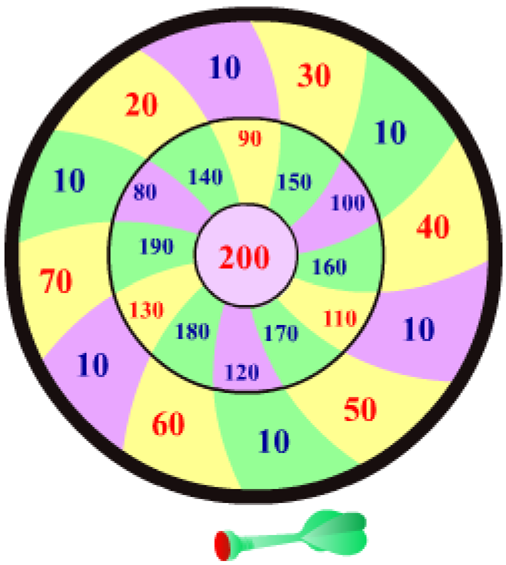
Probability

Observing an Experiment

It is not always possible to tell the exact outcome of a particular action. Take, for example, a dart board.



A dart is repeatedly thrown toward the dartboard, targeting a random number in each throw. We do not know which number is targeted in a particular throw. What we do know is that there is a fixed group of numbers and each time the targeted number is one of them.

We know that the likelihood of occurrence of an unpredictable event is studied under the theory of probability. So, we can say that there is a certain probability for each number to be targeted in the above experiment.

Let us learn more about probability and the meanings of terms associated with it, for example, ‘experiment’ and ‘outcome’.

Did You Know?

The word ‘probability’ has evolved from the Latin word ‘probabilitas’, which can be considered to have the same meaning as the word ‘probity’. In olden days in Europe, ‘probity’ was a measure of authority of a witness in a legal case, and it often correlated with the nobility of the witness.

The modern meaning of probability, however, focuses on the statistical observation of the likelihood of occurrence of an event.

Know More

Probability is widely applicable in daily life and in researches pertaining to different fields. It is an important factor in the diverse worlds of share market, philosophy, artificial intelligence or machine learning, statistics, etc. All gambling is based on probability. In gambling, one considers all possibilities and then tries to predict a result that is most likely to happen. The concept of probability is perhaps the most interesting topic to discuss in mathematics.

Terms Related to Probability

Experiment: When an operation is planned and done under controlled conditions, it is known as an experiment. For example, tossing a coin, throwing a die, drawing a card from a pack of playing cards without seeing, etc., are all experiments. A chance experiment is one in which the result is unknown or not predetermined.

Outcomes: Different results obtained in an experiment are known as outcomes. For example, on tossing a coin, if the result is a head, then the outcome is a head; if the result is a tail, then the outcome is a tail.

Random: An experiment is random if it is done without any conscious decision. For example, drawing a card from a well-shuffled pack of playing cards is a random experiment if it is done without seeing the card or figuring it out by touching.

Trial: A trial is an action or an experiment that results in one or several outcomes. For example, if a coin is tossed five times, then each toss of the coin is called a trial.

Sample space: The set of all possible outcomes of an experiment is called the sample space. It is denoted by the English letter 'S' or Greek letter 'Ω' (omega). In the experiment of tossing a coin, there are only two possible outcomes—a head (H) and a tail (T).

∴ Sample space (S) = {H, T}

Event: The event of an experiment is one or more outcomes of the experiment. For example, tossing a coin and getting a head or a tail is an event. Throwing a die and getting a face marked with an odd number (i.e., 1, 3 or 5) or an even number (2, 4 or 6) is also an event.

Know More

Initially, the word 'probable' meant the same as the word 'approvable' and was used in the same sense to support or approve of opinions and actions. Any action described as 'probable' was considered the most likely and sensible action to be taken by a rational and sensible person.

Whiz Kid

Equally Likely: If each outcome of an experiment has the same probability of occurring, then the outcomes are said to be equally likely outcomes.

Know Your Scientist

Girolamo Cardano (1501−1576) was a great Italian mathematician, physicist, astrologer and gambler. His interest in gambling led him to do more research on the concept of probability and formulate its rules. He was often short of money and kept himself solvent through his gambling skills.

