

## 9.9 Improper Integral

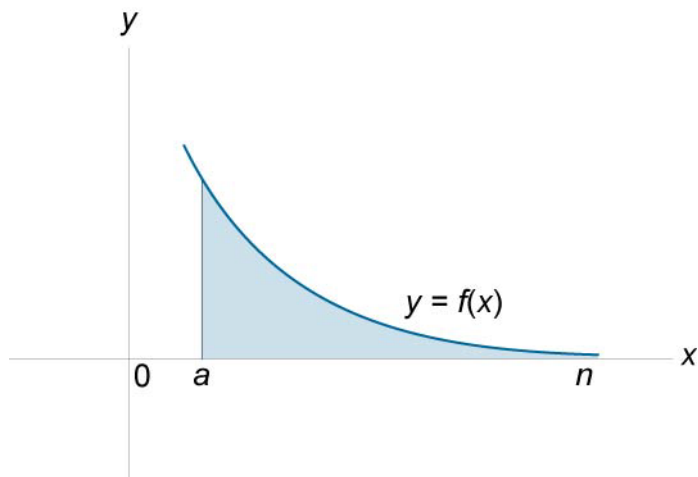
**1070.** The definite integral  $\int_a^b f(x)dx$  is called an **improper integral**

if

- $a$  or  $b$  is infinite,
- $f(x)$  has one or more points of discontinuity in the interval  $[a, b]$ .

**1071.** If  $f(x)$  is a continuous function on  $[a, \infty)$ , then

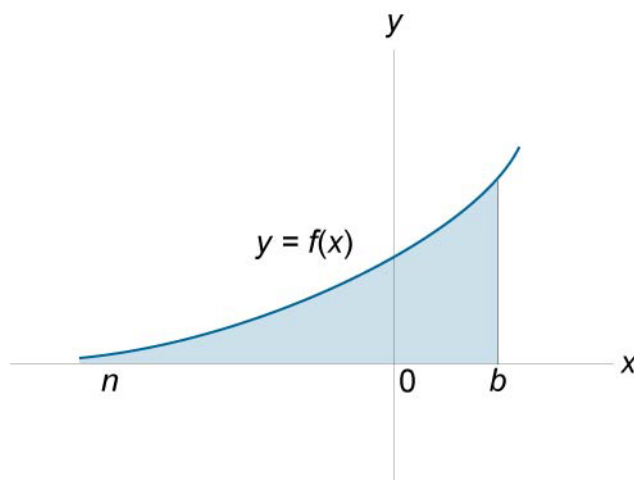
$$\int_a^{\infty} f(x)dx = \lim_{n \rightarrow \infty} \int_a^n f(x)dx.$$



**Figure 184.**

**1072.** If  $f(x)$  is a continuous function on  $(-\infty, b]$ , then

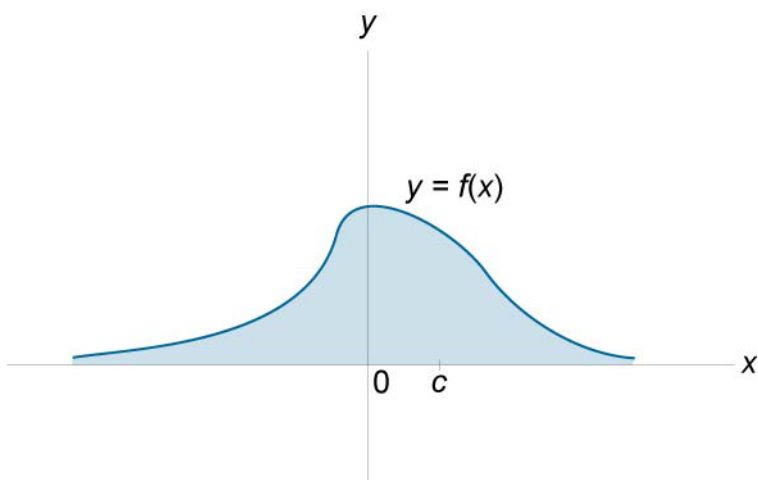
$$\int_{-\infty}^b f(x)dx = \lim_{n \rightarrow -\infty} \int_n^b f(x)dx.$$



**Figure 185.**

Note : The improper integrals in 1071, 1072 are **convergent** if the limits exist and are finite; otherwise the integrals are **divergent**.

