) \qquad f = \frac{I_{CM}\alpha}{R} = \left(\frac{\frac{2}{5}MR^2}{R}\right)\frac{a_{CM}}{R} = \tfrac{2}{5}Ma_{CM}$$

Substituting Equation (2) into Equation (1) gives

$$a_{CM} = \tfrac{5}{7}g \sin \theta$$

which agrees with the result of Example 11.1.

Note that $\Sigma\mathbf{F} = m\mathbf{a}$ applies only if $\Sigma\mathbf{F}$ is the net external force exerted on the sphere and **a** is the acceleration of its center of mass. In the case of our sphere rolling down an incline, even though the frictional force does not change the total kinetic energy of the sphere, it does contribute to $\Sigma\mathbf{F}$ and thus decreases the acceleration of the center of mass. As a result, the final translational kinetic energy is less than it would be in the absence of friction. As mentioned in Example 11.1, some of the initial potential energy is converted to rotational kinetic energy.

QuickLab

Hold a basketball and a tennis ball side by side at the top of a ramp and release them at the same time. Which reaches the bottom first? Does the outcome depend on the angle of the ramp? What if the angle were 90° (that is, if the balls were in free fall)?

Quick Quiz 11.2

Which gets to the bottom first: a ball rolling without sliding down incline A or a box sliding down a frictionless incline B having the same dimensions as incline A?

11.2 THE VECTOR PRODUCT AND TORQUE

2.7 Consider a force **F** acting on a rigid body at the vector position **r** (Fig. 11.7). **The origin *O* is assumed to be in an inertial frame, so Newton's first law is valid in this case.** As we saw in Section 10.6, the *magnitude* of the torque due to this force relative to the origin is, by definition, $rF \sin \phi$, where ϕ is the angle between **r** and **F**. The axis about which **F** tends to produce rotation is perpendicular to the plane formed by **r** and **F**. If the force lies in the xy plane, as it does in Figure 11.7, the torque $\boldsymbol{\tau}$ is represented by a vector parallel to the z axis. The force in Figure 11.7 creates a torque that tends to rotate the body counterclockwise about the z axis; thus the direction of $\boldsymbol{\tau}$ is toward increasing z, and $\boldsymbol{\tau}$ is therefore in the positive z direction. If we reversed the direction of **F** in Figure 11.7, then $\boldsymbol{\tau}$ would be in the negative z direction.

The torque $\boldsymbol{\tau}$ involves the two vectors **r** and **F**, and its direction is perpendicular to the plane of **r** and **F**. We can establish a mathematical relationship between $\boldsymbol{\tau}$, **r**, and **F**, using a new mathematical operation called the **vector product,** or **cross product:**

Torque

$$\boldsymbol{\tau} \equiv \mathbf{r} \times \mathbf{F} \qquad (11.7)$$

[1] Although a coordinate system whose origin is at the center of mass of a rolling object is not an inertial frame, the expression $\tau_{CM} = I\alpha$ still applies in the center-of-mass frame.

We now give a formal definition of the vector product. Given any two vectors **A** and **B**, the **vector product A × B** is defined as a third vector **C**, the magnitude of which is $AB \sin \theta$, where θ is the angle between **A** and **B**. That is, if **C** is given by

$$\mathbf{C} = \mathbf{A} \times \mathbf{B} \tag{11.8}$$

then its magnitude is

$$C \equiv AB \sin \theta \tag{11.9}$$

The quantity $AB \sin \theta$ is equal to the area of the parallelogram formed by **A** and **B**, as shown in Figure 11.8. The *direction* of **C** is perpendicular to the plane formed by **A** and **B**, and the best way to determine this direction is to use the right-hand rule illustrated in Figure 11.8. The four fingers of the right hand are pointed along **A** and then "wrapped" into **B** through the angle θ. The direction of the erect right thumb is the direction of **A × B = C.** Because of the notation, **A × B** is often read "**A** cross **B**"; hence, the term *cross product.*

Some properties of the vector product that follow from its definition are as follows:

1. Unlike the scalar product, the vector product is *not* commutative. Instead, the order in which the two vectors are multiplied in a cross product is important:

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A} \tag{11.10}$$

   Therefore, if you change the order of the vectors in a cross product, you must change the sign. You could easily verify this relationship with the right-hand rule.
2. If **A** is parallel to **B** ($\theta = 0°$ or $180°$), then **A × B** = 0; therefore, it follows that **A × A** = 0.
3. If **A** is perpendicular to **B**, then $|\mathbf{A} \times \mathbf{B}| = AB$.
4. The vector product obeys the distributive law:

$$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C} \tag{11.11}$$

5. The derivative of the cross product with respect to some variable such as t is

$$\frac{d}{dt}(\mathbf{A} \times \mathbf{B}) = \mathbf{A} \times \frac{d\mathbf{B}}{dt} + \frac{d\mathbf{A}}{dt} \times \mathbf{B} \tag{11.12}$$

   where it is important to preserve the multiplicative order of **A** and **B**, in view of Equation 11.10.

It is left as an exercise to show from Equations 11.9 and 11.10 and from the definition of unit vectors that the cross products of the rectangular unit vectors **i**,

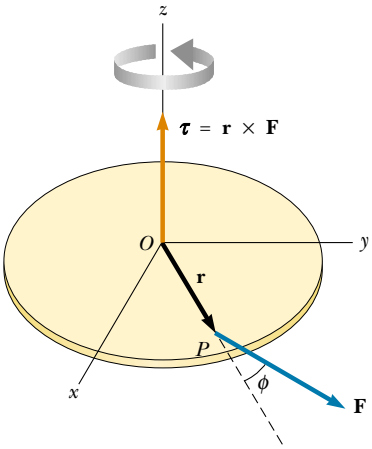


Figure 11.7 The torque vector $\boldsymbol{\tau}$ lies in a direction perpendicular to the plane formed by the position vector **r** and the applied force vector **F**.

Properties of the vector product

Right-hand rule

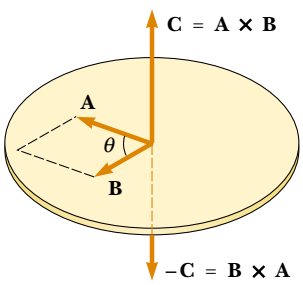


Figure 11.8 The vector product **A × B** is a third vector **C** having a magnitude $AB \sin \theta$ equal to the area of the parallelogram shown. The direction of **C** is perpendicular to the plane formed by **A** and **B**, and this direction is determined by the right-hand rule.

Cross products of unit vectors

$\mathbf{j}$, and $\mathbf{k}$ obey the following rules:

$$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = 0 \quad \textbf{(11.13a)}$$

$$\mathbf{i} \times \mathbf{j} = -\mathbf{j} \times \mathbf{i} = \mathbf{k} \quad \textbf{(11.13b)}$$

$$\mathbf{j} \times \mathbf{k} = -\mathbf{k} \times \mathbf{j} = \mathbf{i} \quad \textbf{(11.13c)}$$

$$\mathbf{k} \times \mathbf{i} = -\mathbf{i} \times \mathbf{k} = \mathbf{j} \quad \textbf{(11.13d)}$$

Signs are interchangeable in cross products. For example, $\mathbf{A} \times (-\mathbf{B}) = -\mathbf{A} \times \mathbf{B}$ and $\mathbf{i} \times (-\mathbf{j}) = -\mathbf{i} \times \mathbf{j}$.

The cross product of any two vectors $\mathbf{A}$ and $\mathbf{B}$ can be expressed in the following determinant form:

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \mathbf{i}\begin{vmatrix} A_y & A_z \\ B_y & B_z \end{vmatrix} - \mathbf{j}\begin{vmatrix} A_x & A_z \\ B_x & B_z \end{vmatrix} + \mathbf{k}\begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix}$$

Expanding these determinants gives the result

$$\mathbf{A} \times \mathbf{B} = (A_y B_z - A_z B_y)\mathbf{i} - (A_x B_z - A_z B_x)\mathbf{j} + (A_x B_y - A_y B_x)\mathbf{k} \quad \textbf{(11.14)}$$

EXAMPLE 11.3 The Cross Product

Two vectors lying in the xy plane are given by the equations $\mathbf{A} = 2\mathbf{i} + 3\mathbf{j}$ and $\mathbf{B} = -\mathbf{i} + 2\mathbf{j}$. Find $\mathbf{A} \times \mathbf{B}$ and verify that $\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$.

Solution Using Equations 11.13a through 11.13d, we obtain

$$\mathbf{A} \times \mathbf{B} = (2\mathbf{i} + 3\mathbf{j}) \times (-\mathbf{i} + 2\mathbf{j})$$

$$= 2\mathbf{i} \times 2\mathbf{j} + 3\mathbf{j} \times (-\mathbf{i}) = 4\mathbf{k} + 3\mathbf{k} = 7\mathbf{k}$$

(We have omitted the terms containing $\mathbf{i} \times \mathbf{i}$ and $\mathbf{j} \times \mathbf{j}$ because, as Equation 11.13a shows, they are equal to zero.)

We can show that $\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$, since

$$\mathbf{B} \times \mathbf{A} = (-\mathbf{i} + 2\mathbf{j}) \times (2\mathbf{i} + 3\mathbf{j})$$

$$= -\mathbf{i} \times 3\mathbf{j} + 2\mathbf{j} \times 2\mathbf{i} = -3\mathbf{k} - 4\mathbf{k} = -7\mathbf{k}$$

Therefore, $\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$.

As an alternative method for finding $\mathbf{A} \times \mathbf{B}$, we could use Equation 11.14, with $A_x = 2$, $A_y = 3$, $A_z = 0$ and $B_x = -1$, $B_y = 2$, $B_z = 0$:

$$\mathbf{A} \times \mathbf{B} = (0)\mathbf{i} - (0)\mathbf{j} + [(2)(2) - (3)(-1)]\mathbf{k} = 7\mathbf{k}$$

Exercise Use the results to this example and Equation 11.9 to find the angle between $\mathbf{A}$ and $\mathbf{B}$.

Answer 60.3°

11.3 ANGULAR MOMENTUM OF A PARTICLE


7.8

Imagine a rigid pole sticking up through the ice on a frozen pond (Fig. 11.9). A skater glides rapidly toward the pole, aiming a little to the side so that she does not hit it. As she approaches a point beside the pole, she reaches out and grabs the pole, an action that whips her rapidly into a circular path around the pole. Just as the idea of linear momentum helps us analyze translational motion, a rotational analog—*angular momentum*—helps us describe this skater and other objects undergoing rotational motion.

To analyze the motion of the skater, we need to know her mass and her velocity, as well as her position relative to the pole. In more general terms, consider a

particle of mass m located at the vector position $\mathbf{r}$ and moving with linear velocity $\mathbf{v}$ (Fig. 11.10).

The instantaneous angular momentum $\mathbf{L}$ of the particle relative to the origin O is defined as the cross product of the particle's instantaneous position vector $\mathbf{r}$ and its instantaneous linear momentum $\mathbf{p}$:

$$\mathbf{L} \equiv \mathbf{r} \times \mathbf{p} \tag{11.15}$$

Angular momentum of a particle

The SI unit of angular momentum is kg·m^2/s. It is important to note that both the magnitude and the direction of $\mathbf{L}$ depend on the choice of origin. Following the right-hand rule, note that the direction of $\mathbf{L}$ is perpendicular to the plane formed by $\mathbf{r}$ and $\mathbf{p}$. In Figure 11.10, $\mathbf{r}$ and $\mathbf{p}$ are in the xy plane, and so $\mathbf{L}$ points in the z direction. Because $\mathbf{p} = m\mathbf{v}$, the magnitude of $\mathbf{L}$ is

$$L = mvr \sin \phi \tag{11.16}$$

where ϕ is the angle between $\mathbf{r}$ and $\mathbf{p}$. It follows that L is zero when $\mathbf{r}$ is parallel to $\mathbf{p}$ ($\phi = 0$ or 180°). In other words, when the linear velocity of the particle is along a line that passes through the origin, the particle has zero angular momentum with respect to the origin. On the other hand, if $\mathbf{r}$ is perpendicular to $\mathbf{p}$ ($\phi = 90°$), then $L = mvr$. At that instant, the particle moves exactly as if it were on the rim of a wheel rotating about the origin in a plane defined by $\mathbf{r}$ and $\mathbf{p}$.

Figure 11.9 As the skater passes the pole, she grabs hold of it. This causes her to swing around the pole rapidly in a circular path.

Quick Quiz 11.3

Recall the skater described at the beginning of this section. What would be her angular momentum relative to the pole if she were skating directly toward it?

In describing linear motion, we found that the net force on a particle equals the time rate of change of its linear momentum, $\Sigma\mathbf{F} = d\mathbf{p}/dt$ (see Eq. 9.3). We now show that the net torque acting on a particle equals the time rate of change of its angular momentum. Let us start by writing the net torque on the particle in the form

$$\sum \boldsymbol{\tau} = \mathbf{r} \times \sum \mathbf{F} = \mathbf{r} \times \frac{d\mathbf{p}}{dt} \tag{11.17}$$

Now let us differentiate Equation 11.15 with respect to time, using the rule given by Equation 11.12:

$$\frac{d\mathbf{L}}{dt} = \frac{d}{dt}(\mathbf{r} \times \mathbf{p}) = \mathbf{r} \times \frac{d\mathbf{p}}{dt} + \frac{d\mathbf{r}}{dt} \times \mathbf{p}$$

Remember, it is important to adhere to the order of terms because $\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$. The last term on the right in the above equation is zero because $\mathbf{v} = d\mathbf{r}/dt$ is parallel to $\mathbf{p} = m\mathbf{v}$ (property 2 of the vector product). Therefore,

$$\frac{d\mathbf{L}}{dt} = \mathbf{r} \times \frac{d\mathbf{p}}{dt} \tag{11.18}$$

Comparing Equations 11.17 and 11.18, we see that

$$\sum \boldsymbol{\tau} = \frac{d\mathbf{L}}{dt} \tag{11.19}$$

The net torque equals time rate of change of angular momentum

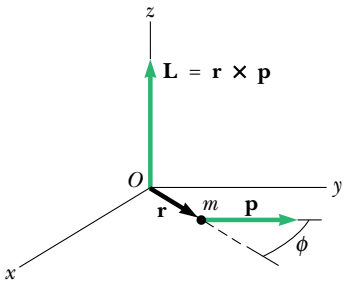


Figure 11.10 The angular momentum $\mathbf{L}$ of a particle of mass m and linear momentum $\mathbf{p}$ located at the vector position $\mathbf{r}$ is a vector given by $\mathbf{L} = \mathbf{r} \times \mathbf{p}$. The value of $\mathbf{L}$ depends on the origin about which it is measured and is a vector perpendicular to both $\mathbf{r}$ and $\mathbf{p}$.

which is the rotational analog of Newton's second law, $\Sigma \mathbf{F} = d\mathbf{p}/dt$. Note that torque causes the angular momentum $\mathbf{L}$ to change just as force causes linear momentum $\mathbf{p}$ to change. This rotational result, Equation 11.19, states that

> the net torque acting on a particle is equal to the time rate of change of the particle's angular momentum.

