04

Determinants

4.01 Introduction

Consider the following pair of equations

$$a_1x+b_1y=c_1$$
$$a_2x+b_2y=c_2,$$

The equations can be solved to find the unique solution if we find $a_1b_2-b_1a_2$. Therefore number $a_1b_2-b_1a_2$ is very important and it can be represented as the matrix obtained from the cofficient of x and y

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$$

The number $a_1b_2-b_1a_2$ which determines uniqueness of solution is associated with the matrix A = $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$ and is called the determinant of A or det A or symbolically we write $|A| = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$. This determinant has two rows and two columns hence it is of order 2.

Note :

1. Only square matrices have determinants
2. A matrix A is said to be Singular matrix if its $|A|=0$
3. For matrix A, $|A|$ is read as determinmant of A and not modulus of A.

4.02 Definition of determinant

Let $A=[a_{ij}]$ is a square matrix of order n we can associate a unique number $|a_{ij}|$ (real or complex) called determinant of the square matrix A, where $a_{ij}=(i,\ j)$ the element of A. it is denoted by $|A|$.

4.03 Value of determinant

(i) Determinant of a matrix of order one

Let $A=[a]$ is a square matrix of order one then determinant of $A=|A|=a$,

For Example : If $A=[3]$ then determinant $A=|A|=|3|=3$

If $A=[3]$ then determinant $A=|A|=|-3|=-3$

(ii) Determinant of a matrix of order two

Let $A=\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$ is a matrix of order 2, then determinant

$$A = |A| = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

$$= a_1 |b_2| - b_1 |a_2|$$

$$= a_1 b_2 - a_2 b_1, \quad \text{value of determinant } A. \qquad (1)$$

$|A|$ = of order 2 = Product of diagonal elements – Product of off-digonal elements.

Example : $A = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}$, then

Determinant $A = |A| = \begin{vmatrix} 2 & 3 \\ -1 & 4 \end{vmatrix} = 2 \cdot (4) - 3\,(-1)$

$$= 8 + 3 = 11.$$

(iii) *Determinant of a matrix of order 3 × 3*

Let $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ is a matrix of order 3, then

Determinant $A = |A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$$= a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$= a_1 (b_2 c_3 - b_3 c_2) - b_1 (a_2 c_3 - a_3 c_2) + c_1 (a_2 b_3 - a_3 b_2) \qquad (2)$$

$$= (a_1 b_2 c_3 + b_1 c_2 a_3 + c_1 a_2 b_3) - (a_3 b_2 c_1 + b_3 c_2 a_1 + c_3 a_2 b_1) \qquad (3)$$

Here numbers $a_1, b_1, c_1;\ a_2, b_2, c_2;\ a_3, b_3, c_3$ are called the elements of the determinant. There are a total of $3^2 = 9$ elements in a matrix of order 3. Thus the determinant of a square matrix of order 3 is the sum of the product of elements a_{ij} in first row with $(-1)^{1+j}$ times the determinant of 2 × 2. Sub-matrix obtained by leaving the first row and column passing through the element.

4.04 Rules to expand third order determinant

(i) Write the elements of first row in consecutive positive and negative sign.

(ii) Multiply first element with the second order determinant obtained by deleting the elements of first row (R_1) and first column (C_1). Then multiply 2nd element and the second order determinant obtained by deleting elements of first row (R_1) and 2nd column (C_2). Now multiply third element and the second order determinant obtained by deleting elements of first row (R_1) and third column (C_3) and third column (C_3). To get the value of the determinant add all the three terms.

(iii) The result will be the value of the determinant of order 3.

Example : Evaluate the determinant $\begin{vmatrix} 1 & 2 & 0 \\ 2 & 3 & 1 \\ 3 & 0 & 2 \end{vmatrix}$

Solution :

$$\begin{vmatrix} 1 & 2 & 0 \\ 2 & 3 & 1 \\ 3 & 0 & 2 \end{vmatrix} = 1\cdot\begin{vmatrix} 3 & 1 \\ 0 & 2 \end{vmatrix} - 2\cdot\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} + 0\cdot\begin{vmatrix} 2 & 3 \\ 3 & 0 \end{vmatrix}$$

$$= 1(3\times 2 - 1\times 0) - 2(2\times 2 - 3\times 1) + 0(2\times 0 - 3\times 3)$$

$$= 1(6) - 2(1) + 0$$

$$= 6 - 2$$

$$= 4.$$

4.05 Sarrus diagram to determine the value of third order determinant

$$|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \left|\begin{array}{ccc|cc} a_1 & b_1 & c_1 & a_1 & b_1 \\ a_2 & b_2 & c_2 & a_2 & b_2 \\ a_3 & b_3 & c_3 & a_3 & b_3 \end{array}\right.$$

$$= (a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3) - (a_3b_2c_1 + b_3c_2a_1 + c_3a_2b_1)$$

Note: To evaluate determinant from Sarrus diagram, Like given diagram, we have substract the sum of product of element of leading diagonal to sum of product of element of non-leading diagonal.

