CHAPTER 9.3

INTEGRAL CALCULUS

1. $\int \frac{x}{x^2+1}\,dx$ is equal to

(A) $\frac{1}{2}\log(x^2+1)$ (B) $\log(x^2+1)$

(C) $\tan^{-1}\frac{x}{2}$ (D) $2\tan^{-1}x$

2. If $F(a) = \frac{1}{\log a}$, $a > 1$ and $F(x) = \int a^2 dx + K$ is equal to

(A) $\frac{1}{\log a}(a^x - a^a + 1)$ (B) $\frac{1}{\log a}(a^x - a^a)$

(C) $\frac{1}{\log a}(a^x + a^a + 1)$ (D) $\frac{1}{\log a}(a^x + a^a - 1)$

3. $\int \frac{dx}{1+\sin x}$ is equal to

(A) $-\cot x + \operatorname{cosec} x + c$ (B) $\cot x + \operatorname{cosec} x + c$

(C) $\tan x - \sec x + c$ (D) $\tan x + \sec x + c$

4. $\int \frac{(3x+1)}{2x^2-2x+3}\,dx$ is equal to

(A) $\frac{3}{4}\log(2x^2-2x+3) + \frac{\sqrt{5}}{2}\tan^{-1}\left(\frac{2x-1}{\sqrt{5}}\right)$

(B) $\frac{4}{3}\log(2x^2-2x+3) + \sqrt{5}\tan^{-1}\left(\frac{2x-1}{\sqrt{5}}\right)$

(C) $\frac{4}{3}\log(2x^2-2x+3) + \frac{2}{\sqrt{5}}\tan^{-1}\left(\frac{2x-1}{\sqrt{5}}\right)$

(D) $\frac{3}{4}\log(2x^2-2x+3) + \frac{2}{\sqrt{5}}\tan^{-1}\left(\frac{2x-1}{\sqrt{5}}\right)$

5. $\int \frac{dx}{1+3\sin^2 x}$ is equal to

(A) $\frac{1}{2}\tan^{-1}(\tan x)$ (B) $2\tan^{-1}(\tan x)$

(C) $\frac{1}{2}\tan^{-1}(2\tan x)$ (D) $2\tan^{-1}\left(\frac{1}{2}\tan x\right)$

6. $\int \frac{2\sin x + 3\cos x}{3\sin x + 4\cos x}\,dx$ is equal to

(A) $\frac{9}{25}x + \frac{1}{25}\log(3\sin x + 4\cos x)$

(B) $\frac{18}{25}x + \frac{2}{25}\log(3\sin x + 4\cos x)$

(C) $\frac{18}{25}x + \frac{1}{25}\log(3\sin x + 4\cos x)$

(D) None of these

7. $\int \sqrt{3+8x-3x^2}\,dx$ is equal to

(A) $\frac{3x-4}{3\sqrt{3}}\sqrt{3+8x-3x^2} - \frac{25}{18\sqrt{3}}\sin^{-1}\left(\frac{3x-4}{5}\right)$

(B) $\frac{3x-4}{6}\sqrt{3+8x-3x^2} + \frac{25\sqrt{3}}{18}\sin^{-1}\left(\frac{3x-4}{5}\right)$

(C) $\frac{3x-4}{6\sqrt{3}}\sqrt{3+8x-3x^2} - \frac{25}{18\sqrt{3}}\sin^{-1}\left(\frac{3x-4}{5}\right)$

(D) None of these

8. $\int \frac{dx}{\sqrt{2x^2+3x+4}}$ is equal to

(A) $\frac{1}{\sqrt{2}}\sin^{-1}\frac{4x+3}{\sqrt{23}}$ (B) $\frac{1}{\sqrt{2}}\sinh^{-1}\frac{4x+3}{\sqrt{23}}$

(C) $\frac{1}{\sqrt{2}}\cosh^{-1}\frac{4x+3}{\sqrt{23}}$ (D) None of these

9. $\int \frac{2x+3}{\sqrt{x^2+x+1}}\, dx$ is equal to

(A) $2\sqrt{x^2+x+1}+2\sinh^{-1}\frac{2x+1}{\sqrt{3}}$

(B) $\sqrt{x^2+x+1}+2\sinh^{-1}\frac{2x+1}{\sqrt{3}}$

(C) $2\sqrt{x^2+x+1}+\sinh^{-1}\frac{2x+1}{\sqrt{3}}$

(D) $2\sqrt{x^2+x+1}-\sinh^{-1}\frac{2x+1}{\sqrt{3}}$

10. $\int \frac{dx}{\sqrt{x-x^2}}$ is equal to

(A) $\sqrt{x-x^2}+c$ (B) $\sin^{-1}(2x-1)+c$

(C) $\log(2x-1)+c$ (D) $\tan^{-1}(2x-1)+c$

11. $\int \frac{1}{(x+1)\sqrt{1-2x-x^2}}\, dx$ is equal to

(A) $\sqrt{2}\cosh^{-1}\left(\frac{\sqrt{2}}{1+x}\right)$ (B) $\frac{1}{\sqrt{2}}\cosh^{-1}\left(\frac{\sqrt{2}}{1+x}\right)$

(C) $-\sqrt{2}\cosh^{-1}\left(\frac{\sqrt{2}}{1+x}\right)$ (D) $-\frac{1}{\sqrt{2}}\cosh^{-1}\left(\frac{\sqrt{2}}{1+x}\right)$

12. $\int \frac{dx}{\sin x+\cos x}$ is equal to

(A) $\frac{1}{\sqrt{2}}\log\tan\left(x+\frac{\pi}{4}\right)$ (B) $\frac{1}{\sqrt{2}}\log\tan\left(\frac{x}{2}+\frac{\pi}{6}\right)$

(C) $\frac{1}{\sqrt{2}}\log\tan\left(\frac{x}{2}+\frac{\pi}{8}\right)$ (D) $\frac{1}{\sqrt{2}}\log\tan\left(\frac{x}{4}+\frac{\pi}{4}\right)$

13. $\int \frac{dx}{\sin(x-a)\sin(x-b)}$ is equal to

(A) $\sin(x-a)\log\sin(x-b)$

(B) $\log\sin\left(\frac{x-a}{x-b}\right)$

(C) $\sin(a-b)\log\left\{\frac{\sin(x-a)}{\sin(x-b)}\right\}$

(D) $\frac{1}{\sin(a-b)}\log\left\{\frac{\sin(x-a)}{\sin(x-b)}\right\}$

14. $\int \frac{dx}{e^x-1}$ is equal to

(A) $\log(e^x-1)$ (B) $\log(1-e^x)$

(C) $\log(e^{-x}-1)$ (D) $\log(1-e^x)$

15. $\int \frac{dx}{1+x+x^2+x^3}$ is equal to

(A) $\frac{1}{2}\left[\log\frac{(x+1)^2}{x^2+1}+\tan^{-1}x\right]$

(B) $\frac{1}{4}\left[\log\frac{(x+1)^2}{x^2+1}+2\tan^{-1}x\right]$

(C) $\frac{1}{2}\left[\log\frac{(x+1)^2}{x^2+1}-2\tan^{-1}x\right]$

(D) None of these

16. $\int \frac{\sin x}{1-\sin x}\, dx$ is equal to

(A) $-x+\sec x+\tan x+k$ (B) $-x+\sec x+\tan x$

(C) $-x+\sec x-\tan x$ (D) $-x-\sec x-\tan x$

17. $\int e^x\{f(x)+f'(x)\}dx$ is equal to

(A) $e^x f'(x)$ (B) $e^x f(x)$

(C) $e^x+f(x)$ (D) None of these

18. The value of $\int e^x\left(\frac{1+\sin x}{1+\cos x}\right)dx$ is

(A) $e^x\tan\frac{x}{2}+c$ (B) $e^x\cot\frac{x}{2}+c$

(C) $e^x\tan x+c$ (D) $e^x\cot x+c$

19. $\int \frac{x^3}{x^2+1}\, dx$ is equal to

(A) $x^2+\log(x^2+1)+c$

(B) $\log(x^2+1)-x^2+c$

(C) $\frac{1}{2}x^2-\frac{1}{2}\log(x^2+1)+c$

(D) $\frac{1}{2}x^2+\frac{1}{2}\log(x^2+1)+c$

20. $\int \sin^{-1}x\, dx$ is equal to

(A) $x\sin^{-1}x+\sqrt{1-x^2}+c$ (B) $x\sin^{-1}x-\sqrt{1-x^2}+c$

(C) $x\sin^{-1}x+\sqrt{1+x^2}+c$ (D) $x\sin^{-1}x-\sqrt{1-x^2}+c$

21. $\int \frac{\sin x+\cos x}{\sqrt{1+\sin 2x}}\, dx$ is equal to

(A) $\sin x$ (B) x

(C) $\cos x$ (D) $\tan x$

22. The value of $\int_0^1 |5x-3|\, dx$ is

(A) –1/2 (B) 13/10

(C) 1/2 (D) 23/10

39. $\int_0^1 \int_x^{\sqrt{x}} (x^2 + y^2)\, dy dx$ is equal to

(A) $\frac{7}{60}$ (B) $\frac{3}{35}$

(C) $\frac{4}{49}$ (D) None of these

40. The value of $\int_0^1 \int_0^{\sqrt{1+x^2}} dy\, dx$ is

(A) $\frac{\pi}{4} \log(\sqrt{2}+1)$ (B) $\frac{\pi}{4} \log(\sqrt{2}-1)$

(C) $\frac{\pi}{2} \log(\sqrt{2}+1)$ (D) None of these

41. If A is the region bounded by the parabolas $y^2 = 4x$ and $x^2 = 4y$, then $\iint_A y dx dy$ is equal to

(A) $\frac{48}{5}$ (B) $\frac{36}{5}$

(C) $\frac{32}{5}$ (D) None of these

42. The area of the region bounded by the curves $x^2 + y^2 = a^2$ and $x + y = a$ in the first quadrant is given by

(A) $\int_0^a \int_{a-x}^{\sqrt{a^2-x^2}} dx dy$ (B) $\int_0^a \int_0^{\sqrt{a^2-x^2}} dx dy$

(C) $\int_{a-x}^{\sqrt{a^2-y^2}} \int_0^a dx dy$ (D) None of these

43. The area bounded by the curves $y = 2\sqrt{x}$, $y = -x$, $x = 1$ and $x = 4$ is given by

(A) 25 (B) $\frac{33}{2}$

(C) $\frac{47}{4}$ (D) $\frac{101}{6}$

44. The area bounded by the curves $y^2 = 9x$, $x - y + 2 = 0$ is given by

(A) 1 (B) $\frac{1}{2}$

(C) $\frac{3}{2}$ (D) $\frac{5}{4}$

45. The area of the cardioid $r = a(1 + \cos\theta)$ is given by

(A) $2\int_{\theta=0}^{\pi} \int_{r=0}^{a(1+\cos\theta)} r dr d\theta$ (B) $2\int_0^{\pi} \int_{r=a}^{a(1+\cos\theta)} r dr d\theta$