It is important to note that Equation 11.19 is valid only if $\Sigma \boldsymbol{\tau}$ and $\mathbf{L}$ are measured about the same origin. (Of course, the same origin must be used in calculating all of the torques.) Furthermore, **the expression is valid for any origin fixed in an inertial frame.**

Angular Momentum of a System of Particles

The total angular momentum of a system of particles about some point is defined as the vector sum of the angular momenta of the individual particles:

$$\mathbf{L} = \mathbf{L}_1 + \mathbf{L}_2 + \cdots + \mathbf{L}_n = \sum_i \mathbf{L}_i$$

where the vector sum is over all n particles in the system.

Because individual angular momenta can change with time, so can the total angular momentum. In fact, from Equations 11.18 and 11.19, we find that the time rate of change of the total angular momentum equals the vector sum of all torques acting on the system, both those associated with internal forces between particles and those associated with external forces. However, the net torque associated with all internal forces is zero. To understand this, recall that Newton's third law tells us that internal forces between particles of the system are equal in magnitude and opposite in direction. If we assume that these forces lie along the line of separation of each pair of particles, then the torque due to each action–reaction force pair is zero. That is, the moment arm d from O to the line of action of the forces is equal for both particles. In the summation, therefore, we see that the net internal torque vanishes. We conclude that the total angular momentum of a system can vary with time only if a net external torque is acting on the system, so that we have

$$\sum \boldsymbol{\tau}_{\text{ext}} = \sum_i \frac{d\mathbf{L}_i}{dt} = \frac{d}{dt} \sum_i \mathbf{L}_i = \frac{d\mathbf{L}}{dt} \tag{11.20}$$

That is,

> the time rate of change of the total angular momentum of a system about some origin in an inertial frame equals the net external torque acting on the system about that origin.

Note that Equation 11.20 is the rotational analog of Equation 9.38, $\Sigma \mathbf{F}_{\text{ext}} = d\mathbf{p}/dt$, for a system of particles.

EXAMPLE 11.4 Circular Motion

A particle moves in the xy plane in a circular path of radius r, as shown in Figure 11.11. (a) Find the magnitude and direction of its angular momentum relative to O when its linear velocity is $\mathbf{v}$.

Solution You might guess that because the linear momentum of the particle is always changing (in direction, not magnitude), the direction of the angular momentum must also change. In this example, however, this is not the case. The magnitude of $\mathbf{L}$ is given by

$$L = mvr \sin 90^\circ = mvr \qquad \text{(for } \mathbf{r} \text{ perpendicular to } \mathbf{v}\text{)}$$

This value of L is constant because all three factors on the right are constant. The direction of $\mathbf{L}$ also is constant, even though the direction of $\mathbf{p} = m\mathbf{v}$ keeps changing. You can visualize this by sliding the vector $\mathbf{v}$ in Figure 11.11 parallel to itself until its tail meets the tail of $\mathbf{r}$ and by then applying the right-hand rule. (You can use $\mathbf{v}$ to determine the direction of $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ because the direction of $\mathbf{p}$ is the same as the direction of $\mathbf{v}$.) Line up your fingers so that they point along $\mathbf{r}$ and wrap your fingers into the vector $\mathbf{v}$. Your thumb points upward and away from the page; this is the direction of $\mathbf{L}$. Hence, we can write the vector expression $\mathbf{L} = (mvr)\mathbf{k}$. If the particle were to move clockwise, $\mathbf{L}$ would point downward and into the page.

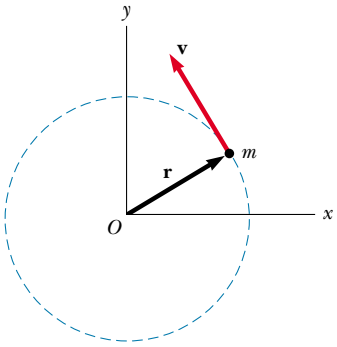


Figure 11.11 A particle moving in a circle of radius r has an angular momentum about O that has magnitude mvr. The vector $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ points *out* of the diagram.

(b) Find the magnitude and direction of $\mathbf{L}$ in terms of the particle's angular speed ω.

Solution Because $v = r\omega$ for a particle rotating in a circle, we can express L as

$$L = mvr = mr^2\omega = I\omega$$

where I is the moment of inertia of the particle about the z axis through O. Because the rotation is counterclockwise, the direction of $\boldsymbol{\omega}$ is along the z axis (see Section 10.1). The direction of $\mathbf{L}$ is the same as that of $\boldsymbol{\omega}$, and so we can write the angular momentum as $\mathbf{L} = I\boldsymbol{\omega} = I\omega\mathbf{k}$.

Exercise A car of mass 1 500 kg moves with a linear speed of 40 m/s on a circular race track of radius 50 m. What is the magnitude of its angular momentum relative to the center of the track?

Answer $3.0 \times 10^6\ \text{kg}\cdot\text{m}^2/\text{s}$

11.4 ANGULAR MOMENTUM OF A ROTATING RIGID OBJECT

Consider a rigid object rotating about a fixed axis that coincides with the z axis of a coordinate system, as shown in Figure 11.12. Let us determine the angular momentum of this object. Each particle of the object rotates in the xy plane about the z axis with an angular speed ω. The magnitude of the angular momentum of a particle of mass m_i about the origin O is $m_i v_i r_i$. Because $v_i = r_i\omega$, we can express the magnitude of the angular momentum of this particle as

$$L_i = m_i r_i^2 \omega$$

The vector $\mathbf{L}_i$ is directed along the z axis, as is the vector $\boldsymbol{\omega}$.

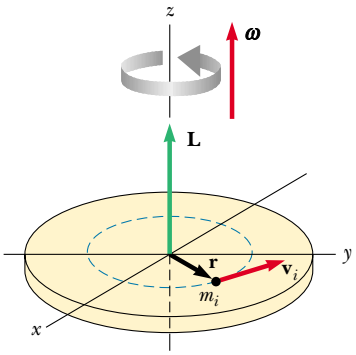


Figure 11.12 When a rigid body rotates about an axis, the angular momentum **L** is in the same direction as the angular velocity $\boldsymbol{\omega}$, according to the expression $\mathbf{L} = I\boldsymbol{\omega}$.

We can now find the angular momentum (which in this situation has only a z component) of the whole object by taking the sum of L_i over all particles:

$$L_z = \sum_i m_i r_i^2 \omega = \left(\sum_i m_i r_i^2\right)\omega$$

$$L_z = I\omega \tag{11.21}$$

where I is the moment of inertia of the object about the z axis.

Now let us differentiate Equation 11.21 with respect to time, noting that I is constant for a rigid body:

$$\frac{dL_z}{dt} = I\frac{d\omega}{dt} = I\alpha \tag{11.22}$$

where α is the angular acceleration relative to the axis of rotation. Because dL_z/dt is equal to the net external torque (see Eq. 11.20), we can express Equation 11.22 as

$$\sum \tau_{\text{ext}} = \frac{dL_z}{dt} = I\alpha \tag{11.23}$$

That is, the net external torque acting on a rigid object rotating about a fixed axis equals the moment of inertia about the rotation axis multiplied by the object's angular acceleration relative to that axis.

Equation 11.23 also is valid for a rigid object rotating about a moving axis provided the moving axis (1) passes through the center of mass and (2) is a symmetry axis.

You should note that if a symmetrical object rotates about a fixed axis passing through its center of mass, you can write Equation 11.21 in vector form as $\mathbf{L} = I\boldsymbol{\omega}$, where **L** is the total angular momentum of the object measured with respect to the axis of rotation. Furthermore, the expression is valid for any object, regardless of its symmetry, if **L** stands for the component of angular momentum along the axis of rotation.[2]

EXAMPLE 11.5 Bowling Ball

Estimate the magnitude of the angular momentum of a bowling ball spinning at 10 rev/s, as shown in Figure 11.13.

Solution We start by making some estimates of the relevant physical parameters and model the ball as a uniform solid sphere. A typical bowling ball might have a mass of 6 kg and a radius of 12 cm. The moment of inertia of a solid sphere about an axis through its center is, from Table 10.2,

$$I = \tfrac{2}{5}MR^2 = \tfrac{2}{5}(6\text{ kg})(0.12\text{ m})^2 = 0.035\text{ kg}\cdot\text{m}^2$$

Therefore, the magnitude of the angular momentum is

[2] In general, the expression $\mathbf{L} = I\boldsymbol{\omega}$ is not always valid. If a rigid object rotates about an arbitrary axis, **L** and $\boldsymbol{\omega}$ may point in different directions. In this case, the moment of inertia cannot be treated as a scalar. Strictly speaking, $\mathbf{L} = I\boldsymbol{\omega}$ applies only to rigid objects of any shape that rotate about one of three mutually perpendicular axes (called *principal axes*) through the center of mass. This is discussed in more advanced texts on mechanics.

$$L = I\omega = (0.035\ \text{kg}\cdot\text{m}^2)(10\ \text{rev/s})(2\pi\ \text{rad/rev})$$
$$= 2.2\ \text{kg}\cdot\text{m}^2/\text{s}$$

Because of the roughness of our estimates, we probably want to keep only one significant figure, and so $L \approx$ $2\ \text{kg}\cdot\text{m}^2/\text{s}$.

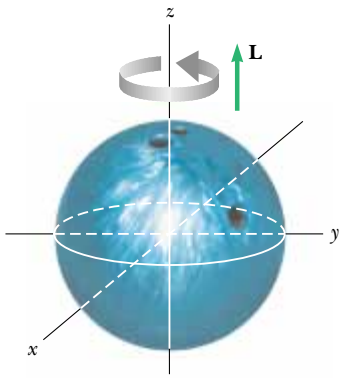


Figure 11.13 A bowling ball that rotates about the z axis in the direction shown has an angular momentum **L** in the positive z direction. If the direction of rotation is reversed, **L** points in the negative z direction.

EXAMPLE 11.6 Rotating Rod

A rigid rod of mass M and length ℓ is pivoted without friction at its center (Fig. 11.14). Two particles of masses m_1 and m_2 are connected to its ends. The combination rotates in a vertical plane with an angular speed ω. (a) Find an expression for the magnitude of the angular momentum of the system.

Solution This is different from the last example in that we now must account for the motion of more than one object. The moment of inertia of the system equals the sum of the moments of inertia of the three components: the rod and the two particles. Referring to Table 10.2 to obtain the expression for the moment of inertia of the rod, and using the expression $I = mr^2$ for each particle, we find that the total moment of inertia about the z axis through O is

$$I = \frac{1}{12}M\ell^2 + m_1\left(\frac{\ell}{2}\right)^2 + m_2\left(\frac{\ell}{2}\right)^2$$
$$= \frac{\ell^2}{4}\left(\frac{M}{3} + m_1 + m_2\right)$$

Therefore, the magnitude of the angular momentum is

$$L = I\omega = \frac{\ell^2}{4}\left(\frac{M}{3} + m_1 + m_2\right)\omega$$

(b) Find an expression for the magnitude of the angular acceleration of the system when the rod makes an angle θ with the horizontal.

Solution If the masses of the two particles are equal, then the system has no angular acceleration because the net torque on the system is zero when $m_1 = m_2$. If the initial angle θ is exactly $\pi/2$ or $-\pi/2$ (vertical position), then the rod will be in equilibrium. To find the angular acceleration of the system at any angle θ, we first calculate the net torque on the system and then use $\Sigma\tau_{\text{ext}} = I\alpha$ to obtain an expression for α.

The torque due to the force $m_1 g$ about the pivot is

$$\tau_1 = m_1 g\frac{\ell}{2}\cos\theta \qquad (\boldsymbol{\tau}_1 \text{ out of page})$$

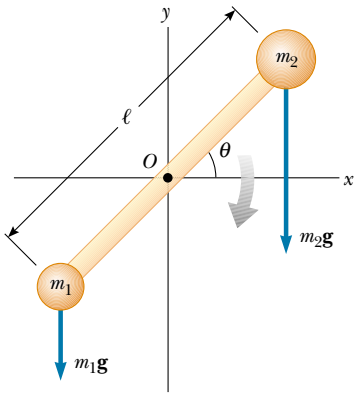


Figure 11.14 Because gravitational forces act on the rotating rod, there is in general a net nonzero torque about O when $m_1 \neq m_2$. This net torque produces an angular acceleration given by $\alpha = \Sigma\tau_{\text{ext}}/I$.

The torque due to the force $m_2 g$ about the pivot is

$$\tau_2 = -m_2 g\frac{\ell}{2}\cos\theta \qquad (\boldsymbol{\tau}_2 \text{ into page})$$

Hence, the net torque exerted on the system about O is

$$\sum\tau_{\text{ext}} = \tau_1 + \tau_2 = \tfrac{1}{2}(m_1 - m_2)g\ell\cos\theta$$

The direction of $\Sigma\boldsymbol{\tau}_{\text{ext}}$ is out of the page if $m_1 > m_2$ and is into the page if $m_2 > m_1$.

To find α, we use $\Sigma\tau_{\text{ext}} = I\alpha$, where I was obtained in part (a):

$$\alpha = \frac{\Sigma\tau_{\text{ext}}}{I} = \frac{2(m_1 - m_2)g\cos\theta}{\ell(M/3 + m_1 + m_2)}$$

Note that α is zero when θ is $\pi/2$ or $-\pi/2$ (vertical position) and is a maximum when θ is 0 or π (horizontal position).

Exercise If $m_2 > m_1$, at what value of θ is ω a maximum?

Answer $\theta = -\pi/2$.

