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COMPLEX NUMBERS AND QUADRATIC EQUATIONS

INTRODUCTION

We know that $x^2 \geq 0$ for all $x \in \mathbf{R}$ i.e. the square of a real number (whether positive, negative or zero) is non-negative. Hence the equations $x^2 = -1$, $x^2 = -5$, $x^2 + 7 = 0$ etc. are not solvable in *real number system*. Thus, there is a need to extend the real number system to a larger system so that we can have solutions of such equations. In fact, our main objective is to solve the quadratic equation $ax^2 + bx + c = 0$, where $a, b, c \in \mathbf{R}$ and the discriminant $= b^2 - 4ac < 0$, which is not possible in real number system. In this chapter, we shall extend the real number system to a larger system called **complex number system** so that the solutions of quadratic equations $ax^2 + bx + c = 0$, where a, b, c are real numbers are possible. We shall also solve quadratic equations with complex coefficients.

5.1 COMPLEX NUMBERS

We know that the equation $x^2 + 1 = 0$ is not solvable in the real number system i.e. it has no real roots. Many mathematicians indicated the square roots of negative numbers, but **Euler** was the first to introduce the symbol i (read 'iota') to represent $\sqrt{-1}$, and he defined $i^2 = -1$.

It follows that i is a solution of the equation $x^2 + 1 = 0$. Also $(-i)^2 = i^2 = -1$. Thus the equation $x^2 + 1 = 0$ has two solutions, $x = \pm i$, where $i = \sqrt{-1}$.

The number i is called an **imaginary number**. In general, the square roots of all negative real numbers are called **imaginary numbers**. Thus $\sqrt{-1}$, $\sqrt{-5}$, $\sqrt{-\frac{9}{4}}$ etc. are all imaginary numbers.

Complex number

A number of the form $a + ib$, where a and b are real numbers, is called a **complex number**.

For example, $3 + 5i$, $-2 + 3i$, $-2 + i\sqrt{5}$, $7 + i\left(-\frac{2}{3}\right)$ are all complex numbers.

The system of numbers $\mathbf{C} = \{z; z = a + ib; a, b \in \mathbf{R}\}$ is called the set of **complex numbers**.

Standard form of a complex number

If a complex number is expressed in the form $a + ib$ where $a, b \in \mathbf{R}$ and $i = \sqrt{-1}$, then it is said to be in the **standard form**.

For example, the complex numbers $2 + 5i$, $-3 + \sqrt{2}i$, $-\frac{2}{3} - 7i$ are all in the standard form.

Real and imaginary parts of a complex number

If $z = a + ib$ ($a, b \in \mathbf{R}$) is a complex number, then a is called the **real part**, denoted by $\text{Re}(z)$ and b is called **imaginary part**, denoted by $\text{Im}(z)$.

For example :

- (i) If $z = 2 + 3i$, then $\operatorname{Re}(z) = 2$ and $\operatorname{Im}(z) = 3$.
- (ii) If $z = -3 + \sqrt{5}i$, then $\operatorname{Re}(z) = -3$ and $\operatorname{Im}(z) = \sqrt{5}$.
- (iii) If $z = 7$, then $z = 7 + 0i$, so that $\operatorname{Re}(z) = 7$ and $\operatorname{Im}(z) = 0$.
- (iv) If $z = -5i$, then $z = 0 + (-5)i$, so that $\operatorname{Re}(z) = 0$ and $\operatorname{Im}(z) = -5$.

Note that imaginary part of a complex number is a real number.

In $z = a + ib$ ($a, b \in \mathbf{R}$), if $b = 0$ then $z = a$, which is a **real number**. If $a = 0$ and $b \neq 0$, then $z = ib$, which is called **purely imaginary number**. If $b \neq 0$, then $z = a + ib$ is **non-real complex number**. Since every real number a can be written as $a + 0i$, we see that $\mathbf{R} \subset \mathbf{C}$ i.e. the set of real numbers $\mathbf{R}$ is a **proper subset** of $\mathbf{C}$, the set of complex numbers.

Note that $\sqrt{3}$, 0 , 2 , π are real numbers; $3 + 2i$, $3 - 2i$ etc. are non-real complex numbers; $2i$, $-\sqrt{2}i$ etc. are purely imaginary numbers.

Equality of two complex numbers

Two complex numbers $z_1 = a + ib$ and $z_2 = c + id$ are called **equal**, written as $z_1 = z_2$, if and only if $a = c$ and $b = d$.

For example, if the complex numbers $z_1 = a + ib$ and $z_2 = -3 + 5i$ are equal, then $a = -3$ and $b = 5$.

5.1.1 Algebra of complex numbers

In this section, we shall define the usual mathematical operations — addition, subtraction, multiplication, division, square, power etc. on complex numbers and will develop the algebra of complex numbers.

Addition of two complex numbers

Let $z_1 = a + ib$ and $z_2 = c + id$ be any two complex numbers, then their sum $z_1 + z_2$ is defined as $z_1 + z_2 = (a + c) + i(b + d)$.

For example, let $z_1 = 2 + 3i$ and $z_2 = -5 + 4i$, then

$$z_1 + z_2 = (2 + (-5)) + (3 + 4)i = -3 + 7i.$$

Properties of addition of complex numbers

(i) Closure property

The sum of two complex numbers is a complex number i.e. if z_1 and z_2 are any two complex numbers, then $z_1 + z_2$ is always a complex number.

(ii) Addition of complex numbers is commutative

If z_1 and z_2 are any two complex numbers, then $z_1 + z_2 = z_2 + z_1$.

(iii) Addition of complex numbers is associative

If z_1 , z_2 and z_3 are any three complex numbers, then

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3).$$

(iv) The existence of additive identity

Let $z = x + iy$, $x, y \in \mathbf{R}$, be any complex number, then

$$(x + iy) + (0 + i0) = (x + 0) + i(y + 0) = x + iy \text{ and}$$

$$(0 + i0) + (x + iy) = (0 + x) + i(0 + y) = x + iy$$

$$\Rightarrow (x + iy) + (0 + i0) = x + iy = (0 + i0) + (x + iy).$$

Therefore, $0 + i0$ acts as the additive identity. It is simply written as 0 .

Thus, $z + 0 = z = 0 + z$ for all complex numbers z .

(v) The existence of additive inverse

For a complex number $z = a + ib$, its negative is defined as

$$-z = (-a) + i(-b) = -a - ib.$$

Note that $z + (-z) = (a - a) + i(b - b) = 0 + i0 = 0$.

Thus $-z$ acts as additive inverse of z .

Subtraction of complex numbers

Let $z_1 = a + ib$ and $z_2 = c + id$ be any two complex numbers, then the subtraction of z_2 from z_1 is defined as

$$\begin{aligned} z_1 - z_2 &= z_1 + (-z_2) \\ &= (a + ib) + (-c - id) \\ &= (a - c) + i(b - d). \end{aligned}$$

For example, let $z_1 = 2 + 3i$ and $z_2 = -1 + 4i$, then

$$\begin{aligned} z_1 - z_2 &= (2 + 3i) - (-1 + 4i) \\ &= (2 + 3i) + (1 - 4i) \\ &= (2 + 1) + (3 - 4)i = 3 - i. \end{aligned}$$

$$\begin{aligned} \text{and } z_2 - z_1 &= (-1 + 4i) - (2 + 3i) \\ &= (-1 + 4i) + (-2 - 3i) \\ &= (-1 - 2) + (4 - 3)i = -3 + i. \end{aligned}$$

Multiplication of two complex numbers

Let $z_1 = a + ib$ and $z_2 = c + id$ be any two complex numbers, then their product $z_1 z_2$ is defined as

$$z_1 z_2 = (ac - bd) + i(ad + bc).$$

Note that intuitively,

$$(a + ib)(c + id) = ac + ibc + iad + i^2 bd ; \text{ now put } i^2 = -1, \text{ thus}$$

$$(a + ib)(c + id) = ac + i(bc + ad) - bd = (ac - bd) + i(ad + bc).$$

For example, let $z_1 = 3 + 7i$ and $z_2 = -2 + 5i$, then

$$\begin{aligned} z_1 z_2 &= (3 + 7i)(-2 + 5i) \\ &= (3 \times (-2) - 7 \times 5) + i(3 \times 5 + 7 \times (-2)) \\ &= -41 + i. \end{aligned}$$

Properties of multiplication of complex numbers**(i) Closure property**

The product of two complex numbers is a complex number *i.e.* if z_1 and z_2 are any two complex numbers, then $z_1 z_2$ is always a complex number.

(ii) Multiplication of complex numbers is commutative

If z_1 and z_2 are any two complex numbers, then $z_1 z_2 = z_2 z_1$.

(iii) Multiplication of complex numbers is associative

If z_1, z_2 and z_3 are any three complex numbers, then $(z_1 z_2) z_3 = z_1 (z_2 z_3)$.