(C) $2\int_0^{\pi/2} \int_{r=0}^{a(1+\cos\theta)} r dr d\theta$ (D) $2\int_0^{\pi/4} \int_{r=0}^{a(1+\cos\theta)} r dr d\theta$

46. The area bounded by the curve $r = \theta\cos\theta$ and the lines $\theta = 0$ and $\theta = \frac{\pi}{2}$ is given by

(A) $\frac{\pi}{4}\left(\frac{\pi^2}{16} - 1\right)$ (B) $\frac{\pi}{16}\left(\frac{\pi^2}{6} - 1\right)$

(C) $\frac{\pi}{16}\left(\frac{\pi^2}{16} - 1\right)$ (D) None of these

47. The area of the lemniscate $r^2 = a^2 \cos 2\theta$ is given by

(A) $4\int_0^{\pi/4} \int_0^{a\sqrt{\cos 2\theta}} r dr d\theta$ (B) $2\int_0^{\pi/2} \int_0^{a\sqrt{\cos 2\theta}} r dr d\theta$

(C) $4\int_0^{\pi/2} \int_0^{a\sqrt{\cos 2\theta}} r dr d\theta$ (D) $2\int_0^{\pi} \int_0^{a\sqrt{\cos 2\theta}} r dr d\theta$

48. The area of the region bounded by the curve $y(x^2 + 2) = 3x$ and $4y = x^2$ is given by

(A) $\int_0^1 \int_{y=0}^{x^2/4} dx dy$ (B) $\int_0^1 \int_{y=0}^{x^2/4} dy dx$

(C) $\int_0^2 \int_{y=x^2/4}^{3x/(x^2+2)} dy dx$ (D) $\int_{y=0}^1 \int_{y=x^2/4}^{3x/(x^2+2)} dx dy$

49. The volume of the cylinder $x^2 + y^2 = a^2$ bounded below by $z = 0$ and bounded above by $z = h$ is given by

(A) $\pi a h$ (B) $\pi a^2 h$

(C) $\frac{1}{3}\pi a^3 h$ (D) None of these

50. $\int_0^1 \int_0^1 \int_0^1 e^{x+y+z} dx dy dz$ is equal to

(A) $(e-1)^3$ (B) $\frac{3}{2}(e-1)$

(C) $(e-1)^2$ (D) None of these

51. $\int_{-1}^1 \int_0^z \int_{x-z}^{x+z} (x + y + z)\, dy\, dx\, dz$ is equal to

(A) 4 (B) –4

(C) 0 (D) None of these

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SOLUTIONS

1. (A) $\int \frac{x}{x^2+1}dx$

Put $x^2+1=t \Rightarrow 2x\,dx=dt$

$\int \frac{x}{x^2+1}dx = \int \frac{1}{2}\cdot\frac{1}{t}dt$

$= \frac{1}{2}\log t = \frac{1}{2}\log(x^2+1)$

2. (A) $F(x)=\int a^x dx + K = \frac{a^x}{\log a}+K$

$\Rightarrow \quad F(a)=\frac{a^a}{\log a}+K$

$K=\frac{1}{\log a}-\frac{a^a}{\log a}=\frac{1-a^a}{\log a}$

$F(x)=\frac{a^x}{\log a}+\frac{1-a^a}{\log a}=\frac{1}{\log a}[a^x-a^a+1]$

3. (C) $\int \frac{dx}{1+\sin x}$

$= \int \frac{dx}{\left(\sin^2\frac{x}{2}+\cos^2\frac{x}{2}\right)+2\sin\frac{x}{2}\cos\frac{x}{2}}$

$=\int \frac{dx}{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2} = \int \frac{\sec^2\frac{x}{2}}{\left(1+\tan\frac{x}{2}\right)^2}dx$

Put $1+\tan\frac{x}{2}=t$

$\Rightarrow \quad \sec^2\frac{x}{2}dx=2dt \quad \Rightarrow \quad \int\frac{2dt}{t^2}dt=-\frac{2}{t}+K$

$=\frac{-2}{1+\tan\frac{x}{2}}+K = \frac{-2\cos\frac{x}{2}}{\cos\frac{x}{2}+\sin\frac{x}{2}}+K$

$=\frac{-2\cos\frac{x}{2}}{\cos\frac{x}{2}+\sin\frac{x}{2}}\times\frac{\cos\frac{x}{2}-\sin\frac{x}{2}}{\cos\frac{x}{2}-\sin\frac{x}{2}}+K$

$=\frac{-2\cos^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}}{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}+K$

$=\frac{-(1+\cos x)+\sin x}{\cos x}+k=\tan x-\sec x-1+K$

$=\tan x-\sec x+c$

4. (A) Let $I=\int\frac{3x+1}{2x^2-2x+3}dx$

Let $3x+1=p(4x-2)+q \quad \Rightarrow \quad p=\frac{3}{4},\ q=\frac{5}{2}$

$I=\frac{3}{4}\int\frac{4x-2}{2x^2-2x+3}dx+\frac{5}{2}\int\frac{dx}{2x^2-2x+3}$

$=\frac{3}{4}\log(2x^2-2x+3)+\frac{5}{4}\int\frac{dx}{\left(x-\frac{1}{2}\right)^2+\left(\frac{\sqrt{5}}{2}\right)^2}$

$=\frac{3}{4}\log(2x^2-2x+3)+\frac{5}{4}\frac{1}{\left(\frac{\sqrt{5}}{2}\right)}\tan^{-1}\frac{x-\frac{1}{2}}{\frac{\sqrt{5}}{2}}$

5. (C) Let $I=\int\frac{dx}{1+3\sin^2 x}$

$=\int\frac{\operatorname{cosec}^2 x\,dx}{\operatorname{cosec}^2 x+3}=\int\frac{\operatorname{cosec}^2 x\,dx}{(1+\cot^2 x)+3}$

Put $\cot x=t \quad \Rightarrow \quad -\operatorname{cosec}^2 x\,dx=dt$

$I=\int\frac{-dt}{4+t^2}=\frac{1}{2}\cot^{-1}\frac{t}{2}=\frac{1}{2}\cot^{-1}\left(\frac{\cot x}{2}\right)$

$=\frac{1}{2}\tan^{-1}(2\tan x)$

6. (C) Let $I=\int\frac{2\sin x+3\cos x}{3\sin x+4\cos x}dx$

Let $(2\sin x+3\cos x)=p(3\cos x-4\sin x)$
$+q(3\sin x+4\cos x)$

$p=\frac{1}{25}\ ,\ q=\frac{18}{25}$

$I=\frac{1}{25}\int\frac{3\cos x-4\sin x}{3\sin x+4\cos x}dx+\frac{18}{25}\int\frac{3\sin x+4\cos x}{3\sin x+4\cos x}dx$

$=\frac{1}{25}\log(3\sin x+4\cos x)+\frac{18}{25}x$

7. (B) $\int\sqrt{3+8x-3x^2}dx=\sqrt{3}\int\sqrt{\left(\frac{5}{3}\right)^2-\left(x-\frac{4}{3}\right)^2}dx$

$=\sqrt{3}\,\frac{1}{2}\left\{\left(x-\frac{4}{3}\right)\sqrt{\left(\frac{5}{3}\right)^2-\left(x-\frac{4}{3}\right)^2}+\left(\frac{5}{3}\right)^2\sin^{-1}\left(\frac{x-\frac{4}{3}}{\frac{5}{3}}\right)\right\}$

$=\frac{3x-4}{6}\sqrt{3+8x-3x^2}+\frac{25\sqrt{3}}{18}\sin^{-1}\frac{3x-4}{5}$

8. (B) $\int\frac{dx}{\sqrt{2x^2+3x+4}}=\frac{1}{\sqrt{2}}\int\frac{dx}{\sqrt{\left(x+\frac{3}{4}\right)^2+\left(\frac{\sqrt{23}}{4}\right)^2}}$

$$= \frac{1}{\sqrt{2}} \sinh^{-1} \frac{x + \frac{3}{4}}{\left(\frac{\sqrt{23}}{4}\right)} = \frac{1}{\sqrt{2}} \sinh^{-1} \frac{4x+3}{\sqrt{23}}$$

9. (B) $\int \frac{2x+3}{\sqrt{x^2+x+1}} dx$

$$= \int \frac{2x+1}{\sqrt{x^2+x+1}} dx + \int \frac{2dx}{\sqrt{x^2+x+1}}$$

$$= \int \frac{2x+1}{\sqrt{x^2+x+1}} dx + 2\int \frac{dx}{\sqrt{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}}$$

$$= \frac{(x^2+x+1)^{1/2}}{\frac{1}{2}} + 2\sinh^{-1} \frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}$$

$$= 2\sqrt{x^2+x+1} + 2\sinh^{-1} \frac{2x+1}{\sqrt{3}}$$

10. (B) $\int \frac{dx}{\sqrt{x}\sqrt{1-x}} = I$

Put $x = \sin^2\theta \Rightarrow dx = 2\sin\theta\cos\theta\, d\theta$

$$I = \int \frac{2\sin\theta\cos\theta}{\sin\theta\sqrt{1-\sin^2\theta}} d\theta = \int \frac{2\sin\theta\cos\theta}{\sin\theta\cos\theta} d\theta$$

$$I = \int 2d\theta = 2\theta + c = 2\sin^{-1}\sqrt{x} + c$$

$$I = \sin^{-1}(2x-1) + c$$

11. (D) Let $I = \int \frac{1}{(x+1)\sqrt{1-2x-x^2}} dx$

Put $x + 1 = \frac{1}{t} \Rightarrow dx = -\frac{1}{t^2} dt$

$$I = \int \frac{-\frac{1}{t^2} dt}{\frac{1}{t}\sqrt{1 - 2\left(\frac{1}{t}-1\right) - \left(\frac{1}{t}-1\right)^2}} = -\int \frac{dt}{\sqrt{2t^2-1}}$$

$$= -\frac{1}{\sqrt{2}} \int \frac{dt}{\sqrt{t^2 - \left(\frac{1}{\sqrt{2}}\right)^2}} = -\frac{1}{\sqrt{2}} \cosh^{-1} \frac{t}{1/\sqrt{2}}$$

$$= -\frac{1}{\sqrt{2}} \cosh^{-1}\left(\frac{\sqrt{2}}{x+1}\right)$$

12. (C) $\int \frac{dx}{\sin x + \cos x}$

$$= \frac{1}{\sqrt{2}} \int \frac{dx}{\sin x \cos\frac{\pi}{4} + \cos x \sin\frac{\pi}{4}}$$

$$= \frac{1}{\sqrt{2}} \int \frac{dx}{\sin\left(x+\frac{\pi}{4}\right)} = \frac{1}{\sqrt{2}} \int \operatorname{cosec}\left(x+\frac{\pi}{4}\right) dx$$

$$= \frac{1}{\sqrt{2}} \left[-\log\cot\frac{1}{2}\left(x+\frac{\pi}{4}\right)\right] = \frac{1}{\sqrt{2}} \log\tan\left(\frac{x}{2}+\frac{\pi}{8}\right)$$