EXAMPLE 11.7 Two Connected Masses

A sphere of mass m_1 and a block of mass m_2 are connected by a light cord that passes over a pulley, as shown in Figure 11.15. The radius of the pulley is R, and the moment of inertia about its axle is I. The block slides on a frictionless, horizontal surface. Find an expression for the linear acceleration of the two objects, using the concepts of angular momentum and torque.

Solution We need to determine the angular momentum of the system, which consists of the two objects and the pulley. Let us calculate the angular momentum about an axis that coincides with the axle of the pulley.

At the instant the sphere and block have a common speed v, the angular momentum of the sphere is $m_1 vR$, and that of the block is $m_2 vR$. At the same instant, the angular momentum of the pulley is $I\omega = Iv/R$. Hence, the total angular momentum of the system is

$$(1) \qquad L = m_1 vR + m_2 vR + I\frac{v}{R}$$

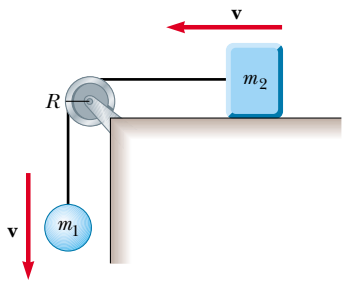


Figure 11.15

Now let us evaluate the total external torque acting on the system about the pulley axle. Because it has a moment arm of zero, the force exerted by the axle on the pulley does not contribute to the torque. Furthermore, the normal force acting on the block is balanced by the force of gravity $m_2\mathbf{g}$, and so these forces do not contribute to the torque. The force of gravity $m_1\mathbf{g}$ acting on the sphere produces a torque about the axle equal in magnitude to $m_1 gR$, where R is the moment arm of the force about the axle. (Note that in this situation, the tension is *not* equal to $m_1 g$.) This is the total external torque about the pulley axle; that is, $\Sigma\tau_{\text{ext}} = m_1 gR$. Using this result, together with Equation (1) and Equation 11.23, we find

$$\sum \tau_{\text{ext}} = \frac{dL}{dt}$$

$$m_1 gR = \frac{d}{dt}\left[(m_1 + m_2)Rv + I\frac{v}{R}\right]$$

$$(2) \qquad m_1 gR = (m_1 + m_2)R\frac{dv}{dt} + \frac{I}{R}\frac{dv}{dt}$$

Because $dv/dt = a$, we can solve this for a to obtain

$$a = \frac{m_1 g}{(m_1 + m_2) + I/R^2}$$

You may wonder why we did not include the forces that the cord exerts on the objects in evaluating the net torque about the axle. The reason is that these forces are internal to the system under consideration, and we analyzed the system as a whole. Only external torques contribute to the change in the system's angular momentum.

11.5 CONSERVATION OF ANGULAR MOMENTUM

7.9 In Chapter 9 we found that the total linear momentum of a system of particles remains constant when the resultant external force acting on the system is zero. We have an analogous conservation law in rotational motion:

Conservation of angular momentum

The total angular momentum of a system is constant in both magnitude and direction if the resultant external torque acting on the system is zero.

This follows directly from Equation 11.20, which indicates that if

$$\sum \boldsymbol{\tau}_{\text{ext}} = \frac{d\mathbf{L}}{dt} = 0 \qquad \textbf{(11.24)}$$

then

$$\mathbf{L} = \text{constant} \qquad \textbf{(11.25)}$$

For a system of particles, we write this conservation law as $\Sigma\,\mathbf{L}_n = \text{constant}$, where the index n denotes the nth particle in the system.

If the mass of an object undergoes redistribution in some way, then the object's moment of inertia changes; hence, its angular speed must change because $L = I\omega$. In this case we express the conservation of angular momentum in the form

$$\mathbf{L}_i = \mathbf{L}_f = \text{constant} \tag{11.26}$$

If the system is an object rotating about a *fixed* axis, such as the z axis, we can write $L_z = I\omega$, where L_z is the component of $\mathbf{L}$ along the axis of rotation and I is the moment of inertia about this axis. In this case, we can express the conservation of angular momentum as

$$I_i\omega_i = I_f\omega_f = \text{constant} \tag{11.27}$$

This expression is valid both for rotation about a fixed axis and for rotation about an axis through the center of mass of a moving system as long as that axis remains parallel to itself. We require only that the net external torque be zero.

Although we do not prove it here, there is an important theorem concerning the angular momentum of an object relative to the object's center of mass:

The resultant torque acting on an object about an axis through the center of mass equals the time rate of change of angular momentum regardless of the motion of the center of mass.

This theorem applies even if the center of mass is accelerating, provided $\boldsymbol{\tau}$ and $\mathbf{L}$ are evaluated relative to the center of mass.

In Equation 11.26 we have a third conservation law to add to our list. We can now state that the energy, linear momentum, and angular momentum of an isolated system all remain constant:

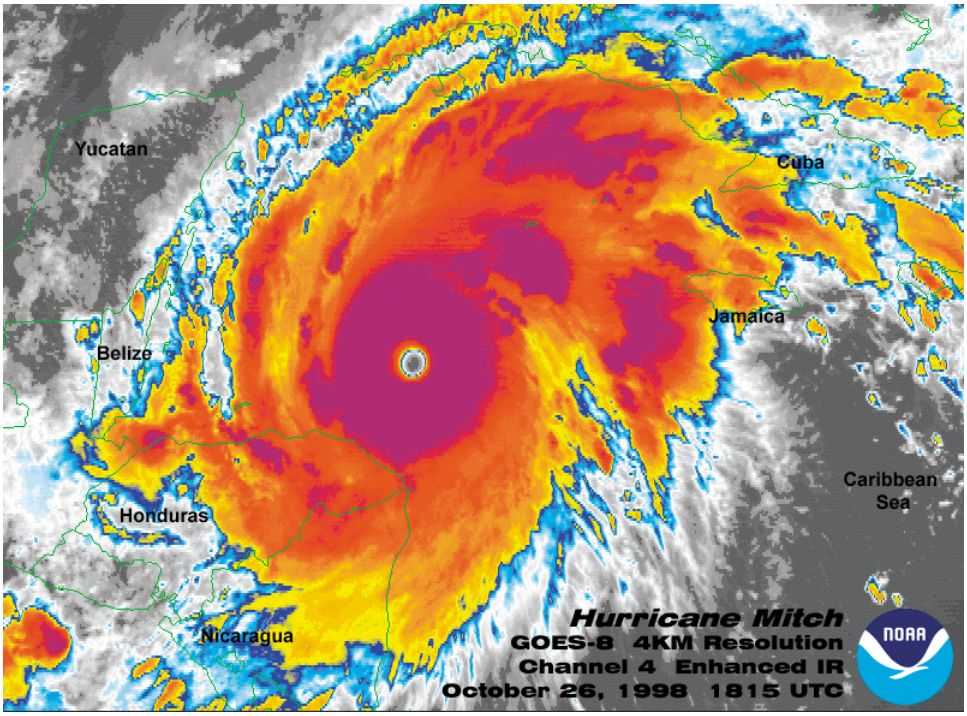
$$\left.\begin{aligned} K_i + U_i &= K_f + U_f \\ \mathbf{p}_i &= \mathbf{p}_f \\ \mathbf{L}_i &= \mathbf{L}_f \end{aligned}\right\} \quad \text{For an isolated system}$$

There are many examples that demonstrate conservation of angular momentum. You may have observed a figure skater spinning in the finale of a program. The angular speed of the skater increases when the skater pulls his hands and feet close to his body, thereby decreasing I. Neglecting friction between skates and ice, no external torques act on the skater. The change in angular speed is due to the fact that, because angular momentum is conserved, the product $I\omega$ remains constant, and a decrease in the moment of inertia of the skater causes an increase in the angular speed. Similarly, when divers or acrobats wish to make several somersaults, they pull their hands and feet close to their bodies to rotate at a higher rate. In these cases, the external force due to gravity acts through the center of mass and hence exerts no torque about this point. Therefore, the angular momentum about the center of mass must be conserved—that is, $I_i\omega_i = I_f\omega_f$. For example, when divers wish to double their angular speed, they must reduce their moment of inertia to one-half its initial value.

Angular momentum is conserved as figure skater Todd Eldredge pulls his arms toward his body. *(© 1998 David Madison)*

Quick Quiz 11.4

A particle moves in a straight line, and you are told that the net torque acting on it is zero about some unspecified point. Decide whether the following statements are true or false: (a) The net force on the particle must be zero. (b) The particle's velocity must be constant.



A color-enhanced, infrared image of Hurricane Mitch, which devastated large areas of Honduras and Nicaragua in October 1998. The spiral, nonrigid mass of air undergoes rotation and has angular momentum. *(Courtesy of NOAA)*

EXAMPLE 11.8 Formation of a Neutron Star

A star rotates with a period of 30 days about an axis through its center. After the star undergoes a supernova explosion, the stellar core, which had a radius of 1.0×10^4 km, collapses into a neutron star of radius 3.0 km. Determine the period of rotation of the neutron star.

Solution The same physics that makes a skater spin faster with his arms pulled in describes the motion of the neutron star. Let us assume that during the collapse of the stellar core, (1) no torque acts on it, (2) it remains spherical, and (3) its mass remains constant. Also, let us use the symbol T for the period, with T_i being the initial period of the star and T_f being the period of the neutron star. The period is the length of time a point on the star's equator takes to make one complete circle around the axis of rotation. The angular speed of a star is given by $\omega = 2\pi/T$. Therefore, because I is proportional to r^2, Equation 11.27 gives

$$T_f = T_i\left(\frac{r_f}{r_i}\right)^2 = (30 \text{ days})\left(\frac{3.0 \text{ km}}{1.0 \times 10^4 \text{ km}}\right)^2$$

$$= 2.7 \times 10^{-6} \text{ days} = 0.23 \text{ s}$$

Thus, the neutron star rotates about four times each second; this result is approximately the same as that for a spinning figure skater.

EXAMPLE 11.9 The Merry-Go-Round

A horizontal platform in the shape of a circular disk rotates in a horizontal plane about a frictionless vertical axle (Fig. 11.16). The platform has a mass $M = 100$ kg and a radius $R = 2.0$ m. A student whose mass is $m = 60$ kg walks slowly from the rim of the disk toward its center. If the angular speed of the system is 2.0 rad/s when the student is at the rim, what is the angular speed when he has reached a point $r = 0.50$ m from the center?

Solution The speed change here is similar to the increase in angular speed of the spinning skater when he pulls his arms inward. Let us denote the moment of inertia of the platform as I_p and that of the student as I_s. Treating the student as a point mass, we can write the initial moment of inertia I_i of the system (student plus platform) about the axis of rotation:

$$I_i = I_{pi} + I_{si} = \tfrac{1}{2}MR^2 + mR^2$$

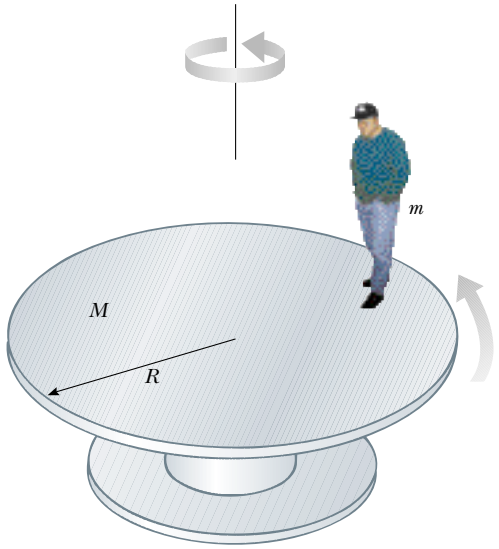


Figure 11.16 As the student walks toward the center of the rotating platform, the angular speed of the system increases because the angular momentum must remain constant.

When the student has walked to the position $r < R$, the moment of inertia of the system reduces to

$$I_f = I_{pf} + I_{sf} = \tfrac{1}{2}MR^2 + mr^2$$

Note that we still use the greater radius R when calculating I_{pf} because the radius of the platform has not changed. Because no external torques act on the system about the axis of rotation, we can apply the law of conservation of angular momentum:

$$I_i\omega_i = I_f\omega_f$$

$$\left(\tfrac{1}{2}MR^2 + mR^2\right)\omega_i = \left(\tfrac{1}{2}MR^2 + mr^2\right)\omega_f$$

$$\omega_f = \left(\frac{\tfrac{1}{2}MR^2 + mR^2}{\tfrac{1}{2}MR^2 + mr^2}\right)\omega_i$$

$$\omega_f = \left(\frac{200 + 240}{200 + 15}\right)(2.0\,\text{rad/s}) = \boxed{4.1\ \text{rad/s}}$$

As expected, the angular speed has increased.

Exercise Calculate the initial and final rotational energies of the system.

Answer $K_i = 880\,\text{J}$; $K_f = 1.8 \times 10^3\,\text{J}$.

Quick Quiz 11.5

Note that the rotational energy of the system described in Example 11.9 increases. What accounts for this increase in energy?

Example 11.10 The Spinning Bicycle Wheel

In a favorite classroom demonstration, a student holds the axle of a spinning bicycle wheel while seated on a stool that is free to rotate (Fig. 11.17). The student and stool are initially at rest while the wheel is spinning in a horizontal plane with an initial angular momentum $\mathbf{L}_i$ that points upward. When the wheel is inverted about its center by 180°, the student and stool start rotating. In terms of $\mathbf{L}_i$, what are the magnitude and the direction of $\mathbf{L}$ for the student plus stool?

Figure 11.17 The wheel is initially spinning when the student is at rest. What happens when the wheel is inverted?

Solution The system consists of the student, the wheel, and the stool. Initially, the total angular momentum of the system $\mathbf{L}_i$ comes entirely from the spinning wheel. As the wheel is inverted, the student applies a torque to the wheel, but this torque is internal to the system. No external torque is acting on the system about the vertical axis. Therefore, the angular momentum of the system is conserved. Initially, we have

$$\mathbf{L}_{\text{system}} = \mathbf{L}_i = \mathbf{L}_{\text{wheel}} \qquad \text{(upward)}$$

After the wheel is inverted, we have $\mathbf{L}_{\text{inverted wheel}} = -\mathbf{L}_i$. For angular momentum to be conserved, some other part of the system has to start rotating so that the total angular momentum remains the initial angular momentum $\mathbf{L}_i$. That other part of the system is the student plus the stool she is sitting on. So, we can now state that

$$\mathbf{L}_f = \mathbf{L}_i = \mathbf{L}_{\text{student+stool}} - \mathbf{L}_i$$

$$\mathbf{L}_{\text{student+stool}} = \boxed{2\mathbf{L}_i}$$

EXAMPLE 11.11 Disk and Stick

A 2.0-kg disk traveling at 3.0 m/s strikes a 1.0-kg stick that is lying flat on nearly frictionless ice, as shown in Figure 11.18. Assume that the collision is elastic. Find the translational speed of the disk, the translational speed of the stick, and the rotational speed of the stick after the collision. The moment of inertia of the stick about its center of mass is 1.33 kg·m^2.