He was also a very good chess player. He wrote a book named *Liber de Ludo Aleae*. In this book about games of chance, he propounded the basic concepts of probability.

Solved Examples

Easy

Example 1:

A fair die is thrown. What is the sample space of this experiment?

Solution:

When a die is thrown, we can have six outcomes, namely, 1, 2, 3, 4, 5 and 6.

We know that sample space is the collection of all possible outcomes of an experiment.

∴ Sample space (S) = {1, 2, 3, 4, 5, 6}

Example 2:

Which of the following are experiments?

i)Tossing a coin

ii)Rolling a six-sided die

iii)Getting a head on a tossed coin

Solution:

Tossing a coin and rolling a six-sided die are experiments, while getting a head on a tossed coin is the outcome of an experiment.

Medium

Example 1:

What is the sample space when two coins are tossed together?

Solution:

When two coins are tossed together, we can get four possible outcomes. These are as follows:

i)A head (H) on one coin and a tail (T) on the other

ii)A head (H) on one coin and a head (H) on the other

iii)A tail (T) on one coin and a head (H) on the other

iv)A tail (T) on one coin and a tail (T) on the other

∴ Sample space (S) = {HT, HH, TH, TT}

Concept of Chance and Probability

In our daily life, various incidents happen and sometimes we know in advance that these incidents will happen. For example, the day after Saturday will be Sunday or the Sun will rise from the east. These are the events which are certain to happen. Similarly, there are events which are impossible such as March comes before February in a year, the apple goes up when dropped from the tree etc.

However, most of the events in our daily life have chances to happen in a particular way and there can be one or more ways in which an event can happen.

For example, India is going to play a cricket match against Bangladesh. Here, the result of this event can occur in various ways whether India will win, lose or draw the game.

Now, can we say India will win the match? Though match between India and Bangladesh is in favour of India, still we cannot say that India will win the match nor can we say that it will lose. This is again a matter of **chance**. We can only say that there is a chance for India to win the match.

Consider one more example now.

Suppose there are five balls of five different colours in a bag - blue, red, yellow, green, and black. Sonu is asked to draw a ball from the bag without looking into it. Can he be certain that he would draw a blue ball? No, it might be any one of the five balls.

Thereafter, Sonu draws one ball at a time without looking into the bag and records the colour of the ball. He then puts that ball inside the bag and again draws a ball. He performs this experiment 20 times and prepares the following table:

Times of drawing a ball	**Colour of the ball**
1	Green
2	Red
3	Green
4	Black
5	Blue
6	Red
7	Red
8	Red
9	Green
10	Green
11	Red
12	Black
13	Blue
14	Red
15	Yellow

16	Red
17	Yellow
18	Yellow
19	Green
20	Black

Can you say that after drawing a green ball, the next ball is always red in colour?

No, the table does not follow any pattern. It is a matter of chance that which colour will come after a particular colour.

The concept of chance and certain event is explained in the given video.

In mathematics, we use probability to find the chance that a particular event can happen by considering all the cases which are possible.

The word probability is often used in day to day conversation also. People often use this word when they talk about the chances of an event to happen. We can often hear people saying that probably he is going to be our next Prime Minister or probably it will rain today. In these sentences, we are talking about the chances of an event to happen.

We can define probability as follows:

Probability is the measure or estimation of likelihood of happening of an event in a particular way.

Probability for an event to happen in a particular way depends upon all possible ways in which that event can happen.

For example, when a dice is rolled, the possible ways (positions) in which we get its top face are six such as 1, 2, 3, 4, 5 and 6. The probability to get any of these numbers on top face depends on all these six ways.

Similarly, probability is applicable in various situations and it can be very helpful to predict the future results.

Let us have a look at the following example.

Example:

Which of the following are certain events, impossible events, or matters of chance?

(i) Water always falls down.

(ii) When a coin is tossed, the outcome is Head.

(iii) Harry is older than his father.

(iv) In the musical chair game, Isha will get the chair.

(v) The size of the Sun is smaller than the size of the Earth.