13. (D) $\int \frac{dx}{\sin(x-a)\sin(x-b)}$

$$= \frac{1}{\sin(a-b)} \int \frac{\sin(a-b)dx}{\sin(x-a)\sin(x-b)}$$

$$= \frac{1}{\sin(a-b)} \int \frac{\sin[(x-b)-(x-a)]}{\sin(x-a)\sin(x-b)} dx$$

$$= \frac{1}{\sin(a-b)} \times \int \frac{\sin(x-b)\cos(x-a) - \cos(x-b)\sin(x-a)}{\sin(x-a)\sin(x-b)} dx$$

$$= \frac{1}{\sin(a-b)} \int [\cot(x-a) - \cot(x-b)] dx$$

$$= \frac{1}{\sin(a-b)} [\log\sin(x-a) - \log\sin(x-b)]\, dx$$

$$= \frac{1}{\sin(a-b)} \log\left\{\frac{\sin(x-a)}{\sin(x-b)}\right\}$$

14. (D) Let $I = \int \frac{dx}{e^x - 1} = \int \frac{e^{-x}\,dx}{1-e^{-x}}$

Put $1 - e^{-x} = t \Rightarrow e^{-x}dx = dt$

$$I = \int \frac{dt}{t} = \log t = \log(1-e^{-x})$$

15. (B) Let $I = \int \frac{dx}{1+x+x^2+x^3}$

$$= \int \frac{dx}{(1+x)(1+x^2)}$$

Let $\frac{1}{(1+x)(1+x^2)} = \frac{A}{1+x} + \frac{Bx+C}{1+x^2}$

$1 = A(1+x^2) + (Bx+C)(1+x)$

Comparing the coefficients of x^2, x and constant terms,

$A + B = 0,\ B + C = 0,\ C + A = 1$

Solving these equations, we get

$A = \frac{1}{2},\ B = -\frac{1}{2},\ C = \frac{1}{2}$

$$I = \frac{1}{2}\int \frac{1}{1+x} dx - \frac{1}{2}\int \frac{x-1}{x^2+1} dx$$

$$= \frac{1}{2}\log(1+x) - \frac{1}{2}\log(x^2+1) + \frac{1}{2}\tan^{-1} x$$

$$= \frac{1}{4}\left[\log\frac{(x+1)^2}{x^2+1} + 2\tan^{-1} x\right]$$

16. (B) Let $I = \int \frac{\sin x}{1 - \sin x}\, dx$

$= \int \frac{1 - (1 - \sin x)}{1 - \sin x} dx$

$= \int \frac{1}{1 - \sin x}\, dx - \int dx = \int \frac{1 + \sin x}{1 - \sin^2 x}\, dx - x$

$= \int \frac{1 + \sin x}{\cos^2 x}\, dx - x = \int (\sec^2 x + \sec x \tan x) dx - x$

$= \tan x + \sec x - x$

17. (B) Let $I = \int e^x \{f(x) + f'(x)\} dx$

$= \int e^x f(x) dx + \int e^x f'(x) dx$

$= \{f(x) e^x - \int f'(x) e^x dx\} + \int e^x f'(x)\, dx = f(x) \cdot e^x$

18. (A) Let $I = \int e^x \left(\frac{1 + \sin x}{1 + \cos x} \right) dx$

$= \int e^x \left(\frac{1 + 2 \sin \frac{x}{2} \cos \frac{x}{2}}{2 \cos^2 \frac{x}{2}} \right) dx$

$= \frac{1}{2} \int e^x \sec^2 \frac{x}{2}\, dx + \int e^x \tan \frac{x}{2}\, dx$

$= \frac{1}{2} \left\{ e^x \cdot 2 \tan \frac{x}{2} - \int e^x \cdot 2 \tan \frac{x}{2}\, dx \right\} + \int e^x \tan \frac{x}{2}\, dx$

$= e^x \tan \frac{x}{2} + c$

19. (C) $I = \int \frac{x^3}{x^2 + 1}\, dx = \int \frac{x \cdot x^2}{x^2 + 1}\, dx$

$= \int \frac{x(x^2 + 1 - 1)}{x^2 + 1}\, dx = \int x dx - \int \frac{x}{x^2 + 1}\, dx$

$= \frac{1}{2} x^2 - \frac{1}{2} \log (x^2 + 1) + c$

20. (A) Let $I = \int \sin^{-1} x\, dx = \int \sin^{-1} x \cdot 1 \cdot dx$

$= \sin^{-1} x \cdot x - \int \frac{1}{\sqrt{1 - x^2}} \cdot x\, dx$

$= x \sin^{-1} x - \int \frac{x}{\sqrt{1 - x^2}}\, dx$

In second part put $1 - x^2 = t^2$

$x dx = - t dt = x \sin^{-1} x + \int dt$

$= x \sin^{-1} x + t = x \sin^{-1} x + \sqrt{1 - x^2} + c$

21. $\int \frac{\sin x + \cos x}{\sqrt{1 + \sin 2x}}\, dx$

$= \int \frac{\sin x + \cos x}{\sqrt{(\sin^2 x + \cos^2 x) + 2 \sin x \cos x}}\, dx$

$= \int \frac{\sin x + \cos x}{\sqrt{(\cos x + \cos x)^2}}\, dx$

$= \int \frac{\sin x + \cos x}{\sin x + \cos x}\, dx = \int dx = x$

22. (D) $\int_0^{3/5} |5x - 3|\, dx = -\int_0^{3/5} |5x - 3|\, dx + \int_{3/5}^1 |5x - 3|\, dx$

$= \left(-\frac{5}{2} x^2 + 3x \right)_0^{3/5} + \left(\frac{5x^2}{2} - 3x \right)_{3/5}^1$

$= \left(-\frac{9}{10} + \frac{9}{5} \right) + \left[\left(\frac{5}{2} - 3 \right) - \left(\frac{9}{10} - \frac{9}{5} \right) \right]$

$= \frac{9}{10} + \left(-\frac{1}{2} + \frac{9}{10} \right) = \frac{13}{10}$

23. (B) $\int_0^1 \frac{dx}{e^x + e^{-x}} = \int_0^1 \frac{e^x\, dx}{e^{2x} + 1}$

Put $e^x = t \quad \Rightarrow \quad e^x dx = dt = \int_1^e \frac{dt}{t^2 + 1} = [\tan^{-1} t]_1^e$

$= \tan^{-1} e - \tan^{-1} 1 = \tan^{-1} e - \frac{\pi}{4}$

24. (D) $\int_0^c x(1 - x)\, dx = \int_0^c (x - x^2) dx$

$= \left(\frac{1}{2} x^2 - \frac{1}{3} x^3 \right)_0^c = \frac{1}{6} c^2 (3 - 2c)$

$\int_0^c x(1 - x) dx = 0 \quad \Rightarrow \quad \frac{1}{6} c^2 (3 - 2c) = 0$

$\Rightarrow \quad c = \frac{3}{2}$

25. (D) Put $x^2 + x = t \quad \Rightarrow \quad (2x + 1) dx = dt$

$\int_0^1 \frac{2x + 1}{\sqrt{x + x^2}}\, dx = \int_0^2 \frac{dt}{\sqrt{t}} = 2(t^{1/2})_0^2 = 2\sqrt{2}$

26. (A) $\int_\pi^\pi x^4 \sin^5 x dx$

Since, $f(-x) = (-x)^4 \sin^5 (-x) = -x^4 \sin^5 x$

$f(x)$ is odd function thus

$\int_\pi^\pi x^4 \sin^5 x\, dx = 0$

27. (A) $\int_0^{\pi/2} \cos^2 x\, dx = \int_0^{\pi/2} \frac{1}{2} (\cos 2x + 1)\, dx$

$$=\frac{1}{2}\left(\frac{1}{2}\sin 2x+x\right)_0^{\pi/2}$$

$$=\frac{1}{2}\left[\frac{1}{2}(\sin\pi-\sin 0)+\left(\frac{\pi}{2}-0\right)\right]$$

$$=\frac{1}{2}\left[\frac{1}{2}(0-0)-0+\frac{\pi}{2}\right]=\frac{\pi}{4}$$

Aliter 1. $\int_0^{\pi/2}\cos^2 x\,dx=\frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\,\Gamma\left(\frac{4}{2}\right)}=\frac{\frac{1}{2}\pi}{2}=\frac{\pi}{4}$

Aliter 2. Use Walli's Rule $\int_0^{\pi/2}\cos^2 x=\frac{1}{2}\cdot\frac{\pi}{2}=\frac{\pi}{4}$

28. (B) Let $I=\int_0^a\sqrt{a^2-x^2}\,dx$

Put $x=a\sin\theta \Rightarrow dx=a\cos\theta\,d\theta$ when $x=0,\ \theta=0,$ when $x=a,\ \theta=\frac{\pi}{2}$

$$I=\int_0^{\pi/2}\sqrt{a^2-a^2\sin^2\theta}\;a\cos\theta\,d\theta$$

$$=a^2\int_0^{\pi/2}\cos^2\theta\,d\theta=a^2\cdot\frac{1}{2}\cdot\frac{\pi}{2}\quad\text{(By Walli's Formula)}$$

$$=\frac{\pi a^2}{4}$$

Aliter: $\int_0^a\sqrt{a^2-x^2}dx$

$$=\left[\frac{1}{2}x\sqrt{a^2-x^2}+\frac{1}{2}a^2\sin^{-1}\frac{x}{a}\right]_0^a=\left[0+\frac{\pi a^2}{4}\right]=\frac{\pi a^2}{4}$$