Solution Because the disk and stick form an isolated system, we can assume that total energy, linear momentum, and angular momentum are all conserved. We have three unknowns, and so we need three equations to solve simultaneously. The first comes from the law of the conservation of linear momentum:

$$p_i = p_f$$

$$m_d v_{di} = m_d v_{df} + m_s v_s$$

$$(2.0\text{ kg})(3.0\text{ m/s}) = (2.0\text{ kg})v_{df} + (1.0\text{ kg})v_s$$

$$(1) \qquad 6.0\text{ kg·m/s} - (2.0\text{ kg})v_{df} = (1.0\text{ kg})v_s$$

Now we apply the law of conservation of angular momentum, using the initial position of the center of the stick as our reference point. We know that the component of angular momentum of the disk along the axis perpendicular to the plane of the ice is negative (the right-hand rule shows that $\mathbf{L}_d$ points into the ice).

$$L_i = L_f$$

$$-rm_d v_{di} = -rm_d v_{df} + I\omega$$

$$-(2.0\text{ m})(2.0\text{ kg})(3.0\text{ m/s}) = -(2.0\text{ m})(2.0\text{ kg})v_{df} + (1.33\text{ kg·m}^2)\omega$$

$$-12\text{ kg·m}^2/\text{s} = -(4.0\text{ kg·m})v_{df} + (1.33\text{ kg·m}^2)\omega$$

$$(2) \qquad -9.0\text{ rad/s} + (3.0\text{ rad/m})v_{df} = \omega$$

We used the fact that radians are dimensionless to ensure consistent units for each term.

Finally, the elastic nature of the collision reminds us that kinetic energy is conserved; in this case, the kinetic energy consists of translational and rotational forms:

$$K_i = K_f$$

$$\tfrac{1}{2}m_d v_{di}^2 = \tfrac{1}{2}m_d v_{df}^2 + \tfrac{1}{2}m_s v_s^2 + \tfrac{1}{2}I\omega^2$$

$$\tfrac{1}{2}(2.0\text{ kg})(3.0\text{ m/s})^2 = \tfrac{1}{2}(2.0\text{ kg})v_{df}^2 + \tfrac{1}{2}(1.0\text{ kg})v_s^2 + \tfrac{1}{2}(1.33\text{ kg·m}^2/\text{s})\omega^2$$

$$(3) \qquad 54\text{ m}^2/\text{s}^2 = 6.0v_{df}^2 + 3.0v_s^2 + (4.0\text{ m}^2)\omega^2$$

In solving Equations (1), (2), and (3) simultaneously, we find that $v_{df} = 2.3$ m/s, $v_s = 1.3$ m/s, and $\omega = -2.0$ rad/s. These values seem reasonable. The disk is moving more slowly than it was before the collision, and the stick has a small translational speed. Table 11.1 summarizes the initial and final values of variables for the disk and the stick and verifies the conservation of linear momentum, angular momentum, and kinetic energy.

Exercise Verify the values in Table 11.1.

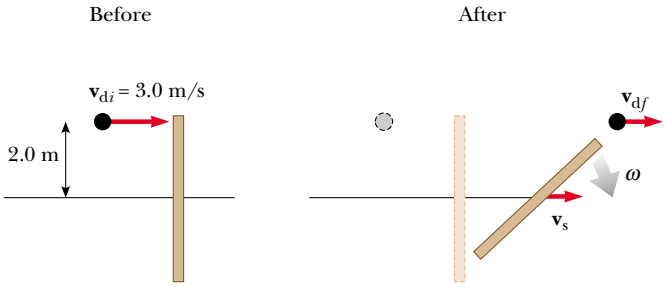


Figure 11.18 Overhead view of a disk striking a stick in an elastic collision, which causes the stick to rotate.

TABLE 11.1 **Comparison of Values in Example 11.11 Before and After the Collision[a]**

	v (m/s)	ω (rad/s)	p (kg·m/s)	L (kg·m^2/s)	K_{trans} (J)	K_{rot} (J)
Before						
Disk	3.0	—	6.0	−12	9.0	—
Stick	0	0	0	0	0	0
Total	—	—	6.0	−12	9.0	0
After						
Disk	2.3	—	4.7	−9.3	5.4	—
Stick	1.3	−2.0	1.3	−2.7	0.9	2.7
Total	—	—	6.0	−12	6.3	2.7

[a] Notice that linear momentum, angular momentum, and total kinetic energy are conserved.

Optional Section

11.6 THE MOTION OF GYROSCOPES AND TOPS

A very unusual and fascinating type of motion you probably have observed is that of a top spinning about its axis of symmetry, as shown in Figure 11.19a. If the top spins very rapidly, the axis rotates about the z axis, sweeping out a cone (see Fig. 11.19b). The motion of the axis about the vertical—known as **precessional motion**—is usually slow relative to the spin motion of the top.

Precessional motion

It is quite natural to wonder why the top does not fall over. Because the center of mass is not directly above the pivot point O, a net torque is clearly acting on the top about O—a torque resulting from the force of gravity $M\mathbf{g}$. The top would certainly fall over if it were not spinning. Because it is spinning, however, it has an angular momentum $\mathbf{L}$ directed along its symmetry axis. As we shall show, the motion of this symmetry axis about the z axis (the precessional motion) occurs because the torque produces a change in the *direction* of the symmetry axis. This is an excellent example of the importance of the directional nature of angular momentum.

The two forces acting on the top are the downward force of gravity $M\mathbf{g}$ and the normal force $\mathbf{n}$ acting upward at the pivot point O. The normal force produces no torque about the pivot because its moment arm through that point is zero. However, the force of gravity produces a torque $\boldsymbol{\tau} = \mathbf{r} \times M\mathbf{g}$ about O, where the direction of $\boldsymbol{\tau}$ is perpendicular to the plane formed by $\mathbf{r}$ and $M\mathbf{g}$. By necessity, the vector $\boldsymbol{\tau}$ lies in a horizontal xy plane perpendicular to the angular momentum vector. The net torque and angular momentum of the top are related through Equation 11.19:

$$\boldsymbol{\tau} = \frac{d\mathbf{L}}{dt}$$

From this expression, we see that the nonzero torque produces a change in angular momentum $d\mathbf{L}$—a change that is in the same direction as $\boldsymbol{\tau}$. Therefore, like the torque vector, $d\mathbf{L}$ must also be at right angles to $\mathbf{L}$. Figure 11.19b illustrates the resulting precessional motion of the symmetry axis of the top. In a time Δt, the change in angular momentum is $\Delta\mathbf{L} = \mathbf{L}_f - \mathbf{L}_i = \boldsymbol{\tau}\,\Delta t$. Because $\Delta\mathbf{L}$ is perpendicular to $\mathbf{L}$, the magnitude of $\mathbf{L}$ does not change ($|\mathbf{L}_i| = |\mathbf{L}_f|$). Rather, what is changing is the *direction* of $\mathbf{L}$. Because the change in angular momentum $\Delta\mathbf{L}$ is in the direction of $\boldsymbol{\tau}$, which lies in the xy plane, the top undergoes precessional motion.

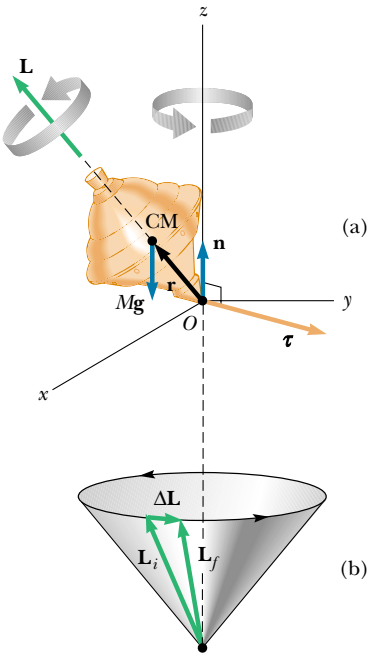


Figure 11.19 Precessional motion of a top spinning about its symmetry axis. (a) The only external forces acting on the top are the normal force $\mathbf{n}$ and the force of gravity $M\mathbf{g}$. The direction of the angular momentum $\mathbf{L}$ is along the axis of symmetry. The right-hand rule indicates that $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} = \mathbf{r} \times M\mathbf{g}$ is in the xy plane. (b). The direction of $\Delta\mathbf{L}$ is parallel to that of $\boldsymbol{\tau}$ in part (a). The fact that $\mathbf{L}_f = \Delta\mathbf{L} + \mathbf{L}_i$ indicates that the top precesses about the z axis.

The essential features of precessional motion can be illustrated by considering the simple gyroscope shown in Figure 11.20a. This device consists of a wheel free to spin about an axle that is pivoted at a distance h from the center of mass of the wheel. When given an angular velocity $\boldsymbol{\omega}$ about the axle, the wheel has an angular momentum $\mathbf{L} = I\boldsymbol{\omega}$ directed along the axle as shown. Let us consider the torque acting on the wheel about the pivot O. Again, the force $\mathbf{n}$ exerted by the support on the axle produces no torque about O, and the force of gravity $M\mathbf{g}$ produces a torque of magnitude Mgh about O, where the axle is perpendicular to the support. The direction of this torque is perpendicular to the axle (and perpendicular to $\mathbf{L}$), as shown in Figure 11.20a. This torque causes the angular momentum to change in the direction perpendicular to the axle. Hence, the axle moves in the direction of the torque—that is, in the horizontal plane.

To simplify the description of the system, we must make an assumption: The total angular momentum of the precessing wheel is the sum of the angular momentum $I\boldsymbol{\omega}$ due to the spinning and the angular momentum due to the motion of

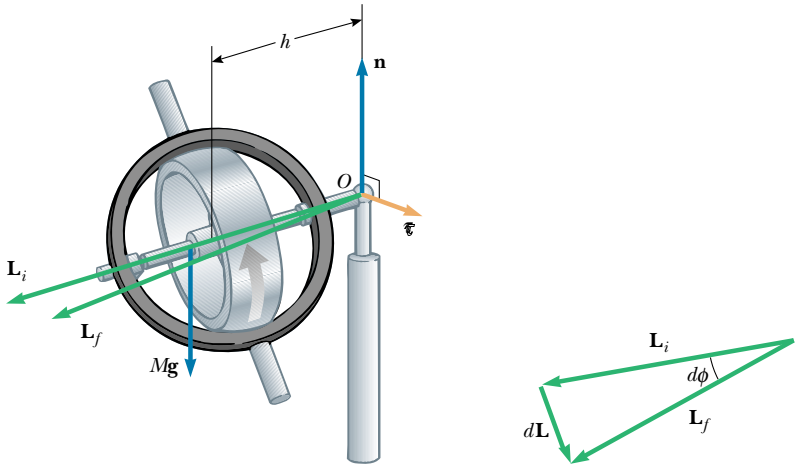
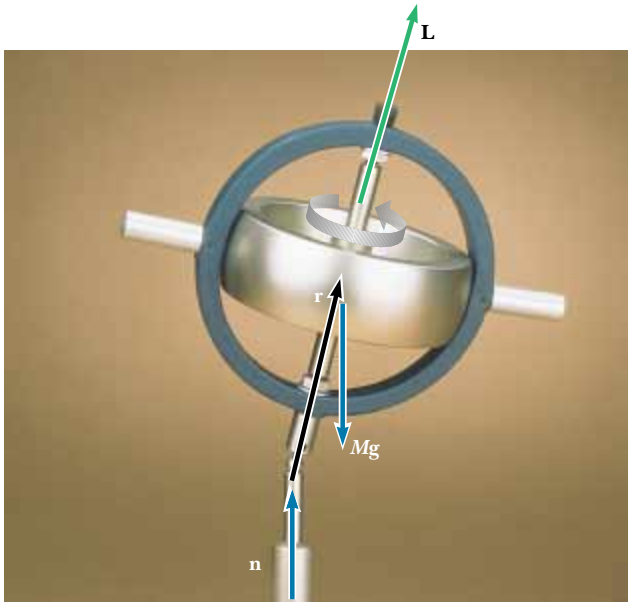


Figure 11.20 (a) The motion of a simple gyroscope pivoted a distance h from its center of mass. The force of gravity $M\mathbf{g}$ produces a torque about the pivot, and this torque is perpendicular to the axle. (b) This torque results in a change in angular momentum $d\mathbf{L}$ in a direction perpendicular to the axle. The axle sweeps out an angle $d\phi$ in a time dt.



This toy gyroscope undergoes precessional motion about the vertical axis as it spins about its axis of symmetry. The only forces acting on it are the force of gravity $M\mathbf{g}$ and the upward force of the pivot $\mathbf{n}$. The direction of its angular momentum $\mathbf{L}$ is along the axis of symmetry. The torque and $\Delta\mathbf{L}$ are directed into the page. *(Courtesy of Central Scientific Company)*

the center of mass about the pivot. In our treatment, we shall neglect the contribution from the center-of-mass motion and take the total angular momentum to be just $I\boldsymbol{\omega}$. In practice, this is a good approximation if $\boldsymbol{\omega}$ is made very large.

In a time dt, the torque due to the gravitational force changes the angular momentum of the system by $d\mathbf{L} = \boldsymbol{\tau}\, dt$. When added vectorially to the original total

angular momentum $I\omega$, this additional angular momentum causes a shift in the direction of the total angular momentum.