(iv) The existence of multiplicative identity

Let $z = x + iy$, $x, y \in \mathbf{R}$, be any complex number, then

$$(x + iy)(1 + i0) = (x.1 - y.0) + i(x.0 + y.1) = x + iy \text{ and}$$

$$(1 + i0)(x + iy) = (1.x - 0.y) + i(1.y + 0.x) = x + iy$$

$$\Rightarrow (x + iy)(1 + i0) = x + iy = (1 + i0)(x + iy).$$

Therefore, $1 + i0$ acts as the multiplicative identity. It is simply written as 1.

Thus $z.1 = z = 1.z$ for all complex numbers z .

(v) Existence of multiplicative inverse

For every non-zero complex number $z = a + ib$, we have the complex number

$\frac{a}{a^2+b^2} - i\frac{b}{a^2+b^2}$ (denoted by z^{-1} or $\frac{1}{z}$) such that

$$z \cdot \frac{1}{z} = 1 = \frac{1}{z} \cdot z$$

(check it)

$\frac{1}{z}$ is called the multiplicative inverse of z .

Note that intuitively, $\frac{1}{a+ib} = \frac{1}{a+ib} \times \frac{a-ib}{a-ib} = \frac{a-ib}{a^2+b^2} = \frac{a}{a^2+b^2} - i\frac{b}{a^2+b^2}$.

(vi) **Multiplication of complex numbers is distributive over addition of complex numbers**

If z_1 , z_2 and z_3 are any three complex numbers, then

$$z_1(z_2 + z_3) = z_1z_2 + z_1z_3$$

$$\text{and } (z_1 + z_2)z_3 = z_1z_3 + z_2z_3.$$

These results are known as **distributive laws**.

Division of complex numbers

Division of a complex number $z_1 = a + ib$ by $z_2 = c + id \neq 0$ is defined as

$$\frac{z_1}{z_2} = z_1 \cdot \frac{1}{z_2} = z_1 \cdot z_2^{-1} = (a + ib) \cdot \left(\frac{c}{c^2 + d^2} - i\frac{d}{c^2 + d^2} \right) = \frac{ac + bd}{c^2 + d^2} + i\frac{bc - ad}{c^2 + d^2}.$$

Note that intuitively,

$$\frac{z_1}{z_2} = \frac{a + ib}{c + id} = \frac{a + ib}{c + id} \times \frac{c - id}{c - id} = \frac{(ac + bd) + i(bc - ad)}{c^2 + d^2}.$$

For example, if $z_1 = 3 + 4i$ and $z_2 = 5 - 6i$, then

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{3+4i}{5-6i} = \frac{3+4i}{5-6i} \times \frac{5+6i}{5+6i} = \frac{(3 \times 5 - 4 \times 6) + (3 \times 6 + 4 \times 5)i}{5^2 - 6^2 \times i^2} \\ &= \frac{-9 + 38i}{25 + 36} = -\frac{9}{61} + \frac{38}{61}i. \end{aligned}$$

Integral powers of a complex number

If z is any complex number, then positive integral powers of z are defined as

$$z^1 = z, z^2 = z.z, z^3 = z^2.z, z^4 = z^3.z \text{ and so on.}$$

If z is any non-zero complex number, then negative integral powers of z are defined as :

$$z^{-1} = \frac{1}{z}, z^{-2} = \frac{1}{z^2}, z^{-3} = \frac{1}{z^3} \text{ etc.}$$

If $z \neq 0$, then $z^0 = 1$.

5.1.2 Powers of i

Integral power of i are defined as :

$$i^0 = 1, i^1 = i, i^2 = -1,$$

$$i^3 = i^2.i = (-1).i = -i,$$

$$i^4 = (i^2)^2 = (-1)^2 = 1,$$

$$i^5 = i^4.i = 1.i = i,$$

$$i^6 = i^4.i^2 = 1.(-1) = -1, \text{ and so on.}$$

$$i^{-1} = \frac{1}{i} = \frac{1}{i} \times \frac{i}{i} = \frac{i}{-1} = -i$$

Remember that $\frac{1}{i} = -i$

$$i^{-2} = \frac{1}{i^2} = \frac{1}{-1} = -1,$$

$$i^{-3} = \frac{1}{i^3} = \frac{1}{i^3} \times \frac{i}{i} = \frac{i}{i^4} = \frac{i}{1} = i$$

$$i^{-4} = \frac{1}{i^4} = \frac{1}{1} = 1, \text{ and so on.}$$

Note that $i^4 = 1$ and $i^{-4} = 1$. It follows that for any integer k ,

$$i^{4k} = 1, i^{4k+1} = i, i^{4k+2} = i^2 = -1, i^{4k+3} = i^3 = -i.$$

Also, we note that $i^2 = -1$ and $(-i)^2 = i^2 = -1$.

Therefore, i and $-i$ are both square roots of -1 . However, by the symbol $\sqrt{-1}$, we shall mean i only i.e. $\sqrt{-1} = i$.

We observe that i and $-i$ are both the solutions of the equation $x^2 + 1 = 0$.

$$\text{Similarly, } (\sqrt{5}i)^2 = (\sqrt{5})^2 i^2 = 5(-1) = -5,$$

$$\text{and } (-\sqrt{5}i)^2 = (-\sqrt{5})^2 i^2 = 5(-1) = -5.$$

Therefore, $\sqrt{5}i$ and $-\sqrt{5}i$ are both square roots of -5 . However, by the symbol $\sqrt{-5}$, we shall mean $\sqrt{5}i$ only i.e. $\sqrt{-5} = \sqrt{5}i$.

In general, if a is any positive real number, then $\sqrt{-a} = \sqrt{a}i$.

We already know that $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$ for all positive real numbers a and b . This result is also true when either $a > 0, b < 0$ or $a < 0, b > 0$. But what if $a < 0, b < 0$? Let us examine :

we note that $i^2 = i \times i = \sqrt{-1} \sqrt{-1} = \sqrt{(-1)(-1)}$ (by assuming $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$ for all real numbers) $= \sqrt{1} = 1$. Thus, we get $i^2 = 1$ which is contrary to the fact that $i^2 = -1$.

Therefore, $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$ is **not true** when a and b are both negative real numbers.

Further, if any of a and b is zero, then $\sqrt{a} \times \sqrt{b} = \sqrt{ab} = 0$.

5.1.3 Identities

If z_1 and z_2 are any two complex numbers, then the following results hold :

$$(i) (z_1 + z_2)^2 = z_1^2 + 2z_1z_2 + z_2^2$$

$$(ii) (z_1 - z_2)^2 = z_1^2 - 2z_1z_2 + z_2^2$$

$$(iii) (z_1 + z_2)(z_1 - z_2) = z_1^2 - z_2^2$$

$$(iv) (z_1 + z_2)^3 = z_1^3 + 3z_1^2z_2 + 3z_1z_2^2 + z_2^3$$

$$(v) (z_1 - z_2)^3 = z_1^3 - 3z_1^2z_2 + 3z_1z_2^2 - z_2^3.$$

Proof. (i) $(z_1 + z_2)^2 = (z_1 + z_2)(z_1 + z_2)$

$$= (z_1 + z_2)z_1 + (z_1 + z_2)z_2 \quad \text{(Distributive law)}$$

$$= z_1^2 + z_2z_1 + z_1z_2 + z_2^2 \quad \text{(Distributive law)}$$

$$= z_1^2 + z_1z_2 + z_1z_2 + z_2^2 \quad \text{(Commutative law)}$$

$$= z_1^2 + 2z_1z_2 + z_2^2.$$

We leave the proofs of the other results for the reader.

5.1.4 Modulus of a complex number

Modulus of a complex number $z = a + ib$, denoted by $\text{mod}(z)$ or $|z|$, is defined as

$$|z| = \sqrt{a^2 + b^2}, \text{ where } a = \text{Re}(z), b = \text{Im}(z).$$

Sometimes, $|z|$ is called **absolute value** of z . Note that $|z| \geq 0$.