29. (D) Let $I=\int_0^{\pi/2}\log(\tan x)\,dx$ ....(1)

$$I=\int_0^{\pi/2}\log\tan\left(\frac{\pi}{2}-x\right)dx$$

$$I=\int_0^{\pi/2}\log(\cot x)....(2)$$

Adding (1) and (2), we get

$$2I=\int_0^{\pi/2}[\log(\tan x)+\log(\cot x)]dx$$

$$=\int_0^{\pi/2}\log(\tan x\cdot\cot x)\,dx$$

$$=\int_0^{\pi/2}\log 1\,dx=0\Rightarrow I=0$$

30. (D) Let $I=\int_0^1 2\sin\left(\frac{\pi t}{2}-\frac{\pi}{4}\right)dt$ ....(i)

$$=\int_0^1 2\sin\left(\frac{\pi}{2}(1-t)-\frac{\pi}{4}\right)dt=\int_0^1 2\sin\left(\frac{\pi}{4}-\frac{\pi}{2}t\right)dt$$

$$=-\int_0^1 2\sin\left(\frac{\pi}{2}t-\frac{\pi}{4}\right)dt=-1$$

$$2I=0\Rightarrow I=0$$

31. (C) Let $I=\int_0^{2a}\frac{f(x)}{f(x)+f(2a-x)}dx$ ....(1)

$$I=\int_0^{2a}\frac{f(2a-x)}{f(2a-x)+f(x)}dx....(2)$$

Adding (1) and (2), we get

$$2I=\int_0^{2a}\frac{f(x)+f(2a-x)}{f(x)+f(2a-x)}dx=\int_0^{2a}1\cdot dx=[x]_0^{2a}=2a$$

$$\Rightarrow I=a$$

32. (C) Let $I=\int_0^1\frac{e^{\sqrt{1-x^2}}}{\sqrt{1-x^2}}x\,dx$

Put $\sqrt{1-x^2}=t$

$$\Rightarrow\frac{1}{2\sqrt{1-x^2}}(-2x)\,dx=dt$$

when $x=0,\ t=1,$ when $x=1,\ t=0$

$$I=\int_1^0-e^t dt=-[e^t]_1^0=-[e^0-e^1]=e-1$$

33. (B) Let $I=\int_0^1\frac{dx}{1-x+x^2}$

$$=\int_0^1\frac{dx}{\left(x-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}=\frac{1}{\frac{\sqrt{3}}{2}}\left[\tan^{-1}\frac{x-\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right]_0^1$$

$$=\frac{2}{\sqrt{3}}\left[\tan^{-1}\frac{1}{\sqrt{3}}-\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)\right]=\frac{2}{\sqrt{3}}\left(\frac{\pi}{6}+\frac{\pi}{6}\right)$$

$$=\frac{2\pi}{3\sqrt{3}}=\frac{2\pi\sqrt{3}}{9}$$

34. (B) Let $I=\int_{-1}^1\frac{|x|}{x}dx=\int_{-1}^0\frac{-x}{x}dx+\int_0^1\frac{x}{x}dx$

$$=\int_{-1}^0-1dx+\int_0^1 1\cdot dx=-[x]_{-1}^0+[x]_0^1$$

$$=-[0-(-1)]+[1-0]=0$$

35. (C) $\int_0^{100\pi}|\sin x|dx=100\int_0^{\pi}|\sin x|dx$

[... $\sin x$ is periodic with period π]

$= 100\int_0^{\pi} \sin x\, dx = 100(-\cos x)_0^{\pi}$

$= 100(-\cos \pi + \cos 0) = 100(1+1) = 200.$

36. (C) Let $I = \int_0^{\pi} \cos^m x \sin nx\, dx = \int_0^{\pi} f(x)\, dx$

Where $f(x) = \cos^m x \sin^n x$

$f(\pi - x) = \cos^m(\pi - x)\sin^n(\pi - x)$

$= (-\cos x)^m (\sin x)^n$

$= -\cos^m x \sin^n x$, if m is odd

$I = \int_0^{\pi} \cos^m x \sin^n x\, dx = 0$, if m is odd

37. (A) Let $I = \int_0^{\pi} xF(\sin x)\, dx$ ....(1)

$= \int_0^{\pi} (x - \pi)F[\sin(\pi - x)]dx$

$I = \int_0^{\pi} (\pi - x)F(\sin x)dx$ ....(2)

Adding (1) and (2), we get

$2I = \int_0^{\pi} \pi F(\sin x)dx$

$\Rightarrow \quad I = \frac{1}{2}\int_0^{\pi} \pi F(\sin x)\, dx$

38. (B) Let $I = \int_0^{\pi/2} \frac{e^x}{2}\left(\sec^2 \frac{x}{2} + 2\tan\frac{x}{2}\right)dx$

$= \int_0^{\pi/2} \frac{1}{2} e^x \sec^2\frac{x}{2}\, dx + \int_0^{\pi/2} e^x \tan\frac{x}{2}\, dx = I_1 + I_2$

Here, $I_1 = \int_0^{\pi/2} \frac{1}{2} e^x \sec^2 \frac{x}{2}\, dx$

$= \left[\frac{1}{2} e^x \cdot 2\tan\frac{x}{2}\right]_0^{\pi/2} - \frac{1}{2}\int_0^{\pi/2} e^x \cdot 2\tan\frac{x}{2}\, dx$

$= \left(e^{\pi/2}\tan\frac{\pi}{4} - 0\right) - \int_0^{\pi/2} e^x \tan\frac{x}{2}\, dx$

$= e^{\pi/2} - I_2$, $I_1 + I_2 = e^{\pi/2}$

$I = I_1 + I_2 = e^{\pi/2}$

39. (B) $\int_0^1 \int_x^{\sqrt{x}} (x^2 + y^2)\, dy\, dx = \int_0^1 \left[x^2 y + \frac{1}{3}y^3\right]_x^{\sqrt{x}} dx$

$= \int_0^1 \left[x^{5/2} + \frac{1}{3}x^{3/2} - x^3 - \frac{1}{3}x^3\right]dx$

$= \left[\frac{2}{7}x^{7/2} + \frac{2}{15}x^{5/2} - \frac{1}{3}x^4\right]_0^1 = \frac{3}{35}$

40. (D) $\int_0^1 \int_0^{\sqrt{1+x^2}} dydx = \int_0^1 [y]_0^{\sqrt{1+x^2}}\, dx$

$= \int_0^1 \sqrt{1+x^2}\, dx$

$= \frac{1}{2}[x\sqrt{1+x^2} + \log(x + \sqrt{1+x^2})]_0^1$

$= \frac{1}{2}[\sqrt{2} + \log(1 + \sqrt{2})]$

41. (A) Let $I = \iint_A ydxdy,$

Solving the given equations $y^2 = 4x$ and $x^2 = 4y$, we get $x = 0,\ x = 4$. The region of integration A is given by

$A = \int_0^4 \int_{x^2/4}^{2\sqrt{x}} ydydx = \int_0^4 \left[\frac{y^2}{2}\right]_{x^2/4}^{2\sqrt{x}} dx$

$= \int_0^4 \frac{1}{2}\left(4x - \frac{x^4}{10}\right)dx = \left[x^2 - \frac{x^5}{160}\right]_0^4 = \frac{48}{5}$

42. (A) The curves are

$x^2 + y^2 = a^2$... ....(i)

$x + y = a$... ....(ii)

The curves (i) and (ii) intersect at A (a, 0) and B (0,a)

The required area $A = \int_{x=0}^{a} \int_{y=a-x}^{\sqrt{a^2 - x^2}} dydx$

43. (D) The given equations of the curves are

$y = 2\sqrt{x}$ i.e., $y^2 = 4x$....(i) $\quad y = -x$....(ii)

If a figure is drawn then from fig. the required area is

$A = \int_1^4 \int_{-x}^{2\sqrt{x}} dydx = \int_1^4 [y]_{-x}^{2\sqrt{x}} = \int_1^4 [2\sqrt{x} + x\,]dx$

$= \left(\frac{32}{3} + 8\right) - \left(\frac{4}{3} + \frac{1}{2}\right) = \frac{101}{6}$

44. (B) The equations of the given curves are

$y^2 = 9x$....(i) $\quad x - y + 2 = 0$....(ii)

The curves (i) and (ii) intersect at

A(1, 3) and B(4, 6)

If a figure is drawn then from fig. the required area is

$A = \int_1^4 \int_{x+2}^{3\sqrt{x}} dydx = \int_1^4 [y]_{x+2}^{3\sqrt{x}}\, dx$

$= \int_1^4 [3\sqrt{x} - (x+2)]dx = \left[2x^{3/2} - \frac{1}{2}x^2 - 2x\right]_1^4$

$= (16 - 8 - 8) - \left(2 - \frac{1}{2} - 2\right) = \frac{1}{2}$

45. (A) The equation of the cardioid is

$r = a(1+\cos\theta)$ ....(i)

If a figure is drawn then from fig. the required area is

Required area $A = 2\int_{\theta=0}^{\pi}\int_{r=0}^{a(1+\cos\theta)} r\,dr\,d\theta$

46. (C) The equation of the given curve is

$r = \theta\cos\theta$....(i)

The required area

$$A = \int_{\theta=0}^{\pi/2}\int_{r=0}^{\theta\cos\theta} r\,dr\,d\theta = \int_0^{\pi/2}\left[\frac{1}{2}r^2\right]_o^{\theta\cos\theta} d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2}\theta^2\cos^2\theta d\theta = \frac{1}{4}\int_0^{\pi/2}\theta^2(1+\cos 2\theta)d\theta$$

$$= \frac{1}{4}\int_0^{\pi/2}\theta^2 d\theta + \frac{1}{4}\int_0^{\pi/2}\theta^2\cos 2\theta\, d\theta$$

$$= \frac{1}{4}\left[\frac{1}{3}\theta^3\right]_0^{\pi/2} + \frac{1}{4}\left[\left(\theta^2\frac{\sin 2\theta}{2}\right)_0^{\pi/2} - \int_0^{\pi/2} 2\theta\frac{\sin 2\theta}{2}d\theta\right]$$

$$= \frac{\pi^3}{96} + \frac{1}{4}\left[-\int_0^{\pi/2}\theta\sin 2\theta\, d\theta\right]$$

$$= \frac{\pi^3}{96} - \frac{1}{4}\left[\left(-\theta\frac{\cos 2\theta}{2}\right)_0^{\pi/2} - \int_0^{\pi/2}\left(-\frac{\cos 2\theta}{2}\right)d\theta\right]$$

$$= \frac{\pi^3}{96} + \frac{1}{4}\left(\frac{-\pi}{4} - 0\right) - \frac{1}{8}\int_0^{\pi/2}\cos 2\theta\, d\theta$$

$$= \frac{\pi^3}{96} - \frac{\pi}{16} - \frac{1}{8}\left(\frac{1}{2}\sin 2\theta\right)_0^{\pi/2} = \frac{\pi}{16}\left(\frac{\pi^2}{16} - 1\right)$$

47. (A) The curve is $r^2 = a^2\cos 2\theta$

If a figure is drawn then from fig. the required area is

$$A = 4\int_{\theta=0}^{\pi/4}\int_{r=0}^{a\sqrt{\cos 2\theta}} r\,dr\,d\theta = 4\int_0^{\pi/4}\left[\frac{1}{2}r^2\right]_0^{a\sqrt{\cos 2\theta}} d\theta$$

$$= 2\int_0^{\pi/4} a^2\cos 2\theta\, d\theta = 2a^2\left[\frac{\sin 2\theta}{2}\right]_0^{\pi/4} = a^2$$

48. (C) The equations of given curves are

$y(x^2+2) = 3x$....(i) and $4y = x^2$....(ii)

The curve (i) and (ii) intersect at A (2, 1).

If a figure is drawn then from fig. the required area is

The required area $A = \int_{x=0}^{2}\int_{y=x^2/4}^{3x/(x^2+2)} dx\,dy$

49. (B) The equation of the cylinder is $x^2 + y^2 = a^2$

The equation of surface CDE is $z = h$.

