JEE ADVANCED

Single Correct Answer Type

1. In triangle ABC, angle A is greater than angle B. If the measures of angles A and B satisfy the equation $3\sin x - 4\sin^3 x - k = 0$, $0 < k < 1$, then the measure of angle C is

   a. $\frac{\pi}{3}$ **b.** $\frac{\pi}{2}$ **c.** $\frac{2\pi}{3}$ **d.** $\frac{5\pi}{6}$

   (IIT-JEE 1990)

2. If the lengths of the sides of triangle are 3, 5, and 7, then the largest angle of the triangle is

   a. $\frac{\pi}{2}$ **b.** $\frac{5\pi}{6}$ **c.** $\frac{2\pi}{3}$ **d.** $\frac{3\pi}{4}$

   (IIT-JEE 1994)

3. In triangle ABC, $\angle B = \pi/3$, and $\angle C = \pi/4$. Let D divide BC internally in the ratio 1:3. Then $\frac{\sin \angle BAD}{\sin \angle CAD}$ equals

   a. $\frac{1}{\sqrt{6}}$ **b.** $\frac{1}{3}$ **c.** $\frac{1}{\sqrt{3}}$ **d.** $\sqrt{\frac{2}{3}}$

   (IIT-JEE 1995)

4. In triangle ABC, $2ac \sin\left(\frac{1}{2}(A - B + C)\right)$ is equal to

   a. $a^2 + b^2 - c^2$ **b.** $c^2 + a^2 - b^2$
   c. $b^2 - c^2 - a^2$ **d.** $c^2 - a^2 - b^2$

   (IIT-JEE 2000)

5. In triangle ABC, let $\angle C = \pi/2$. If r is the inradius and R is circumradius of the triangle, then $2(r + R)$ is equal to

   a. $a + b$ **b.** $b + c$ **c.** $c + a$ **d.** $a + b + c$

   (IIT-JEE 2000)

6. Which of the following pieces of data does NOT uniquely determine an acute-angled triangle ABC (R being the radius of the circumcircle)?

   a. a, $\sin A$, $\sin B$ **b.** a, b, c
   c. a, $\sin B$, R **d.** a, $\sin A$, R

   (IIT-JEE 2002)

7. If the angles of a triangle are in the ratio 4:1:1, then the ratio of the longest side to the perimeter is

   a. $\sqrt{3} : (2 + \sqrt{3})$ **b.** 1:6
   c. $1 : 2 + \sqrt{3}$ **d.** 2:3

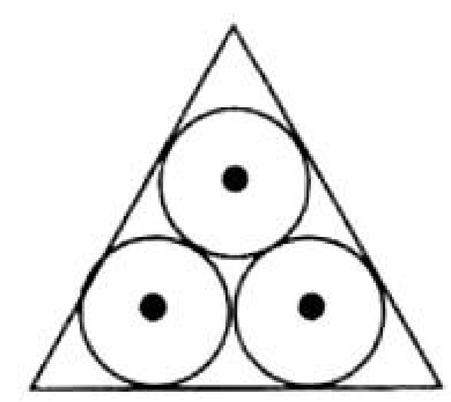
   (IIT-JEE 2003)

8. The side of a triangle are in the ratio $1 : \sqrt{3} : 2$, then the angles of the triangle are in the ratio

   a. 1:3:5 **b.** 2:3:4 **c.** 3:2:1 **d.** 1:2:3

   (IIT-JEE 2004)

9. In an equilateral triangle, three coins of radii 1 unit each are kept so that they touch each other and also the sides of the triangle. The area of the triangle is



   a. $4 + 2\sqrt{3}$ **b.** $6 + 4\sqrt{3}$
   c. $12 + \frac{7\sqrt{3}}{4}$ **d.** $3 + \frac{7\sqrt{3}}{4}$

   (IIT-JEE 2005)

10. In triangle ABC, a, b, c are the lengths of its sides and A, B, C are the angles of triangle ABC. The correct relation is given by

   a. $(b - c)\sin\left(\frac{B - C}{2}\right) = a \cos\frac{A}{2}$

b. $(b-c)\cos\left(\frac{A}{2}\right) = a\sin\frac{B-C}{2}$

c. $(b+c)\sin\left(\frac{B+C}{2}\right) = a\cos\frac{A}{2}$

d. $(b-c)\cos\left(\frac{A}{2}\right) = 2a\sin\frac{B+C}{2}$

(IIT-JEE 2005)

11. One angle of an isosceles triangle is 120° and the radius of its incircle is $\sqrt{3}$. Then the area of the triangle in sq. units is

a. $7 + 12\sqrt{3}$ b. $12 - 7\sqrt{3}$

c. $12 + 7\sqrt{3}$ d. 4π

(IIT-JEE 2006)

12. Let $ABCD$ be a quadrilateral with area 18, side AB parallel to the side CD, and AB = 2CD. Let AD be perpendicular to AB and CD. If a circle is drawn inside the quadrilateral $ABCD$ touching all the sides, then its radius is

a. 3 b. 2 c. $\frac{3}{2}$ d. 1

(IIT-JEE 2007)

13. Let ABC be a triangle such that $\angle ACB = \pi/6$ and let a, b, and c denote the lengths of the side opposite to A, B, and C, respectively. The value(s) of x for which $a = x^2 + x + 1$, $b = x^2 - 1$, and $c = 2x + 1$ is (are)

a. $-(2+\sqrt{3})$ b. $1+\sqrt{3}$

c. $2+\sqrt{3}$ d. $4\sqrt{3}$ **(IIT-JEE 2010)**

14. If the angles A, B and C of a triangle are in an arithmetic progression and if a, b and c denote the lengths of the sides opposite to A, B and C respectively, then the value of the expression $\frac{a}{c}\sin 2C + \frac{c}{a}\sin 2A$ is

a. $\frac{1}{2}$ b. $\frac{\sqrt{3}}{2}$ c. 1 d. $\sqrt{3}$

(IIT-JEE 2010)

15. Let PQR be a triangle of area Δ with $a = 2$, $b = 7/2$, and $c = 5/2$, where a, b, and c are the lengths of the sides of the triangle opposite to the angles at P, Q, and R, respectively. Then $\frac{2\sin P - \sin 2P}{2\sin P + \sin 2P}$ equals

a. $\frac{3}{4\Delta}$ b. $\frac{45}{4\Delta}$ c. $\left(\frac{3}{4\Delta}\right)^2$ d. $\left(\frac{45}{4\Delta}\right)^2$

(IIT-JEE 2012)

Multiple Correct Answers Type

1. There exists a triangle ABC satisfying the conditions

a. $b\sin A = a$, $A < \pi/2$ b. $b\sin A > a$, $A > \pi/2$

c. $b\sin A > a$, $A < \pi/2$ d. $b\sin A < a$, $A < \pi/2$, $b > a$

e. $b\sin A < a$, $A > \pi/2$, $b = a$ **(IIT-JEE 1986)**

2. In a triangle, the lengths of the two larger sides are 10 and 9, respectively. If the angles are in A.P., then the length of the third side can be

a. $5 - \sqrt{6}$ b. $3\sqrt{3}$ c. 5 d. $5 + \sqrt{6}$

(IIT-JEE 1987)

3. If in a triangle PQR, $\sin P$, $\sin Q$, $\sin R$ are in A.P., then

a. the altitudes are in A.P. b. the altitudes are in H.P.

c. the medians are in G.P. d. the medians are in A.P.

(IIT-JEE 1988)

4. Let $A_0A_1A_2A_3A_4A_5$ be a regular hexagon inscribed in a circle of unit radius. Then the product of the lengths of the line segments A_0A_1, A_0A_2, and A_0A_4 is

a. $\frac{3}{4}$ b. $3\sqrt{3}$ c. 3 d. $\frac{3\sqrt{3}}{2}$

(IIT-JEE 1998)

5. In ΔABC, internal angle bisector of $\angle A$ meets side BC in D. $DE \perp AD$ meets AC in E and AB in F. Then

a. AE is H.M. of b and c b. $AD = \frac{2bc}{b+c}\cos\frac{A}{2}$

c. $EF = \frac{4bc}{b+c}\sin\frac{A}{2}$ d. ΔAEF is isosceles

(IIT-JEE 2006)

6. A straight line through the vertex P of a triangle PQR intersects the side QR at the point S and the circumcircle of the triangle PQR at the point T. If S is not the center of the circumcircle, then

a. $\frac{1}{PS}+\frac{1}{ST}<\frac{2}{\sqrt{QS\times SR}}$ b. $\frac{1}{PS}+\frac{1}{ST}>\frac{2}{\sqrt{QS\times SR}}$

c. $\frac{1}{PS}+\frac{1}{ST}<\frac{4}{QR}$ d. $\frac{1}{PS}+\frac{1}{ST}>\frac{4}{QR}$

(IIT-JEE 2008)

7. In a triangle ABC with fixed base BC, the vertex A moves such that

$$\cos B + \cos C = 4\sin^2\frac{A}{2}$$

If a, b, and c denote the lengths of the sides of the triangle opposite to the angles A, B, and C, respectively, then

a. $b + c = 4a$

b. $b + c = 2a$

c. locus of point A is an ellipse

d. locus of point A is a pair of straight lines

(IIT-JEE 2009)

8. In a triangle PQR, P is the largest angle and $\cos P = 1/3$. Further the incircle of the triangle touches the sides PQ, QR, and PR at N, L, and M, respectively, such that the length of PN, QL, and RM are consecutive even integers. Then possible length(s) of the side(s) of the triangle is (are)

a. 16 b. 18 c. 24 d. 22

(JEE Advanced 2013)

Matching Column Type

1. Match the statements/expressions in Column I with the statements/expressions in Column II.

Column I	Column II
(i) $\sum_{i=1}^{\infty} \tan^{-1}\left(\frac{1}{2i^2}\right) = t$, then $\tan t =$	(a) 0
(ii) Sides a, b, c of a triangle ABC are in A.P. and $\cos\theta_1 = \frac{a}{b+c}$, $\cos\theta_2 = \frac{b}{a+c}$, $\cos\theta_3 = \frac{c}{a+b}$, then $\tan^2\left(\frac{\theta_1}{2}\right) + \tan^2\left(\frac{\theta_3}{2}\right) =$	(b) 1
(iii) A line is perpendicular to $x + 2y + 2z = 0$ and passes through (0, 1, 0). The perpendicular distance of this line from the origin is	(c) $\frac{\sqrt{5}}{3}$
	(d) 2/3

(IIT-JEE 2006)

2. Match the statements/expressions in Column I with the statements/expressions in Column II

Column I	Column II
(a) In a triangle ΔXYZ, let a, b and c be the lengths of the sides opposite to the angles X, Y and Z respectively. If $2(a^2 - b^2) = c^2$ and $\lambda = \frac{\sin(X-Y)}{\sin Z}$ then possible values of n for which $\cos(n\pi\lambda) = 0$ is (are)	(p) 1
(b) In a triangle ΔXYZ, let a, b and c be the lengths of the sides opposite to the angles X, Y and Z, respectively. If $1 + \cos 2X - 2\cos 2Y = 2\sin X \sin Y$, then possible value(s) of $\frac{a}{b}$ is (are)	(q) 2
(c) In R^2, let $\sqrt{3}\hat{i} + \hat{j}, \hat{i} + \sqrt{3}\hat{j}$ and $\beta\hat{i} + (1-\beta)\hat{j}$ be the position vectors of X, Y and Z with respect of the origin O, respectively. If the distance of Z from the bisector of the acute angle of $\overrightarrow{OX}$ and $\overrightarrow{OY}$ is $\frac{3}{\sqrt{2}}$, then possible value(s) of $\|\beta\|$ is (are)	(r) 3
(d) Suppose that $F(\alpha)$ denotes the area of the region bounded by $x = 0, x = 2, y^2 = 4x$ and $y = \|\alpha x - 1\| + \|\alpha x - 2\| + \alpha x$, where $\alpha \in \{0, 1\}$. Then the value(s) of $F(\alpha) + \frac{8}{3}\sqrt{2}$, when $\alpha = 0$ and $\alpha = 1$, is (are)	(s) 5
	(t) 6

(JEE Advanced 2015)

Integer Answer Type

1. Let ABC and ABC' be two non-congruent triangles with sides $AB = 4$, $AC = AC' = 2\sqrt{2}$ and angle $B = 30°$. The absolute value of the difference between the areas of these triangles is **(IIT-JEE 2009)**
2. Two parallel chords of a circle of radius 2 are at a distance $\sqrt{3}+1$ apart. If the chord subtend angles $\frac{\pi}{k}$ and $\frac{2\pi}{k}$ at the center, where $k > 0$, then the value of $[k]$ is

   (**Note:** $[k]$ denotes the largest integer less than or equal to k) **(IIT-JEE 2010)**
3. Consider a triangle ABC and let a, b and c denote the lengths of the sides opposite to vertices A, B, and C, respectively. Suppose $a = 6$, $b = 10$, and the area of triangle is $15\sqrt{3}$. If $\angle ACB$ is obtuse and if r denotes the radius of the incircle of the triangle, then the value of r^2 is **(IIT-JEE 2010)**

Fill in the Blanks Type

1. In ΔABC, $\angle A = 90°$ and AD is an altitude. Complete the relation $\frac{BD}{DA} = \frac{AB}{(\cdots)}$. **(IIT-JEE 1980)**
2. ABC is a triangle, P is a point on AB and Q is a point on AC such that $\angle AQP = \angle ABC$. Complete the relation $\frac{\text{Area of } \Delta APQ}{\text{Area of } \Delta ABC} = \frac{(\ldots)}{AC^2}$. **(IIT-JEE 1980)**
3. ABC is a triangle with $\angle B$ greater than $\angle C$. D and E are points on BC such that AD is perpendicular to BC and AE is the bisector of angle A. Complete the relation $\angle DAE = \frac{1}{2}[(\cdots) - \angle C]$. **(IIT-JEE 1980)**
4. The set of all real numbers a such that $a^2 + 2a$, $2a + 3$, and $a^2 + 3a + 8$ are the sides of a triangle is _____. **(IIT-JEE 1985)**
5. In triangle ABC, if $\cot A$, $\cot B$, $\cot C$ are in A.P., then a^2, b^2, c^2 are in ____ progression. **(IIT-JEE 1985)**
6. A polygon of nine sides, each side of length 2, is inscribed in a circle. The radius of the circle is _____. **(IIT-JEE 1987)**

7. If the angles of a triangle are 30° and 45° and the included side is $(\sqrt{3}+1)$ cm, then the area of the triangle is _____. **(IIT-JEE 1988)**

8. If in triangle ABC,
$\frac{2\cos A}{a}+\frac{\cos B}{b}+\frac{2\cos C}{c}=\frac{a}{bc}+\frac{b}{ca}$, then the value of the angle A is _____ degrees. **(IIT-JEE 1993)**

9. In triangle ABC, AD is the altitude from A. If $b > c$, $\angle C = 23°$, and $AD = \frac{abc}{b^2-c^2}$, then $\angle B =$ _____. **(IIT-JEE 1994)**

10. A circle is inscribed in an equilateral triangle of side a. The area of any square inscribed in this circle is _____. **(IIT-JEE 1994)**

11. In triangle ABC, $a:b:c = 4:5:6$. The ratio of the radius of the circumcircle to that of the incircle is _____. **(IIT-JEE 1996)**

Subjective Type

1. ABC is a triangle. D is the middle point of BC. If AD is perpendicular to AC, then prove that
$\cos A\cos C=\frac{2(c^2-a^2)}{3ac}$. **(IIT-JEE 1980)**

2. ABC is a triangle with $AB = AC$. D is any point on the side BC. E and F are points on the sides AB and AC, respectively, such that DE is parallel to AC and DF is parallel to AB. Prove that
$DF + FA + AE + ED = AB + AC$. **(IIT-JEE 1980)**

3. Let the angles A, B, and C of triangle ABC be in A.P. and let $b:c$ be $\sqrt{3}:\sqrt{2}$. Find angle A. **(IIT-JEE 1981)**