For example :

$$(i) \text{ If } z = -3 + 5i, \text{ then } |z| = \sqrt{(-3)^2 + 5^2} = \sqrt{34}.$$

$$(ii) \text{ If } z = 3 - \sqrt{7}i, \text{ then } |z| = \sqrt{3^2 + (-\sqrt{7})^2} = \sqrt{9 + 7} = 4.$$

Properties of modulus of a complex number

If z, z_1 and z_2 are complex numbers, then

$$(i) |-z| = |z| \quad (ii) |z| = 0 \text{ if and only if } z = 0$$

$$(iii) |z_1 z_2| = |z_1| |z_2| \quad (iv) \left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|}, \text{ provided } z_2 \neq 0.$$

Proof. (i) Let $z = a + ib$, where $a, b \in \mathbf{R}$, then $-z = -a - ib$.

$$\therefore |-z| = \sqrt{(-a)^2 + (-b)^2} = \sqrt{a^2 + b^2} = |z|.$$

$$(ii) \text{ Let } z = a + ib, \text{ then } |z| = \sqrt{a^2 + b^2}.$$

$$\text{Now } |z| = 0 \text{ iff } \sqrt{a^2 + b^2} = 0$$

$$\text{i.e. iff } a^2 + b^2 = 0 \text{ i.e. iff } a^2 = 0 \text{ and } b^2 = 0$$

$$\text{i.e. iff } a = 0 \text{ and } b = 0 \text{ i.e. iff } z = 0 + i0$$

$$\text{i.e. iff } z = 0.$$

$$(iii) \text{ Let } z_1 = a + ib, \text{ and } z_2 = c + id, \text{ then}$$

$$z_1 z_2 = (ac - bd) + i(ad + bc).$$

$$\begin{aligned} \therefore |z_1 z_2| &= \sqrt{(ac - bd)^2 + (ad + bc)^2} \\ &= \sqrt{a^2 c^2 + b^2 d^2 - 2abcd + a^2 d^2 + b^2 c^2 + 2abcd} \\ &= \sqrt{(a^2 + b^2)(c^2 + d^2)} \\ &= \sqrt{a^2 + b^2} \sqrt{c^2 + d^2} \quad (\because a^2 + b^2 \geq 0, c^2 + d^2 \geq 0) \\ &= |z_1| |z_2|. \end{aligned}$$

$$(iv) \text{ Here } z_2 \neq 0 \Rightarrow |z_2| \neq 0.$$

$$\text{Let } \frac{z_1}{z_2} = z_3 \Rightarrow z_1 = z_2 z_3 \Rightarrow |z_1| = |z_2 z_3|$$

$$\Rightarrow |z_1| = |z_2| |z_3| \quad (\text{using part (iii)})$$

$$\Rightarrow \frac{|z_1|}{|z_2|} = |z_3| \Rightarrow \frac{|z_1|}{|z_2|} = \left| \frac{z_1}{z_2} \right| \quad \left(\because z_3 = \frac{z_1}{z_2} \right)$$

REMARK

From (iii), on replacing both z_1 and z_2 by z , we get

$$|z z| = |z| |z| \text{ i.e. } |z^2| = |z|^2.$$

$$\text{Similary, } |z^3| = |z^2 z| = |z^2| |z| = |z|^2 |z| = |z|^3 \text{ etc.}$$

5.1.5 Conjugate of a complex number

Conjugate of a complex number $z = a + ib$, denoted by $\bar{z}$, is defined as

$$\bar{z} = a - ib \text{ i.e. } \overline{a + ib} = a - ib.$$

For example :

$$(i) \quad \overline{2 + 5i} = 2 - 5i, \quad \overline{2 - 5i} = 2 + 5i$$

$$(ii) \quad \overline{-3 - 7i} = -3 + 7i, \quad \overline{-3 + 7i} = -3 - 7i.$$

Properties of conjugate of a complex number

If z, z_1 and z_2 are complex numbers, then

$$(i) \quad \overline{(\bar{z})} = z$$

$$(ii) \quad \overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$$

$$(iii) \quad \overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$$

$$(iv) \quad \overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$$

$$(v) \quad \overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}, \text{ provided } z_2 \neq 0$$

$$(vi) \quad |\bar{z}| = |z|$$

$$(vii) \quad z \bar{z} = |z|^2$$

$$(viii) \quad z^{-1} = \frac{\bar{z}}{|z|^2}, \text{ provided } z \neq 0.$$

Proof. (i) Let $z = a + ib$, where $a, b \in \mathbf{R}$, so that $\bar{z} = a - ib$.

$$\therefore \overline{(\bar{z})} = \overline{a - ib} = a + ib = z.$$

(ii) Let $z_1 = a + ib$ and $z_2 = c + id$, then

$$\begin{aligned} \overline{z_1 + z_2} &= \overline{(a + ib) + (c + id)} = \overline{(a + c) + i(b + d)} \\ &= (a + c) - i(b + d) = (a - ib) + (c - id) = \bar{z}_1 + \bar{z}_2. \end{aligned}$$

(iii) Let $z_1 = a + ib$ and $z_2 = c + id$, then

$$\begin{aligned} \overline{z_1 - z_2} &= \overline{(a + ib) - (c + id)} = \overline{(a - c) + i(b - d)} \\ &= (a - c) - i(b - d) = (a - ib) - (c - id) \\ &= \bar{z}_1 - \bar{z}_2. \end{aligned}$$

(iv) Let $z_1 = a + ib$ and $z_2 = c + id$, then

$$\begin{aligned} \overline{z_1 z_2} &= \overline{(a + ib)(c + id)} = \overline{(ac - bd) + i(ad + bc)} \\ &= (ac - bd) - i(ad + bc). \end{aligned}$$

$$\text{Also } \overline{\bar{z}_1 \bar{z}_2} = \overline{(a - ib)(c - id)} = \overline{(ac - bd) - i(ad + bc)}.$$

$$\text{Hence } \overline{z_1 z_2} = \bar{z}_1 \bar{z}_2.$$

(v) Here $z_2 \neq 0 \Rightarrow \bar{z}_2 \neq 0$.

$$\text{Let } \frac{z_1}{z_2} = z_3 \Rightarrow z_1 = z_2 z_3 \Rightarrow \bar{z}_1 = \overline{z_2 z_3}$$

$$\Rightarrow \bar{z}_1 = \bar{z}_2 \bar{z}_3$$

(using part (iv))

$$\Rightarrow \frac{\bar{z}_1}{\bar{z}_2} = \bar{z}_3 \Rightarrow \frac{\bar{z}_1}{\bar{z}_2} = \overline{\left(\frac{z_1}{z_2}\right)}$$

$$\left(\because z_3 = \frac{z_1}{z_2} \right)$$

(vi) Let $z = a + ib$, then $\bar{z} = a - ib$.

$$\therefore |\bar{z}| = \sqrt{a^2 + (-b)^2} = \sqrt{a^2 + b^2} = |z|.$$

(vii) Let $z = a + ib$, then $\bar{z} = a - ib$.

$$\begin{aligned}\therefore z \bar{z} &= (a + ib)(a - ib) \\ &= (aa - b(-b)) + i(a(-b) + ba) \quad (\text{Def. of multiplication}) \\ &= (a^2 + b^2) + i \cdot 0 \\ &= a^2 + b^2 = \left(\sqrt{a^2 + b^2}\right)^2 = |z|^2.\end{aligned}$$

Remember that $(a + ib)(a - ib) = a^2 + b^2$.

(viii) Let $z = a + ib \neq 0$, then $|z| \neq 0$.

$$\begin{aligned}\therefore z \bar{z} &= (a + ib)(a - ib) = a^2 + b^2 = |z|^2 \\ \Rightarrow \frac{z \bar{z}}{|z|^2} &= 1 \Rightarrow \frac{\bar{z}}{|z|^2} = \frac{1}{z} = z^{-1}\end{aligned}$$

$$\text{Thus, } z^{-1} = \frac{\bar{z}}{|z|^2}, \text{ provided } z \neq 0.$$

REMARK

From (iv), on replacing both z_1 and z_2 by z , we get

$$\overline{z z} = \bar{z} \bar{z} \text{ i.e. } \overline{z^2} = (\bar{z})^2.$$

Similarly, $\overline{(z^3)} = \overline{(z^2 z)} = (\bar{z}^2) \bar{z} = (\bar{z})^2 \bar{z} = (\bar{z})^3$ etc.

NOTE

The order relations 'greater than' and 'less than' are not defined for complex numbers i.e. the inequalities $2 + 3i > -2 + 5i$, $4i \geq 1 - 2i$, $-1 + 3i < 5$ etc. are meaningless.

ILLUSTRATIVE EXAMPLES

Example 1. A student says

$$1 = \sqrt{1} = \sqrt{(-1)(-1)} = \sqrt{-1} \sqrt{-1} = i \cdot i = i^2 = -1. \text{ Thus } 1 = -1.$$

Where is the fault?

Solution. $1 = \sqrt{1} = \sqrt{(-1)(-1)}$ is true, but $\sqrt{(-1)(-1)} = \sqrt{-1} \sqrt{-1}$ is wrong.

Because if both a, b are negative real numbers, then $\sqrt{a} \sqrt{b} = \sqrt{ab}$ is not true.

Example 2. If $z = \sqrt{37} + \sqrt{-19}$, find $\text{Re}(z)$, $\text{Im}(z)$, $\bar{z}$ and $|z|$.