If a figure is drawn then from fig. the required area is

Thus the equation volume is $V = 4\int_A z\,dx\,dy$

$$= 4\int_0^a\int_0^{\sqrt{a^2-x^2}} h\,dy\,dx = 4h\int_0^a [y]_0^{\sqrt{a^2-x^2}} dx = 4h\int_0^a \sqrt{a^2-x^2}\,dx$$

Let $x = a\sin\theta$, $\Rightarrow$ $dx = a\cos\theta\, d\theta$,

Volume $V = 4h\int_0^{\pi/2}\sqrt{a^2 - a^2\sin^2\theta}\cdot a\cos\theta\, d\theta$

$$= 4ha^2\int_0^{\pi/2}\cos^2\theta\, d\theta = 4ha^2\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \pi a^2 h.$$

50. (A) $\int_0^1\int_0^1\int_0^1 e^{x+y+z}\,dx\,dy\,dz$

$$= \int_0^1\int_0^1 [e^{x+y+z}]_0^1\, dy\,dz = \int_0^1\int_0^1 [e^{1+y+z} - e^{y+z}]dy\,dz$$

$$= \int_0^1 [e^{1+y+z} - e^{y+z}]_0^1\, dz$$

$$= \int_0^1 [(e^{2+z} - e^{1+z}) - (e^{1+z} - e^z)]dz$$

$$= \int_0^1 (e^{2+z} - 2e^{1+z} + e^z)dz = [e^{2+z} - 2e^{1+z} + e^z]_0^1$$

$$= (e^3 - 2e^2 + e) - (e^2 - 2e + 1)$$

$$= e^3 - 3e^2 + 3e - 1 = (e-1)^3$$

51. (C) $\int_{-1}^{1}\int_0^z\int_{x-z}^{x+z}(x+y+z)\,dy\,dx\,dz$

$$= \int_{-1}^{1}\int_0^z\left[\frac{(x+y+z)^2}{2}\right]_{x-y}^{x+z} dx\,dz$$

$$= \int_{-1}^{1}\int_0^z\left[\frac{(2x+2z)^2}{2} - \left(\frac{2x}{2}\right)^2\right]dx\,dz$$

$$= 2\int_{-1}^{1}\left[\int_0^3((x+z)^2 - x^2)dx\right]dz = 2\int_{-1}^{1}\left[\frac{(x+z)^3}{3} - \frac{x^3}{3}\right]_0^z dz$$

$$= \frac{2}{3}\int_{-1}^{1}[(2z)^3 - z^3 - z]dz = \frac{2}{3}\int_{-1}^{1}6z^3 dz = 4\left[\frac{z^4}{4}\right]_{-1}^{1}$$

$$= 4\left(\frac{1}{4} - \frac{1}{4}\right) = 0$$

****

10. The integration of $f(z) = x^2 + ixy$ from A(1, 1) to B(2, 4) along the straight line AB joining the two points is

(A) $\frac{-29}{3} + i11$ (B) $\frac{29}{3} - i11$

(C) $\frac{23}{5} + i6$ (D) $\frac{23}{5} - i6$

11. $\int_c \frac{e^{2z}}{(z+1)^4} dz = ?$ where c is the circle of $|z| = 3$

(A) $\frac{4\pi i}{9} e^{-3}$ (B) $\frac{4\pi i}{9} e^{3}$

(C) $\frac{4\pi i}{3} e^{-1}$ (D) $\frac{8\pi i}{3} e^{-2}$

12. $\int_c \frac{1-2z}{z(z-1)(z-2)} dz = ?$ where c is the circle $|z| = 1.5$

(A) $2 + i6\pi$ (B) $4 + i3\pi$

(C) $1 + i\pi$ (D) $i3\pi$

13. $\int_c (z - z^2) dz = ?$ where c is the upper half of the circle $z = 1$

(A) $\frac{-2}{3}$ (B) $\frac{2}{3}$

(C) $\frac{3}{2}$ (D) $\frac{-3}{2}$

14. $\int_c \frac{\cos \pi z}{z-1} dz = ?$ where c is the circle $|z| = 3$

(A) $i2\pi$ (B) $-i2\pi$

(C) $i6\pi^2$ (D) $-i6\pi^2$

15. $\int_c \frac{\sin \pi z^2}{(z-2)(z-1)} dz = ?$ where c is the circle $|z| = 3$

(A) $i6\pi$ (B) $i2\pi$

(C) $i4\pi$ (D) 0

16. The value of $\frac{1}{2\pi i} \int_c \frac{\cos \pi z}{z^2 - 1} dz$ around a rectangle with vertices at $2 \pm i$, $-2 \pm i$ is

(A) 6 (B) $i2e$

(C) 8 (D) 0

Statement for Q. 17–18:

$f(z_0) = \int_c \frac{3z^2 + 7z + 1}{(z - z_0)} dz$, where c is the circle $x^2 + y^2 = 4$.

17. The value of $f(3)$ is

(A) 6 (B) $4i$

(C) $-4i$ (D) 0

18. The value of $f'(1 - i)$ is

(A) $7(\pi + i2)$ (B) $6(2 + i\pi)$

(C) $2\pi(5 + i13)$ (D) 0

Statement for 19–21:

Expand the given function in Taylor's series.

19. $f(z) = \frac{z-1}{z+1}$ about the points $z = 0$

(A) $1 + 2(z + z^2 + z^3......)$ (B) $-1 - 2(z - z^2 + z^3......)$

(C) $-1 + 2(z - z^2 + z^3......)$ (D) None of the above

20. $f(z) = \frac{1}{z+1}$ about $z = 1$

(A) $\frac{-1}{2}\left[1 - \frac{1}{2}(z-1) + \frac{1}{2^2}(z-1)^2.......\right]$

(B) $\frac{1}{2}\left[1 - \frac{1}{2}(z-1) + \frac{1}{2^2}(z-1)^2.......\right]$

(C) $\frac{1}{2}\left[1 + \frac{1}{2}(z-1) + \frac{1}{2^2}(z-1)^2.......\right]$

(D) None of the above

21. $f(z) = \sin z$ about $z = \frac{\pi}{4}$

(A) $\frac{1}{\sqrt{2}}\left[1 + \left(z - \frac{\pi}{4}\right) - \frac{1}{2!}\left(z - \frac{\pi}{4}\right)^2 - ....\right]$

(B) $\frac{1}{\sqrt{2}}\left[1 + \left(z - \frac{\pi}{4}\right) + \frac{1}{2!}\left(z - \frac{\pi}{4}\right)^2 + ...\right]$

(C) $\frac{1}{\sqrt{2}}\left[1 - \left(z - \frac{\pi}{4}\right) - \frac{1}{2!}\left(z - \frac{\pi}{4}\right)^2 - ....\right]$

(D) None of the above

22. If $|z + 1| < 1$, then z^{-2} is equal to

(A) $1 + \sum_{n=1}^{\infty} (n+1)(z+1)^{n-1}$

(B) $1 + \sum_{n=1}^{\infty} (n+1)(z+1)^{n+1}$

(C) $1 + \sum_{n=1}^{\infty} n(z+1)^{n}$

(D) $1 + \sum_{n=1}^{\infty} (n+1)(z+1)^{n}$

Statement for Q. 23–25.

Expand the function $\frac{1}{(z-1)(z-2)}$ in Laurent's series for the condition given in question.

23. $1<|z|<2$

(A) $\frac{1}{z}+\frac{2}{z^2}+\frac{3}{z^3}+.......$

(B) $...-z^{-3}-z^{-2}-z^{-1}-\frac{1}{2}-\frac{1}{4}z-\frac{1}{8}z^2-\frac{1}{18}z^3-...$

(C) $\frac{1}{z^2}+\frac{3}{z^2}+\frac{7}{z^4}...........$

(D) None of the above

24. $|z|>2$

(A) $\frac{6}{z}+\frac{13}{z^2}+\frac{20}{z^3}+........$

(B) $\frac{1}{z}+\frac{8}{z^2}+\frac{13}{z^3}+.........$

(C) $\frac{1}{z^2}+\frac{3}{z^3}+\frac{7}{z^4}+.........$

(D) $\frac{2}{z^2}-\frac{3}{z^3}+\frac{4}{z^4}-........$

25. $|z|<1$

(A) $1+3z\frac{+7}{2}z^2+\frac{15}{4}z^2.....$

(B) $\frac{1}{2}+\frac{3}{4}z+\frac{7}{8}z^2+\frac{15}{16}z^3...$

(C) $\frac{1}{4}+\frac{3}{4}+\frac{z^2}{8}+\frac{z^3}{16}.......$

(D) None of the above

26. If $|z-1|<1$, the Laurent's series for $\frac{1}{z(z-1)(z-2)}$ is

(A) $-(z-1)-\frac{(z-1)^3}{2!}-\frac{(z-1)^5}{5!}-...........$

(B) $-(z-1)^{-1}-\frac{(z-1)^3}{2!}-\frac{(z-1)^5}{5!}-.........$

(C) $-(z-1)-(z-1)^3-(z-1)^5-..........$

(D) $-(z-1)^{-1}-(z-1)-(z-1)^3-(z-1)^5-.........$

27. The Laurent's series of $\frac{1}{z(e^z-1)}$ for $|z|<2$ is

(A) $\frac{1}{z^2}+\frac{1}{2z}+\frac{1}{12}+6z+\frac{1}{720}z^2+..........$

(B) $\frac{1}{z^2}-\frac{1}{2z}+\frac{1}{12}-\frac{1}{720}z^2+..........$

(C) $\frac{1}{z}+\frac{1}{12}+\frac{1}{634}z^2+\frac{1}{720}z^2+..........$

(D) None of the above

28. The Laurent's series of $f(z)=\frac{z}{(z^2+1)(z^2+4)}$ is, where $|z|<1$

(A) $\frac{1}{4}z-\frac{5}{16}z^3+\frac{21}{64}z^5..........$

(B) $\frac{1}{2}+\frac{1}{4}z^2+\frac{5}{16}z^4+\frac{21}{64}z^6..........$

(C) $\frac{1}{2}z-\frac{3}{4}z^3+\frac{15}{8}z^5..........$

(D) $\frac{1}{2}+\frac{1}{2}z^2+\frac{3}{4}z^4+\frac{15}{8}z^6..........$

29. The residue of the function $\frac{1-e^{Zz}}{z^4}$ at its pole is

(A) $\frac{4}{3}$

(B) $\frac{-4}{3}$

(C) $\frac{-2}{3}$

(D) $\frac{2}{3}$

30. The residue of $z\cos\frac{1}{z}$ at $z=0$ is

(A) $\frac{1}{2}$

(B) $\frac{-1}{2}$

(C) $\frac{1}{3}$

(D) $\frac{-1}{3}$

31. $\int_c \frac{1-2z}{z(1-z)(z-2)}dz=?$ where c is $|z|=1.5$

(A) $-i3\pi$

(B) $i3\pi$

(C) 2

(D) -2

32. $\int_c \frac{z\ \cos z}{\left(z-\frac{\pi}{2}\right)}dz=?$ where c is $|z-1|=1$

(A) 6π

(B) -6π

(C) $i2\pi$

(D) None of the above

33. $\int_c z^2 e^{\frac{1}{z}}dz=?$ where c is $|z|=1$

(A) $i3\pi$

(B) $-i3\pi$

C) $\frac{i\pi}{3}$

(D) None of the above

34. $\int_0^{2\pi}\frac{d\theta}{2+\cos\theta}=?$

(A) $\frac{-2\pi}{\sqrt{2}}$

(B) $\frac{2\pi}{\sqrt{3}}$

(C) $2\pi\sqrt{2}$

(D) $-2\pi\sqrt{3}$

35. $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)(x^2+b^2)} dx = ?$

(A) $\frac{\pi ab}{a+b}$ (B) $\frac{\pi(a+b)}{ab}$

(C) $\frac{\pi}{a+b}$ (D) $\pi(a+b)$

36. $\int_{0}^{\infty} \frac{dx}{1+x^6} = ?$

(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{2}$

(C) $\frac{2\pi}{3}$ (D) $\frac{\pi}{3}$

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SOLUTIONS

1. (C) Since, $f(z) = u + iv = \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2}$; $z \neq 0$

$\Rightarrow \quad u = \frac{x^3 - y^3}{x^2+y^2}$; $\quad v = \frac{x^3+y^3}{x^2+y^2}$

Cauchy Riemann equations are

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

By differentiation the value of $\frac{\partial u}{\partial x}, \frac{\partial y}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ at$(0,0)$

we get $\frac{0}{0}$, so we apply first principle method.