4. The exradii r_1, r_2, and r_3 of ΔABC are in H.P. Show that its sides a, b, and c are in A.P. **(IIT-JEE 1983)**

5. For triangle ABC, it is given that $\cos A + \cos B + \cos C = 3/2$. Prove that the triangle is equilateral. **(IIT-JEE 1984)**

6. With usual notation, if in triangle ABC,
$\frac{b+c}{11}=\frac{c+a}{12}=\frac{a+b}{13}$, then prove that
$\frac{\cos A}{7}=\frac{\cos B}{19}=\frac{\cos C}{25}$. **(IIT-JEE 1984)**

7. In triangle ABC, the median to the side BC is of length $\frac{1}{\sqrt{11-6\sqrt{3}}}$ and it divides the angle A into angles 30° and 45°. Find the length of the side BC. **(IIT-JEE 1985)**

8. If in triangle ABC, $\cos A\cos B + \sin A\sin B\sin C = 1$. Show that $a:b:c = 1:1:\sqrt{2}$. **(IIT-JEE 1986)**

9. Three circles touch one another externally. The tangents at their points of contact meet at a point whose distance from a point of contact is 4. Find the ratio of the product of the radii to the sum of the radii of the circles. **(IIT-JEE 1992)**

10. Consider the following statements concerning triangle ABC:
    (i) The sides a, b, and c and area (Δ) are rational
    (ii) a, $\tan\frac{B}{2}$, and $\tan\frac{C}{2}$ are rational
    (iii) a, $\sin A$, $\sin B$, and $\sin C$ are rational
    Prove that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i). **(IIT-JEE 1994)**

11. Let $A_1, A_2, \ldots, A_n$ be the vertices of an n-sided regular polygon such that $\frac{1}{A_1A_2}=\frac{1}{A_1A_3}+\frac{1}{A_1A_4}$. Find the value of n. **(IIT-JEE 1994)**

12. The sides of a triangle are three consecutive natural numbers and its largest angle is twice the smallest one. Determine the sides of the triangle. **(IIT-JEE 1997)**

13. In a triangle of base a, the ratio of the other two sides is $r\ (<1)$. Show that the altitude of the triangle is less than or equal to $\frac{ar}{1-r^2}$. **(IIT-JEE 1997)**

14. Let A, B, C be three angles such that $A=\frac{\pi}{4}$ and $\tan B\tan C = p$. Find all possible values of p such that A, B, C are angles of a triangle. **(IIT-JEE 1997)**

15. Prove that a triangle ABC is equilateral if and only if $\tan A + \tan B + \tan C = 3\sqrt{3}$ **(IIT-JEE 1998)**

16. Let ABC be a triangle having O and I as its circumcenter and incentre, respectively. If R and r are the circumradius and the inradius, respectively, then prove that $(OI)^2 = R^2 - 2Rr$. Further show that the triangle AIO is a right-angled triangle if and only if a is arithmetic mean of b and c. **(IIT-JEE 1999)**

17. Let ABC be a triangle with incenter I and inradius r. Let D, E, and F be the feet of the perpendiculars from I to the sides BC, CA, and AB, respectively. If r_1, r_2, and r_3 are the radii of circles inscribed in the quadrilaterals $AFIE$, $BDIF$, and $CEID$, respectively, prove that
$$\frac{r_1}{r-r_1}+\frac{r_2}{r-r_2}+\frac{r_3}{r-r_3}=\frac{r_1r_2r_3}{(r-r_1)(r-r_2)(r-r_3)}.$$
(IIT-JEE 2000)

18. If Δ is the area of a triangle with side lengths a, b, and c, then show that $\Delta\le\frac{1}{4}\sqrt{(a+b+c)\,abc}$. Also show that the equality occurs in the above inequality if and only if $a = b = c$. **(IIT-JEE 2001)**

19. If I_n is the area of n-sided regular polygon inscribed in a circle of unit radius and O_n be the area of the polygon circumscribing the given circle, prove that
$$I_n=\frac{O_n}{2}\left(1+\sqrt{1-\left(\frac{2I_n}{n}\right)^2}\right).$$
(IIT-JEE 2003)

Answer Key

JEE Advanced

Single Correct Answer Type

1. c.	**2.** c.	**3.** a.	**4.** b.	**5.** a.
6. d.	**7.** a.	**8.** d.	**9.** b.	**10.** b.
11. c.	**12.** b.	**13.** b.	**14.** d.	**15.** c.

Multiple Correct Answers Type

1. a., d. **2.** a., d. **3.** b. **4.** c.
5. a., b., c., d. **6.** b., d. **7.** b., c.
8. b., d.

Matching Column Type

1. (ii) – (d)
2. (a) – (p), (r), (s); (b) – (p)

Integer Answer Type

1. (4) **2.** (3) **3.** (3)

Fill in the Blanks Type

1. AC **2.** AP^2 **3.** $\angle B$ **4.** $a > 5$
5. A.P. **6.** cosec($\pi/9$) **7.** $\frac{\sqrt{3}+1}{2}$ **8.** 90°
9. 113° **10.** $a/6$ sq. units **11.** 16/7

Subjective Type

3. 75° **7.** 2 units **9.** 16 : 1 **13.** 4, 5, 6

Hints and Solutions

JEE Advanced

Single Correct Answer Type

1. c. Given that $A > B$

and $\quad 3\sin x - 4\sin^3 x - k = 0,\ 0 < k < 1$

$\Rightarrow \quad \sin 3x = k$

As A and B satisfy above equation (given), we have

$\sin 3A = k,\ \sin 3B = k$

or $\quad \sin 3A - \sin 3B = 0$

$$\Rightarrow \quad 2\cos\frac{3A+3B}{2}\sin\frac{3A-3B}{2} = 0$$

$$\Rightarrow \quad \cos\left(\frac{3A+3B}{2}\right) = 0 \text{ or } \sin\left(\frac{3A-3B}{2}\right) = 0$$

$$\Rightarrow \quad \frac{3A+3B}{2} = 90° \text{ or } \frac{3A-3B}{2} = 0$$

$\Rightarrow \quad A + B = 60°$ or $A = B$

But given that $A > B$, therefore $A \neq B$

Thus, $A + B = 60°$

But $\quad A + B + C = 180°$

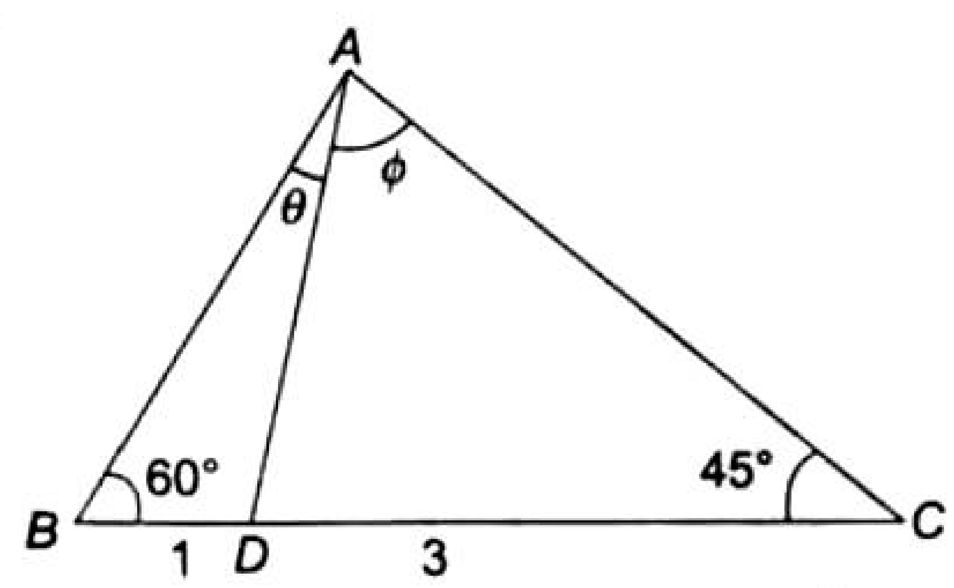
$$\therefore \quad C = 180° - 60° = 120° = \frac{2\pi}{3}$$

2. c. Let $a = 3$, $b = 5$, $c = 7$. Then the largest angle is opposite to the longest side, i.e., $\angle C$. Therefore,

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{9 + 25 - 49}{2 \times 3 \times 5} = \frac{-1}{2}$$

or $\quad C = \dfrac{2\pi}{3}$

3. a.



Applying the sine rule in ΔABD and ΔADC and eliminating common side, we get

$$\frac{AD}{\sin 60°} = \frac{BD}{\sin\theta} \text{ and } \frac{AD}{\sin 45°} = \frac{DC}{\sin\phi}$$

Dividing, we get $\dfrac{DC}{BD} = \dfrac{\sin 60°}{\sin 45°}\dfrac{\sin\phi}{\sin\theta}$

$$\text{or} \quad \frac{3}{1} = \frac{\left(\sqrt{3}/2\right)}{\left(1/\sqrt{2}\right)}\frac{\sin\phi}{\sin\theta}$$

$$\text{or} \quad \frac{\sin\theta}{\sin\phi} = \frac{\sqrt{3}}{\sqrt{2}\times 3} = \frac{1}{\sqrt{6}}$$

4. b. We know that $A + B + C = 180°$

or $\quad A + C - B = 180° - 2B$

Now, $2ac\sin\left[\frac{1}{2}(A - B + C)\right]$

$$= 2ac\sin(90° - B) = 2ac\cos B$$

$$= \frac{2ac\left(a^2 + c^2 - b^2\right)}{2ac} = a^2 + c^2 - b^2$$

5. a. Since ΔABC is right angled at C, circum-radius, $R = \dfrac{c}{2}$

Now, $r = (s - c)\tan(C/2) = (s - c)\tan(\pi/4) = s - c$

Thus, $2(r + R) = 2r + 2R = 2s - c = a + b$

6. d. We know by sine law in ΔABC, we have

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin(\pi - A - B)} = 2R$$

or $$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin(A + B)} = 2R$$

a. If we know a, $\sin A$, $\sin B$, we can find b, c, and the value of angles A, B, and C.

b. With a, b, c, we can find $\angle A$, $\angle B$, $\angle C$ using the cosine law.

c. a, $\sin B$, R are given, so $\sin A$, b and hence $\sin(A + B)$ and then C can be found.

d. If we know a, $\sin A$, R, then we know only the ratio $\frac{b}{\sin B}$ or $\frac{c}{\sin(A+B)}$; we cannot determine the values of b, c, $\sin B$, $\sin C$ separately. Therefore, the triangle cannot be determined uniquely in this case.

7. a. Given that $4A + A + A = 180°$ or $A = 30°$

Angles are 120°, 30°, 30°

$$\Rightarrow \frac{\sin 120°}{a} = \frac{\sin 30°}{b} = \frac{\sin 30°}{c} = 2R \text{ (say)}$$

$$\Rightarrow \frac{a}{a+b+c} = \frac{\sin 120°}{\sin 120° + \sin 30° + \sin 30°}$$

$$= \frac{\sqrt{3}}{2+\sqrt{3}}$$

8. d. Sides are in the ratio $1 : \sqrt{3} : 2$

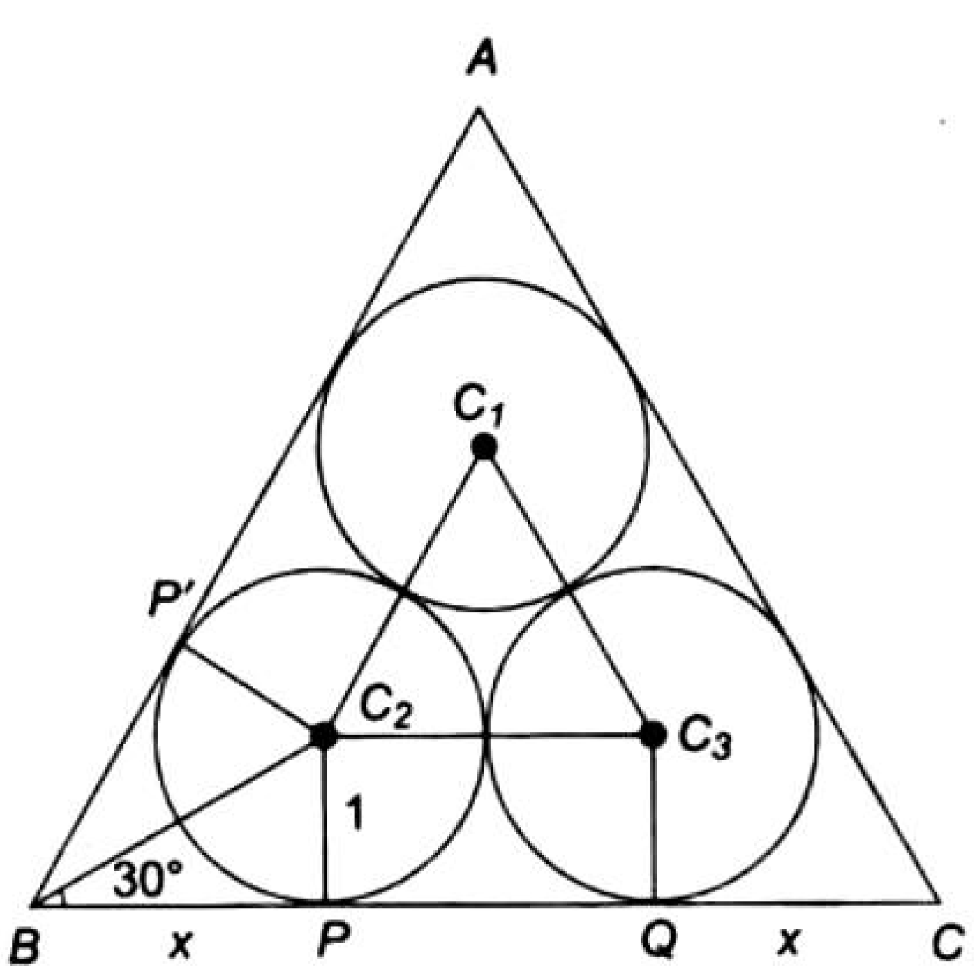
Let $a = k$, $b = \sqrt{3}\,k$, and $c = 2k$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{\sqrt{3}}{2} \Rightarrow A = \frac{\pi}{6}$$

$$\cos B = \frac{c^2 + a^2 - b^2}{2ac} = \frac{1}{2} \Rightarrow B = \frac{\pi}{3}$$

$$\Rightarrow C = \pi - \overline{A + B} = \frac{\pi}{2} \quad \Rightarrow A:B:C = 1:2:3$$

9. b. The situation is as shown in figure.