Solution. Given $z = \sqrt{37} + \sqrt{-19} = \sqrt{37} + i\sqrt{19}$.

$$\therefore \text{Re}(z) = \sqrt{37} \text{ and } \text{Im}(z) = \sqrt{19}.$$

$$\bar{z} = \overline{\sqrt{37} + i\sqrt{19}} = \sqrt{37} - i\sqrt{19}.$$

$$|z| = \sqrt{(\sqrt{37})^2 + (\sqrt{19})^2} = \sqrt{37+19} = \sqrt{56} = 2\sqrt{14}.$$

Example 3. If $4x + i(3x - y) = 3 - 6i$ and x, y are real numbers, then find the values of x and y .
(NCERT)

Solution. Given $4x + i(3x - y) = 3 - 6i$

$$\Rightarrow 4x + i(3x - y) = 3 + i(-6).$$

Equating real and imaginary parts on both sides, we get

$$4x = 3 \text{ and } 3x - y = -6$$

$$\Rightarrow x = \frac{3}{4} \text{ and } 3 \times \frac{3}{4} - y = -6$$

$$\Rightarrow x = \frac{3}{4} \text{ and } y = 6 + \frac{9}{4} = \frac{33}{4}.$$

$$\text{Hence } x = \frac{3}{4} \text{ and } y = \frac{33}{4}.$$

Example 4. For what real values of x and y are the following numbers equal

$$(i) (1 + i) y^2 + (6 + i) \text{ and } (2 + i) x$$

$$(ii) x^2 - 7x + 9yi \text{ and } y^2i + 20i - 12 ?$$

Solution. (i) Given $(1 + i) y^2 + (6 + i) = (2 + i) x$

$$\Rightarrow (y^2 + 6) + i(y^2 + 1) = 2x + ix$$

$$\Rightarrow y^2 + 6 = 2x \text{ and } y^2 + 1 = x$$

$$\Rightarrow x = 5 \text{ and } y^2 = 4 \Rightarrow x = 5 \text{ and } y = \pm 2.$$

Hence, the required values of x and y are

$$x = 5, y = 2; x = 5, y = -2.$$

$$(ii) \text{ Given } x^2 - 7x + 9yi = y^2i + 20i - 12$$

$$\Rightarrow (x^2 - 7x) + i(9y) = (-12) + i(y^2 + 20)$$

$$\Rightarrow x^2 - 7x = -12 \text{ and } 9y = y^2 + 20$$

$$\Rightarrow x^2 - 7x + 12 = 0 \text{ and } y^2 - 9y + 20 = 0$$

$$\Rightarrow (x - 4)(x - 3) = 0 \text{ and } (y - 5)(y - 4) = 0$$

$$\Rightarrow x = 4, 3 \text{ and } y = 5, 4.$$

Hence, the required values of x and y are

$$x = 4, y = 5; x = 4, y = 4; x = 3, y = 5; x = 3, y = 4.$$

Example 5. Express each of the following in the standard form $a + ib$:

$$(i) \left(\frac{1}{3} + i\frac{7}{3} \right) + \left(4 + i\frac{1}{3} \right) - \left(-\frac{4}{3} + i \right)$$

$$(ii) 3(7 + i7) + i(7 + i7)$$

(NCERT)

(NCERT)

$$(iii) (-2 + \sqrt{-3})(-3 + 2\sqrt{-3})$$

$$(iv) \frac{(3 + i\sqrt{5})(3 - i\sqrt{5})}{(\sqrt{3} + \sqrt{2}i) - (\sqrt{3} - i\sqrt{2})}.$$

(NCERT)

$$\textbf{Solution.} (i) \left(\frac{1}{3} + i\frac{7}{3} \right) + \left(4 + i\frac{1}{3} \right) - \left(-\frac{4}{3} + i \right)$$

$$= \left(\frac{1}{3} + i\frac{7}{3} \right) + \left(4 + i\frac{1}{3} \right) + \left(\frac{4}{3} - i \right)$$

$$= \left(\frac{1}{3} + 4 + \frac{4}{3} \right) + i \left(\frac{7}{3} + \frac{1}{3} - 1 \right) = \frac{17}{3} + \frac{5}{3}i.$$

$$(ii) 3(7 + i7) + i(7 + i7) = (21 + 21i) + (7i + 7i^2)$$

$$= 21 + 21i + 7i + 7(-1) = (21 - 7) + (21 + 7)i$$

$$= 14 + 28i.$$

$$(iii) (-2 + \sqrt{-3})(-3 + 2\sqrt{-3}) = (-2 + \sqrt{3}i)(-3 + 2\sqrt{3}i)$$

$$= (6 - 2\sqrt{3}\sqrt{3}) + (-3\sqrt{3} - 4\sqrt{3})i$$

$$= 0 - 7\sqrt{3}i.$$

$$\begin{aligned}
 (iv) \quad \frac{(3 + i\sqrt{5})(3 - i\sqrt{5})}{(\sqrt{3} + \sqrt{2}i) - (\sqrt{3} - i\sqrt{2})} &= \frac{(3)^2 + (\sqrt{5})^2}{\sqrt{2}i + \sqrt{2}i} & ((a + ib)(a - ib) = a^2 + b^2) \\
 &= \frac{9 + 5}{2\sqrt{2}i} = \frac{14}{2\sqrt{2}} \cdot \frac{1}{i} = \frac{7}{\sqrt{2}}(-i) & \left(\because \frac{1}{i} = -i\right) \\
 &= 0 - \frac{7}{\sqrt{2}}i.
 \end{aligned}$$

Example 6. Express the following in the form $a + ib$:

$$\begin{aligned}
 (i) \quad (-i)(2i) \left(-\frac{1}{8}i\right)^3 & \quad \text{(NCERT)} & (ii) \quad i^{102} & (iii) \quad i^{-39} & \quad \text{(NCERT)} \\
 (iv) \quad (-\sqrt{-1})^{31} & (v) \quad i^9 + i^{19} & \quad \text{(NCERT)} & (vi) \quad i^{35} + \frac{1}{i^{35}}.
 \end{aligned}$$

Solution. (i) $(-i)(2i) \left(-\frac{1}{8}i\right)^3 = (-1)^4 \times 2 \times \left(\frac{1}{8}\right)^3 \times i^5$

$$\begin{aligned}
 &= 1 \times 2 \times \frac{1}{512} \times i^4 \times i \\
 &= \frac{1}{256} \times 1 \times i = 0 + \frac{1}{256}i.
 \end{aligned}$$

$$\begin{aligned}
 (ii) \quad i^{102} &= i^{4 \times 25 + 2} = i^2 & (\because i^{4k+2} = i^2, k \in \mathbf{I}) \\
 &= -1 = -1 + i0.
 \end{aligned}$$

$$\begin{aligned}
 (iii) \quad i^{-39} &= i^{4 \times (-10) + 1} = i & (\because i^{4k+1} = i, k \in \mathbf{I}) \\
 &= 0 + i.
 \end{aligned}$$

$$\begin{aligned}
 (iv) \quad (-\sqrt{-1})^{31} &= (-i)^{31} = (-1)^{31} i^{31} \\
 &= -i^{4 \times 7 + 3} = -i^3 & (\because i^{4k+3} = i^3, k \in \mathbf{I}) \\
 &= -i^2 \cdot i = -(-1)i = i = 0 + i.
 \end{aligned}$$

$$\begin{aligned}
 (v) \quad i^9 + i^{19} &= i^{2 \times 4 + 1} + i^{4 \times 4 + 3} = i + i^3 \\
 &= i + i^2 \cdot i = i + (-1)i = 0 = 0 + i0.
 \end{aligned}$$

$$\begin{aligned}
 (vi) \quad i^{35} + \frac{1}{i^{35}} &= i^{35} + i^{-35} = i^{4 \times 8 + 3} + i^{4 \times (-9) + 1} \\
 &= i^3 + i = i^2 i + i = (-1)i + i \\
 &= 0 = 0 + i0.
 \end{aligned}$$

Example 7. Express each of the following in the standard form $a + ib$:

$$(i) \quad (1 - i)^4 \quad \text{(NCERT)} \quad (ii) \quad \left(-2 - \frac{1}{3}i\right)^3 \quad \text{(NCERT)}$$

$$(iii) \quad (2i - i^2)^2 + (1 - 3i)^3 \quad (iv) \quad \left(i^{18} + \left(\frac{1}{i}\right)^{25}\right)^3 \quad \text{(NCERT)}$$

$$(v) \quad (1 + i)^6 + (1 - i)^3 \quad \text{(NCERT Exemplar Problems)}$$

Solution. (i) $(1 - i)^4 = ((1 - i)^2)^2 = (1 + i^2 - 2i)^2$

$$\begin{aligned}
 &= (1 + (-1) - 2i)^2 = (-2i)^2 = 4i^2 \\
 &= 4(-1) = -4 = -4 + i0.
 \end{aligned}$$

$$\begin{aligned}
 \text{(ii)} \quad \left(-2 - \frac{1}{3}i\right)^3 &= (-1)^3 \left(2 + \frac{1}{3}i\right)^3 \\
 &= -\left[2^3 + 3 \times 2^2 \times \frac{1}{3}i + 3 \times 2 \times \left(\frac{1}{3}i\right)^2 + \left(\frac{1}{3}i\right)^3\right] \\
 &= -\left[8 + 4i + \frac{2}{3}i^2 + \frac{1}{27}i^3\right] \\
 &= -\left[8 + 4i + \frac{2}{3}(-1) + \frac{1}{27}(-i)\right] \\
 &= -\left[\frac{22}{3} + \frac{107}{27}i\right] = -\frac{22}{3} - \frac{107}{27}i.
 \end{aligned}$$