(vi) When a dice is thrown, any one of the numbers among 1, 2, 3, 4, 5, and 6 shows up on the top face.

Solution:

The events **(i)** and **(vi)** are certain.

Water always falls down. It does not go up. A dice contains the number 1, 2, 3, 4, 5, and 6. Thus, when a dice is thrown, any one of the above numbers must show up on the top face.

The events **(iii)** and **(v)** are impossible to happen.

A son cannot be older than his father and the size of the Sun is greater than that of Earth.

The events **(ii)** and **(iv)** are matters of chance.

When a coin is tossed, the outcome may either be Head or Tail. In the musical chair game, Isha may or may not get the chair.

Sample Space and Events of Experiments

Sample Space

- The set of all possible outcomes of a random experiment is called the s**ample space** associated with the experiment. Sample space is denoted by the symbol **S**.
- E.g., consider the experiment of tossing a coin. In this experiment, there are two possible outcomes—a head or a tail. Thus, the sample space of this experiment is **S** = {H, T}.

- Each element of the sample space is called a **sample point**. In other words, we can say, each outcome of the random experiment is called a **sample point**.

- In order to understand sample space and sample point in a better way, let us go through the following video.

Event and Its Types

- Any subset E of a sample space S is called an **event**.

- E.g., if we throw a die twice, then the sample space so obtained is given by
  S = {(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6), (3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6), (4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6), (5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6), (6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)}

  If we consider set E as the outcomes in which both the throws show the same number, then it is given by

  E = {(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)}.

  Since E is a subset of S, E is called an event of S. In this case, the event E is defined as, “Same number appeared in both throws”.

- An event E of a sample space S is said to have occurred if the outcome $\omega \in E$. On the other hand, if $\omega \notin E$, then we say that the event E has not occurred.

- Consider the experiment of throwing a die twice. Let E and F denote respectively the events “the sum of the appeared numbers is 8” and “the sum of the appeared numbers is 12”.

  Then, E = {(2, 6), (3, 5), (4, 4), (5, 3), (6, 2)}

  And F = {(6, 6)}

  Now, if in a trial, we get the outcome as (3, 5), then we can say that the event E has occurred and the event F has not occurred because $(3, 5) \in E$ and $(3, 5) \notin F$.

- Events can be classified into various types on the basis of the elements they have.

- The event of a sample space S which contains no sample point is called an **impossible event**. It is denoted by an empty set Φ.

  E.g., the event “The sum of the appeared numbers on the twice throw of a die is less than 2” is an impossible event since there is no outcome related to this event.

- The event of a sample space which contains all the sample points is called a **sure event**. The entire sample space S is called a sure event.

- An event containing only one sample point is called a **simple event** or an **elementary event**. In fact, a sample space S containing *n* elements has *n* simple events.

  On throwing a die two times, we get 36 outcomes. So, there are 36 simple events. They can be written as follows:

  $E_1 = \{(1, 1)\}$, $E_2 = \{(1, 2)\}$ ….. $E_7 = \{(2, 1)\}$, $E_8 = \{(2, 2)\}$ ….. $E_{36} = \{(6, 6)\}$

- An event containing more than one sample point is called a **compound event**.
  On throwing a die two times, let E denote the event "The sum of the appeared numbers is a multiple of 3", and then we have

  E = {(1, 2), (1, 5), (2, 1), (2, 4), (3, 3), (3, 6), (4, 2), (4, 5), (5, 1), (5, 4), (6, 3), (6, 6)}.

  Here, the event E contains 12 sample points. So, E is a compound event.

Solved Examples

Example 1:

A boy has two bags B_1 and B_2, a coin and a die. Bag B_1 contains 4 equal-sized red balls and bag B_2 contains 5 equal-sized green balls. He takes a ball from bag B_1. Then, he tosses the coin. Thereafter, he takes a ball from bag B_2. Finally, he throws the die. Write the sample space for this experiment. Find how many sample points are there in the sample space.

Solution:

Let us denote a red ball as R and a green ball as G.

It is given that the two bags B_1 and B_2 contain 4 red balls and 5 green balls respectively.