For the circle with center C_2, BP and BP' are two tangents to the circle; therefore BC_2 must be the bisector of $\angle B$. But $\angle B = 60°$ (as ΔABC is an equilateral triangle). Thus,

$$\angle C_2BP = 30°$$

$$\tan 30° = \frac{1}{x} \quad \text{or} \quad x = \sqrt{3}$$

$$BC = BP + PQ + QC = x + 2 + x = 2 + 2\sqrt{3}$$

$(\because PQ = C_2C_3 = 2)$

$$\therefore \quad \text{Area of } \Delta ABC = \frac{\sqrt{3}}{4}\left(2 + 2\sqrt{3}\right)^2$$

$$= \sqrt{3}\left(1 + 3 + 2\sqrt{3}\right)$$

$$= 4\sqrt{3} + 6 \text{ sq. units}$$

10. b. Let us consider $\frac{b-c}{a}$.

$$\frac{b-c}{a} = \frac{\sin B - \sin C}{\sin A}$$

$$= \frac{2\cos\left(\frac{B+C}{2}\right)\sin\left(\frac{B-C}{2}\right)}{2\sin(A/2)\cos(A/2)}$$

$$= \frac{\sin(A/2)\sin\left(\frac{B-C}{2}\right)}{\sin(A/2)\cos(A/2)}$$

$$= \frac{\sin\left(\frac{B-C}{2}\right)}{\cos(A/2)}$$

$$\therefore \quad (b-c)\cos\left(\frac{A}{2}\right) = a\sin\left(\frac{B-C}{2}\right)$$

11. c. Let the two equal sides be x. By applying the sine rule in ΔABC, we get

$$\frac{x}{\sin 30°} = \frac{a}{\sin 120°} \Rightarrow a = x\sqrt{3}$$

$$\Rightarrow \quad \Delta = \frac{1}{2}(x)(x)\sin 120° = \frac{\sqrt{3}}{4}x^2$$

$$\text{Also, } \sqrt{3} = \frac{\Delta}{s} \Rightarrow \frac{(2x+a)}{2}\sqrt{3} = \frac{\sqrt{3}}{4}x^2$$

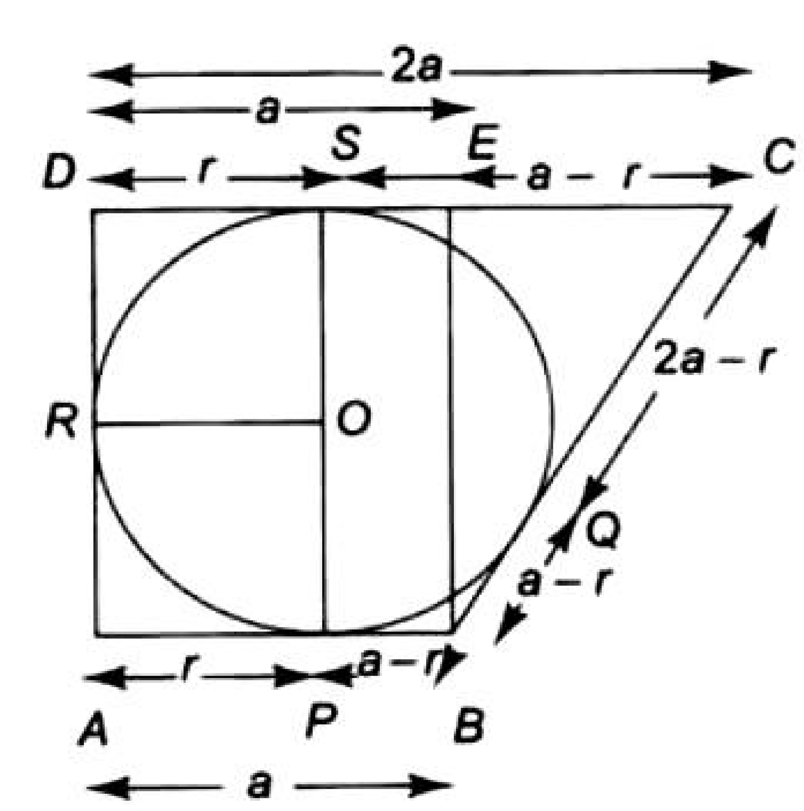
$$\Rightarrow \quad x = 2(2 + \sqrt{3}) \qquad \text{(Using } a = x\sqrt{3}\text{)}$$

$$\Rightarrow \quad \Delta = \frac{\sqrt{3}}{4} \times 4\left(4 + 3 + 4\sqrt{3}\right)$$

$$\Rightarrow \quad \Delta = 7\sqrt{3} + 12 \text{ sq. units}$$

12. b. Given $AB \parallel CD$, $CD = 2AB$. Let $AB = a$. Then $CD = 2a$. Let the radius of the circle be r.

Let the circle touches AB at P, BC at Q, AD at R, and CD at S. Then $AR = AP = r$, $BP = BQ = a - r$, $DR = DS = r$, and $CQ = CS = 2a - r$.



In ΔBEC,

$$BC^2 = BE^2 + EC^2$$

$\Rightarrow \quad (a - r + 2a - r)^2 = (2r)^2 + (a)^2$

or $\quad 9a^2 + 4r^2 - 12\,ar = 4r^2 + a^2$

or $\quad a = \dfrac{3}{2}r \qquad$ (i)

Also, Area (quad. $ABCD$) = 18

or $\quad$ Area (quad. $ABED$) + Area (ΔBCE) = 18

or $\quad a \times 2r + \dfrac{1}{2} \times a \times 2r = 18$

or $\quad ar = 6 \quad$ or $\quad \dfrac{3r^2}{2} = 6 \qquad$ [using Eq. (i)]

or $\quad r^2 = 4 \quad$ or $\quad r = 2$

13. b. Using cosine rule of $\angle C$, we get

$$\frac{\sqrt{3}}{2} = \frac{(x^2+x+1)^2+(x^2-1)^2-(2x+1)^2}{2(x^2+x+1)(x^2-1)}$$

or $\quad \sqrt{3} = \dfrac{2x^2+2x-1}{x^2+x+1}$

or $\quad (\sqrt{3}-2)x^2+(\sqrt{3}-2)x+(\sqrt{3}+1)=0$

or $\quad x = \dfrac{(2-\sqrt{3})\pm\sqrt{3}}{2(\sqrt{3}-2)}$

or $\quad x = -(2+\sqrt{3}), 1+\sqrt{3}$ or $x = 1+\sqrt{3}$ as $(x > 0)$.

14. d. Since angles of ΔABC are in A.P., $2B = A + C$

Also, $A + B + C = 180°$

$\therefore \quad B = 60°$

$\therefore \quad \dfrac{a}{c}\sin 2C + \dfrac{c}{a}\sin 2A = 2\sin A\cos C + 2\sin C\cos A$

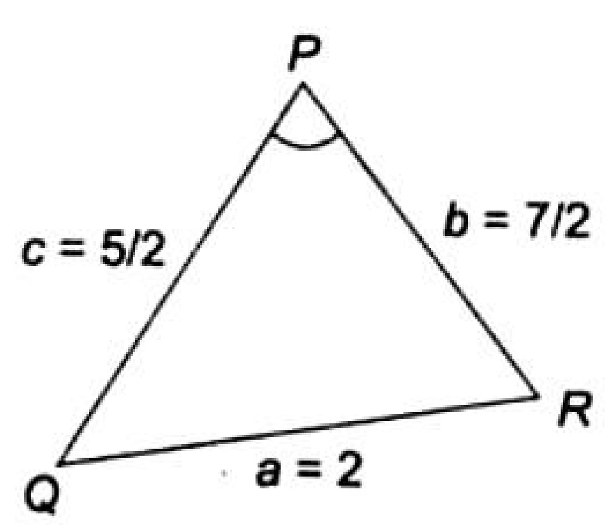
$$= 2\sin(A + C) = 2\sin B = 2 \times \frac{\sqrt{3}}{2} = \sqrt{3}.$$

15. c. $\dfrac{2\sin P - 2\sin P\cos P}{2\sin P + 2\sin P\cos P}$

$$= \frac{1-\cos P}{1+\cos P} = \frac{2\sin^2\frac{P}{2}}{2\cos^2\frac{P}{2}} = \tan^2\frac{P}{2}$$

$$= \frac{(s-b)(s-c)}{s(s-a)}$$

$$= \frac{((s-b)(s-c))^2}{\Delta^2} = \frac{\left(\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\right)^2}{\Delta^2} = \left(\frac{3}{4\Delta}\right)^2$$



Multiple Correct Answers Type

1. a., d.

In ΔABC, we have $\dfrac{\sin A}{a} = \dfrac{\sin B}{b}$ or $a\sin B = b\sin A$

a. $b\sin A = a \Rightarrow a\sin B = a \Rightarrow \sin B = 1 \Rightarrow B = \dfrac{\pi}{2}$

Since $A < \pi/2$, ΔABC is possible.

b. $b\sin A > a \Rightarrow a\sin B > a$

$\Rightarrow \sin B > 1$, which is not possible.

c. $b\sin A > a, A < \pi/2$

$\Rightarrow \quad a\sin B > a$

$\Rightarrow \quad \sin B > 1$, which is not possible

d. $b\sin A < a \Rightarrow a\sin B < a$

Hence, $\sin B < 1$, so value of $\angle B$ exists.

Now, $b > a \Rightarrow B > A$. Since $A < \pi/2$. ΔABC is possible when $B > \pi/2$.

e. Since $b = a$, we have $B = A$. But $A > \pi/2$.

Therefore, $B > \pi/2$. But this is not possible for any triangle.

2. a., d. The angles of the triangle are in A.P.

Now one angle is greater than 60° and other is less than 60°.

Let $A > 60°$.

Thus, larger angles are A and B.

$\therefore \quad a = 10$ and $b = 9$

$\angle B = 60°$

Using the cosine formula, $\cos B = \dfrac{a^2+c^2-b^2}{2ac}$, we get

$$\cos 60° = \frac{100 + c^2 - 81}{2 \times 10 \times c}$$

or $\quad \dfrac{1}{2} = \dfrac{19+c^2}{2\times 10c}$

or $\quad c^2 - 10c + 19 = 0$

or $\quad c = \dfrac{10 \pm \sqrt{100-76}}{2} = 5 \pm \sqrt{6}$

Given that $a = 10$, $b = 9$ are the longer sides. Therefore

$c = 5 \pm \sqrt{6}$ (both the values are less than 9 and 10)

3. b. In ΔPQR, let d_1, d_2, d_3 be the altitudes on QR, RP, and PQ, respectively. Then,

Area (ΔPQR) = $\Delta = \dfrac{1}{2}pd_1 = \dfrac{1}{2}qd_2 = \dfrac{1}{2}rd_3$

or $\quad d_1 = \dfrac{2\Delta}{p}, d_2 = \dfrac{2\Delta}{q}, d_3 = \dfrac{2\Delta}{r}$

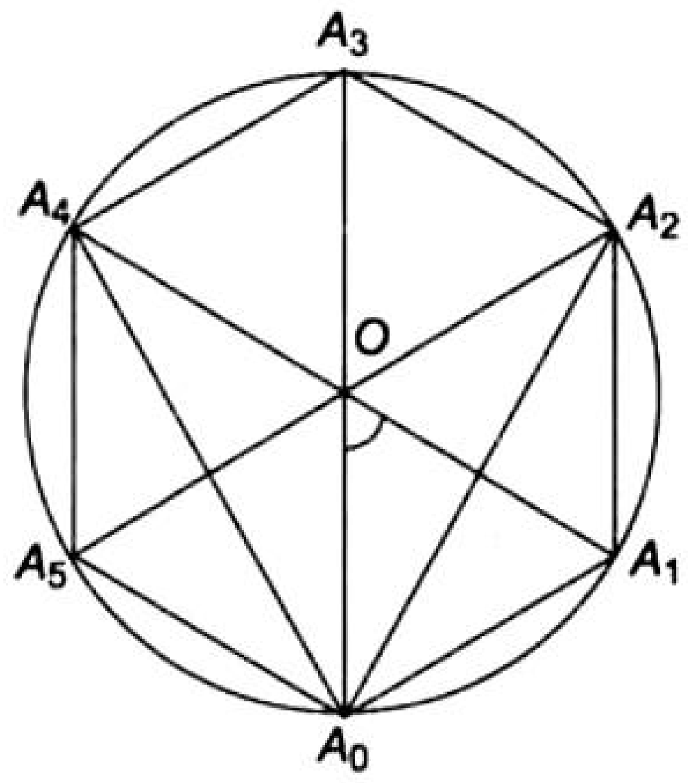
or $\quad d_1 = \dfrac{2\Delta}{2R\sin P}, d_2 = \dfrac{2\Delta}{2R\sin Q},$

$d_3 = \dfrac{2\Delta}{2R\sin R}$ [using sine law]

Hence, d_1, d_2, d_3 are in H.P.

(as given that $\sin P, \sin Q, \sin R$ are in A.P.)

4. c.



Given that $A_0A_1A_2A_3A_4A_5$ is a regular hexagon inscribed in a circle of radius 1.

$$\angle A_0OA_1 = \frac{360°}{6} = 60°$$

But in ΔOA_0A_1, $OA_0 = OA_1 = 1$

$\therefore \quad \angle OA_0A_1 = \angle OA_1A_0 = 60°$

Therefore, ΔOA_0A_1 is an equilateral triangle.

$$A_0A_1 = 1 = A_1A_2 = A_2A_3 = A_3A_4 = A_4A_5 = A_5A_0$$

$$\angle A_0A_1A_2 = 120°$$

Using cosine rule, we get

$$\cos 120° = \frac{(A_0A_1)^2 + (A_1A_2)^2 - (A_0A_2)^2}{2(A_0A_1)(A_1A_2)}$$

or $\quad -\frac{1}{2} = \frac{1+1-(A_0A_2)^2}{2\times1\times1} \qquad$ or $A_0A_2 = \sqrt{3}$

Thus, by symmetry $A_0A_4 = \sqrt{3}$

$\Rightarrow \quad A_0A_1 \cdot A_0A_2 \cdot A_0A_4 = 1 \times \sqrt{3} \times \sqrt{3} = 3$

5. a., b., c., d.

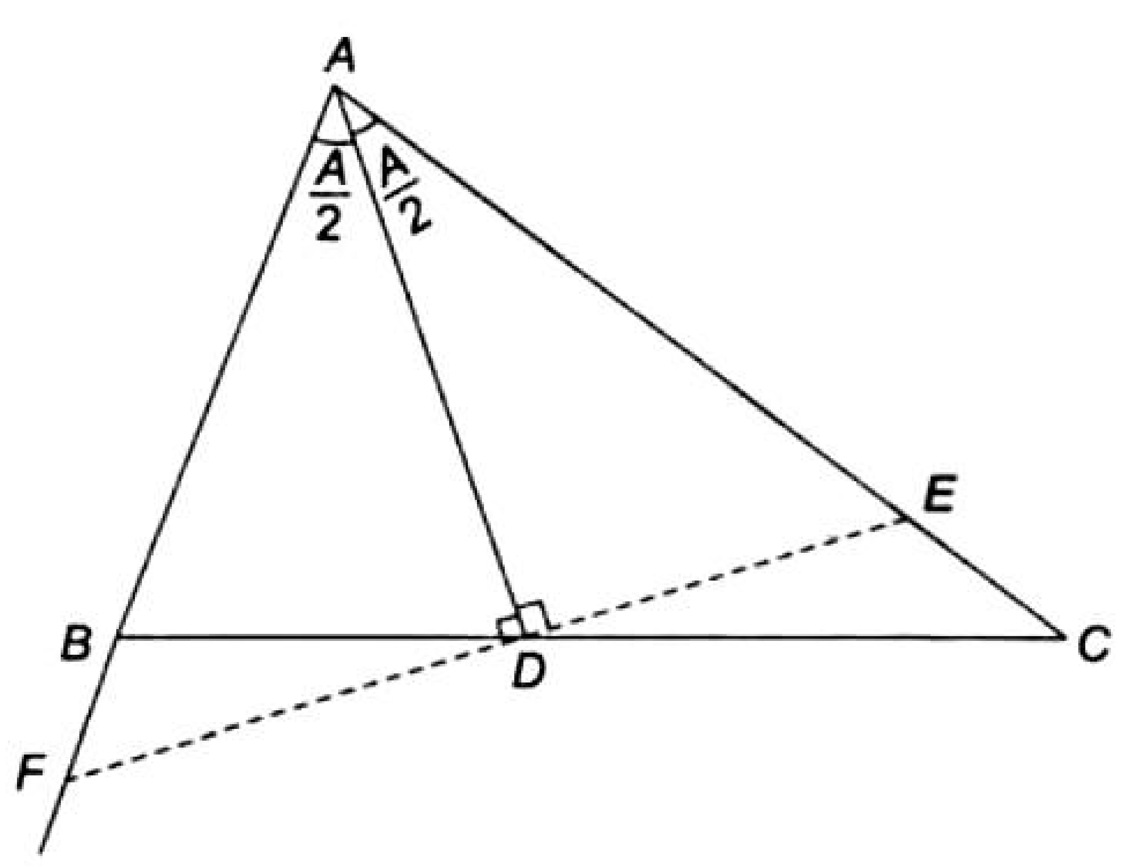
In ΔAFE, AD is angle bisector of $\angle A$. and $AD \perp EF$.

$\therefore \quad D$ is midpoint of EF and ΔAEF is isosceles triangle.