$$\begin{aligned}
 \text{(iii)} \quad (2i - i^2)^2 + (1 - 3i)^3 &= (2i + 1)^2 + (1 - 3i)^3 \\
 &= (4i^2 + 4i + 1) + (1 - 9i + 27i^2 - 27i^3) \\
 &= -4 + 4i + 1 + 1 - 9i - 27 + 27i = -29 + 22i.
 \end{aligned}$$

$$\begin{aligned}
 \text{(iv)} \quad \left(i^{18} + \left(\frac{1}{i}\right)^{25}\right)^3 &= \left(i^{4 \times 4 + 2} + (-i)^{25}\right)^3 & \left(\because \frac{1}{i} = -i\right) \\
 &= \left(i^2 + (-1)^{25}i^{25}\right)^3 = (-1 - i^{4 \times 6 + 1})^3 \\
 &= (-1 - i)^3 = (-1)^3(1 + i)^3 \\
 &= -[1 + 3i + 3i^2 + i^3] \\
 &= -[1 + 3i - 3 - i] = -(-2 + 2i) \\
 &= 2 - 2i.
 \end{aligned}$$

$$\begin{aligned}
 \text{(v)} \quad (1 + i)^6 &= ((1 + i)^2)^3 = (1 + i^2 + 2i)^3 = (1 - 1 + 2i)^3 = (2i)^3 \\
 &= 8i^3 = 8(-i) = -8i
 \end{aligned}$$

$$\begin{aligned}
 \text{and} \quad (1 - i)^3 &= 1 - i^3 - 3i + 3i^2 = 1 - (-i) - 3i + 3(-1) \\
 &= -2 - 2i
 \end{aligned}$$

$$\therefore (1 + i)^6 + (1 - i)^3 = -8i + (-2 - 2i) = -2 - 10i.$$

Example 8. Find the multiplicative inverse of $\sqrt{5} + 3i$.

(NCERT)

Solution. Let $z = \sqrt{5} + 3i$,

then $\bar{z} = \sqrt{5} - 3i$ and $|z|^2 = (\sqrt{5})^2 + 3^2 = 5 + 9 = 14$.

We know that the multiplicative inverse of z is given by

$$z^{-1} = \frac{\bar{z}}{|z|^2} = \frac{\sqrt{5} - 3i}{14} = \frac{\sqrt{5}}{14} - \frac{3}{14}i.$$

Alternatively

$$\begin{aligned}
 z^{-1} &= \frac{1}{z} = \frac{1}{\sqrt{5} + 3i} = \frac{1}{\sqrt{5} + 3i} \times \frac{\sqrt{5} - 3i}{\sqrt{5} - 3i} \\
 &= \frac{\sqrt{5} - 3i}{(\sqrt{5})^2 - (3i)^2} = \frac{\sqrt{5} - 3i}{5 - 9(-1)} = \frac{\sqrt{5} - 3i}{14} = \frac{\sqrt{5}}{14} - \frac{3}{14}i.
 \end{aligned}$$

Example 9. Express the following in the form $a + ib$:

$$\text{(i)} \quad \frac{i}{1+i}$$

$$\text{(ii)} \quad \frac{5 + \sqrt{2}i}{1 - \sqrt{2}i}$$

(NCERT)

$$\text{(iii)} \quad \left(\frac{1}{1-4i} - \frac{2}{1+i}\right)\left(\frac{3-4i}{5+i}\right) \quad (\text{NCERT})$$

$$\text{(iv)} \quad \frac{(1+i)(3+i)}{3-i} - \frac{(1-i)(3-i)}{3+i}.$$

Solution. (i) $\frac{i}{1+i} = \frac{i}{1+i} \times \frac{1-i}{1-i} = \frac{i-i^2}{1-i^2} = \frac{i-(-1)}{1-(-1)} = \frac{1+i}{2} = \frac{1}{2} + \frac{1}{2}i.$

(ii) $\frac{5 + \sqrt{2}i}{1 - \sqrt{2}i} = \frac{5 + \sqrt{2}i}{1 - \sqrt{2}i} \times \frac{1 + \sqrt{2}i}{1 + \sqrt{2}i} = \frac{5 + 5\sqrt{2}i + \sqrt{2}i + 2i^2}{1^2 - (\sqrt{2}i)^2}$
 $= \frac{5 + 6\sqrt{2}i - 2}{1 - 2(-1)} = \frac{3 + 6\sqrt{2}i}{3} = 1 + 2\sqrt{2}i.$

(iii) $\left(\frac{1}{1-4i} - \frac{2}{1+i}\right)\left(\frac{3-4i}{5+i}\right) = \frac{1+i-2+8i}{(1-4i)(1+i)} \times \frac{3-4i}{5+i}$
 $= \frac{-1+9i}{1+i-4i+4} \times \frac{3-4i}{5+i} = \frac{(-1+9i)(3-4i)}{(5-3i)(5+i)}$
 $= \frac{-3+4i+27i+36}{25+5i-15i+3} = \frac{33+31i}{28-10i}$
 $= \frac{33+31i}{28-10i} \times \frac{28+10i}{28+10i} = \frac{33 \times 28 + 330i + 31 \times 28i - 310}{(28)^2 - (10i)^2}$
 $= \frac{924 - 310 + (330 + 868)i}{784 - 100(-1)} = \frac{614 + 1198i}{884} = \frac{307}{442} + \frac{599}{442}i.$

(iv) $\frac{(1+i)(3+i)}{3-i} - \frac{(1-i)(3-i)}{3+i} = \frac{(1+i)(3+i)(3+i) - (1-i)(3-i)(3-i)}{(3-i)(3+i)}$
 $= \frac{(1+i)(8+6i) - (1-i)(8-6i)}{9-i^2}$
 $= \frac{(2+14i) - (2-14i)}{9+1} = \frac{28i}{10} = 0 + \frac{14}{5}i.$

Example 10. (i) If $\frac{(1+i)^2}{2-i} = x + iy$, then find the value of $x + y$. (NCERT Exemplar Problems)

(ii) If $\left(\frac{1+i}{1-i}\right)^3 - \left(\frac{1-i}{1+i}\right)^3 = x + iy$, then find (x, y) . (NCERT Exemplar Problems)

(iii) If $\left(\frac{1-i}{1+i}\right)^{100} = a + ib$, then find (a, b) . (NCERT Exemplar Problems)

Solution. (i) $x + iy = \frac{(1+i)^2}{2-i} = \frac{1+i^2+2i}{2-i} = \frac{1-1+2i}{2-i} = \frac{2i}{2-i}$
 $= \frac{2i}{2-i} \times \frac{2+i}{2+i} = \frac{4i+2i^2}{2^2-i^2} = \frac{4i+2(-1)}{4-(-1)}$
 $= \frac{-2+4i}{5} = -\frac{2}{5} + \frac{4}{5}i.$

$$\Rightarrow x = -\frac{2}{5} \text{ and } y = \frac{4}{5}.$$

$$\therefore x + y = -\frac{2}{5} + \frac{4}{5} = \frac{2}{5}.$$

(ii) We have, $\frac{1+i}{1-i} = \frac{1+i}{1-i} \times \frac{1+i}{1+i} = \frac{(1+i)^2}{1^2-i^2}$
 $= \frac{1+i^2+2i}{1-(-1)} = \frac{1-1+2i}{2} = \frac{2i}{2} = i$

$$\therefore \frac{1-i}{1+i} = \frac{1}{i} = -i \quad \left(\because \frac{1}{i} = -i \right)$$

$$\begin{aligned}\text{Given } x + iy &= \left(\frac{1+i}{1-i}\right)^3 - \left(\frac{1-i}{1+i}\right)^3 = i^3 - (-i)^3 = i^3 + i^3 \\ &= 2i^3 = 2(-i) = 0 - 2i\end{aligned}$$

$$\Rightarrow x = 0 \text{ and } y = -2.$$

Hence, the ordered pair (x, y) is $(0, -2)$.

$$\begin{aligned}\text{(iii) Given } a + ib &= \left(\frac{1-i}{1+i}\right)^{100} = (-i)^{100} && \text{(see part (ii))} \\ &= (-1)^{100} (i)^{4 \times 25} = 1 \times 1 = 1 = 1 + 0i\end{aligned}$$

$$\Rightarrow a = 1 \text{ and } b = 0.$$

Hence, the ordered pair (a, b) is $(1, 0)$.