The vector diagram in Figure 11.20b shows that in the time dt, the angular momentum vector rotates through an angle $d\phi$, which is also the angle through which the axle rotates. From the vector triangle formed by the vectors $\mathbf{L}_i$, $\mathbf{L}_f$, and $d\mathbf{L}$, we see that

$$\sin(d\phi) \approx d\phi = \frac{dL}{L} = \frac{(Mgh)\,dt}{L}$$

where we have used the fact that, for small values of any angle θ, $\sin\theta \approx \theta$. Dividing through by dt and using the relationship $L = I\omega$, we find that the rate at which the axle rotates about the vertical axis is

$$\omega_p = \frac{d\phi}{dt} = \frac{Mgh}{I\omega} \tag{11.28}$$

Precessional frequency

The angular speed ω_p is called the **precessional frequency.** This result is valid only when $\omega_p \ll \omega$. Otherwise, a much more complicated motion is involved. As you can see from Equation 11.28, the condition $\omega_p \ll \omega$ is met when $I\omega$ is great compared with Mgh. Furthermore, note that the precessional frequency decreases as ω increases—that is, as the wheel spins faster about its axis of symmetry.

Quick Quiz 11.6

How much work is done by the force of gravity when a top precesses through one complete circle?

Optional Section

11.7 ANGULAR MOMENTUM AS A FUNDAMENTAL QUANTITY

We have seen that the concept of angular momentum is very useful for describing the motion of macroscopic systems. However, the concept also is valid on a submicroscopic scale and has been used extensively in the development of modern theories of atomic, molecular, and nuclear physics. In these developments, it was found that the angular momentum of a system is a fundamental quantity. The word *fundamental* in this context implies that angular momentum is an intrinsic property of atoms, molecules, and their constituents, a property that is a part of their very nature.

To explain the results of a variety of experiments on atomic and molecular systems, we rely on the fact that the angular momentum has discrete values. These discrete values are multiples of the fundamental unit of angular momentum $\hbar = h/2\pi$, where h is called Planck's constant:

$$\text{Fundamental unit of angular momentum} = \hbar = 1.054 \times 10^{-34}\ \text{kg}\cdot\text{m}^2/\text{s}$$

Let us accept this postulate without proof for the time being and show how it can be used to estimate the angular speed of a diatomic molecule. Consider the O_2 molecule as a rigid rotor, that is, two atoms separated by a fixed distance d and rotating about the center of mass (Fig. 11.21). Equating the angular momentum to the fundamental unit $\hbar$, we can estimate the lowest angular speed:

$$I_{CM}\omega \approx \hbar \qquad \text{or} \qquad \omega \approx \frac{\hbar}{I_{CM}}$$

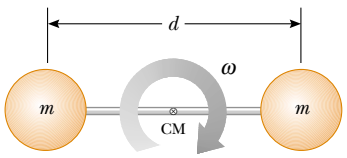


Figure 11.21 The rigid-rotor model of a diatomic molecule. The rotation occurs about the center of mass in the plane of the page.

In Example 10.3, we found that the moment of inertia of the O_2 molecule about this axis of rotation is $1.95 \times 10^{-46}\ \text{kg}\cdot\text{m}^2$. Therefore,

$$\omega \approx \frac{\hbar}{I_{\text{CM}}} = \frac{1.054 \times 10^{-34}\ \text{kg}\cdot\text{m}^2/\text{s}}{1.95 \times 10^{-46}\ \text{kg}\cdot\text{m}^2} = 5.41 \times 10^{11}\ \text{rad/s}$$

Actual angular speeds are multiples of this smallest possible value.

This simple example shows that certain classical concepts and models, when properly modified, might be useful in describing some features of atomic and molecular systems. A wide variety of phenomena on the submicroscopic scale can be explained only if we assume discrete values of the angular momentum associated with a particular type of motion.

The Danish physicist Niels Bohr (1885–1962) accepted and adopted this radical idea of discrete angular momentum values in developing his theory of the hydrogen atom. Strictly classical models were unsuccessful in describing many properties of the hydrogen atom. Bohr postulated that the electron could occupy only those circular orbits about the proton for which the orbital angular momentum was equal to $n\hbar$, where n is an integer. That is, he made the bold assumption that orbital angular momentum is quantized. From this simple model, the rotational frequencies of the electron in the various orbits can be estimated (see Problem 43).

Summary

The **total kinetic energy** of a rigid object rolling on a rough surface without slipping equals the rotational kinetic energy about its center of mass, $\frac{1}{2}I_{\text{CM}}\omega^2$, plus the translational kinetic energy of the center of mass, $\frac{1}{2}Mv_{\text{CM}}{}^2$:

$$K = \tfrac{1}{2}I_{\text{CM}}\omega^2 + \tfrac{1}{2}Mv_{\text{CM}}{}^2 \qquad \textbf{(11.4)}$$

The **torque** $\boldsymbol{\tau}$ due to a force $\mathbf{F}$ about an origin in an inertial frame is defined to be

$$\boldsymbol{\tau} \equiv \mathbf{r} \times \mathbf{F} \qquad \textbf{(11.7)}$$

Given two vectors $\mathbf{A}$ and $\mathbf{B}$, the **cross product** $\mathbf{A} \times \mathbf{B}$ is a vector $\mathbf{C}$ having a magnitude

$$C \equiv AB \sin\theta \qquad \textbf{(11.9)}$$

where θ is the angle between $\mathbf{A}$ and $\mathbf{B}$. The direction of the vector $\mathbf{C} = \mathbf{A} \times \mathbf{B}$ is perpendicular to the plane formed by $\mathbf{A}$ and $\mathbf{B}$, and this direction is determined by the right-hand rule.

The **angular momentum** $\mathbf{L}$ of a particle having linear momentum $\mathbf{p} = m\mathbf{v}$ is

$$\mathbf{L} \equiv \mathbf{r} \times \mathbf{p} \qquad \textbf{(11.15)}$$

where $\mathbf{r}$ is the vector position of the particle relative to an origin in an inertial frame.

The **net external torque** acting on a particle or rigid object is equal to the time rate of change of its angular momentum:

$$\sum \boldsymbol{\tau}_{\text{ext}} = \frac{d\mathbf{L}}{dt} \qquad \textbf{(11.20)}$$

The z *component* of **angular momentum** of a rigid object rotating about a fixed z axis is

$$L_z = I\omega \qquad \textbf{(11.21)}$$

where I is the moment of inertia of the object about the axis of rotation and ω is its angular speed.

The **net external torque** acting on a rigid object equals the product of its moment of inertia about the axis of rotation and its angular acceleration:

$$\sum \tau_{\text{ext}} = I\alpha \qquad \textbf{(11.23)}$$

If the net external torque acting on a system is zero, then the total angular momentum of the system is constant. Applying this **law of conservation of angular momentum** to a system whose moment of inertia changes gives

$$I_i\omega_i = I_f\omega_f = \text{constant} \qquad \textbf{(11.27)}$$

Questions

1. Is it possible to calculate the torque acting on a rigid body without specifying a center of rotation? Is the torque independent of the location of the center of rotation?
2. Is the triple product defined by $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$ a scalar or a vector quantity? Explain why the operation $(\mathbf{A}\cdot\mathbf{B})\times\mathbf{C}$ has no meaning.
3. In some motorcycle races, the riders drive over small hills, and the motorcycles become airborne for a short time. If a motorcycle racer keeps the throttle open while leaving the hill and going into the air, the motorcycle tends to nose upward. Why does this happen?
4. If the torque acting on a particle about a certain origin is zero, what can you say about its angular momentum about that origin?
5. Suppose that the velocity vector of a particle is completely specified. What can you conclude about the direction of its angular momentum vector with respect to the direction of motion?
6. If a single force acts on an object, and the torque caused by that force is nonzero about some point, is there any other point about which the torque is zero?
7. If a system of particles is in motion, is it possible for the total angular momentum to be zero about some origin? Explain.
8. A ball is thrown in such a way that it does not spin about its own axis. Does this mean that the angular momentum is zero about an arbitrary origin? Explain.
9. In a tape recorder, the tape is pulled past the read-and-write heads at a constant speed by the drive mechanism. Consider the reel from which the tape is pulled—as the tape is pulled off it, the radius of the roll of remaining tape decreases. How does the torque on the reel change with time? How does the angular speed of the reel change with time? If the tape mechanism is suddenly turned on so that the tape is quickly pulled with a great force, is the tape more likely to break when pulled from a nearly full reel or a nearly empty reel?
10. A scientist at a hotel sought assistance from a bellhop to carry a mysterious suitcase. When the unaware bellhop rounded a corner carrying the suitcase, it suddenly moved away from him for some unknown reason. At this point, the alarmed bellhop dropped the suitcase and ran off. What do you suppose might have been in the suitcase?
11. When a cylinder rolls on a horizontal surface as in Figure 11.3, do any points on the cylinder have only a vertical component of velocity at some instant? If so, where are they?
12. Three objects of uniform density—a solid sphere, a solid cylinder, and a hollow cylinder—are placed at the top of an incline (Fig. Q11.12). If they all are released from rest at the same elevation and roll without slipping, which object reaches the bottom first? Which reaches it last? You should try this at home and note that the result is independent of the masses and the radii of the objects.

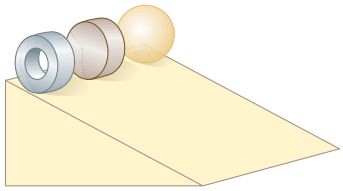


Figure Q11.12 Which object wins the race?

13. A mouse is initially at rest on a horizontal turntable mounted on a frictionless vertical axle. If the mouse begins to walk around the perimeter, what happens to the turntable? Explain.
14. Stars originate as large bodies of slowly rotating gas. Because of gravity, these regions of gas slowly decrease in size. What happens to the angular speed of a star as it shrinks? Explain.
15. Often, when a high diver wants to execute a flip in midair, she draws her legs up against her chest. Why does this make her rotate faster? What should she do when she wants to come out of her flip?
16. As a tether ball winds around a thin pole, what happens to its angular speed? Explain.

17. Two solid spheres—a large, massive sphere and a small sphere with low mass—are rolled down a hill. Which sphere reaches the bottom of the hill first? Next, a large, low-density sphere and a small, high-density sphere having the same mass are rolled down the hill. Which one reaches the bottom first in this case?
18. Suppose you are designing a car for a coasting race—the cars in this race have no engines; they simply coast down a hill. Do you want to use large wheels or small wheels? Do you want to use solid, disk-like wheels or hoop-like wheels? Should the wheels be heavy or light?
19. Why do tightrope walkers carry a long pole to help themselves keep their balance?
20. Two balls have the same size and mass. One is hollow, whereas the other is solid. How would you determine which is which without breaking them apart?
21. A particle is moving in a circle with constant speed. Locate one point about which the particle's angular momentum is constant and another about which it changes with time.
22. If global warming occurs over the next century, it is likely that some polar ice will melt and the water will be distributed closer to the equator. How would this change the moment of inertia of the Earth? Would the length of the day (one revolution) increase or decrease?

PROBLEMS

1, 2, 3 = straightforward, intermediate, challenging = full solution available in the *Student Solutions Manual and Study Guide*
WEB = solution posted at **http://www.saunderscollege.com/physics/** = Computer useful in solving problem = Interactive Physics
= paired numerical/symbolic problems

Section 11.1 Rolling Motion of a Rigid Object

WEB 1. A cylinder of mass 10.0 kg rolls without slipping on a horizontal surface. At the instant its center of mass has a speed of 10.0 m/s, determine (a) the translational kinetic energy of its center of mass, (b) the rotational energy about its center of mass, and (c) its total energy.

2. A bowling ball has a mass of 4.00 kg, a moment of inertia of 1.60×10^{-2} kg·m^2, and a radius of 0.100 m. If it rolls down the lane without slipping at a linear speed of 4.00 m/s, what is its total energy?
3. A bowling ball has a mass M, a radius R, and a moment of inertia $\frac{2}{5}MR^2$. If it starts from rest, how much work must be done on it to set it rolling without slipping at a linear speed v? Express the work in terms of M and v.
4. A uniform solid disk and a uniform hoop are placed side by side at the top of an incline of height h. If they are released from rest and roll without slipping, determine their speeds when they reach the bottom. Which object reaches the bottom first?
5. (a) Determine the acceleration of the center of mass of a uniform solid disk rolling down an incline making an angle θ with the horizontal. Compare this acceleration with that of a uniform hoop. (b) What is the minimum coefficient of friction required to maintain pure rolling motion for the disk?
6. A ring of mass 2.40 kg, inner radius 6.00 cm, and outer radius 8.00 cm rolls (without slipping) up an inclined plane that makes an angle of $\theta = 36.9°$ (Fig. P11.6). At the moment the ring is at position $x = 2.00$ m up the plane, its speed is 2.80 m/s. The ring continues up the plane for some additional distance and then rolls back down. It does not roll off the top end. How far up the plane does it go?

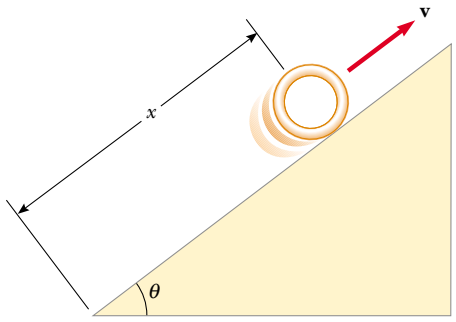


Figure P11.6

7. A metal can containing condensed mushroom soup has a mass of 215 g, a height of 10.8 cm, and a diameter of 6.38 cm. It is placed at rest on its side at the top of a 3.00-m-long incline that is at an angle of 25.0° to the horizontal and is then released to roll straight down. Assuming energy conservation, calculate the moment of inertia of the can if it takes 1.50 s to reach the bottom of the incline. Which pieces of data, if any, are unnecessary for calculating the solution?
8. A tennis ball is a hollow sphere with a thin wall. It is set rolling without slipping at 4.03 m/s on the horizontal section of a track, as shown in Figure P11.8. It rolls around the inside of a vertical circular loop 90.0 cm in diameter and finally leaves the track at a point 20.0 cm below the horizontal section. (a) Find the speed of the ball at the top of the loop. Demonstrate that it will not fall from the track. (b) Find its speed as it leaves the track. (c) Suppose that static friction between the ball and the track was negligible, so that the ball slid instead of rolling. Would its speed

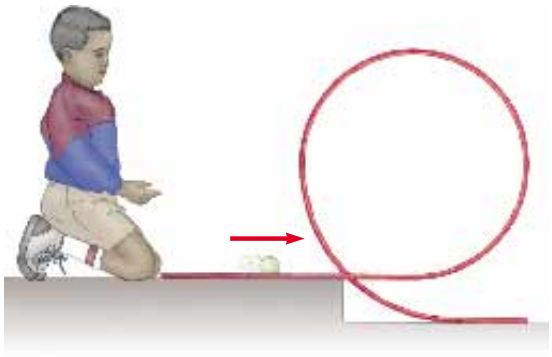


Figure P11.8

then be higher, lower, or the same at the top of the loop? Explain.