We have the following observations while the boy performs the given experiment.

- When the boy takes out a ball from bag B_1, it will always be a red ball (R).

- When the boy tosses the coin, it can be either a head (H) or a tail (T).

- When the boy takes out a ball from bag B_2, it will always be a green ball (G).

- When the boy throws the die, any of the numbers 1, 2, 3, 4, 5 and 6 can appear on the die.

Hence, the sample space (S) of the experiment is given as

S = {RHG1, RHG2, RHG3, RHG4, RHG5, RHG6, RTG1, RTG2, RTG3, RTG4, RTG5, RTG6}

Clearly, the sample space S contains 12 sample points.

Example 2:

With respect to the sample space “All the months of the year 2008”, the events E, F, G and H are defined as

E: “Month/Months having 31 days”

F: “Month/Months having 28 days”

G: “Month/Months having 25 to 35 days”

H: “Month/Months having 29 days”

Classify the above events as impossible, sure, simple or compound.

Solution:

We have the sample space as

S = {January, February, March, April, May, June, July, August, September, October, November,

December}

And E = {January, March, May, July, August, October, December}

2008 is a leap year. So, the number of days in the month of February is 29.

∴H = {February}

And F = Φ

G = {January, February, March, April, May, June, July, August, September, October, November,

December}

With respect to the given sample space:

1. F is an impossible event.
2. G is a sure event.
3. H is a simple event.
4. Both E and G are compound events.

Probability Of Events

Suppose Shashank throws a dice. There are six different outcomes. The outcomes of an experiment or a collection of outcomes makes an **event.**

In this example, getting the number 1 on the top face of the dice is an event. Similarly, getting the other numbers (2, 3, 4, 5, or 6) are also known as events.

Can we tell what will be the probability of getting 2 on the top face of the dice?

There are six possible outcomes and all are equally likely to occur. **The probability is the ratio of getting an outcome to the total number of outcomes.**

Probability of an event

$$= \frac{\text{Number of outcomes that make an event}}{\text{Total number of outcomes of the experiment}}$$

Mathematically,

$$P(A) = \frac{n(A)}{n(S)}$$

Here,

P(*A*) represents the probability of an event *A*.

n(*S*) represents the total number of outcomes or number of elements in sample space.

n(*A*) represents the number of outcomes that make event A or the number of elements in set *A*.

It should be noted that sample space S is the universal set here, so all elements of set A belong to set S i.e, $A \subseteq S$.

Now, let us find the solution of above discussed case.

In the above example, the total number of outcomes are 6.

∴ Probability of getting a number 2 on the top face = 1/6

What is the probability of getting 6 on the top face of the dice?

Since all the outcomes are equally likely to occur,

∴ Probability of getting a number 6 on the top face $= \frac{1}{6}$

Similarly, for other numbers (1, 3, 4, and 5) as well, the probability of showing up on the top face is $\frac{1}{6}$.

This is how we can find out the probability of the occurrence of an outcome in an experiment.

Can we calculate the probability of the occurrence of a multiple of 3 on the top face of the dice?

Yes, we can.

Consider the multiples of 3 out of six possible outcomes. The multiples of 3 are 3 and 6 out of six possible outcomes.

Thus, probability of getting a multiple of 3

$$= \frac{\text{Number of multiples of 3}}{\text{Total number of possible outcomes}}$$

$$= \frac{2}{6} = \frac{1}{3} \quad \frac{\text{Multiples of 3}}{\text{Total number of possible outcomes}} = \frac{2}{6} = \frac{1}{3}$$

Properties of probability:

Property 1: Probability of a certain event is 1 and probability of an impossible event is 0.

Probability of an impossible event is denoted by P(ϕ) and probability of a certain event is denoted by P(S).

Therefore, $P(\phi) = 0$ and $P(S) = 1$.

Proof:

We have

$$P(A) = \frac{n(A)}{n(S)}$$

Impossible event means that there is no way in which the event can occur. So, number of outcomes making event A will be 0 or set A will be empty set i.e., ϕ.

$$\therefore P(\phi) = \frac{n(\phi)}{n(S)} = \frac{0}{n(S)} = 0$$

Certain event means that there is only one possibility. So, the number of outcomes in sample space S as well as in set A will be equal i.e., 1.

$$\therefore P(A) = \frac{n(A)}{n(S)} = \frac{n(S)}{n(S)} = 1$$

Hence proved.