Example 11. (i) If $(1 + i)z = (1 - i)\bar{z}$, then show that $z = -i\bar{z}$. (NCERT Exemplar Problems)

$$\text{(ii) If } z_1 = 2 - i \text{ and } z_2 = -2 + i, \text{ find } \operatorname{Re} \left(\frac{z_1 z_2}{\bar{z}_1} \right). \quad \text{(NCERT)}$$

Solution. (i) Given $(1 + i)z = (1 - i)\bar{z}$

$$\Rightarrow \frac{z}{\bar{z}} = \frac{1-i}{1+i} = \frac{1-i}{1+i} \times \frac{1-i}{1-i} = \frac{(1-i)^2}{1^2 - i^2} = \frac{1 + i^2 - 2i}{1 - (-1)}$$

$$\Rightarrow \frac{z}{\bar{z}} = \frac{1-1-2i}{2} = -\frac{2i}{2} = -i$$

$$\Rightarrow z = -i\bar{z}.$$

$$\begin{aligned}\text{(ii)} \quad \frac{z_1 z_2}{\bar{z}_1} &= \frac{(2-i)(-2+i)}{\overline{2-i}} = \frac{-4+2i+2i-i^2}{2+i} = \frac{-4+4i-(-1)}{2+i} \\ &= \frac{-3+4i}{2+i} = \frac{-3+4i}{2+i} \times \frac{2-i}{2-i} \\ &= \frac{-6+3i+8i-4i^2}{2^2-i^2} = \frac{-6+11i-4(-1)}{4-(-1)} \\ &= \frac{-2+11i}{5} = -\frac{2}{5} + \frac{11}{5}i.\end{aligned}$$

$$\therefore \operatorname{Re} \left(\frac{z_1 z_2}{\bar{z}_1} \right) = -\frac{2}{5}.$$

Example 12. (i) Find the conjugate of $\frac{(3-2i)(2+3i)}{(1+2i)(2-i)}$ (NCERT)

(ii) Find the modulus of $\frac{1+i}{1-i} - \frac{1-i}{1+i}$ (NCERT)

(iii) Find the modulus of $\frac{(2-3i)^2}{-1+5i}$

$$\begin{aligned}\text{Solution. (i) Let } z &= \frac{(3-2i)(2+3i)}{(1+2i)(2-i)} = \frac{6+9i-4i+6}{2-i+4i+2} \\ &= \frac{12+5i}{4+3i} = \frac{12+5i}{4+3i} \times \frac{4-3i}{4-3i} = \frac{48-36i+20i+15}{4^2-(3i)^2} \\ &= \frac{63-16i}{16-9(-1)} = \frac{63-16i}{25} = \frac{63}{25} - \frac{16}{25}i.\end{aligned}$$

$$\therefore \text{Conjugate of } z = \frac{63}{25} + \frac{16}{25}i.$$

Example 32. If α and β are different complex numbers with $|\beta| = 1$, then find $\left| \frac{\beta - \alpha}{1 - \bar{\alpha}\beta} \right|$.
(NCERT)

Solution. We have $\left| \frac{\beta - \alpha}{1 - \bar{\alpha}\beta} \right| = \left| \frac{(\beta - \alpha)\bar{\beta}}{(1 - \bar{\alpha}\beta)\bar{\beta}} \right|$ (Note this step)

$$= \left| \frac{(\beta - \alpha)\bar{\beta}}{\bar{\beta} - \bar{\alpha}\beta\bar{\beta}} \right| = \left| \frac{(\beta - \alpha)\bar{\beta}}{\bar{\beta} - \bar{\alpha}} \right| \quad (\text{Given } |\beta| = 1 \Rightarrow |\beta|^2 = 1 \Rightarrow \beta\bar{\beta} = 1)$$

$$= \frac{|(\beta - \alpha)\bar{\beta}|}{|\bar{\beta} - \bar{\alpha}|} \quad \left(\because \left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|} \right)$$

$$= \frac{|\beta - \alpha| |\bar{\beta}|}{|\bar{\beta} - \bar{\alpha}|} \quad (\because \overline{z_1 - z_2} = \overline{z_1} - \overline{z_2})$$

$$= \frac{|\beta - \alpha| |\beta|}{|\bar{\beta} - \bar{\alpha}|} \quad (\because |\bar{z}| = |z|)$$

$$= |\beta| = 1 \quad (|\beta| = 1, \text{ given})$$

Hence, $\left| \frac{\beta - \alpha}{1 - \bar{\alpha}\beta} \right| = 1$.

Example 33. If $|z_1| = |z_2| = \dots = |z_n| = 1$, prove that

$$|z_1 + z_2 + \dots + z_n| = \left| \frac{1}{z_1} + \frac{1}{z_2} + \dots + \frac{1}{z_n} \right|. \quad (\text{NCERT Exemplar Problems})$$

Solution. Given $|z_1| = |z_2| = \dots = |z_n| = 1$

$$\Rightarrow |z_1|^2 = |z_2|^2 = \dots = |z_n|^2 = 1$$

$$\Rightarrow z_1 \bar{z}_1 = 1, z_2 \bar{z}_2 = 1, \dots, z_n \bar{z}_n = 1$$

$$\Rightarrow \frac{1}{z_1} = \bar{z}_1, \frac{1}{z_2} = \bar{z}_2, \dots, \frac{1}{z_n} = \bar{z}_n.$$

$$\therefore \left| \frac{1}{z_1} + \frac{1}{z_2} + \dots + \frac{1}{z_n} \right| = \left| \bar{z}_1 + \bar{z}_2 + \dots + \bar{z}_n \right|$$

$$= \left| \overline{z_1 + z_2 + \dots + z_n} \right| = |z_1 + z_2 + \dots + z_n|. \quad (\because |\bar{z}| = |z|)$$

EXERCISE 5.1

Very short answer type questions (1 to 31) :

1. Evaluate the following :

(i) $\sqrt{-9} \times \sqrt{-4}$

(ii) $\sqrt{(-9)(-4)}$

(iii) $\sqrt{-25} \times \sqrt{16}$

(iv) $3\sqrt{-16}\sqrt{-25}$

(v) $\sqrt{-16} + 3\sqrt{-25} + \sqrt{-36} - \sqrt{-625}$.

2. If $z = -3 - i$, find $\text{Re}(z)$, $\text{Im}(z)$, $\bar{z}$ and $|z|$.

3. If $z^2 = -i$, then is it true that $z = \pm \frac{1}{\sqrt{2}}(1 - i)$?

4. If $z^2 = -3 + 4i$, then is it true that $z = \pm(1 + 2i)$?

5. If $i = \sqrt{-1}$, then show that $(x + 1 + i)(x + 1 - i) = x^2 + 2x + 2$.

6. Find real values of x and y if

(i) $2y + (3x - y)i = 5 - 2i$

(ii) $(3x - 1) + (\sqrt{3} + 2y)i = 5$

(iii) $(3y - 2) + i(7 - 2x) = 0$.

7. If $x, y \in \mathbf{R}$ and $(5y - 2) + i(3x - y) = 3 - 7i$, find the values of x and y .
8. If x, y are reals and $(3y + 2) + i(x + 3y) = 0$, find the values of x and y .
9. If x, y are reals and $(1 - i)x + (1 + i)y = 1 - 3i$, find the values of x and y .
10. For any two complex numbers z_1 and z_2 , prove that

$$\operatorname{Re}(z_1 z_2) = \operatorname{Re}(z_1) \operatorname{Re}(z_2) - \operatorname{Im}(z_1) \operatorname{Im}(z_2).$$
 (NCERT)
11. For any complex number z , prove that