Section 11.2 The Vector Product and Torque

9. Given $\mathbf{M} = 6\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $\mathbf{N} = 2\mathbf{i} - \mathbf{j} - 3\mathbf{k}$, calculate the vector product $\mathbf{M} \times \mathbf{N}$.

10. The vectors 42.0 cm at 15.0° and 23.0 cm at 65.0° both start from the origin. Both angles are measured counterclockwise from the x axis. The vectors form two sides of a parallelogram. (a) Find the area of the parallelogram. (b) Find the length of its longer diagonal.

WEB **11.** Two vectors are given by $\mathbf{A} = -3\mathbf{i} + 4\mathbf{j}$ and $\mathbf{B} = 2\mathbf{i} + 3\mathbf{j}$. Find (a) $\mathbf{A} \times \mathbf{B}$ and (b) the angle between $\mathbf{A}$ and $\mathbf{B}$.

12. For the vectors $\mathbf{A} = -3\mathbf{i} + 7\mathbf{j} - 4\mathbf{k}$ and $\mathbf{B} = 6\mathbf{i} - 10\mathbf{j} + 9\mathbf{k}$, evaluate the expressions (a) $\cos^{-1}\ (\mathbf{A}\cdot\mathbf{B}/AB)$ and (b) $\sin^{-1}\ (|\mathbf{A} \times \mathbf{B}|/AB)$. (c) Which give(s) the angle between the vectors?

13. A force of $\mathbf{F} = 2.00\mathbf{i} + 3.00\mathbf{j}$ N is applied to an object that is pivoted about a fixed axis aligned along the z coordinate axis. If the force is applied at the point $\mathbf{r} = (4.00\mathbf{i} + 5.00\mathbf{j} + 0\mathbf{k})$ m, find (a) the magnitude of the net torque about the z axis and (b) the direction of the torque vector $\boldsymbol{\tau}$.

14. A student claims that she has found a vector $\mathbf{A}$ such that $(2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}) \times \mathbf{A} = (4\mathbf{i} + 3\mathbf{j} - \mathbf{k})$. Do you believe this claim? Explain.

15. Vector $\mathbf{A}$ is in the negative y direction, and vector $\mathbf{B}$ is in the negative x direction. What are the directions of (a) $\mathbf{A} \times \mathbf{B}$ and (b) $\mathbf{B} \times \mathbf{A}$?

16. A particle is located at the vector position $\mathbf{r} = (\mathbf{i} + 3\mathbf{j})$ m, and the force acting on it is $\mathbf{F} = (3\mathbf{i} + 2\mathbf{j})$ N. What is the torque about (a) the origin and (b) the point having coordinates (0, 6) m?

17. If $|\mathbf{A} \times \mathbf{B}| = \mathbf{A}\cdot\mathbf{B}$, what is the angle between $\mathbf{A}$ and $\mathbf{B}$?

18. Two forces $\mathbf{F}_1$ and $\mathbf{F}_2$ act along the two sides of an equilateral triangle, as shown in Figure P11.18. Point O is the intersection of the altitudes of the triangle. Find a third force $\mathbf{F}_3$ to be applied at B and along BC that will make the total torque about the point O be zero. Will the total torque change if $\mathbf{F}_3$ is applied not at B, but rather at any other point along BC?

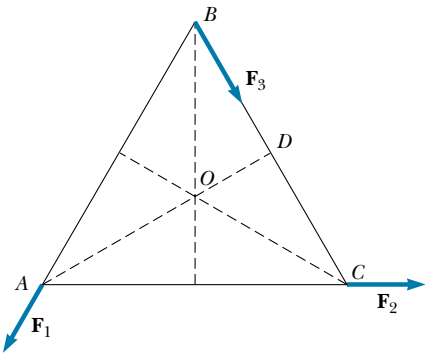


Figure P11.18

Section 11.3 Angular Momentum of a Particle

19. A light, rigid rod 1.00 m in length joins two particles—with masses 4.00 kg and 3.00 kg—at its ends. The combination rotates in the xy plane about a pivot through the center of the rod (Fig. P11.19). Determine the angular momentum of the system about the origin when the speed of each particle is 5.00 m/s.

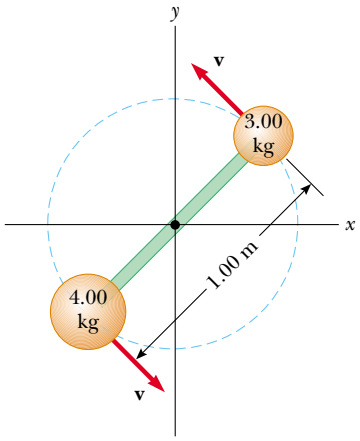


Figure P11.19

20. A 1.50-kg particle moves in the xy plane with a velocity of $\mathbf{v} = (4.20\mathbf{i} - 3.60\mathbf{j})$ m/s. Determine the particle's angular momentum when its position vector is $\mathbf{r} = (1.50\mathbf{i} + 2.20\mathbf{j})$ m.

WEB **21.** The position vector of a particle of mass 2.00 kg is given as a function of time by $\mathbf{r} = (6.00\mathbf{i} + 5.00t\mathbf{j})$ m. Determine the angular momentum of the particle about the origin as a function of time.

22. A conical pendulum consists of a bob of mass m in motion in a circular path in a horizontal plane, as shown in Figure P11.22. During the motion, the supporting wire of length ℓ maintains the constant angle θ with the vertical. Show that the magnitude of the angular momen-

tum of the mass about the center of the circle is

$$L = (m^2 g \ell^3 \sin^4 \theta / \cos \theta)^{1/2}$$

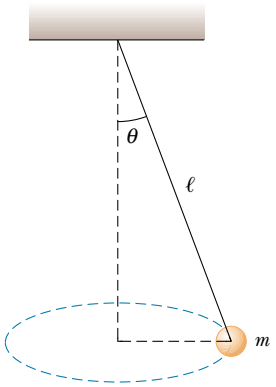


Figure P11.22

23. A particle of mass m moves in a circle of radius R at a constant speed v, as shown in Figure P11.23. If the motion begins at point Q, determine the angular momentum of the particle about point P as a function of time.

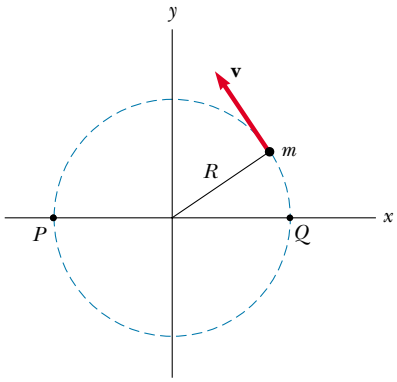


Figure P11.23

24. A 4.00-kg mass is attached to a light cord that is wound around a pulley (see Fig. 10.20). The pulley is a uniform solid cylinder with a radius of 8.00 cm and a mass of 2.00 kg. (a) What is the net torque on the system about the point O? (b) When the mass has a speed v, the pulley has an angular speed $\omega = v/R$. Determine the total angular momentum of the system about O. (c) Using the fact that $\boldsymbol{\tau} = d\mathbf{L}/dt$ and your result from part (b), calculate the acceleration of the mass.

25. A particle of mass m is shot with an initial velocity $\mathbf{v}_i$ and makes an angle θ with the horizontal, as shown in Figure P11.25. The particle moves in the gravitational field of the Earth. Find the angular momentum of the particle about the origin when the particle is (a) at the origin, (b) at the highest point of its trajectory, and (c) just about to hit the ground. (d) What torque causes its angular momentum to change?

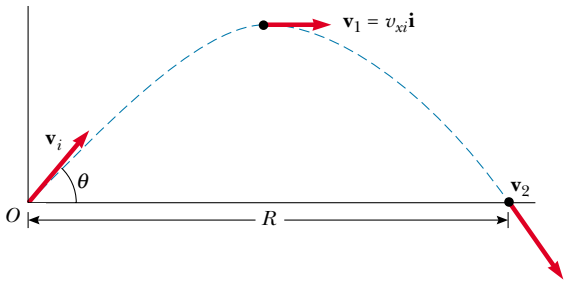


Figure P11.25

26. Heading straight toward the summit of Pike's Peak, an airplane of mass 12 000 kg flies over the plains of Kansas at a nearly constant altitude of 4.30 km and with a constant velocity of 175 m/s westward. (a) What is the airplane's vector angular momentum relative to a wheat farmer on the ground directly below the airplane? (b) Does this value change as the airplane continues its motion along a straight line? (c) What is its angular momentum relative to the summit of Pike's Peak?

27. A ball of mass m is fastened at the end of a flagpole connected to the side of a tall building at point P, as shown in Figure P11.27. The length of the flagpole is ℓ, and θ is the angle the flagpole makes with the horizontal. Suppose that the ball becomes loose and starts to fall. Determine the angular momentum (as a function of time) of the ball about point P. Neglect air resistance.

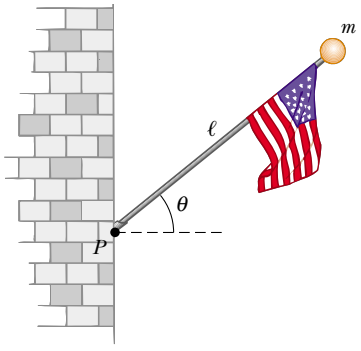


Figure P11.27

28. A fireman clings to a vertical ladder and directs the nozzle of a hose horizontally toward a burning building. The rate of water flow is 6.31 kg/s, and the nozzle speed is 12.5 m/s. The hose passes between the fireman's feet, which are 1.30 m vertically below the nozzle. Choose the origin to be inside the hose between the fireman's

feet. What torque must the fireman exert on the hose? That is, what is the rate of change of angular momentum of the water?

Section 11.4 Angular Momentum of a Rotating Rigid Object

29. A uniform solid sphere with a radius of 0.500 m and a mass of 15.0 kg turns counterclockwise about a vertical axis through its center. Find its vector angular momentum when its angular speed is 3.00 rad/s.

30. A uniform solid disk with a mass of 3.00 kg and a radius of 0.200 m rotates about a fixed axis perpendicular to its face. If the angular speed is 6.00 rad/s, calculate the angular momentum of the disk when the axis of rotation (a) passes through its center of mass and (b) passes through a point midway between the center and the rim.

31. A particle with a mass of 0.400 kg is attached to the 100-cm mark of a meter stick with a mass of 0.100 kg. The meter stick rotates on a horizontal, frictionless table with an angular speed of 4.00 rad/s. Calculate the angular momentum of the system when the stick is pivoted about an axis (a) perpendicular to the table through the 50.0-cm mark and (b) perpendicular to the table through the 0-cm mark.

32. The hour and minute hands of Big Ben, the famous Parliament Building tower clock in London, are 2.70 m and 4.50 m long and have masses of 60.0 kg and 100 kg, respectively. Calculate the total angular momentum of these hands about the center point. Treat the hands as long thin rods.

Section 11.5 Conservation of Angular Momentum

33. A cylinder with a moment of inertia of I_1 rotates about a vertical, frictionless axle with angular velocity ω_i. A second cylinder that has a moment of inertia of I_2 and initially is not rotating drops onto the first cylinder (Fig. P11.33). Because of friction between the surfaces, the two eventually reach the same angular speed ω_f. (a) Calculate ω_f. (b) Show that the kinetic energy of the system decreases in this interaction and calculate the ratio of the final rotational energy to the initial rotational energy.

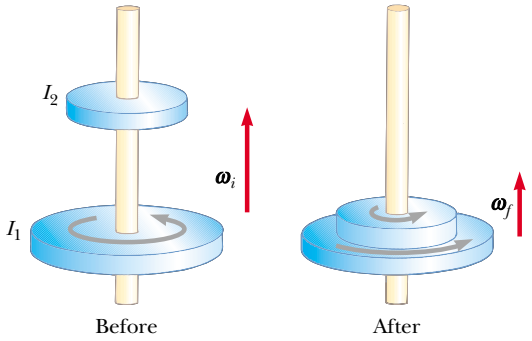


Figure P11.33

34. A playground merry-go-round of radius $R = 2.00$ m has a moment of inertia of $I = 250\ \text{kg}\cdot\text{m}^2$ and is rotating at 10.0 rev/min about a frictionless vertical axle. Facing the axle, a 25.0-kg child hops onto the merry-go-round and manages to sit down on its edge. What is the new angular speed of the merry-go-round?

35. A student sits on a freely rotating stool holding two weights, each of which has a mass of 3.00 kg. When his arms are extended horizontally, the weights are 1.00 m from the axis of rotation and he rotates with an angular speed of 0.750 rad/s. The moment of inertia of the student plus stool is $3.00\ \text{kg}\cdot\text{m}^2$ and is assumed to be constant. The student pulls the weights inward horizontally to a position 0.300 m from the rotation axis. (a) Find the new angular speed of the student. (b) Find the kinetic energy of the rotating system before and after he pulls the weights inward.

36. A uniform rod with a mass of 100 g and a length of 50.0 cm rotates in a horizontal plane about a fixed, vertical, frictionless pin passing through its center. Two small beads, each having a mass 30.0 g, are mounted on the rod so that they are able to slide without friction along its length. Initially, the beads are held by catches at positions 10.0 cm on each side of center; at this time, the system rotates at an angular speed of 20.0 rad/s. Suddenly, the catches are released, and the small beads slide outward along the rod. Find (a) the angular speed of the system at the instant the beads reach the ends of the rod and (b) the angular speed of the rod after the beads fly off the rod's ends.

37. WEB A 60.0-kg woman stands at the rim of a horizontal turntable having a moment of inertia of $500\ \text{kg}\cdot\text{m}^2$ and a radius of 2.00 m. The turntable is initially at rest and is free to rotate about a frictionless, vertical axle through its center. The woman then starts walking around the rim clockwise (as viewed from above the system) at a constant speed of 1.50 m/s relative to the Earth. (a) In what direction and with what angular speed does the turntable rotate? (b) How much work does the woman do to set herself and the turntable into motion?