Therefore, ΔAFE is an isosceles triangle. Now,

To find AD,

Area (ΔABC) = Area (ΔABD) + Area (ΔADC)



$\Rightarrow \quad \frac{1}{2} bc \sin A = \frac{1}{2} c\, AD \sin\frac{A}{2} + \frac{1}{2} b\, AD \sin\frac{A}{2}$

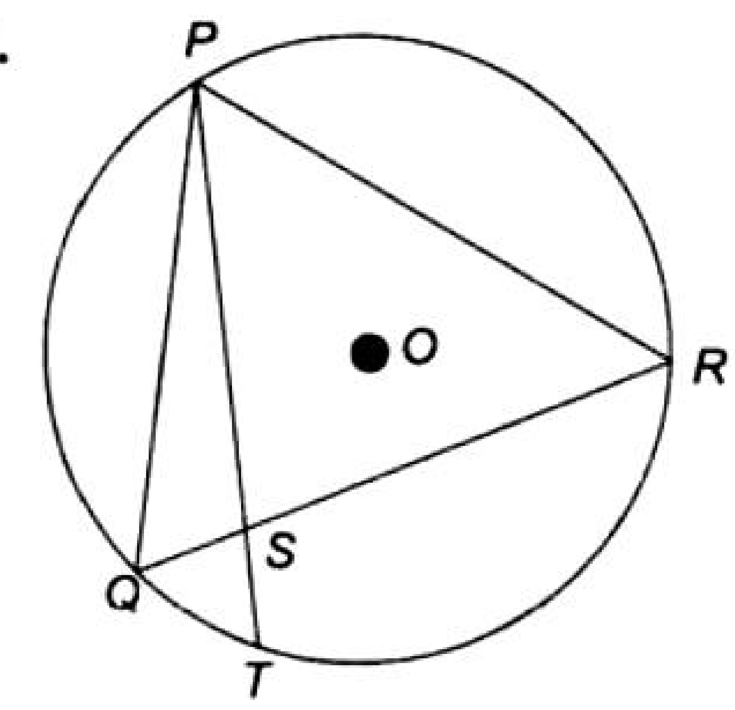
$\Rightarrow \quad AD = \dfrac{2bc\cos\frac{A}{2}}{b+c}$

Also, $AD = AE\cos\frac{A}{2}$ (From ΔADE)

$\Rightarrow \quad AE = \frac{2bc}{b+c}$ = H.M. of b and c

Again $EF = 2\,DE = 2\,AD\tan\frac{A}{2} = \dfrac{4bc\sin\frac{A}{2}}{b+c}$

6. b., d.



$PS \times ST = QS \times SR$ (Secant property of circle)

Now A.M. > G.M.

$$\Rightarrow \quad \frac{\frac{1}{PS}+\frac{1}{ST}}{2} > \sqrt{\frac{1}{PS}\times\frac{1}{ST}} \qquad (1)$$

$$\Rightarrow \quad \frac{1}{PS}+\frac{1}{ST} > \frac{2}{\sqrt{QS\times SR}}$$

Also, $\frac{QS+SR}{2} > \sqrt{QS\times SR}$ (2)

or $\quad \frac{1}{\sqrt{QS\times SR}} > \frac{2}{QR}$

or $\quad \frac{1}{PS}+\frac{1}{ST} > \frac{4}{QR}$ (From (1) and (2))

7. b., c.

$$\cos B + \cos C = 4\sin^2\frac{A}{2}$$

$\Rightarrow \quad 2\cos\frac{B+C}{2}\cos\frac{B-C}{2} = 4\sin^2\frac{A}{2}$

$\Rightarrow \quad 2\cos\frac{B-C}{2} = 4\sin\frac{A}{2} \qquad \left(\because \cos\frac{B+C}{2} = \sin\frac{A}{2}\right)$

$\Rightarrow \quad 2\cos\frac{A}{2}\cos\frac{B-C}{2} = 4\sin\frac{A}{2}\cos\frac{A}{2}$

$\Rightarrow \quad 2\sin\frac{B+C}{2}\cos\frac{B-C}{2} = 2\sin A$

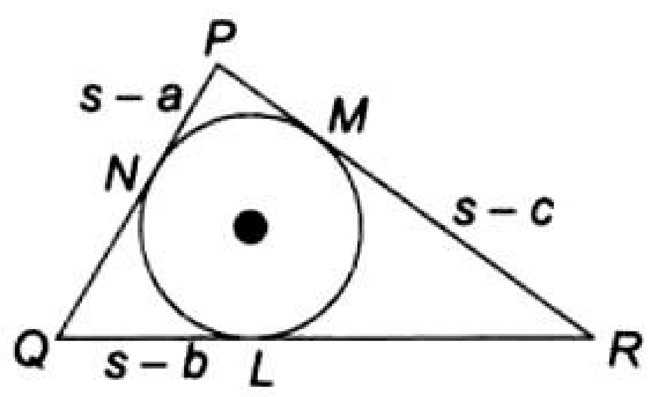
$\Rightarrow \quad \sin B + \sin C = 2\sin A$

$\Rightarrow \quad b + c = 2a$ (Using sine rule)

Thus sum of two variable sides b and c is constant '$2a$'.

So locus of vertex A is ellipse with vertices B and C as its foci.

8. b., d.



Let $s - a = 2k - 2$, $s - b = 2k$, $s - c = 2k + 2$,
$k \in I, k > 1$

Adding we get,

$s = 6k$. So, $a = 4k + 2$, $b = 4k$, $c = 4k - 2$

Now, $\cos P = \frac{1}{3} \Rightarrow \frac{b^2 + c^2 - a^2}{2bc} = \frac{1}{3}$

$\Rightarrow \quad 3\,[(4k)^2 + (4k-2)^2 - (4k+2)^2]$
$= 2 \times 4k\,(4k - 2)$

or $\quad 3\,[16k^2 - 4\,(4k) \times 2] = 8k\,(4k - 2)$

or $\quad 48k^2 - 96k = 32k^2 - 16k$ or $16k^2 = 80k$ or $k = 5$

So, sides are 22, 20, 18.

Matching Column Type

1. (ii) – (d) $\cos \theta_1 = \frac{a}{b+c}$

$$\Rightarrow \quad \cos \theta_1 = \frac{1 - \tan^2 \frac{\theta_1}{2}}{1 + \tan^2 \frac{\theta_1}{2}} = \frac{a}{b+c}$$

$$\Rightarrow \quad \tan^2 \frac{\theta_1}{2} = \frac{b + c - a}{b + c + a}$$

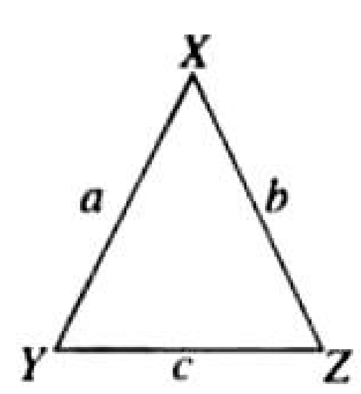
Also, $\cos \theta_3 = \dfrac{1 - \tan^2 \frac{\theta_3}{2}}{1 + \tan^2 \frac{\theta_3}{2}} = \dfrac{c}{a+b}$

$$\Rightarrow \quad \tan^2 \frac{\theta_3}{2} = \frac{a + b - c}{a + b + c}$$

$$\therefore \quad \tan^2 \frac{\theta_1}{2} + \tan^2 \frac{\theta_3}{2} = \frac{2b}{3b} = \frac{2}{3}$$

Note: Solutions of the remaining parts are given in their respective chapters.

2. (a) – (p), (r), (s)



Given $2(a^2 - b^2) = c^2$

$\Rightarrow \quad 2\,(\sin^2 X - \sin^2 Y) = \sin^2 Z$

$\Rightarrow \quad 2 \sin(X + Y) \sin(X - Y) = \sin^2 Z$

$\Rightarrow \quad 2 \sin(\pi - Z) \sin(X - Y) = \sin^2 Z$

$$\Rightarrow \quad \sin(X - Y) = \frac{\sin Z}{2} \qquad \text{...(i)}$$

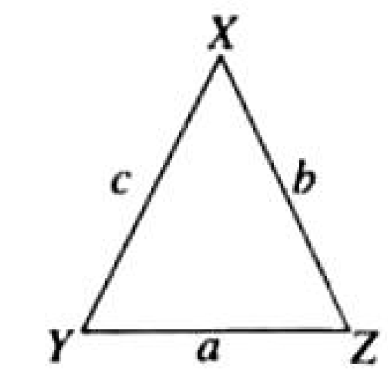
$$\therefore \quad \lambda = \frac{\sin(X - Y)}{\sin Z} = \frac{1}{2}$$

Now $\cos(n\pi\lambda) = 0$

$$\Rightarrow \quad \cos\left(\frac{n\pi}{2}\right) = 0$$

$\therefore \quad n = 1, 3, 5$

(b) – (p)



$1 + \cos 2X - 2 \cos 2Y = 2 \sin X \sin Y$

$2 \cos^2 X - 2 \cos 2Y = 2 \sin X \sin Y$

$1 - \sin^2 X - 1 + 2 \sin^2 Y = \sin X \sin Y$

$\sin^2 X + \sin X \sin Y = 2 \sin^2 Y$

$\sin X (\sin X + \sin Y) = 2 \sin^2 Y$

$\Rightarrow \quad a(a + b) = 2b^2$

$\Rightarrow \quad a^2 + ab - 2b^2 = 0$

$$\Rightarrow \quad \left(\frac{a}{b}\right)^2 + \frac{a}{b} - 2 = 0$$

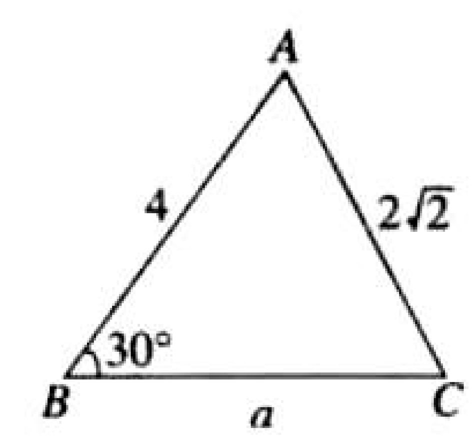
$$\Rightarrow \quad \frac{a}{b} = -2, 1$$

$$\Rightarrow \quad \frac{a}{b} = 1$$

Note: Solutions of the remaining parts are given in their respective chapters.

Integer Answer Type

1. (4)



$$\cos 30° = \frac{a^2 + 16 - 8}{2 \times a \times 4}$$

$$\Rightarrow \quad \frac{\sqrt{3}}{2} = \frac{a^2 + 8}{8a}$$

$\Rightarrow \quad a^2 - 4\sqrt{3}\,a + 8 = 0$

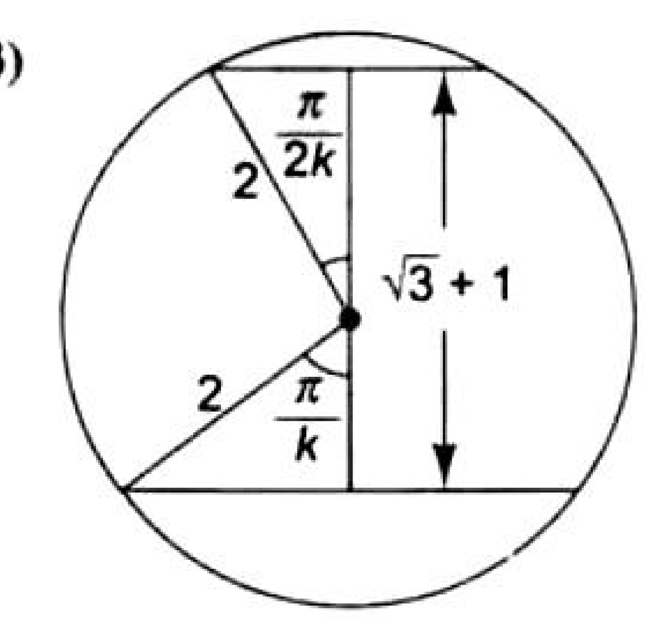
$\Rightarrow \quad a_1 + a_2 = 4\sqrt{3}$, $a_1 a_2 = 8$

$\Rightarrow \quad |a_1 - a_2| = 4$

$$\Rightarrow \quad \left|\frac{1}{2} a_1 \times 4 \sin 30° - \frac{1}{2} a_2 \times 4 \sin 30°\right|$$
$$= 4 \times \frac{1}{2} \times 4 \sin 30°$$

$\Rightarrow \quad |\Delta_1 - \Delta_2| = 4$

2. (3)



$$2 \cos \frac{\pi}{2k} + 2 \cos \frac{\pi}{k} = \sqrt{3} + 1$$

or $\cos\frac{\pi}{2k}+\cos\frac{\pi}{k}=\frac{\sqrt{3}+1}{2}$

Let $\frac{\pi}{k}=\theta$. Then,

$$\cos\theta+\cos\frac{\theta}{2}=\frac{\sqrt{3}+1}{2}$$

or $2\cos^2\frac{\theta}{2}-1+\cos\frac{\theta}{2}=\frac{\sqrt{3}+1}{2}$

or $2t^2+t-\frac{\sqrt{3}+3}{2}=0$ [where $\cos(\theta/2)=t$]

or $t=\frac{-1\pm\sqrt{1+4(3+\sqrt{3})}}{4}$

$$=\frac{-1\pm(2\sqrt{3}+1)}{4}=\frac{-2-2\sqrt{3}}{4},\frac{\sqrt{3}}{2}$$

$\Rightarrow$ $t=\cos\frac{\theta}{2}\in[-1,1]\ \therefore \cos\frac{\theta}{2}=\frac{\sqrt{3}}{2}$

$\Rightarrow$ $\frac{\theta}{2}=\frac{\pi}{6}\Rightarrow k=3.$

3. (3) $\Delta=\frac{1}{2}ab\sin C\Rightarrow\sin C=\frac{2\Delta}{ab}=\frac{2\times15\sqrt{3}}{6\times10}=\frac{\sqrt{3}}{2}$

$\Rightarrow$ $C=120°$

$\Rightarrow$ $c=\sqrt{a^2+b^2-2ab\cos C}$

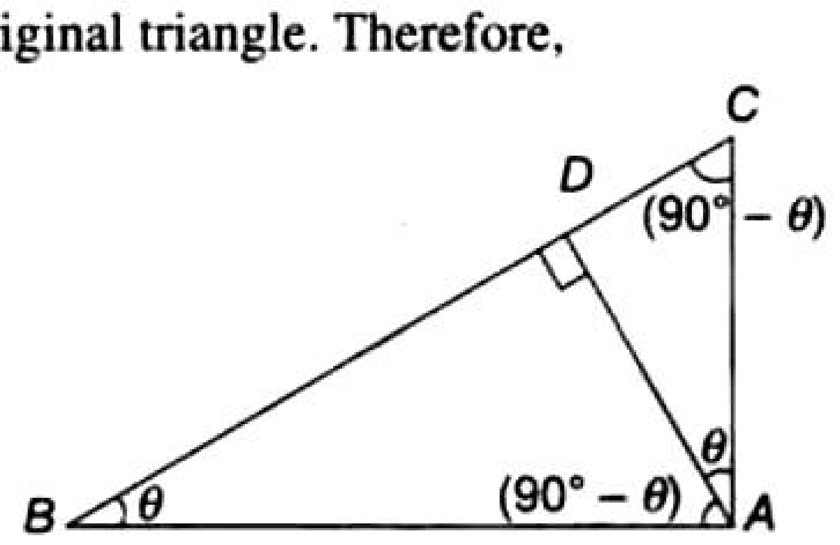
$$=\sqrt{6^2+10^2-2\times6\times10\times\cos120°}=14$$

Now $r=\frac{\Delta}{s}$

$\Rightarrow$ $r^2=\frac{225\times3}{\left(\frac{6+10+14}{2}\right)^2}=3$

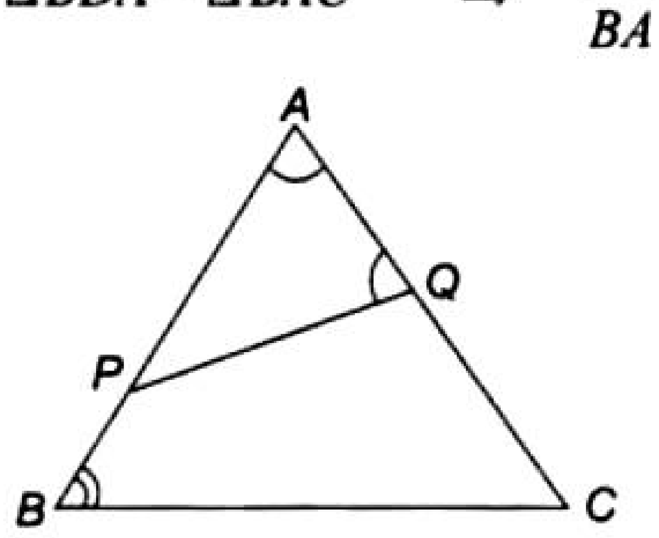
Fill in the Blanks Type

1. We know that altitude from right vertex to hypotenuse in right-angled triangle divides it into two triangles each being similar to the original triangle. Therefore,



$\Delta BDA\sim\Delta BAC \Rightarrow \frac{DB}{BA}=\frac{AB}{BC}=\frac{DA}{AC}$

2.