$$\operatorname{Re}(z) = \frac{z + \bar{z}}{2} \text{ and } \operatorname{Im}(z) = \frac{z - \bar{z}}{2i}.$$
12. If z is a complex number, show that $\frac{z - \bar{z}}{2i}$ is real.
 Express the following (13 to 20) complex numbers in the standard form $a + ib$:
13. (i) $(-5i)\left(\frac{1}{8}i\right)$ (NCERT) (ii) $(5i)\left(-\frac{3}{5}i\right)$. (NCERT)
14. (i) $(1 - i) - (-1 + 6i)$ (NCERT) (ii) $\left(\frac{1}{5} + i\frac{2}{5}\right) - \left(4 + i\frac{5}{2}\right)$. (NCERT)
15. (i) $(-2 + 3i) + 3\left(-\frac{1}{2}i + 1\right) - (2i)$ (ii) $(7 + i5)(7 - i5)$.
16. (i) $(-\sqrt{3} + \sqrt{-2})(2\sqrt{3} - i)$ (NCERT) (ii) $(-5 + 3i)^2$.
17. (i) $\left(\frac{1}{2} + 2i\right)^3$ (ii) $(5 - 3i)^3$. (NCERT)
18. (i) $\left(\frac{1}{3} + 3i\right)^3$ (NCERT) (ii) $(\sqrt{5} + 7i)(\sqrt{5} - 7i)^2$.
19. (i) i^{99} (ii) i^{-35} . (NCERT)
20. (i) $(-\sqrt{-4})^3$ (ii) $i + i^2 + i^3 + i^4$.
21. Find the value of $(-1 + \sqrt{-3})^2 + (-1 - \sqrt{-3})^2$.
22. If n is any integer, then find the value of
 (i) $(-\sqrt{-1})^{4n+3}$ (ii) $\frac{i^{4n+1} - i^{4n-1}}{2}$. (NCERT Exemplar Problems)
23. Find the multiplicative inverse of $-i$. (NCERT)
24. Express the following numbers in the form $a + ib$, $a, b \in \mathbf{R}$:
 (i) $\frac{i}{1+i}$ (ii) $\frac{1-i}{1+i}$.
25. If $(a + ib)(c + id) = A + iB$, then show that $(a^2 + b^2)(c^2 + d^2) = A^2 + B^2$.
26. Find the modulus of the following complex numbers :
 (i) $(3 - 4i)(-5 + 12i)$ (ii) $\frac{5 - 12i}{-3 + 4i}$.
27. Find the modulus of the following :
 (i) $\frac{(2 - 3i)^2}{4 + 3i}$ (ii) $(\sqrt{7} - 3i)^3$.
28. (i) If $z = 3 - \sqrt{7}i$, then find $|z^{-1}|$.
 (ii) If $z = x + iy$, $x, y \in \mathbf{R}$, then find $|iz|$.
29. Find the conjugate of i^7 .
30. Write the conjugate of $(2 + 3i)(1 - 2i)$ in the form $a + ib$, $a, b \in \mathbf{R}$.
31. Solve for x : $|1 + i|^x = 2$.
 Express the following (32 and 33) complex numbers in the standard form $a + ib$:
32. (i) $i^{55} + i^{60} + i^{65} + i^{70}$ (ii) $\frac{i + i^2 + i^4}{1 + i^2 + i^4}$.

44. If $a + ib = \frac{(x + i)^2}{2x^2 + 1}$, then prove that $a^2 + b^2 = \frac{(x^2 + 1)^2}{(2x^2 + 1)^2}$.

45. If $z = 1 + 2i$, then find the value of $z^3 + 7z^2 - z + 16$.

46. Show that if $\left| \frac{z - 5i}{z + 5i} \right| = 1$, then z is a real number.

47. (i) If $z = x + iy$ and $\left| \frac{z - 2}{z - 3} \right| = 2$, show that $3(x^2 + y^2) - 20x + 32 = 0$.

(NCERT Exemplar Problems)

(ii) If $z = x + iy$ and $\frac{|z - 1 - i| + 4}{3|z - 1 - i| - 2} = 1$, show that $x^2 + y^2 - 2x - 2y = 7$.

48. Find the least positive integral value of n for which $\left(\frac{1+i}{1-i} \right)^n$ is a real number.

49. Find the real value of θ such that $\frac{1 + i \cos \theta}{1 - 2i \cos \theta}$ is a real number.

50. If z is a complex number such that $|z| = 1$, prove that $\frac{z-1}{z+1}$ ($z \neq -1$) is purely imaginary.
What is the exception?

(NCERT Exemplar Problems)

51. If z_1, z_2 and z_3 are complex numbers such that $|z_1| = |z_2| = |z_3| = \left| \frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} \right| = 1$, then find the value of $|z_1 + z_2 + z_3|$.

(NCERT Exemplar Problems)

5.2 ARGAND PLANE

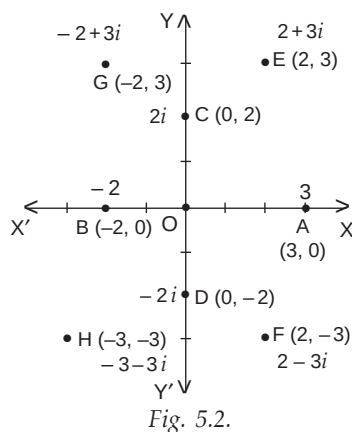
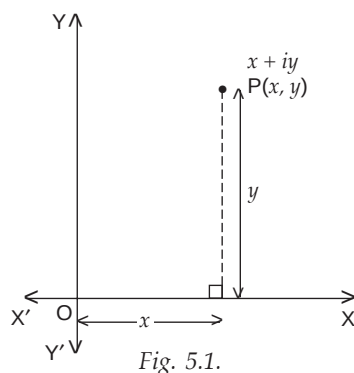
We know that corresponding to every real number there exists a unique point on the number line (called real axis) and conversely corresponding to every point on the line there exists a unique real number *i.e.* there is a one-one correspondence between the set $\mathbf{R}$ of real numbers and the points on the real axis.

In a similar way, corresponding to every ordered pair (x, y) of real numbers there exists a unique point P in the co-ordinate plane with x as **abscissa** and y as **ordinate** of the point P and conversely corresponding to every point P in the plane there exists a unique ordered pair of real numbers. Thus, there is a one-one correspondence between the set of ordered pairs $\{(x, y); x, y \in \mathbf{R}\}$ and the points in the co-ordinate plane.

The point P with co-ordinates (x, y) is said to represent the complex number $z = x + iy$.

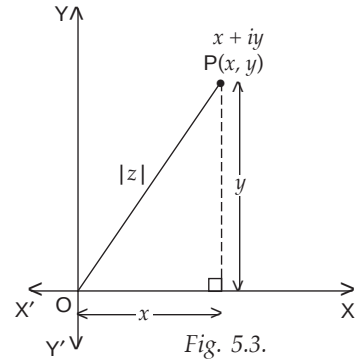
It follows that the complex number $z = x + iy$ can be uniquely represented by the point $P(x, y)$ in the co-ordinate plane and conversely corresponding to the point $P(x, y)$ in the plane there exists a unique complex number $z = x + iy$. The co-ordinate plane that represents the complex numbers is called the **complex plane** or **Argand plane**.

The complex numbers $3, -2, 2i, -2i, 2 + 3i, 2 - 3i, -2 + 3i$ and $-2 - 3i$ which correspond to the ordered pairs $(3, 0), (-2, 0), (0, 2), (0, -2), (2, 3), (2, -3), (-2, 3)$ and $(-2, -3)$ respectively have been represented geometrically in the co-ordinate plane by the points A, B, C, D, E, F, G and H respectively in fig. 5.2.



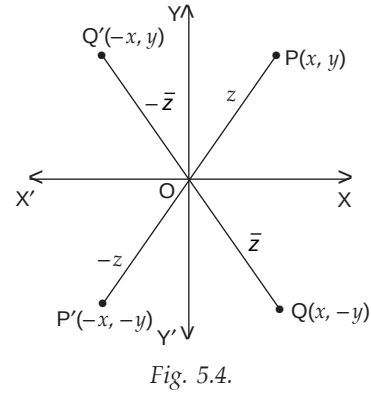
Note that every real number $x = x + 0i$ is represented by point $(x, 0)$ lying on x -axis, and every purely imaginary number iy is represented by point $(0, y)$ lying on y -axis. Consequently, x -axis is called the **real axis** and y -axis is called the **imaginary axis**.

If the point $P(x, y)$ represents the complex number $z = x + iy$, then the distance between the points P and the origin $O(0, 0) = \sqrt{x^2 + y^2} = |z|$. Thus, the modulus of z i.e. $|z|$ is the distance between points P and O (see fig. 5.3).



Geometric representation of $-z$, $\bar{z}$ and $-\bar{z}$. If $z = x + iy$, $x, y \in \mathbf{R}$ is represented by the point $P(x, y)$ in the complex plane, then the complex numbers $-z$, $\bar{z}$, $-\bar{z}$ are represented by the points $P'(-x, -y)$, $Q(x, -y)$ and $Q'(-x, y)$ respectively in the complex plane (see fig. 5.4).

Geometrically, the point $Q(x, -y)$ is the mirror image of the point $P(x, y)$ in the real axis. Thus, conjugate of z i.e. $\bar{z}$ is the mirror image of z in the x -axis.



5.3 POLAR REPRESENTATION OF COMPLEX NUMBERS

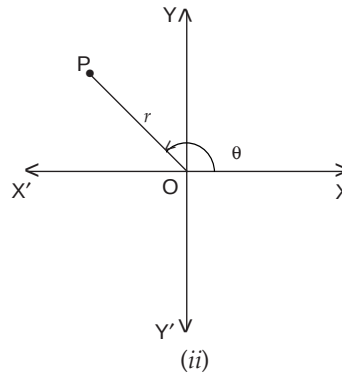
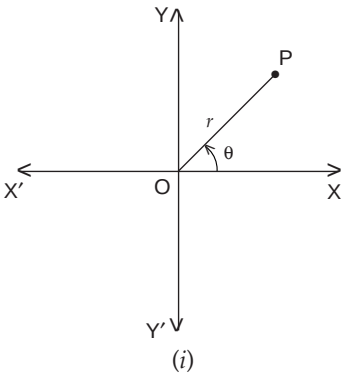
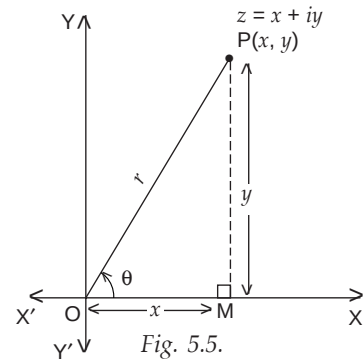
Let the point $P(x, y)$ represent the non-zero complex number $z = x + iy$ in the Argand plane. Let the directed line segment OP be of length $r (> 0)$ and θ be the radian measure of the angle which OP makes with the positive direction of x -axis (shown in fig. 5.5). Then $r = \sqrt{x^2 + y^2} = |z|$ and is called **modulus** of z ; and θ is called **amplitude** or **argument** of z and is written as **amp(z)** or **arg(z)**.