38. A puck with a mass of 80.0 g and a radius of 4.00 cm slides along an air table at a speed of 1.50 m/s, as shown in Figure P11.38a. It makes a glancing collision

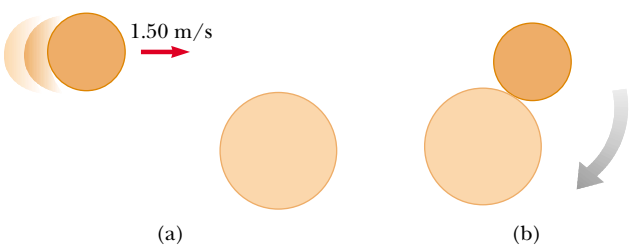


Figure P11.38

with a second puck having a radius of 6.00 cm and a mass of 120 g (initially at rest) such that their rims just touch. Because their rims are coated with instant-acting glue, the pucks stick together and spin after the collision (Fig. P11.38b). (a) What is the angular momentum of the system relative to the center of mass? (b) What is the angular speed about the center of mass?

39. A wooden block of mass M resting on a frictionless horizontal surface is attached to a rigid rod of length ℓ and of negligible mass (Fig. P11.39). The rod is pivoted at the other end. A bullet of mass m traveling parallel to the horizontal surface and normal to the rod with speed v hits the block and becomes embedded in it. (a) What is the angular momentum of the bullet–block system? (b) What fraction of the original kinetic energy is lost in the collision?

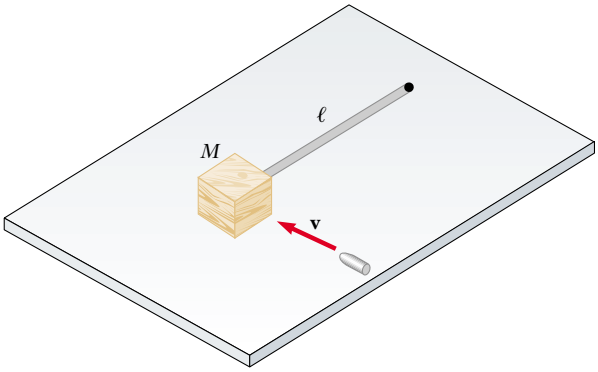


Figure P11.39

40. A space station shaped like a giant wheel has a radius of 100 m and a moment of inertia of 5.00×10^8 kg·m^2. A crew of 150 are living on the rim, and the station's rotation causes the crew to experience an acceleration of $1g$ (Fig. P11.40). When 100 people move to the center of the station for a union meeting, the angular speed changes. What acceleration is experienced by the managers remaining at the rim? Assume that the average mass of each inhabitant is 65.0 kg.

41. A wad of sticky clay of mass m and velocity $\mathbf{v}_i$ is fired at a solid cylinder of mass M and radius R (Fig. P11.41). The cylinder is initially at rest and is mounted on a fixed horizontal axle that runs through the center of mass. The line of motion of the projectile is perpendicular to the axle and at a distance d, less than R, from the center. (a) Find the angular speed of the system just after the clay strikes and sticks to the surface of the cylinder. (b) Is mechanical energy conserved in this process? Explain your answer.

42. Suppose a meteor with a mass of 3.00×10^{13} kg is moving at 30.0 km/s relative to the center of the Earth and strikes the Earth. What is the order of magnitude of the maximum possible decrease in the angular speed of the Earth due to this collision? Explain your answer.

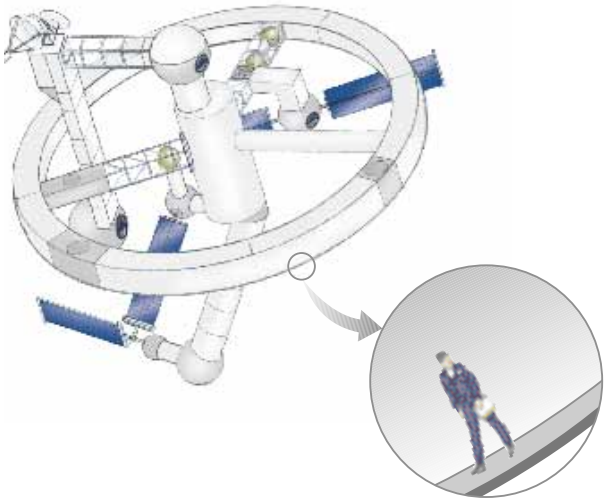


Figure P11.40

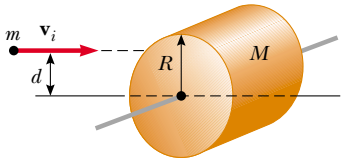


Figure P11.41

(Optional)

Section 11.7 Angular Momentum as a Fundamental Quantity

43. In the Bohr model of the hydrogen atom, the electron moves in a circular orbit of radius 0.529×10^{-10} m around the proton. Assuming that the orbital angular momentum of the electron is equal to $h/2\pi$, calculate (a) the orbital speed of the electron, (b) the kinetic energy of the electron, and (c) the angular speed of the electron's motion.

ADDITIONAL PROBLEMS

44. Review Problem. A rigid, massless rod has three equal masses attached to it, as shown in Figure P11.44. The rod is free to rotate in a vertical plane about a frictionless axle perpendicular to the rod through the point P, and it is released from rest in the horizontal position at $t = 0$. Assuming m and d are known, find (a) the moment of inertia of the system about the pivot, (b) the torque acting on the system at $t = 0$, (c) the angular acceleration of the system at $t = 0$, (d) the linear acceleration of the mass labeled "3" at $t = 0$, (e) the maximum

kinetic energy of the system, (f) the maximum angular speed attained by the rod, (g) the maximum angular momentum of the system, and (h) the maximum speed attained by the mass labeled "2."

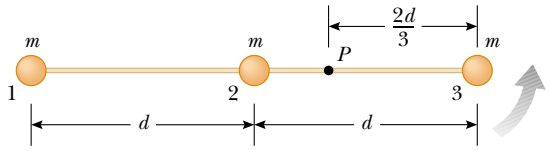


Figure P11.44

45. A uniform solid sphere of radius r is placed on the inside surface of a hemispherical bowl having a much greater radius R. The sphere is released from rest at an angle θ to the vertical and rolls without slipping (Fig. P11.45). Determine the angular speed of the sphere when it reaches the bottom of the bowl.

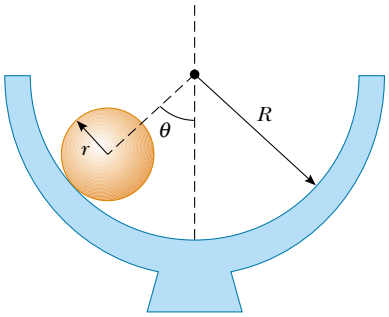


Figure P11.45

46. A 100-kg uniform horizontal disk of radius 5.50 m turns without friction at 2.50 rev/s on a vertical axis through its center, as shown in Figure P11.46. A feedback mechanism senses the angular speed of the disk, and a drive motor at A ensures that the angular speed remains constant. While the disk turns, a 1.20-kg mass on top of the disk slides outward in a radial slot. The 1.20-kg mass starts at the center of the disk at time $t = 0$ and moves outward with a constant speed of 1.25 cm/s relative to the disk until it reaches the edge at $t = 440$ s. The sliding mass experiences no friction. Its motion is constrained by a brake at B so that its radial speed remains constant. The constraint produces tension in a light string tied to the mass. (a) Find the torque as a function of time that the drive motor must provide while the mass is sliding. (b) Find the value of this torque at $t = 440$ s, just before the sliding mass finishes its motion. (c) Find the power that the drive motor must deliver as a function of time. (d) Find the value of the power when the sliding mass is just reaching the end of the slot. (e) Find the string tension as a function of time. (f) Find the work done by the drive motor during the 440-s motion. (g) Find the work done by the string brake on the sliding mass. (h) Find the total work done on the system consisting of the disk and the sliding mass.

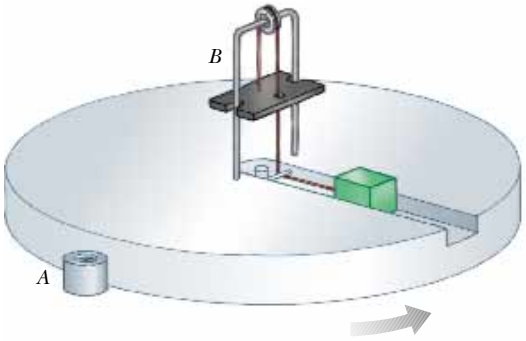


Figure P11.46

47. A string is wound around a uniform disk of radius R and mass M. The disk is released from rest when the string is vertical and its top end is tied to a fixed bar (Fig. P11.47). Show that (a) the tension in the string is one-third the weight of the disk, (b) the magnitude of the acceleration of the center of mass is $2g/3$, and (c) the speed of the center of mass is $(4gh/3)^{1/2}$ as the disk descends. Verify your answer to part (c) using the energy approach.

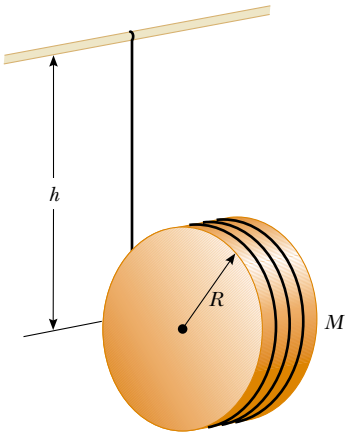


Figure P11.47

48. Comet Halley moves about the Sun in an elliptical orbit, with its closest approach to the Sun being about 0.590 AU and its greatest distance from the Sun being 35.0 AU (1 AU = the average Earth–Sun distance). If the comet's speed at its closest approach is 54.0 km/s,

what is its speed when it is farthest from the Sun? The angular momentum of the comet about the Sun is conserved because no torque acts on the comet. The gravitational force exerted by the Sun on the comet has a moment arm of zero.

49. A constant horizontal force **F** is applied to a lawn roller having the form of a uniform solid cylinder of radius R and mass M (Fig. P11.49). If the roller rolls without slipping on the horizontal surface, show that (a) the acceleration of the center of mass is $2\mathbf{F}/3M$ and that (b) the minimum coefficient of friction necessary to prevent slipping is $F/3Mg$. (*Hint:* Consider the torque with respect to the center of mass.)

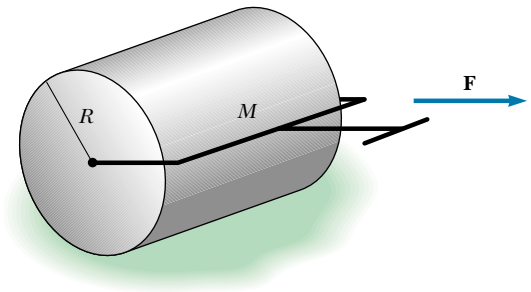


Figure P11.49

50. A light rope passes over a light, frictionless pulley. A bunch of bananas of mass M is fastened at one end, and a monkey of mass M clings to the other (Fig. P11.50).

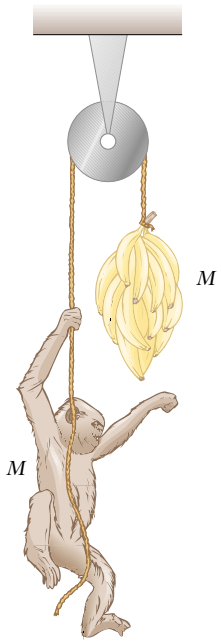


Figure P11.50

The monkey climbs the rope in an attempt to reach the bananas. (a) Treating the system as consisting of the monkey, bananas, rope, and pulley, evaluate the net torque about the pulley axis. (b) Using the results to part (a), determine the total angular momentum about the pulley axis and describe the motion of the system. Will the monkey reach the bananas?

51. A solid sphere of mass m and radius r rolls without slipping along the track shown in Figure P11.51. The sphere starts from rest with its lowest point at height h above the bottom of a loop of radius R, which is much larger than r. (a) What is the minimum value that h can have (in terms of R) if the sphere is to complete the loop? (b) What are the force components on the sphere at point P if $h = 3R$?

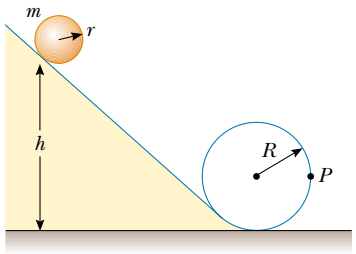


Figure P11.51

52. A thin rod with a mass of 0.630 kg and a length of 1.24 m is at rest, hanging vertically from a strong fixed hinge at its top end. Suddenly, a horizontal impulsive force $(14.7\mathbf{i})$ N is applied to it. (a) Suppose that the force acts at the bottom end of the rod. Find the acceleration of the rod's center of mass and the horizontal force that the hinge exerts. (b) Suppose that the force acts at the midpoint of the rod. Find the acceleration of this point and the horizontal hinge reaction. (c) Where can the impulse be applied so that the hinge exerts no horizontal force? (This point is called the *center of percussion*.)

53. At one moment, a bowling ball is both sliding and spinning on a horizontal surface such that its rotational kinetic energy equals its translational kinetic energy. Let v_{CM} represent the ball's center-of-mass speed relative to the surface. Let v_r represent the speed of the topmost point on the ball's surface relative to the center of mass. Find the ratio v_{CM}/v_r.

54. A projectile of mass m moves to the right with speed v_i (Fig. P11.54a). The projectile strikes and sticks to the end of a stationary rod of mass M and length d that is pivoted about a frictionless axle through its center (Fig. P11.54b). (a) Find the angular speed of the system right after the collision. (b) Determine the fractional loss in mechanical energy due to the collision.