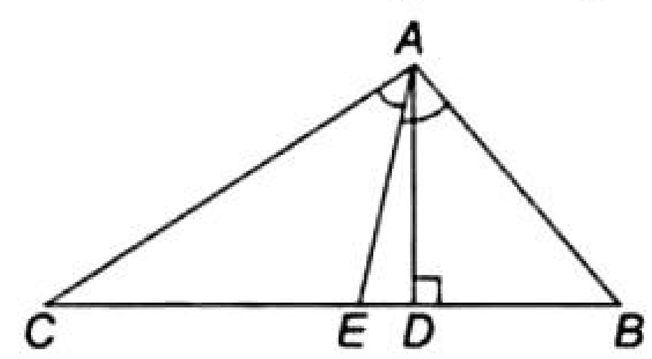
In ΔAPQ and ΔABC, $\angle A=\angle A$ (common),
$\angle AQP=\angle ABC$ (given)

$\therefore$ $\Delta APQ\sim\Delta ACB \Rightarrow \frac{AP}{AC}=\frac{AQ}{AB}$

$\Rightarrow$ $\frac{\text{Area }(\Delta APQ)}{\text{Area}(\Delta ACB)}=\frac{\frac{1}{2}AP\ AQ\sin(\angle A)}{\frac{1}{2}AB\ AC\sin(\angle A)}=\frac{AP^2}{AC^2}$

3. We have $\angle BAE=\angle CAE$ (given)
and $\angle ADB=\angle ADC=90°$ (given)
Now, $\angle DAE=\angle A-\angle BAD-\angle EAC$

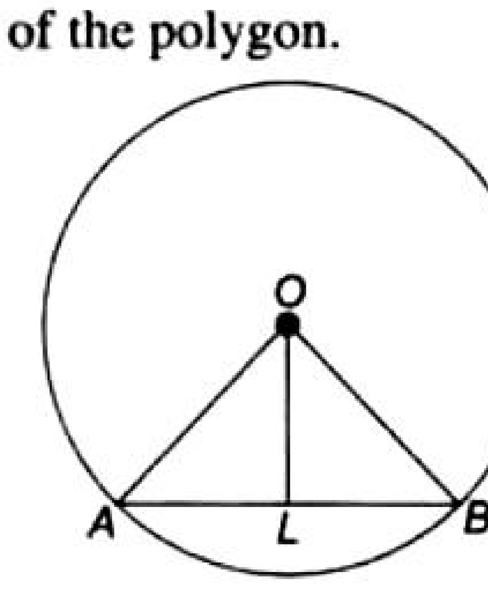
$$=\angle A-(90°-\angle B)-\frac{1}{2}\angle A=\frac{1}{2}(\angle B-\angle C)$$



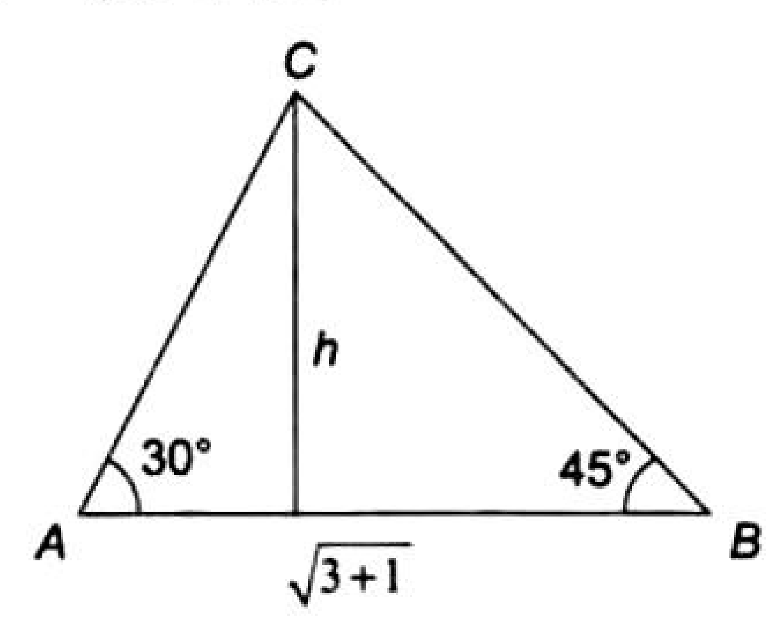
4. If a^2+2a, $2a+3$, a^2+3a+8 are sides of a triangle, then sum of any two sides is greater than the third side.
Let $x=a^2+2a$; $y=2a+3$; $z=a^2+3a+8$.
Then $x+y>z$
$\Rightarrow$ $a^2+4a+3>a^2+3a+8$
$\Rightarrow$ $a>5$ (i)
From $y+z>x \Rightarrow a^2+5a+11>a^2+2a$
$\Rightarrow$ $3a>-11 \Rightarrow a>-11/3$ (ii)
$z+x>y \Rightarrow 2a^2+5a+8>2a+3$
$\Rightarrow$ $2a^2+3a+5>0$
Here coefficient of $a^2>0$ and $D=9-40=-$ve
Therefore, it is true for all values of a.
Combining Eqs. (i) and (ii), we get $a>5$.

5. $\cot A$, $\cot B$, $\cot C$ are in A.P. Thus,
$\cot B-\cot A=\cot C-\cot B$
or $\frac{\sin(A-B)}{\sin A\sin B}=\frac{\sin(B-C)}{\sin B\sin C}$
or $\sin(A-B)\sin(A+B)=\sin(B+C)\sin(B-C)$
or $\sin^2A-\sin^2B=\sin^2B-\sin^2C$
or $a^2-b^2=b^2-c^2$
Hence, a^2, b^2, c^2 are in A.P.

6. Let $AB=2$ units be one of the sides of the polygon.
Then $\angle AOB=2\pi/9$ where O is the center of the circle.
If $OL\perp AB$, then $AL=1$ and $\angle AOL=\pi/9$
$\therefore$ Radius of the circle
$=OA$
$=AL\operatorname{cosec}\pi/9$
$=\operatorname{cosec}\pi/9$.



7. $\angle C=180°-(30°+45°)$



By the sine rule in ΔABC, we have

$$\frac{a}{\sin 30^\circ} = \frac{b}{\sin 45^\circ} = \frac{\sqrt{3}+1}{\sin 105^\circ}$$

or $$\frac{a}{1/2} = \frac{b}{1/\sqrt{2}} = \frac{\sqrt{3}+1}{\frac{\sqrt{3}}{2}\frac{1}{2}+\frac{1}{2}\frac{1}{\sqrt{2}}}$$

or $$\frac{a}{\sqrt{2}} = \frac{b}{2} = 1$$

or $$a = \sqrt{2},\ b = 2$$

$\therefore$ $$\Delta = \frac{1}{2}ab\sin C$$

$$= \frac{1}{2}\sqrt{2}\,(2)\times\sin(105^\circ)$$

$$= \sqrt{2}\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right) = \frac{\sqrt{3}+1}{2} \text{ sq. units}$$

8. In ΔABC, we have

$$\frac{2\cos A}{a} + \frac{\cos B}{b} + \frac{2\cos C}{c} = \frac{a}{bc} + \frac{b}{ca}$$

or $$\frac{2\left(\frac{b^2+c^2-a^2}{2bc}\right)}{a} + \frac{\left(\frac{a^2+c^2-b^2}{2ac}\right)}{b} + \frac{2\left(\frac{b^2+a^2-c^2}{2ab}\right)}{c} = \frac{a^2+b^2}{abc}$$

or $$2b^2 + 2c^2 - 2a^2 + a^2 + c^2 - b^2 + 2a^2 + 2b^2 - 2c^2 = 2a^2 + 2b^2$$

or $$b^2 + c^2 = a^2$$

Hence, the triangle is right angled at A, i.e.,
$\angle A = 90^\circ$.

9. $$\Delta = \frac{1}{2}a \times AD$$

$\Rightarrow$ $$AD = \frac{2\Delta}{a} = \frac{abc}{b^2-c^2} \quad \text{[given]}$$

or $$\frac{2\Delta}{a} = \frac{4R\Delta}{b^2-c^2}$$

or $$2Ra = b^2 - c^2$$

or $$\sin A = \sin^2 B - \sin^2 C = \sin(B+C)\sin(B-C) = \sin A\sin(B-C)$$

or $$\sin(B-C) = 1 \quad \Rightarrow \quad \angle B - \angle C = \frac{\pi}{2}$$

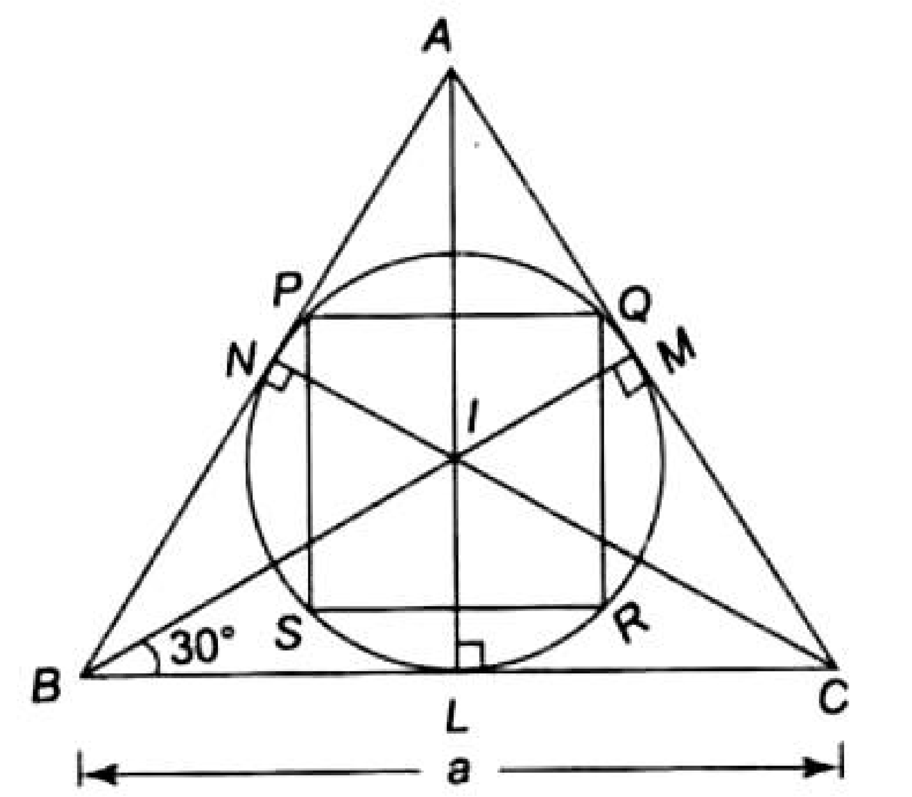
or $$\angle B = \frac{\pi}{2} + \angle C = 90^\circ + 23^\circ = 113^\circ$$

10. Given that ABC is an equilateral triangle of side a and r is the radius of the circle inscribed in it.

In ΔIBL, we get

$$\tan 30^\circ = \frac{r}{a/2} \text{ or } r = \frac{a}{2\sqrt{3}}$$

If $PQRS$ is the square inscribed in circle of radius r, then



Side of square $= 2(r\sin 45^\circ)$

$$= \frac{2r}{\sqrt{2}} = \frac{2}{\sqrt{2}} \times \frac{a}{2\sqrt{3}} = \frac{a}{\sqrt{6}}$$

$\therefore$ Area of square $= \left(\frac{a}{6}\right)$ sq. units.

11. $a = 4k,\ b = 5k,\ c = 6k$

$$s = \frac{15}{2}k,\ s-a = \frac{7}{2}k,\ s-b = \frac{5}{2}k,$$

$$s-c = \frac{3}{2}k$$

$\Rightarrow$ $$\Delta^2 = 15 \times 7 \times 5 \times 3\left(\frac{k}{2}\right)^4$$

or $$\Delta = 15\sqrt{7}\left(\frac{k}{2}\right)^2$$

Also, $$r = \frac{\Delta}{s} = \frac{15\sqrt{7}(k/2)^2}{(15/2)k} = \sqrt{7}\,\frac{k}{2}$$

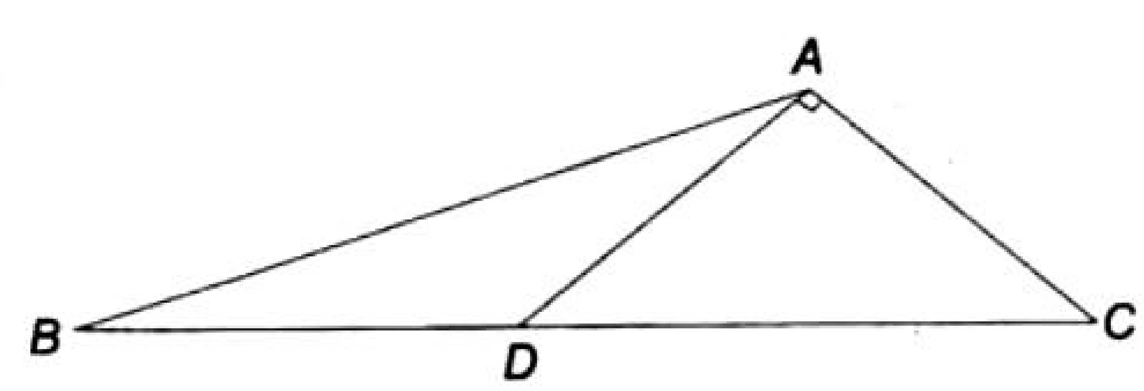
$\Rightarrow$ $$R = \frac{abc}{4\Delta} = \frac{4\times 5\times 6\times k^3}{4\times 15\sqrt{7}\times(k^2/4)}$$

$$= \frac{8}{\sqrt{7}}k$$

$\therefore$ $$\frac{R}{r} = \frac{16}{7}$$

Subjective Type

1.



From ΔABC, $$\cos A = \frac{b^2+c^2-a^2}{2bc} \quad \text{(i)}$$

From ΔCAD, $$\cos C = \frac{AC}{CD} = \frac{b}{(a/2)} = \frac{2b}{a} \quad \text{(ii)}$$

From ΔABD, $$\frac{BD}{\sin(A-90^\circ)} = \frac{AB}{\sin\angle ADB}$$

$\Rightarrow$ $$\frac{a/2}{-\cos A} = \frac{c}{\sin(90^\circ+C)}$$

or $\dfrac{a}{-2\cos A} = \dfrac{c}{\cos C}$

or $\cos A = \dfrac{a\cos C}{-2c} = \dfrac{a}{-2c}\,\dfrac{2b}{a} = -\dfrac{b}{c}$ [from Eq. (ii)]

From Eq. (i), we get $\dfrac{b^2 + c^2 - a^2}{2bc} = \dfrac{-b}{c}$ (iii)

$\Rightarrow$ $b^2 + c^2 - a^2 = -2b^2$ or $c^2 - a^2 = -3b^2$

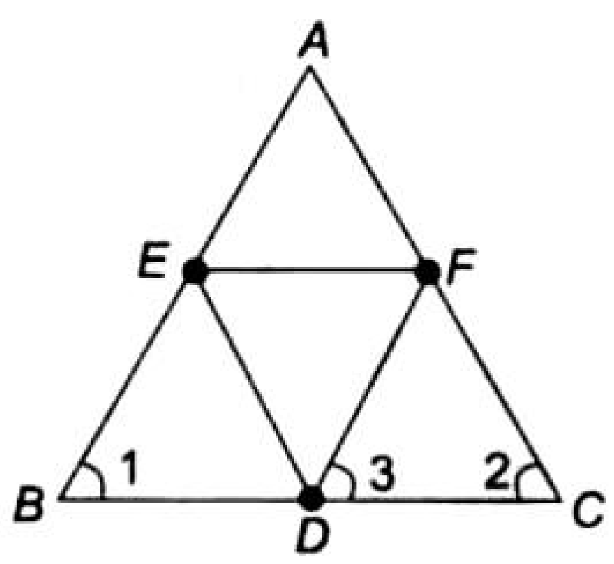
Now, $\cos A \cos C = \dfrac{b^2 + c^2 - a^2}{2bc}\,\dfrac{2b}{a}$

$= \dfrac{b^2 + c^2 - a^2}{ca} = \dfrac{3b^2 + 3(c^2 - a^2)}{3ca}$

$= \dfrac{a^2 - c^2 + 3(c^2 - a^2)}{3ca}$ [from Eq. (iii)]

$= \dfrac{2(c^2 - a^2)}{3ca}$

2.