At the origin,

$$\frac{\partial u}{\partial x} = \lim_{h\to 0} \frac{u(0+h,\ 0) - u(0,0)}{h} = \lim_{h\to 0} \frac{h^3/h^2}{h} = 1$$

$$\frac{\partial u}{\partial v} = \lim_{h\to 0} \frac{u(0, 0+k) - u(0,0)}{k} = \lim_{k\to 0} \frac{-k^3/k^2}{k} = -1$$

$$\frac{\partial v}{\partial x} = \lim_{h\to 0} \frac{v(0+h, 0) - v(0,0)}{h} = \lim_{h\to 0} \frac{h^3/h^2}{h} = 1$$

$$\frac{\partial v}{\partial y} = \lim_{k\to 0} \frac{v(0,\ 0+k), v(0,0)}{k} = \lim_{k\to 0} \frac{k^3/k^2}{k} = 1$$

Thus, we see that $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

Hence, Cauchy-Riemann equations are satisfied at $z = 0$.

Again, $f'(0) = \lim_{z\to 0} \frac{f(z) - f(0)}{z}$

$$= \lim_{z\to 0} \left[\frac{(x^3 - y^3) + i(x^3+y^3)}{(x^2+y^2)} \frac{1}{(x+iy)} \right]$$

Now let $z \to 0$ along $y = x$, then

$$f'(0) = \lim_{z\to 0} \left[\frac{(x^3-y^3) + i(x^3+y^3)}{(x^2+y^2)} \frac{1}{(x+iy)} \right] = \frac{2i}{2(1+i)} = \frac{1+i}{2}$$

Again let $z \to 0$ along $y = 0$, then

$$f'(0) = \lim_{x\to 0} \left[\frac{x^3 + i(x^3)}{(x^2)} \frac{1}{x} \right] = 1 + i$$

So we see that $f'(0)$ is not unique. Hence $f'(0)$ does not exist.

2. (A) Since, $f'(z) = \frac{df}{dz} = \lim_{\Delta z\to 0} \frac{\Delta f}{\Delta z}$

or $f'(z) = \lim_{\Delta z\to 0} \frac{\Delta u + i\Delta v}{\Delta x + i\Delta y}$ ....(1)

Now, the derivative $f'(z)$ exits of the limit in equation (1) is unique i.e. it does not depends on the path along which $\Delta z \to 0$.

Let $\Delta z \to 0$ along a path parallel to real axis

$\Rightarrow \quad \Delta y = 0 \therefore \Delta z \to 0 \quad \Rightarrow \quad \Delta x \to 0$

Now equation (1)

$$f'(z) = \lim_{\Delta x \to 0} \frac{\Delta u + i\Delta v}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} + i \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x}$$

$$f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \qquad ....(2)$$

Again, let $\Delta z \to 0$ along a path parallel to imaginary axis, then $\Delta x \to 0$ and $\Delta z \to 0 \to \Delta y \to 0$

Thus from equation (1)

$$\phi'(z) = \lim_{\Delta y \to 0} \frac{\Delta z + i\Delta v}{i\Delta y} = \lim_{\Delta y \to 0} \frac{\Delta u}{i\Delta y} + i \lim_{\Delta y \to 0} \frac{\Delta v}{i\Delta z} = \frac{\partial u}{i\partial y} + \frac{\partial v}{\partial y}$$

$$f'(z) = \frac{-i\partial u}{\partial y} + \frac{\partial v}{\partial y} \qquad ....(3)$$

Now, for existence of $f'(z)$ R.H.S. of equation (2) and (3) must be same i.e.,

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial v}{\partial x} = \frac{-\partial u}{\partial y}$$

$$f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}$$

3. (A) Given $f(z) = x^2 + iy^2$ since, $f(z) = u + iv$

Here $u = x^2$ and $v = y^2$

Now, $u = x^2 \quad \Rightarrow \quad \frac{\partial u}{\partial x} = 2x$ and $\frac{\partial u}{\partial y} = 0$

and $v = y^2 \quad \Rightarrow \quad \frac{\partial v}{\partial x} = 0$ and $\frac{\partial v}{\partial y} = 2y$

we know that $\quad f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \qquad ....(1)$

and $f'(z) = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x} ....(2)$

Now, equation (1) gives $f'(z) = 2x \qquad ....(3)$

and equation (2) gives $f'(z) = 2y \qquad ....(4)$

Now, for existence of $f'(z)$ at any point is necessary that the value of $f'(z)$ most be unique at that point, whatever be the path of reaching at that point

From equation (3) and (4) $2x = 2y$

Hence, $f'(z)$ exists for all points lie on the line $x = y$.

4. (B) $\frac{\partial u}{\partial x} = 2(1-y)$; $\frac{\partial^2 u}{\partial x^2} = 0 \qquad ....(1)$

$\frac{\partial u}{\partial y} = -2x$; $\frac{\partial^2 u}{\partial y^2} = 0 \qquad ....(2)$

$\Rightarrow \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, Thus u is harmonic.

Now let v be the conjugate of u then

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$$

(by Cauchy-Riemann equation)

$\Rightarrow \quad dv = 2x\,dx + 2(1-y)dy$

On integrating $v = x^2 - y^2 + 2y + C$

5. (C) Given $f(z) = u + i\,v \qquad ....(1)$

$\Rightarrow \quad if(z) = -v + iu \qquad ....(2)$

add equation (1) and (2)

$\Rightarrow \quad (1+i)f(z) = (u-v) + i(u+v)$

$\Rightarrow \quad F(z) = U + iV$

where, $F(z) = (1+i)f(z)$; $U = (u-v)$; $V = u+v$

Let $F(z)$ be an analytic function.

Now, $U = u - v = e^x(\cos y - \sin y)$

$\frac{\partial U}{\partial x} = e^x(\cos y - \sin y) \quad$ and $\quad \frac{\partial U}{\partial y} = e^x(-\sin y - \cos y)$

Now, $dV = \frac{-\partial U}{\partial y}dx + \frac{\partial U}{\partial x}dy ....(3)$

$= e^x(\sin y + \cos y)dx + e^x(\cos y - \sin y)dy$

$= d[e^x(\sin y + \cos y)]$

on integrating $V = e^x(\sin y + \cos y) + c_1$

$F(z) = U + iV = e^x(\cos y - \sin y) + ie^x(\sin y + \cos y) + ic_1$

$= e^x(\cos y + i\sin y) + ie^x(\cos y + i\sin y) + ic_1$

$F(z) = (1+i)e^{x+iy} + ic_1 = (1+i)e^z + ic_1$

$(1+i)f(z) = (1+i)e^z + ic_1$

$$\Rightarrow \quad f(z) = e^z + \frac{i}{1+i}c_1 = e^z + c_1\frac{i(1-i)}{(1+i)(1-i)} = e^z + \frac{(i+1)}{2}c_1$$

$\Rightarrow \quad f(z) = e^z + (1+i)c$

6. (C) $u = \sinh x \cos y$

$\frac{\partial u}{\partial x} = \cosh x \cos y = \phi(x, y)$

and $\frac{\partial u}{\partial y} = -\sinh x \sin y = \psi(x, y)$

by Milne's Method

$f'(z) = \phi(z, 0) - i\psi(z, 0) = \cosh z - i \cdot 0 = \cosh z$

On integrating $f(z) = \sinh z +$ constant

$\Rightarrow \quad f(z) = w = \sinh z + ic$

(As u does not contain any constant, the constant c is in the function x and hence i.e. in w).

7. (A) $\frac{\partial v}{\partial x} = 2y = h(x, y)$, $\frac{\partial v}{\partial y} = 2x = g(x, y)$

by Milne's Method $f'(z) = g(z, 0) + ih(z, 0) = 2z + i\,0 = 2z$

On integrating $f(z) = z^2 + c$

8. (D) $\frac{\partial v}{\partial y} = \frac{-(x^2+y^2)-(x-y)2y}{(x^2+y^2)^2}$

$= \frac{y^2-x^2-2xy}{(x^2+y^2)^2} = g(x, y)$

$\frac{\partial v}{\partial x} = \frac{(x^2+y^2)-(x-y)2x}{(x^2+y^2)^2} = \frac{y^2-x^2+2xy}{(x^2+y^2)^2} = h(x, y)$

By Milne's Method

$f'(z) = g(z,0) + ih(z,0) = -\frac{1}{z^2} + i\left(-\frac{1}{z^2}\right) = -(1+i)\frac{1}{z^2}$

On integrating

$f(z) = (1+i)\int \frac{1}{z^2}\, dz + c = (1+i)\frac{1}{z} + c$

9. (A) $\frac{\partial u}{\partial x} = \frac{2\cos 2x(\cosh 2y - \cos 2x) - 2\sin^2 2x}{(\cosh 2y - \cos 2x)^2}$

$= \frac{2\cos 2x \cosh 2y - 2}{(\cosh 2y - \cos 2y)^2} = \phi(x, y)$

$\frac{\partial u}{\partial y} = \frac{2\sin 2x \sinh 2y}{(\cosh 2y - \cos 2x)^2} = \psi(x, y)$

By Milne's Method

$f'(z) = \phi(z,0) - i\psi(z,0)$

$= \frac{2\cos 2z - 2}{(1-\cos 2z)^2} - i(0) = \frac{-2}{1-\cos 2z} = -\operatorname{cosec}^2 z$