From figure 5.5, we see that

$$x = r \cos \theta \text{ and } y = r \sin \theta$$

$$\therefore z = x + iy = r \cos \theta + ir \sin \theta = r(\cos \theta + i \sin \theta).$$

Thus, $z = r(\cos \theta + i \sin \theta)$. This form of z is called **polar form** of the complex number z .



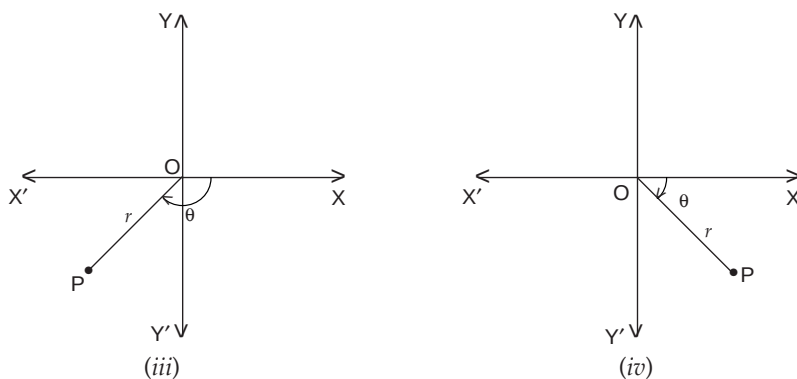


Fig. 5.6.

For any non-zero complex number z , there corresponds only one value of θ in $-\pi < \theta \leq \pi$ (see fig. 5.6). The unique value of θ such that $-\pi < \theta \leq \pi$ is called **principal value of amplitude** or **argument**.

Thus every (non-zero) complex number $z = x + iy$ can be uniquely expressed as $z = r(\cos \theta + i \sin \theta)$ where $r > 0$ and $-\pi < \theta \leq \pi$ and conversely, for every $r > 0$ and θ such that $-\pi < \theta \leq \pi$, we get a unique (non-zero) complex number $z = r(\cos \theta + i \sin \theta) = x + iy$.

Note that the complex number zero cannot be put into the form $r(\cos \theta + i \sin \theta)$ and so, the argument of zero complex number does not exist.

REMARK

If we take origin as the pole and the positive direction of the x -axis as the initial line, then the point P is uniquely determined by the ordered pair of real numbers (r, θ) , called the **polar co-ordinates** of the point P (see fig. 5.6).

ILLUSTRATIVE EXAMPLES

Example 1. Convert the following complex numbers in the polar form and represent them in Argand plane :

- | | | |
|----------------------------|------------------------------|--------------------------------|
| (i) $\sqrt{3} + i$ (NCERT) | (ii) $-\sqrt{3} + i$ (NCERT) | (iii) $-1 - i\sqrt{3}$ (NCERT) |
| (iv) $2 - 2i$ | (v) -3 (NCERT) | (vi) $-5i$. |

Solution. (i) Let $z = \sqrt{3} + i = r(\cos \theta + i \sin \theta)$.

Then $r \cos \theta = \sqrt{3}$ and $r \sin \theta = 1$.

On squaring and adding, we get

$$r^2 (\cos^2 \theta + \sin^2 \theta) = (\sqrt{3})^2 + 1^2$$

$$\Rightarrow r^2 = 4 \Rightarrow r = 2.$$

$$\therefore \cos \theta = \frac{\sqrt{3}}{2} \text{ and } \sin \theta = \frac{1}{2}.$$

The value of θ such that $-\pi < \theta \leq \pi$ and satisfying both the above equations is given by $\theta = \frac{\pi}{6}$.

Hence, $z = 2 \left(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6} \right)$, which is the required polar form.

The complex number $z = \sqrt{3} + i$ is represented in fig. 5.7.

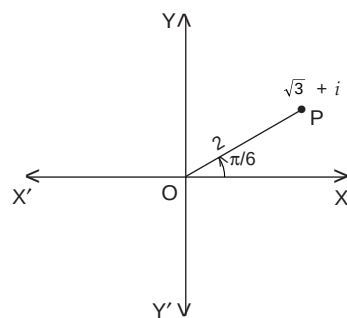


Fig. 5.7.

ANSWERS**EXERCISE 5.1**

1. (i) -6 (ii) 6 (iii) $20i$ (iv) -60 (v) 0
 2. -3 ; -1 ; $-3 + i$; $\sqrt{10}$. 3. Yes 4. Yes
 6. (i) $x = \frac{1}{6}$, $y = \frac{5}{2}$ (ii) $x = 2$, $y = -\frac{\sqrt{3}}{2}$ (iii) $x = \frac{7}{2}$, $y = \frac{2}{3}$
 7. $x = -2$, $y = 1$ 8. $x = 2$, $y = -\frac{2}{3}$ 9. $x = 2$, $y = -1$
 13. (i) $\frac{5}{8} + i0$ (ii) $3 + i0$ 14. (i) $2 - 7i$ (ii) $-\frac{19}{5} - \frac{21}{10}i$
 15. (i) $1 - \frac{1}{2}i$ (ii) $74 + i0$ 16. (i) $(-6 + \sqrt{2}) + \sqrt{3}(1 + 2\sqrt{2})i$ (ii) $16 - 30i$
 17. (i) $-\frac{47}{8} - \frac{13}{2}i$ (ii) $-10 - 198i$ 18. (i) $-\frac{242}{27} - 26i$ (ii) $54\sqrt{5} - 378i$
 19. (i) $0 - i$ (ii) $0 + i$ 20. (i) $0 + 8i$ (ii) $0 + i0$
 21. -4 22. (i) i (ii) i 23. i
 24. (i) $\frac{1}{2} + \frac{1}{2}i$ (ii) $0 - i$ 26. (i) 65 (ii) $\frac{13}{5}$
 27. (i) $\frac{13}{5}$ (ii) 64 28. (i) $\frac{1}{4}$ (ii) $\sqrt{x^2 + y^2}$
 29. i 30. $8 + i$ 31. 2
 32. (i) $0 + i0$ (ii) $0 + i$ 33. (i) $1 - i$ (ii) $16 + i0$
 34. (i) $\frac{2}{13} + \frac{3}{13}i$ (ii) $\frac{4}{25} + \frac{3}{25}i$ (iii) $\frac{3}{16} - \frac{\sqrt{7}}{16}i$
 35. (i) $\frac{21}{25} - \frac{47}{25}i$ (ii) $-\frac{1}{4} - \frac{\sqrt{3}}{4}i$ (iii) $\frac{2}{5} + \frac{29}{5}i$
 (iv) $\frac{8}{65} + \frac{1}{65}i$ (v) $\frac{1}{2} + \frac{1}{2}i$ (vi) $\frac{63}{25} - \frac{16}{25}i$
 36. (i) $\frac{40}{41} - \frac{9}{41}i$; $\frac{40}{41} + \frac{9}{41}i$; 1 (ii) $1 + i$; $1 - i$; $\sqrt{2}$
 (iii) $-1 + i$; $-1 - i$; $\sqrt{2}$ (iv) $-9 + 46i$; $-9 - 46i$; $\sqrt{2197}$
 37. (i) $0 + \frac{1}{2}i$ (iii) 1 (iv) 2^n 38. (i) $\frac{11}{5}$ (ii) $\frac{1}{5}$ (iii) 0
 39. (i) $x = \frac{5}{13}$, $y = \frac{14}{13}$ (ii) $x = 6$, $y = 1$ (iii) $x = \frac{2}{21}$, $y = -\frac{8}{21}$
 40. (i) $\frac{3}{2} - 2i$ is the only solution (ii) all purely imaginary numbers
 45. $-17 + 24i$ 48. 2
 49. $(2n + 1)\frac{\pi}{2}$, $n \in \mathbb{I}$ 50. Exception is $z = 1$ 51. 1

EXERCISE 5.2

1. (i) True (ii) True (iii) True (iv) True (v) True
 2. 0 3. $-\theta$
 4. (i) $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$ (ii) $1 - i$ (iii) $0 - 3i$ (iv) $\frac{5}{2} + \frac{5\sqrt{3}}{2}i$ 5. $-2\sqrt{3} + 2i$