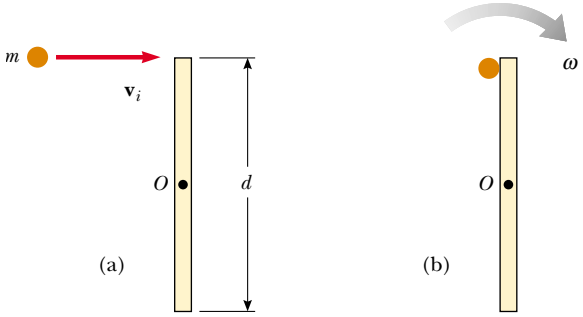


Figure P11.54

55. A mass m is attached to a cord passing through a small hole in a frictionless, horizontal surface (Fig. P11.55). The mass is initially orbiting with speed v_i in a circle of radius r_i. The cord is then slowly pulled from below, and the radius of the circle decreases to r. (a) What is the speed of the mass when the radius is r? (b) Find the tension in the cord as a function of r. (c) How much work W is done in moving m from r_i to r? (*Note:* The tension depends on r.) (d) Obtain numerical values for v, T, and W when $r = 0.100$ m, $m = 50.0$ g, $r_i = 0.300$ m, and $v_i = 1.50$ m/s.

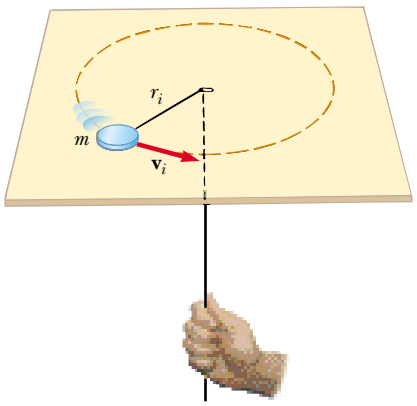


Figure P11.55

56. A bowler releases a bowling ball with no spin, sending it sliding straight down the alley toward the pins. The ball continues to slide for some distance before its motion becomes rolling without slipping; of what order of magnitude is this distance? State the quantities you take as data, the values you measure or estimate for them, and your reasoning.

57. Following Thanksgiving dinner, your uncle falls into a deep sleep while sitting straight up and facing the television set. A naughty grandchild balances a small spherical grape at the top of his bald head, which itself has the shape of a sphere. After all of the children have had time to giggle, the grape starts from rest and rolls down your uncle's head without slipping. It loses contact with your uncle's scalp when the radial line joining it to the center of curvature makes an angle θ with the vertical. What is the measure of angle θ?

58. A thin rod of length h and mass M is held vertically with its lower end resting on a frictionless horizontal surface. The rod is then released to fall freely. (a) Determine the speed of its center of mass just before it hits the horizontal surface. (b) Now suppose that the rod has a fixed pivot at its lower end. Determine the speed of the rod's center of mass just before it hits the surface.

WEB **59.** Two astronauts (Fig. P11.59), each having a mass of 75.0 kg, are connected by a 10.0-m rope of negligible mass. They are isolated in space, orbiting their center of mass at speeds of 5.00 m/s. (a) Treating the astronauts as particles, calculate the magnitude of the angular momentum and (b) the rotational energy of the system. By pulling on the rope, one of the astronauts shortens the distance between them to 5.00 m. (c) What is the new angular momentum of the system? (d) What are the astronauts' new speeds? (e) What is the new rotational energy of the system? (f) How much work is done by the astronaut in shortening the rope?

60. Two astronauts (see Fig. P11.59), each having a mass M, are connected by a rope of length d having negligible mass. They are isolated in space, orbiting their center of mass at speeds v. Treating the astronauts as particles, calculate (a) the magnitude of the angular momentum and (b) the rotational energy of the system. By pulling on the rope, one of the astronauts shortens the distance between them to $d/2$. (c) What is the new angular momentum of the system? (d) What are the astronauts' new speeds? (e) What is the new rotational energy of the system? (f) How much work is done by the astronaut in shortening the rope?

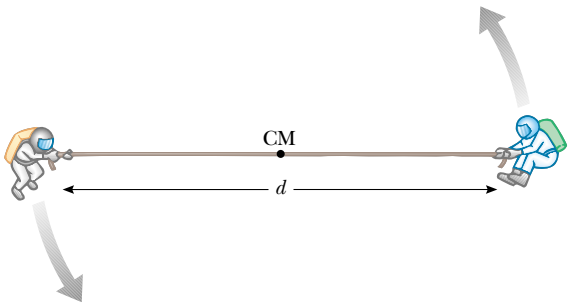


Figure P11.59 Problems 59 and 60.

61. A solid cube of wood of side $2a$ and mass M is resting on a horizontal surface. The cube is constrained to ro-

tate about an axis AB (Fig. P11.61). A bullet of mass m and speed v is shot at the face opposite $ABCD$ at a height of $4a/3$. The bullet becomes embedded in the cube. Find the minimum value of v required to tip the cube so that it falls on face $ABCD$. Assume $m \ll M$.

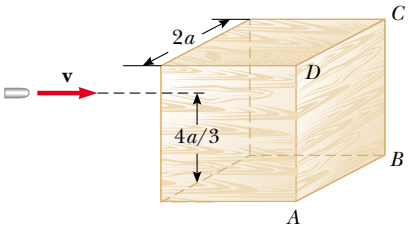


Figure P11.61

62. A large, cylindrical roll of paper of initial radius R lies on a long, horizontal surface with the open end of the paper nailed to the surface. The roll is given a slight shove ($v_i \approx 0$) and begins to unroll. (a) Determine the speed of the center of mass of the roll when its radius has diminished to r. (b) Calculate a numerical value for this speed at $r = 1.00$ mm, assuming $R = 6.00$ m. (c) What happens to the energy of the system when the paper is completely unrolled? (*Hint:* Assume that the roll has a uniform density and apply energy methods.)

63. A spool of wire of mass M and radius R is unwound under a constant force $\mathbf{F}$ (Fig. P11.63). Assuming that the spool is a uniform solid cylinder that does not slip, show that (a) the acceleration of the center of mass is $4\mathbf{F}/3M$ and that (b) the force of friction is to the *right* and is equal in magnitude to $F/3$. (c) If the cylinder starts from rest and rolls without slipping, what is the speed of its center of mass after it has rolled through a distance d?

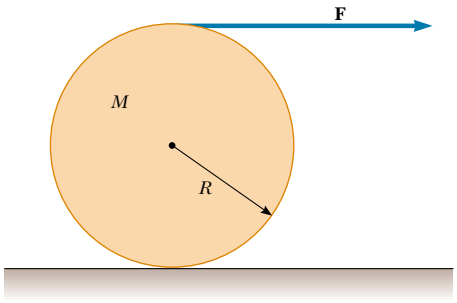


Figure P11.63

64. A uniform solid disk is set into rotation with an angular speed ω_i about an axis through its center. While still rotating at this speed, the disk is placed into contact with a horizontal surface and released, as shown in Figure P11.64. (a) What is the angular speed of the disk once pure rolling takes place? (b) Find the fractional loss in kinetic energy from the time the disk is released until the time pure rolling occurs. (*Hint:* Consider torques about the center of mass.)

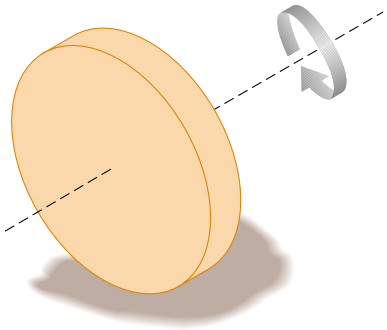


Figure P11.64 Problems 64 and 65.

65. Suppose a solid disk of radius R is given an angular speed ω_i about an axis through its center and is then lowered to a horizontal surface and released, as shown in Problem 64 (see Fig. P11.64). Furthermore, assume that the coefficient of friction between the disk and the surface is μ. (a) Show that the time it takes for pure rolling motion to occur is $R\omega_i/3\mu g$. (b) Show that the distance the disk travels before pure rolling occurs is $R^2\omega_i^2/18\mu g$.

66. A solid cube of side $2a$ and mass M is sliding on a frictionless surface with uniform velocity $\mathbf{v}$, as shown in Figure P11.66a. It hits a small obstacle at the end of the table; this causes the cube to tilt, as shown in Figure

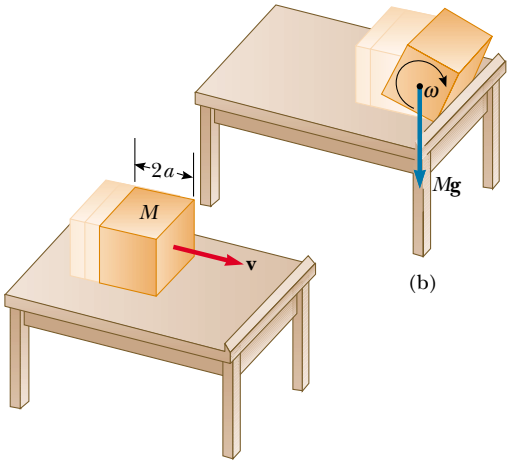


Figure P11.66

P11.66b. Find the minimum value of **v** such that the cube falls off the table. Note that the moment of inertia of the cube about an axis along one of its edges is $8Ma^2/3$. (*Hint:* The cube undergoes an inelastic collision at the edge.)

67. A plank with a mass $M = 6.00$ kg rides on top of two identical solid cylindrical rollers that have $R = 5.00$ cm and $m = 2.00$ kg (Fig. P11.67). The plank is pulled by a constant horizontal force of magnitude $F = 6.00$ N applied to the end of the plank and perpendicular to the axes of the cylinders (which are parallel). The cylinders roll without slipping on a flat surface. Also, no slipping occurs between the cylinders and the plank. (a) Find the acceleration of the plank and that of the rollers. (b) What frictional forces are acting?

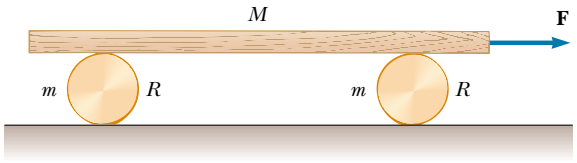


Figure P11.67

68. A spool of wire rests on a horizontal surface as in Figure P11.68. As the wire is pulled, the spool does not slip at the contact point *P*. On separate trials, each one of the forces $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$, and $\mathbf{F}_4$ is applied to the spool. For each one of these forces, determine the direction in which the spool will roll. Note that the line of action of $\mathbf{F}_2$ passes through *P*.

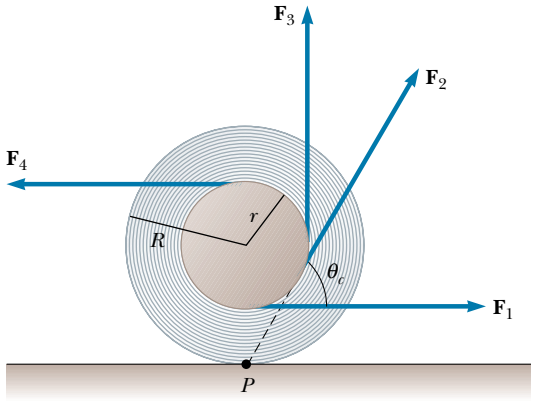


Figure P11.68 Problems 68 and 69.

69. The spool of wire shown in Figure P11.68 has an inner radius *r* and an outer radius *R*. The angle θ between the applied force and the horizontal can be varied. Show that the critical angle for which the spool does not slip and remains stationary is

$$\cos\theta_c = \frac{r}{R}$$

(*Hint:* At the critical angle, the line of action of the applied force passes through the contact point.)

70. In a demonstration that employs a ballistics cart, a ball is projected vertically upward from a cart moving with constant velocity along the horizontal direction. The ball lands in the catching cup of the cart because both the cart and the ball have the same horizontal component of velocity. Now consider a ballistics cart on an incline making an angle θ with the horizontal, as shown in Figure P11.70. The cart (including its wheels) has a mass *M*, and the moment of inertia of each of the two wheels is $mR^2/2$. (a) Using conservation of energy considerations (assuming that there is no friction between the cart and the axles) and assuming pure rolling motion (that is, no slipping), show that the acceleration of the cart along the incline is

$$a_x = \left(\frac{M}{M + 2m}\right) g \sin\theta$$

(b) Note that the *x* component of acceleration of the ball released by the cart is $g \sin\theta$. Thus, the *x* component of the cart's acceleration is *smaller* than that of the ball by the factor $M/(M + 2m)$. Use this fact and kinematic equations to show that the ball overshoots the cart by an amount Δx, where

$$\Delta x = \left(\frac{4m}{M + 2m}\right)\left(\frac{\sin\theta}{\cos^2\theta}\right)\frac{v_{yi}^2}{g}$$

and v_{yi} is the initial speed of the ball imparted to it by the spring in the cart. (c) Show that the distance *d* that the ball travels measured along the incline is

$$d = \frac{2v_{yi}^2}{g}\,\frac{\sin\theta}{\cos^2\theta}$$

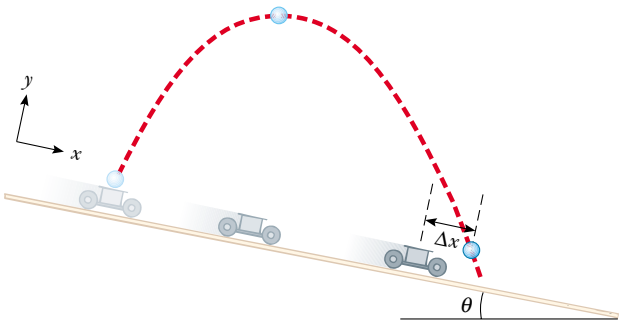


Figure P11.70

Answers to Quick Quizzes

11.1 There is very little resistance to motion that can reduce the kinetic energy of the rolling ball. Even though there is friction between the ball and the floor (if there were not, then no rotation would occur, and the ball would slide), there is no relative motion of the two surfaces (by the definition of "rolling"), and so kinetic friction cannot reduce K. (Air drag and friction associated with deformation of the ball eventually stop the ball.)

11.2 The box. Because none of the box's initial potential energy is converted to rotational kinetic energy, at any time $t > 0$ its translational kinetic energy is greater than that of the rolling ball.

11.3 Zero. If she were moving directly toward the pole, $\mathbf{r}$ and $\mathbf{p}$ would be antiparallel to each other, and the sine of the angle between them is zero; therefore, $L = 0$.

11.4 Both (a) and (b) are false. The net force is not necessarily zero. If the line of action of the net force passes through the point, then the net torque about an axis passing through that point is zero even though the net force is not zero. Because the net force is not necessarily zero, you cannot conclude that the particle's velocity is constant.

11.5 The student does work as he walks from the rim of the platform toward its center.

11.6 Because it is perpendicular to the precessional motion of the top, the force of gravity does no work. This is a situation in which a force causes motion but does no work.