On integrating

$f(z) = -\int \operatorname{cosec}^2 z\, dz + ic = \cot z + ic$

10. $x = at + b,\ y = ct + d$

On A, $z = 1 + i$ and On B, $z = 2 + 4i$

Let $z = 1 + i$ corresponds to $t = 0$

and $z = 2 + 4i$ corresponding to $t = 1$

then, $t = 0 \quad \Rightarrow \quad x = b,\ y = d$

$\Rightarrow \quad b = 1,\ d = 1$

and $t = 1 \quad \Rightarrow \quad x = a + b,\ y = c + d$

$\Rightarrow \quad 2 = a + 1,\ 4 = c + 1 \Rightarrow \quad a = 1,\ c = 3$

AB is , $y = 3t + 1 \Rightarrow \quad dx = dt\ ;\ dy = 3\,dt$

$\int_c f(z)dz = \int_c (x^2 + ixy)(dx + idy)$

$= \int_{t=0}^{1} [(t+1)^2 + i(t+1)(3t+1)][dt + 3i\,dt]$

$= \int_0^1 [(t^2+2t+1) + i(3t^2+4t+1)](1+3i)dt$

$= (1+3i)\left[\frac{t^3}{3} + t^2 + t + i(t^3 + 2t^2 + t)\right]_0^1 = -\frac{29}{3} + 11i$

11. (D) We know by the derivative of an analytic function that

$f''(z_o) = \frac{n!}{2\pi i}\int_c \frac{f(z)\,dz}{(z-z_o)^{n+1}}$ or $\int_c \frac{f(z)\,dz}{(z-z_o)^{n+1}} = \frac{2\pi i}{n!} f^n(z_o)$

Taking $n = 3$, $\int_c \frac{f(z)\,dz}{(z-z_o)^4} = \frac{\pi i}{3} f''(z_o)$ ....(1)

Given $f_c \frac{e^{2z}dz}{(z+1)^4} = \int_c \frac{e^{2z}dz}{[z-(-1)]^4}$

Taking $f(z) = e^{2z}$, and $z_o = -1$ in (1), we have

$\int_c \frac{e^{2z}dz}{(z+1)^4} = \frac{\pi i}{3} f'''(-1)$....(2)

Now, $f(z) = e^{2z} \quad \Rightarrow \quad f'''(z) = 8e^{2z}$

$\Rightarrow \quad f'''(-1) = 8e^{-2}$

equation (2) have

$\Rightarrow \quad \int_c \frac{e^{2z}dz}{(z+1)^4} = \frac{8\pi i}{3} e^{-2}$ ....(3)

If is the circle $|z| = 3$

Since, $f(z)$ is analytic within and on $|z| = 3$

$\int_{|z|=3} \frac{e^{2z}dz}{(z+1)^4} = \frac{8\pi i}{3} e^{-z}$

12. (D) Since, $\frac{1-2z}{z(z-1)(z-2)} = \frac{1}{2z} + \frac{1}{z-1} - \frac{3}{2(z-2)}$

$\int_c \frac{1-2z}{z(z-1)(z-2)} dz = \frac{1}{2} I_1 + I_2 - \frac{3}{2} I_3$....(1)

Since, $z = 0$ is the only singularity for $I_1 = \int_c \frac{1}{z} dz$ and it lies inside $|z| = 1.5$, therefore by Cauchy's integral Formula

$I_1 = \int_c \frac{1}{z} dz = 2\pi i$ ....(2)

$\left[f(z_o) = \frac{1}{2\pi i}\int_c \frac{f(z)\,dz}{z - z_o}\right]$ [Here $f(z) = 1 = f(z_o)$ and $z_o = 0$]

Similarly, for $I_2 = \int_c \frac{1}{z-1} dz$, the singular point $z = 1$ lies inside $|z| = 1.5$, therefore $I_2 = 2\pi i$....(3)

For $I_3 = \int_c \frac{1}{z-2} dz$, the singular point $z = 2$ lies outside the circle $|z| = 1.5$, so the function $f(z)$ is analytic everywhere in c i.e. $|z| = 1.5$, hence by Cauchy's integral theorem

$I_3 = \int_c \frac{1}{z-2} dz = 0$....(4)

using equations (2), (3), (4) in (1), we get

$\int_c \frac{1-2z}{z(z-1)(z-2)} dz = \frac{1}{2}(2\pi i) + 2\pi i - \frac{3}{2}(0) = 3\pi i$

13. (B) Given contour c is the circle $|z| = 1$

$\Rightarrow \quad z = e^{i\theta} \quad \Rightarrow \quad dz = ie^{i\theta}d\theta$

Now, for upper half of the circle, $0 \le \theta \le \pi$

$$\int_c (z - z^2)dz = \int_{\theta=0}^{\pi} (e^{i\theta} - e^{2i\theta})ie^{i\theta}d\theta$$

$$= i\int_0^{\pi} (e^{2i\theta} - e^{3i\theta})d\theta = i\left[\frac{e^{2i\theta}}{2i} - \frac{e^{3i\theta}}{3i}\right]_0^{\pi}$$

$$= i \cdot \frac{1}{i}\left[\frac{1}{2}\cdot(e^{2\pi i} - 1) - \frac{1}{3}(e^{3\pi x} - 1)\right] = \frac{2}{3}$$

14. (B) Let $f(z) = \cos \pi z$ then $f(z)$ is analytic within and on $|z| = 3$, now by Cauchy's integral formula

$$f(z_o) = \frac{1}{2\pi i}\int_c \frac{f(z)}{z - z_o}dz \Rightarrow \int_c \frac{f(z)\,dz}{z - z_o} = 2\pi i f(z_o)$$

take $f(z) = \cos \pi z$, $z_o = 1$, we have

$$\int_{|z|=3} \frac{\cos \pi z}{z - 1}dz = 2\pi i f(1) = 2\pi i \cos \pi = -2\pi i$$

15. (D) $\displaystyle\int_c \frac{\sin \pi z^2}{(z-1)(z-2)}dz$

$$= \int_c \frac{\sin \pi z^2}{z-2}dz - \int_c \frac{\sin \pi z^2}{z-1}dz$$

$= 2\pi i f(2) - 2\pi i f(1)$ since, $f(z) = \sin \pi z^2$

$\Rightarrow \quad f(2) = \sin 4\pi = 0$ and $f(1) = \sin \pi = 0$

16. (D) Let, $I = \dfrac{1}{2\pi i}\displaystyle\int_c \frac{1}{z^2 - 1}\cos \pi z\, dz$

$$= \frac{1}{2\cdot 2\pi i}\int_c\left(\frac{1}{z-1} - \frac{1}{z+1}\right)\cos \pi z\, dz$$

Or $I = \dfrac{1}{4\pi i}\displaystyle\int_c\left(\frac{\cos nz}{z-1} - \frac{\cos nz}{z+1}\right)dz$

17. (D) $f(3) = \displaystyle\int_c \frac{3z^2 + 7z + 1}{z - 3}dz$, since $z_o = 3$ is the only singular point of $\dfrac{3z^2 + 7z + 1}{z - 3}$ and it lies outside the circle $x^2 + y^2 = 4$ i.e., $|z| = 2$, therefore $\dfrac{3z^2 + 7z + 1}{z - 3}$ is analytic everywhere within c.

Hence by Cauchy's theorem—

$$f(3) = \int_c \frac{3z^2 + 7z + 1}{z - 3}dz = 0$$

18. (C) The point $(1 - i)$ lies within circle $|z| = 2$ (... the distance of $1 - i$ i.e., (1, 1) from the origin is $\sqrt{2}$ which is less than 2, the radius of the circle).

Let $\phi(z) = 3z^2 + 7z + 1$ then by Cauchy's integral formula

$$\int_c \frac{3z^2 + 7z + 1}{z - z_o}dz = 2\pi i\phi(z_o)$$

$\Rightarrow \quad f(z_o) = 2\pi i\phi(z_o) \quad \Rightarrow \quad f'(z_o) = 2\pi i\phi'(z_o)$

and $f''(z_o) = 2\pi i\ \phi''(z_o)$

since, $\phi(z) = 3z^2 + 7z + 1$

$\Rightarrow \quad \phi'(z) = 6z + 7$ and $\phi''(z) = 6$

$f'(1 - i) = 2\pi i[6(1 - i) + 7] = 2\pi(5 + 13i)$

19. (C) $f(z) = \dfrac{z - 1}{z + 1} = 1 - \dfrac{2}{z + 1}$

$\Rightarrow \quad f(0) = -1,\ f(1) = 0$

$\Rightarrow \quad f'(z) = \dfrac{2}{(z+1)^2} \quad \Rightarrow \quad f'(0) = 2;$

$f''(z) = \dfrac{-4}{(z-1)^3} \quad \Rightarrow \quad f''(0) = -4;$

$f'''(z) = \dfrac{12}{(z+1)^4} \quad \Rightarrow \quad f'''(0) = 12;$ and so on.

Now, Taylor series is given by

$$f(z) = f(z_0) + (z - z_0)f'(z_0) + \frac{(z - z_0)^2}{2!}f''(z_0) + \frac{(z - z_0)^3}{3!}f'''(z_0) + ...$$

about $z = 0$

$$f(z) = -1 + z(2) + \frac{z^2}{2!}(-4) + \frac{z^3}{3!}(12) + ...$$

$= -1 + 2z - 2z^2 + 2z^3....$

$f(z) = -1 + 2(z - z^2 + z^3 ....)$

20. (B) $f(z) = \dfrac{1}{z + 1} \quad \Rightarrow \quad f(1) = \dfrac{1}{2}$

$f'(z) = \dfrac{-1}{(z+1)^2} \quad \Rightarrow \quad f'(1) = \dfrac{-1}{4}$

$f''(z) = \dfrac{2}{(z+1)^3} \quad \Rightarrow \quad f''(1) = \dfrac{1}{4}$

$f'''(z) = \dfrac{-6}{(z+1)^4} \quad \Rightarrow \quad f'''(1) = -\dfrac{3}{8}$ and so on.

Taylor series is

$$f(z) = f(z_0) + (z - z_0)f'(z_0) + \frac{(z - z_0)^2}{2!}f''(z_0) + \frac{(z - z_0)^3}{3!}f'''(z_0) + ...$$

about $z = 1$

$$f(z) = \frac{1}{2} + (z - 1)\left(\frac{-1}{4}\right) + \frac{(z-1)^2}{2!}\left(\frac{1}{4}\right) + \frac{(z-1)^3}{3!}\left(-\frac{3}{8}\right) + ...$$

$$= \frac{1}{2} - \frac{1}{2^2}(z - 1) + \frac{1}{2^3}(z - 1)^2 - \frac{1}{2^4}(z - 1)^3 + ...$$

or $f(z) = \dfrac{1}{2}\left[1 - \dfrac{1}{2}(z - 1) + \dfrac{1}{2^2}(z - 1)^2 - \dfrac{1}{2^3}(z - 1)^3 + ...\right]$

21. (A) $f(z) = \sin z \quad \Rightarrow \quad f\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$f'(z) = \cos z \quad \Rightarrow \quad f'\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

$f''(z) = -\sin z \quad \Rightarrow \quad f''\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$

$f'''(z) = -\cos z \quad \Rightarrow \quad f'''\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$ and so on.