Let us consider few examples based on this property.

Consider the events **"son being older than his father", "Saturday comes before Friday in a week"** and **"taking water from an empty mug".** All these are impossible events and thus, the probability of occurrence of each of these events is 0.

Now, consider the events **"Sun is larger than earth**" and **"Sunday comes before Monday in a week".** Both of these are certain events and thus, the probability of occurrence each of these events is 1.

Property 2: If the sample space S is a finite set and A is an event of S then probability of event A lies between 0 and 1 both inclusive.

Therefore, $0 \leq P(A) \leq 1$.

Proof:

Since $A \subseteq S$, we have

$\phi \subseteq A \subseteq S$

$\Rightarrow n(\phi) \leq n(A) \leq n(S)$

$\Rightarrow 0 \le n(A) \le n(S)$ $[n(\phi) = 0]$

On dividing the inequality by $n(S)$, we get

$$\frac{0}{n(S)} \le \frac{n(A)}{n(S)} \le \frac{n(S)}{n(S)}$$

$\Rightarrow 0 \le P(A) \le 1$

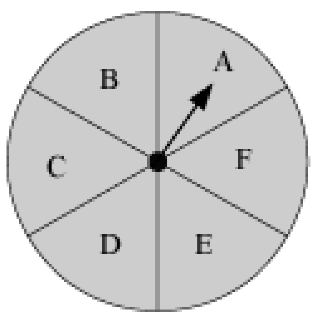
Hence proved.

These are very important properties related to probability which prove to be very helpful at times.

Now, let us have a look at some examples.

Example 1:

The given figure shows a wheel in which six English alphabets are written in six equal sectors of the wheel. Suppose we spin the wheel. What is the possibility of the pointer stopping in the sector containing alphabet A?



Solution:

The total number of possible outcomes is 6. The pointer can stop at six different sectors (A, B, C, D, E, F).

Thus, probability of pointer stopping in the sector containing A = 1/6

Example 2:

A bag has 6 blue and 4 red balls. A ball is drawn from the bag without looking into the bag.

1. **What is the probability of getting a blue ball?**

2. **What is the probability of getting a red ball?**

Solution:

In a bag, there are 6 blue and 4 red balls.

∴ Total number of outcomes = 6 + 4 = 10

1. Getting a blue ball consists of 6 outcomes, since there are 6 blue balls.

Probability of getting a blue ball $= \frac{6}{10} = \frac{3}{5}$

2. Getting a red ball consists of 4 outcomes, since there are 4 red balls.

Probability of getting a red ball $= \frac{4}{10} = \frac{2}{5}$

Example 3:

When a dice is thrown, what is the probability of getting

(a) A prime number

(b) An even number

(c) An odd number

(d) A number less than or equal to 2

(e) A number more than or equal to 4

Solution:

When a dice is thrown, the total number of outcomes is 6.

(a) The prime numbers out of six possible outcomes are 2, 3, and 5. Thus, getting a prime number consists of 3 outcomes.

∴ Probability of getting a prime number $= \frac{3}{6} = \frac{1}{2}$

(b) Out of the possible outcomes, the even numbers are 2, 4, and 6. Thus, the number of outcomes of getting an even numbers is 3.

∴ Probability of getting an even number $= \frac{3}{6} = \frac{1}{2}$

(c) The odd numbers are 1, 3, and 5. Thus, the number of outcomes of getting an odd number is 3.

∴ Probability of getting an odd number $= \frac{3}{6} = \frac{1}{2}$

(d) The numbers less than or equal to 2 are 1 and 2. Thus, there are 2 possible

outcomes.

∴ Required probability $= \frac{2}{6} = \frac{1}{3}$

(e) The numbers more than or equal to 4 are 4, 5, and 6. Thus, there are 3 possible

outcomes.

∴ Required probability $= \frac{3}{6} = \frac{1}{2}$