**1073.** 
$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^c f(x)dx + \int_c^{\infty} f(x)dx$$



**Figure 186.**

If for some real number  $c$ , both of the integrals in the right side are convergent, then the integral  $\int_{-\infty}^{\infty} f(x)dx$  is also **convergent**; otherwise it is **divergent**.

**1074. Comparison Theorems**

Let  $f(x)$  and  $g(x)$  be continuous functions on the closed interval  $[a, \infty)$ . Suppose that  $0 \leq g(x) \leq f(x)$  for all  $x$  in  $[a, \infty)$ .

- If  $\int_a^{\infty} f(x)dx$  is convergent, then  $\int_a^{\infty} g(x)dx$  is also convergent,
- If  $\int_a^{\infty} g(x)dx$  is divergent, then  $\int_a^{\infty} f(x)dx$  is also divergent.

**1075. Absolute Convergence**

If  $\int_a^{\infty} |f(x)|dx$  is convergent, then the integral  $\int_a^{\infty} f(x)dx$  is absolutely convergent.

**1076. Discontinuous Integrand**

Let  $f(x)$  be a function which is continuous on the interval  $[a, b)$  but is discontinuous at  $x = b$ . Then

$$\int_a^b f(x)dx = \lim_{\varepsilon \rightarrow 0^+} \int_a^{b-\varepsilon} f(x)dx$$

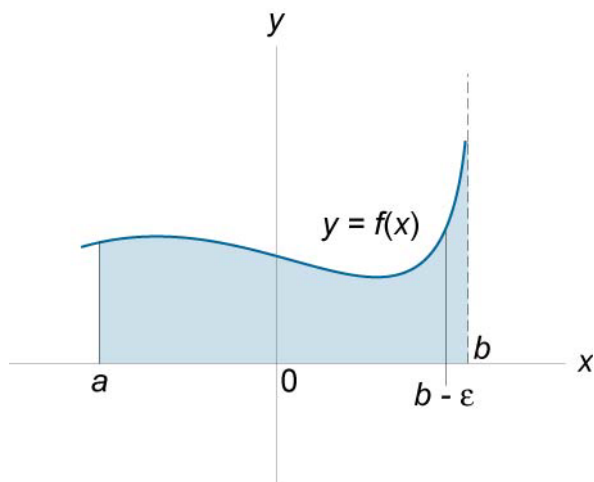


Figure 187.

**1077.** Let  $f(x)$  be a continuous function for all real numbers  $x$  in the interval  $[a, b]$  except for some point  $c$  in  $(a, b)$ . Then

$$\int_a^b f(x) dx = \lim_{\varepsilon \rightarrow 0^+} \int_a^{c-\varepsilon} f(x) dx + \lim_{\delta \rightarrow 0^+} \int_{c+\delta}^b f(x) dx.$$

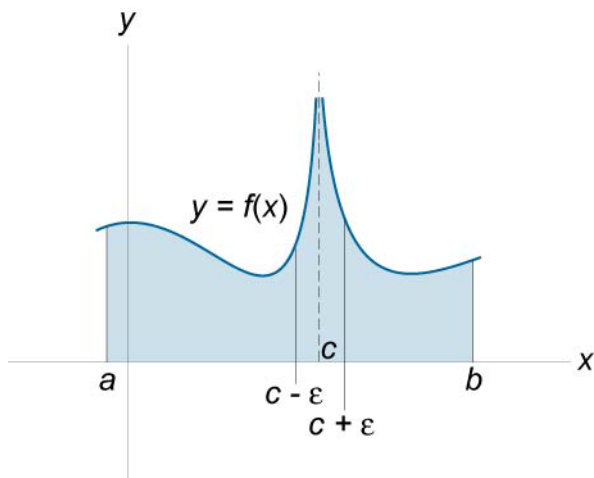


Figure 188.