Taylor series is given by

$$f(z) = f(z_0) + (z - z_0)f'(z_0) + \frac{(z-z_0)^2}{2!}f''(z_0) + \frac{(z-z_0)^3}{3!}f'''(z_0) + \ldots.$$

about $z = \frac{\pi}{4}$

$$f(z) = \frac{1}{\sqrt{2}} + \left(z - \frac{\pi}{4}\right)\frac{1}{\sqrt{2}} + \frac{\left(z-\frac{\pi}{4}\right)^2}{2!}\left(-\frac{1}{\sqrt{2}}\right) + \frac{\left(z-\frac{\pi}{4}\right)^3}{3!}\left(-\frac{1}{\sqrt{2}}\right) + \ldots$$

$$f(z) = \frac{1}{\sqrt{2}}\left[1 + \left(z - \frac{\pi}{4}\right) - \frac{1}{2!}\left(z-\frac{\pi}{4}\right)^2 - \frac{1}{3!}\left(z - \frac{\pi}{4}\right)^3 - \ldots\right]$$

22. (D) Let $f(z) = z^{-2} = \frac{1}{z^2} = \frac{1}{[1-(1+z)]^2}$

$f(z) = [1-(1+z)]^{-2}$

Since, $|1+z| < 1$, so by expanding R.H.S. by binomial theorem, we get

$$f(z) = 1 + 2(1+z) + 3(1+z)^2 + 4(1+z)^3 + \ldots + (n+1)(1+z)^n + \ldots$$

or $f(z) = z^{-2} = 1 + \sum_{n=1}^{\infty}(n+1)(z+1)^n$

23. (B) Here $f(z) = \frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}$ ....(1)

Since, $|z| > 1 \quad \Rightarrow \quad \frac{1}{|z|} < 1$ and $|z| < 2 \quad \Rightarrow \quad \frac{|z|}{2} < 1$

$$\frac{1}{z-1} = \frac{1}{z\left(1-\frac{1}{z}\right)} = \frac{1}{z}\left(1 - \frac{1}{z}\right)^{-1}$$

$$= \frac{1}{z}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \ldots\right)$$

and $\frac{1}{z-2} = \frac{-1}{2}\left(1 - \frac{z}{2}\right)^{-1} = -\frac{1}{2}\left[1 + \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{9} + \ldots\right]$

equation (1) gives—

$$f(z) = -\frac{1}{2}\left(1 + \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{9} + ..\right) - \frac{1}{z}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \ldots\right)$$

or $f(z) = \ldots - z^{-4} - z^{-2} - z^{-1} - \frac{1}{2} - \frac{1}{4}z - \frac{1}{8}z^2 - \frac{1}{18}z^3 - \ldots$

24. (C) $\frac{2}{|z|} < 1 \quad \Rightarrow \quad \frac{1}{|z|} < \frac{1}{2} < 1 \quad \Rightarrow \quad \frac{1}{|z|} < 1$

$$\frac{1}{z-1} = \frac{1}{z}\left(1 - \frac{1}{z}\right)^{-1} = \frac{1}{2}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \ldots\right)$$

and $\frac{1}{z-2} = \frac{1}{z}\left(1 - \frac{2}{z}\right)^{-1} = \frac{1}{z}\left(1 + \frac{2}{z} + \frac{4}{z^2} + \frac{8}{z^3} + \ldots.\right)$

Laurent's series is given by

$$f(z) = \frac{1}{z}\left(1 + \frac{2}{z} + \frac{4}{z^2} + \frac{98}{z^3} + ..\right) - \frac{1}{z}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + ..\right)$$

$$= \frac{1}{z}\left(\frac{1}{z} + \frac{3}{z^2} + \frac{7}{z^3} + \ldots\right)$$

$$\Rightarrow \quad f(z) = \frac{1}{z^2} + \frac{3}{z^3} + \frac{7}{z^4} + \ldots$$

25. (B) $|z| < 1$, $\frac{1}{z-2} - \frac{1}{z-1} = -\frac{1}{2}\left(1 - \frac{z}{2}\right)^{-1} + (1-z)^{-1}$

$$= -\frac{1}{2}\left[1 + \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} + \ldots\right] + (1 + z + z^2 + z^3 + \ldots)$$

$$f(z) = \frac{1}{2} + \frac{3}{4}z + \frac{7}{8}z^2 + \frac{15}{16}z^3 + \ldots$$

26. (D) Since, $\frac{1}{z(z-1)(z-2)} = \frac{1}{2z} - \frac{1}{z-1} + \frac{1}{2(z-2)}$

For $|z-1| < 1$ Let $z - 1 = u$

$\Rightarrow \quad z = u + 1$ and $|u| < 1$

$$\frac{1}{z(z-1)(z-2)} = \frac{1}{2z} - \frac{1}{z-1} + \frac{1}{2(z-2)}$$

$$= \frac{1}{2(u+1)} - \frac{1}{u} + \frac{1}{2(u-1)} = \frac{1}{2}(1+u)^{-1} - u^{-1} - \frac{1}{2}(1-u)^{-1}$$

$$= \frac{1}{2}[1 - u + u^2 - u^3 + \ldots] - u^{-1} - \frac{1}{2}(1 + u + u^2 + u^3 + \ldots)$$

$$= \frac{1}{2}(-2u - 2u^3 - \ldots) - u^{-1} = -u - u^3 - u^5 - \ldots - u^{-1}$$

Required Laurent's series is

$$f(z) = -(z-1)^{-1} - (z-1) - (z-1)^3 - (z-1)^5 - \ldots$$

27. (B) Let $f(z) = \frac{1}{z(e^z - 1)}$

$$= \frac{1}{z\left[1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \ldots - 1\right]}$$

$$\int_c f(z)dz = 2\pi i \times \frac{1}{6} = \frac{1}{3}\pi i$$

34. (B) Let $z = e^{i\theta} \quad \Rightarrow \quad d\theta = \frac{-idz}{z} \;;\; z \le \theta \le 2\pi$

and $\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$

$$\int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = \int_c \frac{\frac{-idz}{z}}{2 + \frac{1}{2}\left(z+\frac{1}{z}\right)}; \quad c: |z| = 1$$

$$= -2i\int_c \frac{dz}{z^2+4z+1}$$

Let $f(z) = \frac{1}{z^2+4z+1}$

$f(z)$ has poles at $z = -2+\sqrt{3}, -2-\sqrt{3}$ out of these only $z = -2+\sqrt{3}$ lies inside the circle $c: |z| = 1$

$\int_c f(z)dz = 2\pi i$(Residue at $z = -2+\sqrt{3}$)

Now, residue at $z = -2+\sqrt{3}$

$$= \lim_{z\to -2+\sqrt{3}} (z+2-\sqrt{3})f(z) = \lim_{z\to -2+\sqrt{3}} \frac{1}{(z+2+\sqrt{3})} = \frac{1}{2\sqrt{3}}$$

$$\int_c f(z)dz = 2\pi i \times \frac{1}{2\sqrt{3}} = \frac{\pi i}{\sqrt{3}}$$

$$\int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = -2i \times \frac{\pi i}{\sqrt{3}} = \frac{2\pi}{\sqrt{3}}$$

35. (C) $I = \int_c \frac{z^2}{(z^2+a^2)(z^2+b^2)}dz = \int_c f(z)dz$

where c is be semi circle r with segment on real axis from $-R$ to R.

The poles are $z = \pm ia$, $z = \pm ib$. Here only $z = ia$ and $z = ib$ lie within the contour c

$\int_c f(z)dz = 2\pi i$

(sum of residues at $z = ia$ and $z = ib$)

Residue at $z = ia$,

$$= \lim_{z\to ia} (z-ia)\frac{z^2}{(z-ia)(z-ia)(z^2+b^2)} = \frac{a}{2i(a^2-b^2)}$$

Residue at $z = ib$

$$= \lim_{z\to ib} (z-ib)\frac{z^2}{(z-ia)(z+ia)(z+ib)(z-ib)} = \frac{-b}{2i(a^2-b^2)}$$

$$\int_c f(z)dz = \int_r f(z)dz + \int_{-R}^{R} f(z)dz$$

$$= \frac{2\pi i}{2i(a^2-b^2)}(a-b) = \frac{\pi}{a+b}$$

Now $\int_r f(z)dz = \int_0^{\pi} \frac{ie^{2i\theta} iRe^{i\theta}d\theta}{(R^2e^{2i\theta}+a^2)(R^2e^{2i\theta}+b^2)}$

$$= \int_0^{\pi} \frac{\frac{e^{3i\theta}}{R}d\theta}{\left(e^{2i\theta}+\frac{a^2}{R^2}\right)\left(e^{2i\theta}+\frac{b^2}{R^2}\right)}$$

Now when $R\to\infty$, $\int_r b(z)dz = 0$

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)(x^2+b^2)}dz = \frac{\pi}{a+b}$$

36. (C) Let $I = \int_c \frac{dz}{1+z^6} = \int_c f(z)dz$

c is the contour containing semi circle r of radius R and segment from $-R$ to R.

For poles of $f(z)$, $1+z^6 = 0$

$\Rightarrow \quad z = (-1)^{\pi/6} = e^{i(2n+1)\pi/6}$

where $n = 0, 1, 2, 3, 4, 5, 6$

Only poles $z = \frac{-\sqrt{3}+i}{2}, i, \frac{\sqrt{3}+i}{2}$ lie in the contour

Residue at $z = \frac{+\sqrt{3}+i}{2}$

$$= \frac{1}{(z_1-z_2)(z_1-z_3)(z_1-z_4)(z_1-z_5)(z_1-z_6)}$$

$$= \frac{1}{3i(1+\sqrt{3}\,i)} = \frac{1-\sqrt{3}i}{12i}$$

Residue at $z = i$ is $\frac{1}{6i}$

Residue at $z = \frac{1+\sqrt{3}i}{12i}$ is $= \frac{1}{3i(1-\sqrt{3}i)} = \frac{1+\sqrt{3}i}{12i}$

$$\int_c f(z)dz = \int_r f(z)dz + \int_{-R}^{R} f(z)dz$$

$$= \frac{2\pi i}{12i}(1-\sqrt{3}i+1+\sqrt{3}i+2i) = \frac{2\pi}{3}$$

or $\int_r f(z)dz + \int_{-R}^{R} f(z)dz = \frac{2\pi}{3}$....(1)

Now $\int_c f(z)dz = \int_0^{\pi} \frac{iRe^{i\theta}d\theta}{1+R^6e^{6i\theta}} = \int_0^{\pi} \frac{\frac{ie^{i\theta}d\theta}{R^5}}{\frac{1}{R^6}+e^{6i\theta}}$

where $R\to\infty$, $\int_r f(z)dz \to 0$

$(1) \to \int_0^{\infty} \frac{ax}{1+x^6} = \frac{2\pi}{3}$

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