Given that $AB = AC$

$\therefore$ $\angle 1 = \angle 2$ (i)

But $AB \parallel DF$ (given) and BC is transversal

$\therefore$ $\angle 1 = \angle 3$ (ii)

From Eqs. (i) and (ii), we get

$\angle 2 = \angle 3 \Rightarrow DF = CF$

Similarly, we can prove

$DE = BE$

Now, $DF + FA + AE + ED$

$= CF + FA + AE + BE = AC + AB$

3. If A, B, C are in A.P., then $A + C = 2B$.

But $A + B + C = 180°$

or $3B = 180°$ or $B = 60°$

We have $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

or $\dfrac{\sin B}{\sin C} = \dfrac{b}{c} = \dfrac{\sqrt{3}}{\sqrt{2}}$

$\Rightarrow$ $\sin C = \dfrac{\sqrt{2}}{\sqrt{3}} \sin 60° = \dfrac{1}{\sqrt{2}}$ or $C = 45°$

$\therefore$ $A = 180° - B - C = 75°$

4. r_1, r_2, r_3 are in H.P.

$\Rightarrow$ $1/r_1, 1/r_2, 1/r_3$ are in A.P.

$\Rightarrow$ $\dfrac{s-a}{\Delta}, \dfrac{s-a}{\Delta}, \dfrac{s-c}{\Delta}$ are in A.P.

$\Rightarrow$ $s - a, s - b, s - c$ are in A.P.

$\Rightarrow$ a, b, c are in A.P.

5. Given that in ΔABC

$\cos A + \cos B + \cos C = \dfrac{3}{2}$

$\Rightarrow$ $\dfrac{b^2 + c^2 - a^2}{2bc} + \dfrac{a^2 + c^2 - b^2}{2ac} + \dfrac{a^2 + b^2 - c^2}{2ab}$

$= \dfrac{3}{2}$

or $ab^2 + ac^2 - a^3 + a^2b + bc^2 - b^3 + ca^2 + b^2c - c^3$

$= 3abc$

or $ab^2 + ac^2 + bc^2 + ba^2 + ca^2 + cb^2 - 6abc$

$= a^3 + b^3 + c^3 - 3abc$

or $a(b - c)^2 + b(c - a)^2 + c(a - b)^2$

$= \left(\dfrac{a+b+c}{2}\right)\left[(a-b)^2 + (b-c)^2 + (c-a)^2\right]$

or $(a + b - c)(a - b)^2 + (b + c - a)(b - c)^2$

$+ (c + a - b)(c - a)^2 = 0$ (i)

We know that $a + b > c$, $b + c > a$, $c + a > b$, i.e., the sum of any two sides of a triangle is greater than the third side.

Therefore, each side on the L.H.S. of Eq. (i) has positive coefficient multiplied by a perfect square, each must be separately zero. Thus,

$a - b = 0;\ b - c = 0;\ c - a = 0 \Rightarrow a = b = c$

Hence, ABC is an equilateral triangle.

Alternative Method:

$\cos A + \cos B + \cos C = \dfrac{3}{2}$

$\Rightarrow$ $2\cos\dfrac{A+B}{2}\cos\dfrac{A-B}{2} + 1 - 2\sin^2\dfrac{C}{2} = \dfrac{3}{2}$

$\Rightarrow$ $2\sin\dfrac{C}{2}\cos\dfrac{A-B}{2} - \dfrac{1}{2} - 2\sin^2\dfrac{C}{2} = 0$

$\Rightarrow$ $2\sin^2\dfrac{C}{2} - 2\cos\dfrac{A-B}{2}\sin\dfrac{C}{2} + \dfrac{1}{2} = 0$ (1)

Now $\sin\dfrac{C}{2}$ is real

$\therefore$ $D \geq 0$

$\therefore$ $4\cos^2\dfrac{A-B}{2} - 4 \times 2 \times \dfrac{1}{2} \geq 0$

$\therefore$ $\cos^2\dfrac{A-B}{2} \geq 1$

$\therefore$ $\cos^2\dfrac{A-B}{2} = 1$

$\therefore$ $\cos\dfrac{A-B}{2} = 1$

$\therefore$ $\dfrac{A-B}{2} = 0$

$\therefore$ $A = B$

For $A = B$ from eqn (i), we get

$4\sin^2\dfrac{C}{2} - 4\sin\dfrac{C}{2} + 1 = 0$

or $\sin\dfrac{C}{2} = \dfrac{1}{2}$

or $C = 60°$

$\therefore$ $A = B = 60°$

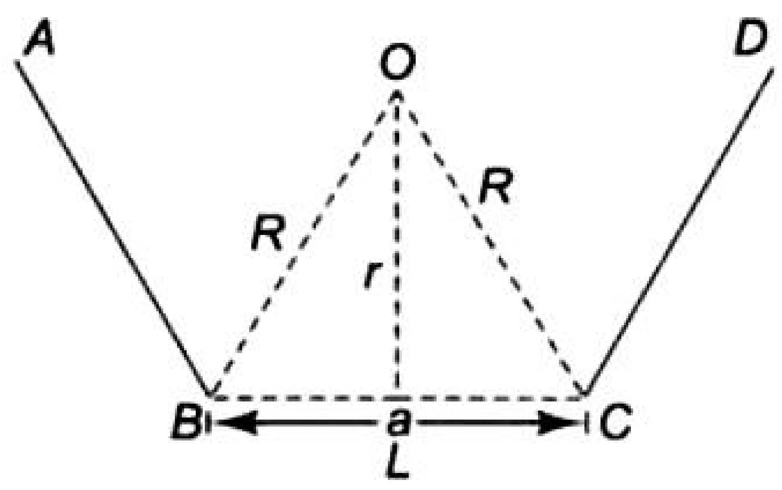
Hence triangle is equilateral.

6. Let $\dfrac{b+c}{11} = \dfrac{c+a}{12} = \dfrac{a+b}{13} = k$

$\therefore$ $b + c = 11k$ (i)

$c + a = 12k$ (ii)

$a + b = 13k$ (iii)



Adding the above three equations, we get

$2(a + b + c) = 36k$

or $a + b + c = 18k$ (iv)

From Eqs. (i) and (iv), $a = 7k$

From Eqs. (ii) and (iv), $b = 6k$

From Eqs. (iii) and (iv), $c = 5k$

$$\cos A = \frac{b^2+c^2-a^2}{2bc} = \frac{36k^2+25k^2-49k^2}{2\times 6k\times 5k}$$

$$= \frac{12k^2}{60k^2} = \frac{1}{5}$$

$$\cos B = \frac{c^2+a^2-b^2}{2ca} = \frac{25k^2+49k^2-36k^2}{2\times 5k\times 7k}$$

$$= \frac{38k^2}{70k^2} = \frac{19}{35}$$

$$\cos C = \frac{a^2+b^2-c^2}{2ab} = \frac{49k^2+36k^2-25k^2}{2\times 7k\times 6k}$$

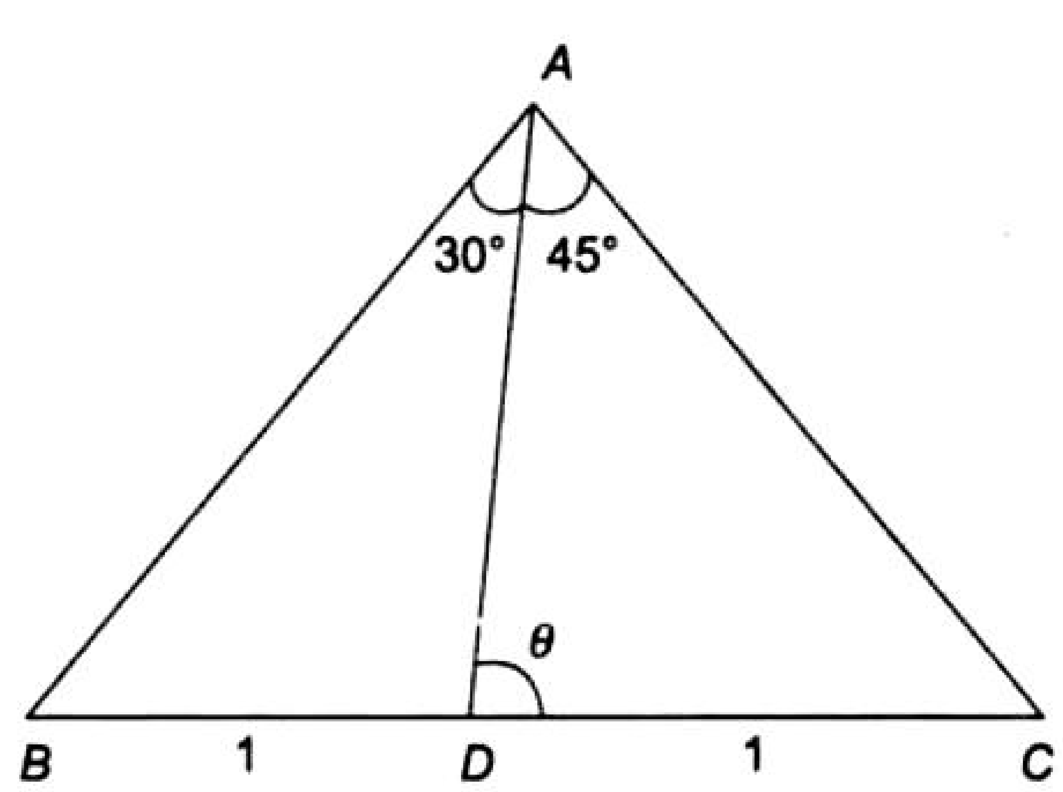
$$= \frac{60k^2}{84k^2} = \frac{5}{7}$$

$\therefore$ $\dfrac{\cos A}{(1/5)} = \dfrac{\cos B}{(19/35)} = \dfrac{\cos C}{(5/7)}$

or $\dfrac{\cos A}{(7/35)} = \dfrac{\cos B}{(19/35)} = \dfrac{\cos C}{(25/35)}$

or $\dfrac{\cos A}{7} = \dfrac{\cos B}{19} = \dfrac{\cos C}{25}$

7. We know that if in ΔABC, AD divides BC in the ratio $m{:}n$ with $\angle ADC = \theta$ and $\angle BAD = \alpha$.



If $\angle CAD = \beta$, then m–n theorem states that

$(m + n)\cot\theta = m\cot\alpha - n\cot\beta$ (i)

By applying Eq. (i) in ΔABC, we get

$(1 + 1)\cot\theta = 1\cot 30° - 1\cot 45°$

or $2\cot\theta = \sqrt{3}-1$

or $\cot\theta = \dfrac{\sqrt{3}-1}{2}$ or $\tan\theta = \dfrac{2}{\sqrt{3}-1}$

But $\theta = B + 30°$, we get

$$\tan(B+30°) = \frac{2}{\sqrt{3}-1} = \frac{2(\sqrt{3}+1)}{3-1} = \sqrt{3}+1$$

or $\dfrac{\tan B + \frac{1}{\sqrt{3}}}{1-\frac{1}{\sqrt{3}}\tan B} = \sqrt{3}+1$

or $\dfrac{\sqrt{3}\tan B + 1}{\sqrt{3}-\tan B} = \sqrt{3}+1$

or $\sqrt{3}\tan B + 1 = 3 + \sqrt{3} - (\sqrt{3}+1)\tan B$

or $(2\sqrt{3}+1)\tan B = 2+\sqrt{3}$

or $\tan B = \dfrac{2+\sqrt{3}}{2\sqrt{3}+1}$

or $\cot B = \dfrac{2\sqrt{3}+1}{2+\sqrt{3}}\times\dfrac{2-\sqrt{3}}{2-\sqrt{3}} = \dfrac{4\sqrt{3}-6+2-\sqrt{3}}{4-3}$

$= 3\sqrt{3}-4$

or $\text{cosec}^2 B = 1 + 27 + 16 - 24\sqrt{3} = 44 - 24\sqrt{3}$

or $\sin^2 B = \dfrac{1}{4(11-6\sqrt{3})}$

or $\sin B = \dfrac{1}{2\sqrt{11-6\sqrt{3}}}$ (i)

Now in ΔABD, by applying the sine law, we get

$$\frac{BD}{\sin 30°} = \frac{AD}{\sin B}$$

or $BD = \dfrac{AD}{\sin B}\times\sin 30° = \dfrac{\frac{1}{\sqrt{11-6\sqrt{3}}}}{\frac{1}{2\sqrt{11-6\sqrt{3}}}}\times\dfrac{1}{2} = 1$

$\therefore$ $BC = 2\,BD = 2$ units

8. Given $\cos A\cos B + \sin A\sin B\sin C = 1$

or $\sin C = \dfrac{1-\cos A\cos B}{\sin A\sin B} \le 1$ [by using Eq. (i)]

or $1 \le \cos A\cos B + \sin A\sin B$

or $\cos(A - B) \ge 1$

or $\cos(A - B) = 1$

or $A - B = 0$ or $A = B$

$\Rightarrow$ $\sin C = \dfrac{1-\cos^2 A}{\sin^2 A} = \dfrac{\sin^2 A}{\sin^2 A} = 1$

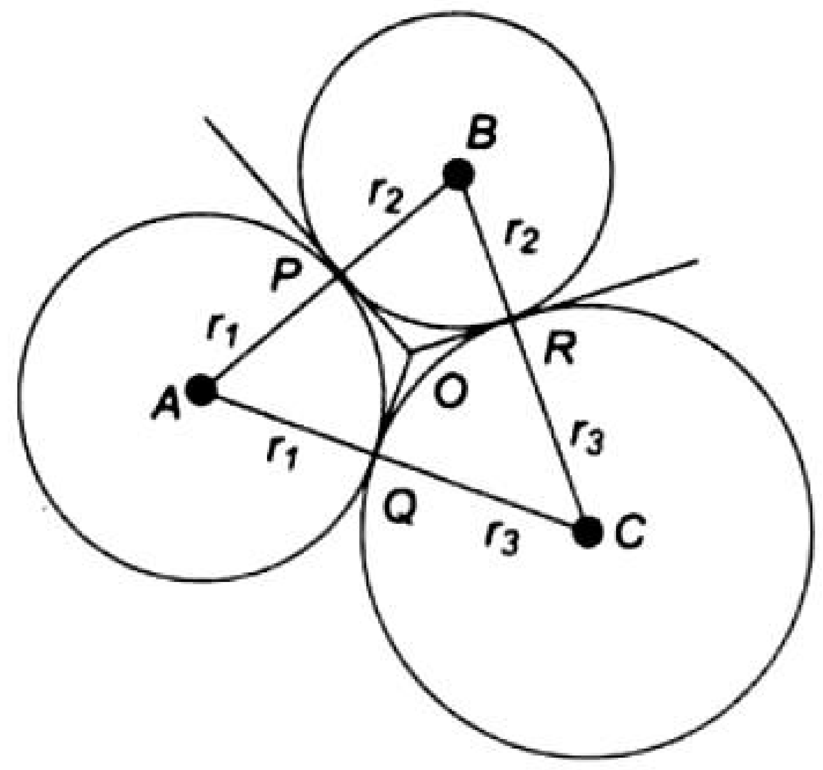
or $C = 90°$ $\Rightarrow$ $A + B = 90°$ or $A = B = 45°$ [by using Eq. (ii)]

$\Rightarrow$ $a{:}b{:}c = \sin A{:}\sin B{:}\sin C = 1{:}1{:}\sqrt{2}$

9. Let us consider three circles with centers at A, B, and C with radii r_1, r_2, and r_3, respectively, which touch each other externally at P, Q, and R. Let the common tangents at P, Q, and R meet each other at O. Then

$$OP = OQ = OR = 4 \text{ (given)}$$

(lengths of tangents from a point to a circle are equal)



Also, $OP \perp AB$, $OQ \perp AC$, $OR \perp BC$

Therefore, O is the incenter of ΔABC.

Thus, for ΔABC,

$$s = \frac{(r_1 + r_2) + (r_2 + r_3) + (r_3 + r_1)}{2}$$

$$= r_1 + r_2 + r_3$$

$\Rightarrow$ $\Delta = \sqrt{(r_1 + r_2 + r_3) \cdot r_1 \cdot r_2 \cdot r_3}$ (Heron's formula)

Now, $r = \frac{\Delta}{s}$

or $4 = \frac{\sqrt{(r_1 + r_2 + r_3) r_1 r_2 r_3}}{r_1 + r_2 + r_3}$

$$= \frac{\sqrt{r_1 r_2 r_3}}{\sqrt{r_1 + r_2 + r_3}}$$

or $\frac{r_1 r_2 r_3}{r_1 + r_2 + r_3} = \frac{16}{1}$

or $r_1 r_2 r_3 : (r_1 + r_2 + r_3) = 16:1$

10. Given that

i. The sides a, b, c, and area Δ are rational.

ii. a, $\tan B/2$, $\tan C/2$ are rational

iii. a, $\sin A$, $\sin B$, $\sin C$ are rational

To prove that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i)

i. a, b, c, and Δ are rational.

$\Rightarrow$ $s = \frac{a+b+c}{2}$ is also rational

$\Rightarrow$ $\tan\left(\frac{B}{2}\right) = \frac{\Delta}{s(s-b)}$ is also rational

and $\tan\left(\frac{C}{2}\right) = \frac{\Delta}{s(s-c)}$ is also rational

Hence, (i) $\Rightarrow$ (ii)

ii. a, $\tan B/2$, $\tan C/2$ are rational

$\Rightarrow$ $\sin B = \frac{2\tan B/2}{1+\tan^2 B/2}$

and $\sin C = \frac{2\tan(B/2)}{1+\tan^2(B/2)}$ are rational

and $\tan(A/2) = \tan\left[90 - \left(\frac{B}{2} + \frac{C}{2}\right)\right]$

$$= \cot\left(\frac{B}{2} + \frac{C}{2}\right)$$

$$= \frac{1}{\tan\left(\frac{B}{2} + \frac{C}{2}\right)}$$

$= \frac{1 - \tan(B/2)\tan(C/2)}{\tan(B/2) + \tan(C/2)}$ is rational

$\therefore$ $\sin A = \frac{2\tan A/2}{1+\tan^2 A/2}$ is rational

Hence, (ii) $\Rightarrow$ (iii)

iii. a, $\sin A$, $\sin B$, $\sin C$ are rational

But $\frac{a}{\sin A} = 2R$

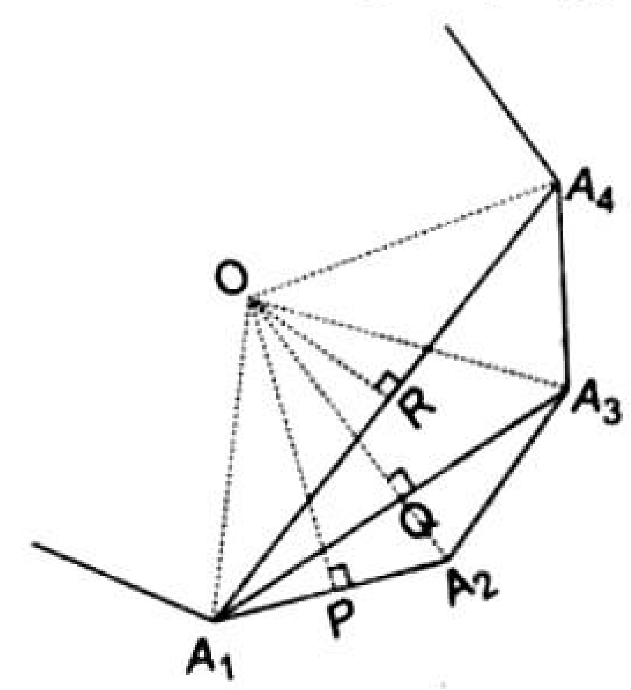
Hence, R is rational

$\therefore$ $b = 2R \sin B$, $c = 2R \sin C$ are rational

$\therefore$ $\Delta = \frac{1}{2} bc \sin A$ is rational

Hence, (iii) $\Rightarrow$ (i)

11. Let a be the side of n sided regular polygon $A_1A_2A_3A_4 \ldots A_n$



$\therefore$ Angle subtended by each side at centre $= \frac{2\pi}{n}$

Also $OA_1 = OA_2 = OA_3 = \ldots = OA_n = r$ (Say)

In ΔA_1OA_2, using cosine rule

$$A_1A_2^2 = r^2 + r^2 - 2r^2 \cos\frac{2\pi}{n} = 2r^2\left(1 - \cos\frac{2\pi}{n}\right)$$

$$= 4r^2 \sin^2\frac{\pi}{n}$$

$\therefore$ $A_1A_2 = 2r\sin\frac{\pi}{n}$

Similarly in ΔA_1OA_3, $A_1A_3 = 2r\sin\frac{2\pi}{n}$

and in ΔA_1OA_4, $A_1A_4 = 2r\sin\frac{3\pi}{n}$

But given that $\frac{1}{A_1A_2} = \frac{1}{A_1A_3} + \frac{1}{A_1A_4}$

$\Rightarrow$ $\frac{1}{2\sin\frac{\pi}{n}} = \frac{1}{2\sin\frac{2\pi}{n}} + \frac{1}{2\sin\frac{3\pi}{n}}$

$\Rightarrow$ $2\sin\frac{2\pi}{n}\sin\frac{3\pi}{n} = 2\sin\frac{\pi}{n}\sin\frac{3\pi}{n} + 2\sin\frac{\pi}{n}\sin\frac{2\pi}{n}$

$\Rightarrow$ $\cos\frac{\pi}{n} - \cos\frac{5\pi}{n} = \cos\frac{2\pi}{n} - \cos\frac{4\pi}{n} + \cos\frac{\pi}{n} - \cos\frac{3\pi}{n}$

$$\Rightarrow \quad \cos\frac{5\pi}{n} + \cos\frac{2\pi}{n} = \cos\frac{4\pi}{n} + \cos\frac{3\pi}{n}$$

$$\Rightarrow \quad 2\cos\frac{7\pi}{2n}\cos\frac{3\pi}{2n} = 2\cos\frac{7\pi}{2n}\cos\frac{\pi}{2n}$$

$$\Rightarrow \quad \cos\frac{7\pi}{2n}\left(\cos\frac{3\pi}{2n} - \cos\frac{\pi}{2n}\right) = 0$$

$$\Rightarrow \quad 2\cos\frac{7\pi}{2n}\sin\frac{\pi}{n}\sin\frac{\pi}{2n} = 0$$

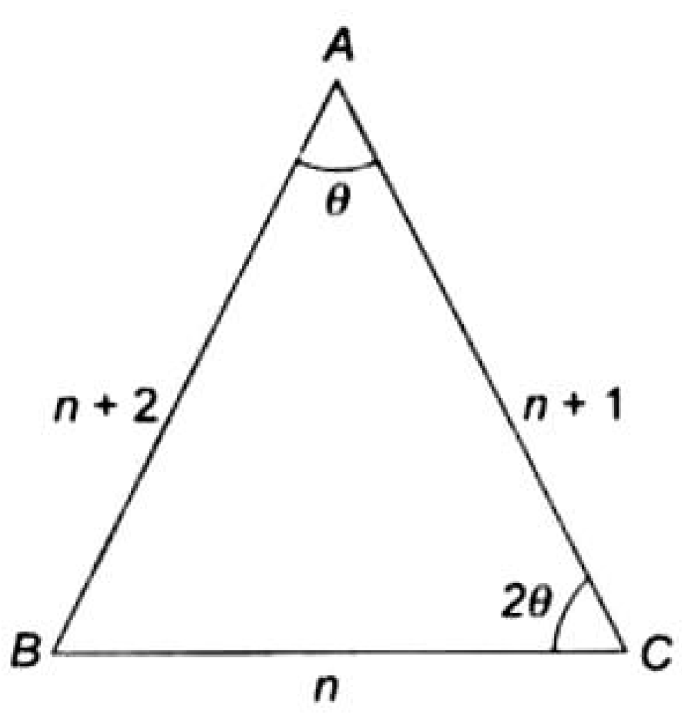
$$\Rightarrow \quad \frac{7\pi}{2n} = (2k+1)\frac{\pi}{2} \text{ or } \frac{\pi}{n} = k\pi \text{ or } \frac{\pi}{2n} = k\pi$$

$$\Rightarrow \quad n = \frac{7}{2k+1} \text{ or } n = \frac{1}{k} \text{ or } n = \frac{1}{2k}$$

$\therefore \quad n = 7$ for $k = 0$

12. Let $a = n$, $b = n + 1$, $c = n + 2$, $n \in N$

Let the smallest angle $\angle A = \theta$, then the greatest angle $\angle C = 2\theta$.



In ΔABC, by applying the sine law, we get

$$\frac{\sin\theta}{n} = \frac{\sin 2\theta}{n+2}$$

or
$$\frac{\sin\theta}{n} = \frac{2\sin\theta\cos\theta}{n+2}.$$

or
$$\frac{1}{n} = \frac{2\cos\theta}{n+2} \qquad [\text{as } \sin\theta \neq 0]$$

$$\Rightarrow \quad \cos\theta = \frac{n+2}{2n} \qquad \text{(i)}$$

In ΔABC, by the cosine law, we get

$$\cos\theta = \frac{(n+1)^2 + (n+2)^2 - n^2}{2(n+1)(n+2)} \qquad \text{(ii)}$$

Comparing the value of $\cos\theta$ from Eqs. (i) and (ii), we get

$$\frac{(n+1)^2 + (n+2)^2 - n^2}{2(n+1)(n+2)} = \frac{n+2}{2n}$$

or $\quad (n+2)^2(n+1) = n(n+2)^2 + n(n+1)^2 - n^3$

or $\quad n(n+2)^2 + (n+2)^2 = n(n+2)^2 + n(n+1)^2 - n^3$

or $\quad n^2 + 4n + 4 = n^3 + 2n^2 + n - n^3$

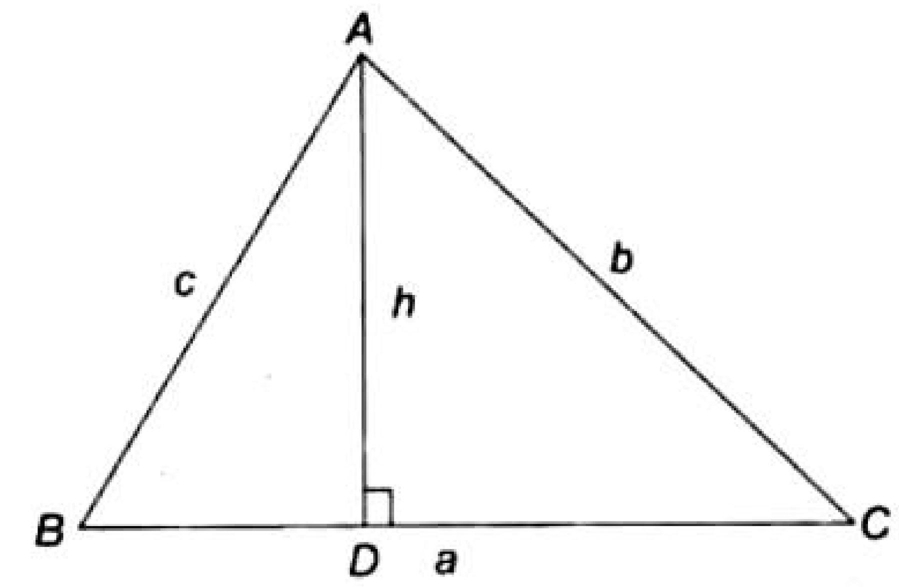
or $\quad n^2 - 3n - 4 = 0$

or $\quad (n+1)(n-4) = 0$

or $\quad n = 4 \quad [\text{as } n \neq -1]$

Therefore, the sides of the triangle are 4, 4 + 1, 4 + 2, i.e., 4, 5, 6.

13. Given that in ΔABC, base $= a$ and $\frac{c}{b} = r$. We have to find the attitude h.



We have, in ΔABD,

$$h = c\sin B = \frac{c\,a\sin B}{a}$$

$$= \frac{c\sin A\sin B}{\sin(B+C)}$$

$$= \frac{c\sin A\sin B\sin(B-C)}{\sin(B+C)\sin(B-C)}$$

$$= \frac{c\sin A\sin B\sin(B-C)}{\sin^2 B - \sin^2 C}$$

$$= \frac{c\frac{a}{k}\frac{b}{k}\sin(B-C)}{\frac{b^2}{k^2} - \frac{c^2}{k^2}}$$

$$= \frac{abc\sin(B-C)}{b^2 - c^2}$$

$$= \frac{a\left(\frac{c}{b}\right)\sin(B-C)}{1 - \left(\frac{c}{b}\right)^2}$$

$$= \frac{ar\sin(B-C)}{1 - r^2}$$

$$\leq \frac{ar}{1-r^2} \qquad [\because \sin(B-C) \leq 1]$$

$$\therefore \quad h \leq \frac{ar}{1-r^2}$$

14. We have in ΔABC, $A = \frac{\pi}{4}$ and $\tan B\tan C = p$

We know that in ΔABC,

$$\tan A + \tan B + \tan C = \tan A\tan B\tan C$$

$$\therefore \quad \tan\frac{\pi}{4} + \tan B + \tan C = \tan\frac{\pi}{4} \times p$$

$$\therefore \quad \tan B + \tan C = p - 1$$

Thus $\tan B$ and $\tan C$ are roots of equation

$$f(x) = x^2 - (p-1)x + p = 0 \qquad \text{(i)}$$

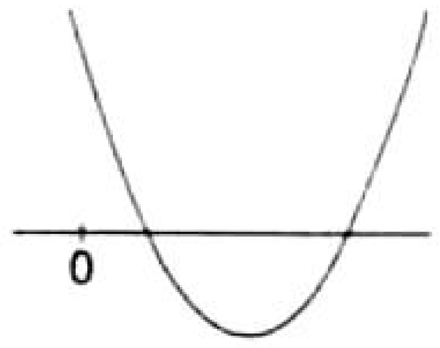
Case I:

When B and C both are acute angles.

$\therefore$ both roots of the above equation are positive

$\therefore \quad f(0) > 0$ or $p > 0 \qquad$ (ii)

and $\quad \frac{p-1}{2} > 0 \ \therefore p > 1 \qquad$ (iii)



Also $D \ge 0$

$\Rightarrow \quad (p-1)^2 - 4p \ge 0$

$\Rightarrow \quad p^2 - 6p + 1 \ge 0$

$\Rightarrow \quad [p - (3 - 2\sqrt{2})][p - (3 + 2\sqrt{2})] \ge 0$

$\Rightarrow \quad p \le 3 - 2\sqrt{2}$ or $p \ge 3 + 2\sqrt{2} \qquad$ (iv)

From (ii), (iii) and (iv), we get $p \ge 3 + 2\sqrt{2}$

Case II:

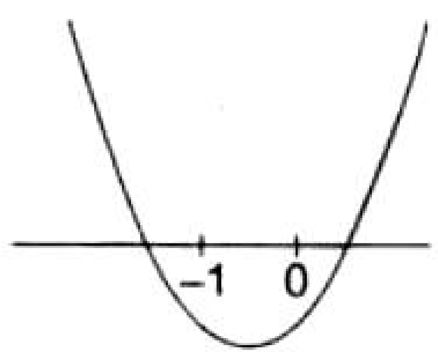
When one of the angles is obtuse.

We have $B + C = \frac{3\pi}{4}$

If B is obtuse then $\frac{\pi}{2} < B < \frac{3\pi}{4}$

Thus $-\infty < \tan B < -1$

Thu one root of eq. (i) is less than -1 and other root is greater than 0.



$\therefore \quad f(0) < 0$ and $f(-1) < 0$

$\therefore \quad p < 0$ and $1 + p - 1 + p < 0$ or $p < 0 \qquad$ (v)

From (iv) and (v), we get $p \in (-\infty, 0) \cup [3 + 2\sqrt{2}, \infty)$

15. Let A, B, C be an equilateral triangle, then

$A = B = C = 60°$

$\Rightarrow \quad \tan A + \tan B + \tan C = 3\tan 60° = 3\sqrt{3}$

Conversely suppose that $\tan A + \tan B + \tan C = 3\sqrt{3}$

Also $\frac{\tan A + \tan B + \tan C}{3} \ge \sqrt[3]{\tan A \tan B \tan C}$

[since A.M. ≥ G.M.]

$\Rightarrow \quad \tan A \tan B \tan C \ge \sqrt[3]{\tan A \tan B \tan C}$

$\Rightarrow \quad \tan^2 A \tan^2 B \tan^2 C \ge 27 \qquad$ [cubing both sides]

$\Rightarrow \quad \tan A \tan B \tan C \ge 3\sqrt{3}$

$\Rightarrow \quad \tan A + \tan B + \tan C \ge 3\sqrt{3}$

Now A.M. = G.M. when $\tan A = \tan B = \tan C$.

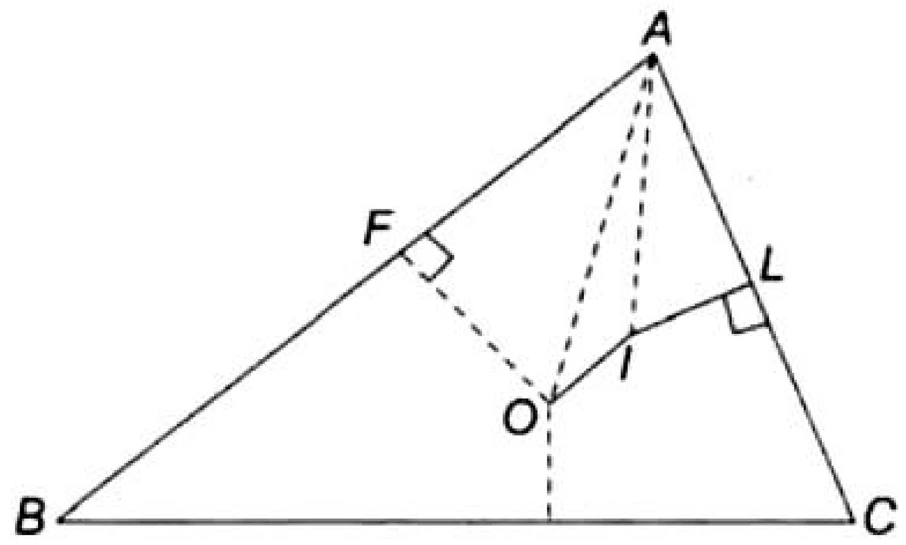
Thus when $\tan A + \tan B + \tan C = 3\sqrt{3}$, triangle is equilateral.

16. Let O be the circumcenter and OF be the perpendicular to AB. Let I be the incenter and IE be the perpendicular to AC. Then,

$\angle OAF = 90° - C.$

$\Rightarrow \quad \angle OAI = \angle IAF - \angle OAF$

$$= \frac{A}{2} - (90° - C) = \frac{A}{2} + C - \frac{A+B+C}{2} = \frac{C-B}{2}$$



Also, $AI = \frac{IE}{\sin\frac{A}{2}} = \frac{r}{\sin\frac{A}{2}} = 4R\sin\frac{B}{2}\sin\frac{C}{2}$

Hence in ΔOAI, $OI^2 = OA^2 + AI^2 - 2OA\, AI \cos\angle OAI$

$$= R^2 + 16R^2\sin^2\frac{B}{2}\sin^2\frac{C}{2} - 8R^2\sin\frac{B}{2}\sin\frac{C}{2}\cos\frac{C-B}{2}$$

$$\Rightarrow \quad \frac{OI^2}{R^2} = 1 + 16\sin^2\frac{B}{2}\sin^2\frac{C}{2} - 8\sin\frac{B}{2}\sin\frac{C}{2}\left(\cos\frac{B}{2}\cos\frac{C}{2} + \sin\frac{B}{2}\sin\frac{C}{2}\right)$$

$$= 1 - 8\sin\frac{B}{2}\sin\frac{C}{2}\left(\cos\frac{B}{2}\cos\frac{C}{2} - \sin\frac{B}{2}\sin\frac{C}{2}\right)$$

$$= 1 - 8\sin\frac{B}{2}\sin\frac{C}{2}\cos\frac{B+C}{2}$$

$$= 1 - 8\sin\frac{B}{2}\sin\frac{C}{2}\sin\frac{A}{2}$$

$$\therefore \quad OI = R\sqrt{1 - 8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}} = \sqrt{R^2 - 2Rr}$$

Again if ΔAIO is right angled triangle, then

$\Rightarrow \quad OA^2 = OI^2 + IA^2$

$\Rightarrow \quad R^2 = R^2 - 2Rr + \frac{r^2}{\sin^2 A/2}$

$\Rightarrow \quad 2R\sin^2(A/2) = r$

$\Rightarrow \quad 2\,\frac{abc}{4\Delta}\cdot\frac{(s-b)(s-c)}{bc} = \frac{\Delta}{s}$

$\Rightarrow \quad a(s-b)(s-c) = 2(s-a)(s-b)(s-b)(s-c)$

$\Rightarrow \quad a = 2s - 2a$

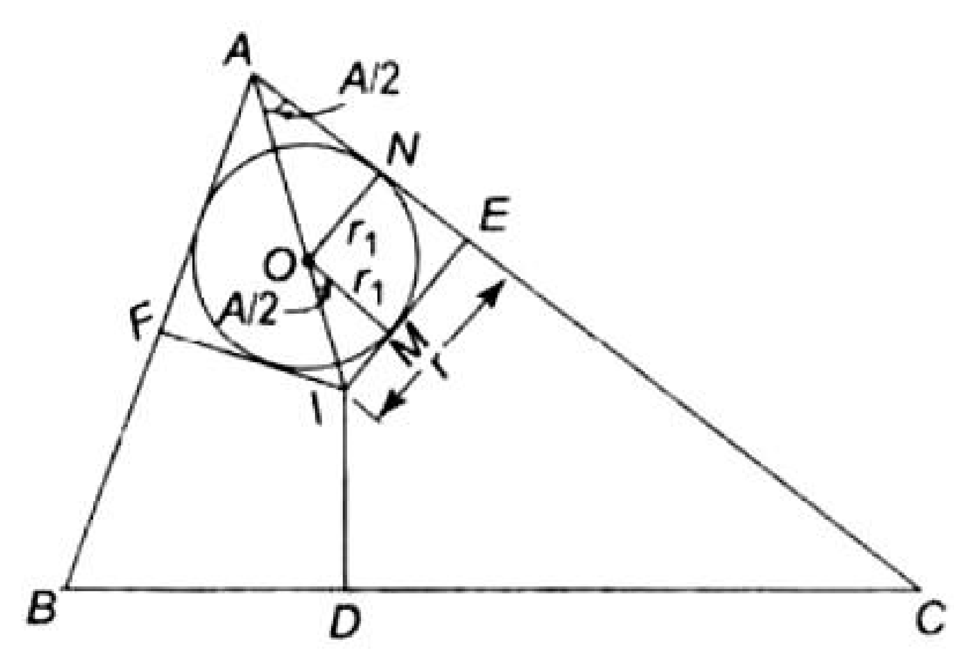
$\Rightarrow \quad a = b + c - a$

$\Rightarrow \quad b + c = 2a$

$\Rightarrow \quad a, b, c$ are in A.P.

$\Rightarrow \quad a$ is A.M. of b and c.

17.



I is incenter of ΔABC.

r_1 is the radius of the circle inscribed in quadrilateral $AFIE$.

From the figure, AF and AE are tangents to the circle.

Also IE and IF are tangent to the circle.

So AI is passing through the center of the circle by symmetry.

Now ON is parallel to IE.

So ΔANO and ΔOMI are similar.

So in ΔOMI, we have $\cot\frac{A}{2} = \frac{OM}{IM} = \frac{r_1}{r - r_1}$

Similarly from other circles we get

$\cot\frac{B}{2} = \frac{r_2}{r - r_2}$ and $\cot\frac{C}{2} = \frac{r_3}{r - r_3}$

Now in ΔABC, we know that

$$\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}=\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}$$

$$\Rightarrow \quad \frac{r_1}{r-r_1}\cdot\frac{r_2}{r-r_2}\cdot\frac{r_3}{r-r_3}=\frac{r_1}{r-r_1}+\frac{r_2}{r-r_2}+\frac{r_3}{r-r_3}$$

18. We have to prove that $\Delta \le \frac{1}{4}\sqrt{(a+b+c)\,abc}$

or $\quad \Delta \le \frac{1}{4}\sqrt{2s\,abc}$

or $\quad \Delta^2 \le \frac{1}{16}2s\,abc$

or $\quad \Delta^2 \le \frac{1}{16}2s\,\Delta 4R$

or $\quad rs \le \frac{1}{2}sR$

Hence, $R \ge 2r$ [which is always true in Δ]

Alternative Method:

In triangle, sum of two sides is greater than the third side.

So $a+b>c$, $b+c>a$ and $c+a>b$

Now consider quantities $a+b-c$, $b+c-a$, $c+a-b$.

Using A. M. ≥ G.M., we get

$$\frac{(a+b-c)+(b+c-a)}{2}\ge\sqrt{(a+b-c)(b+c-a)}$$

or $\quad b\ge\sqrt{(a+b-c)(b+c-a)}$

Similarly we get $c\ge\sqrt{(c+a-b)(b+c-a)}$

and $a\ge\sqrt{(a+b-c)(c+a-b)}$

Multiplying we get $abc \ge (a+b-c)\,(b+c-a)\,(c+a-b)$

$\Rightarrow \quad abc \ge (2s-2a)\,(2s-2b)(2s-2c)$

$\Rightarrow \quad sabc \ge 8s(s-a)\,(s-b)(s-c)$

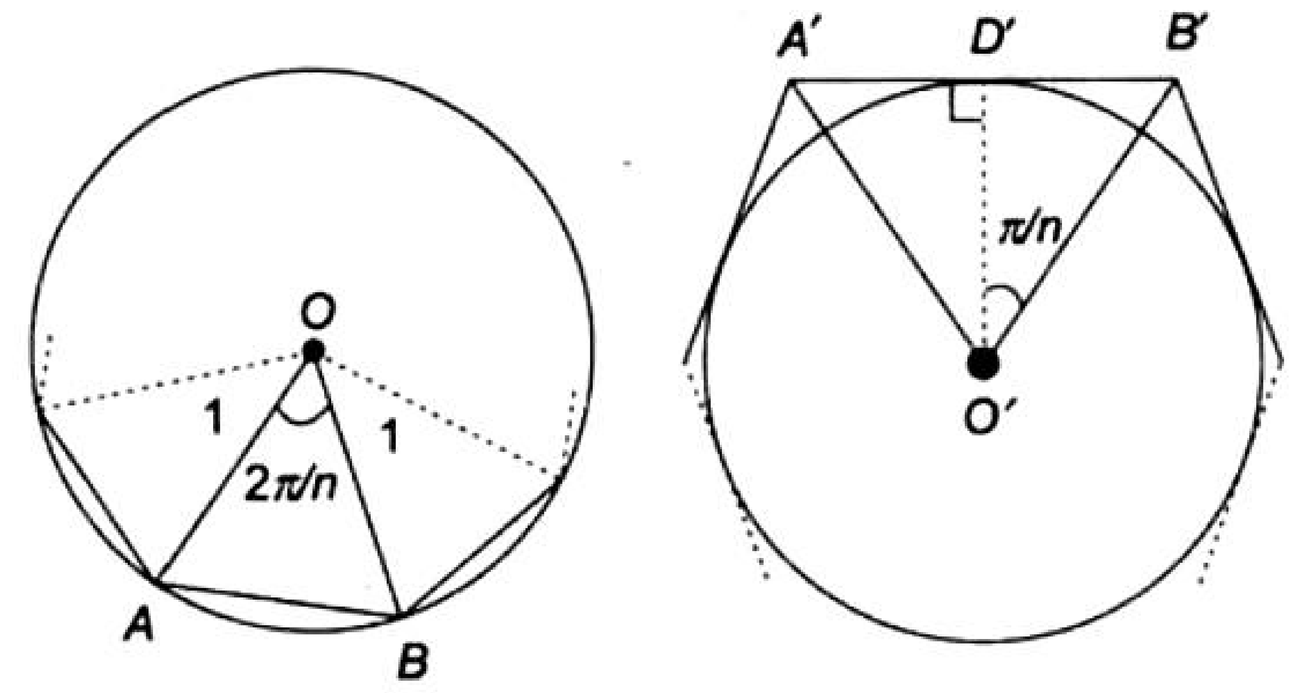
$\Rightarrow \quad (a+b+c)\,abc \ge 16\Delta^2$

$\Rightarrow \quad \Delta \le \frac{1}{4}\sqrt{(a+b+c)\,abc}$

19. Let OAB be one triangle out of n of a n-sided polygon inscribed in a circle of radius 1. Then,

$$\angle AOB=\frac{2\pi}{n}$$

$$OA=OB=1$$



$\therefore \quad$ Area $(\Delta AOB)=\frac{1}{2}\,1\times 1\times\sin\left(\frac{2\pi}{n}\right)$

$$=\frac{1}{2}\sin\left(\frac{2\pi}{n}\right)$$

Area of n-sided polygon $= I_n = \frac{n}{2}\sin\frac{2\pi}{n}$ (i)

Similarly, let $O'A'B'$ be one of the triangles out of n of n-sided polygon escribing the circle of unit radius.

Then in $\Delta O'B'A'$, $\cos\frac{\pi}{n}=\frac{1}{O'B'}$

or $\quad O'B'=\sec\frac{\pi}{n}$

Area $(\Delta O'B'A')=\frac{1}{2}\,(O'B')^2\sin\left(\frac{2\pi}{n}\right)$

$$=\frac{1}{2}\sec^2\left(\frac{\pi}{n}\right)\sin\left(\frac{2\pi}{n}\right)$$

Therefore, the area of n-sided polygon is given by

$$(O_n)=\frac{n}{2}\sec^2\frac{\pi}{n}\sin\frac{2\pi}{n} \quad \text{(ii)}$$

From Eqs. (i) and (ii), we get

$$\frac{I_n}{O_n}=\frac{\frac{n}{2}\sin\frac{2\pi}{n}}{\frac{n}{2}\sec^2\frac{\pi}{n}\sin\frac{2\pi}{n}}=\frac{1}{\sec^2\frac{\pi}{n}}$$

or $\quad I_n=\left(\cos^2\frac{\pi}{n}\right)O_n$

$$=\frac{O_n}{2}\left[1+\cos\left(\frac{2\pi}{n}\right)\right]$$

$$=\frac{O_n}{2}\left[1+\sqrt{1-\sin^2\frac{2\pi}{n}}\right]$$

$$=\frac{O_n}{2}\left[1-\sqrt{1-\left(\frac{2I_n}{n}\right)^2}\right]$$