Example : Determinant $\begin{vmatrix} 1 & 2 & -1 \\ 3 & 5 & 7 \\ 2 & 4 & 6 \end{vmatrix} = \left|\begin{array}{ccc|cc} 1 & 2 & -1 & 1 & 2 \\ 3 & 5 & 7 & 3 & 5 \\ 2 & 4 & 6 & 2 & 4 \end{array}\right.$

$$= (30 + 28 - 12) - (-10 + 28 + 36)$$

$$= 46 - 54 = -8.$$

4.06 Difference between matrix and determinant

(i) Matrix is a proper representation of number and does not have a numerical value while determinant has a unique numerical value.

(ii) Matrix can be of any order while determinants are square matrices where number of rows and columns are same.

(iii) If we change the number of rows and columns of the matrix we get a new matrix whereas the value of determinant unchanged.

4.07 Minors and cofactors of a determinant

Minors : Minor of an element a_{ij} of a determinant is the determinant obtained by deleting its ith row and jth column in which element a_{ij} lies. Minor of an element a_{ij} is denoted by A_{ij}.

Example : $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$. Here element a_2, lies in the second row and first column then leaving the second row and first column in Δ we get the respective determinant.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \text{ or } \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix}$$ which is the minor of element a_2

similarly the minor of element c_3 of Δ will be

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \text{ or } \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

Minor of an element of a determinant of order $n(n \geq 2)$ is a determinatn of order $n - 1$.

Example : The minor of element 1 in the determinant $\begin{vmatrix} -3 & 2 \\ 1 & 5 \end{vmatrix}$ is $|\,2\,|$.

The minor of element 3 in the determinant $\begin{vmatrix} 1 & -2 & 3 \\ 7 & 0 & 5 \\ -3 & -1 & 4 \end{vmatrix}$ is $\begin{vmatrix} 7 & 0 \\ -3 & -1 \end{vmatrix}$ and element 7 is $\begin{vmatrix} -2 & 3 \\ -1 & 4 \end{vmatrix}$

Cofactor : Cofactor of an element a_{ij}, denoted by F_{ij} is defined by

$$F_{ij} = (-1)^{i+j} \text{ Minors}$$

$$\Rightarrow \qquad F_{ij} = (-1)^{i+j} A_{ij},$$

here A_{ij} and F_{ij} denotes the Minors and Cofactors of element a_{ij}

i.e., $$F_{ij} = \begin{cases} A_{ij} & ; \ i+j \text{ is even} \\ -A_{ij} & ; \ i+j \text{ is odd} \end{cases}$$

Example: If $$\Delta = \begin{vmatrix} 7 & 4 & -1 \\ -2 & 3 & 0 \\ 1 & -5 & 2 \end{vmatrix}$$ then

Cofactor of 7 $= (-1)^{1+1} \begin{vmatrix} 3 & 0 \\ -5 & 2 \end{vmatrix} = 6 - 0 = 6$

Cofactor of 5 $= (-1)^{3+2} \begin{vmatrix} 7 & -1 \\ -2 & 0 \end{vmatrix} = -(0-2) = 2$

Cofactor of 4 $= (-1)^{1+2} \begin{vmatrix} -2 & 0 \\ 1 & 2 \end{vmatrix} = -(-4) = 4$

Note: For easy calculation in a matrix of order 2 and 3 the signs of elements to find the cofactor is

$$\begin{vmatrix} + & - \\ - & + \end{vmatrix}, \begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

4.08 Expansion of determinants

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$ is a determinant of third order

Expanding along first row we get

$$\Delta = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$= a_{11}A_{11} - a_{12}A_{12} + a_{13}A_{13}$, where A_{11}, A_{12} and A_{13} are the minors of corresponding elements

$= a_{11}F_{11} + a_{12}F_{12} + a_{13}F_{13}$, where F_{11}, F_{12} and F_{13} are the cofactors of corresponding elements

Similarly we can see that

$$\Delta = a_{21}F_{21} + a_{22}F_{22} + a_{23}F_{23}$$

$$\Delta = a_{11}F_{11} + a_{21}F_{21} + a_{31}F_{31}$$

$$\Delta = a_{13}F_{13} + a_{23}F_{23} + a_{33}F_{33} \quad \text{etc}$$

Thus the value of the determinants is the sum of elements with its corresponding cofactors.

Note:

(i) The expansion can be done along any row or column in determinant.

(ii) This rule is valid for any type of determinant.

(iii) Expansion should be done with any row or column with maximum zeroes.

Illustrative Examples

Example 1. Evaluate the determinant $\begin{vmatrix} 2 & 4 \\ 2 & -3 \end{vmatrix}$

Solution : $\begin{vmatrix} 2 & 4 \\ 2 & -3 \end{vmatrix} = (-6) - (8) = -14.$

Example 2. Evaluate the determinant $\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}$

Solution : $\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} = (\cos^2\theta) - (-\sin^2\theta)$

$$= \cos^2\theta + \sin^2\theta = 1.$$

Example 3. Evaluate the determinant $\begin{vmatrix} 3 & 11 & -1 \\ 5 & 2 & 0 \\ 10 & 3 & 0 \end{vmatrix}$

Solution : $\begin{vmatrix} 3 & 11 & -1 \\ 5 & 2 & 0 \\ 10 & 3 & 0 \end{vmatrix}$ expanding alogn third column

$$= -1\begin{vmatrix} 5 & 2 \\ 10 & 3 \end{vmatrix} - 0 + 0 = -(15-20) = 5.$$

Example 4. If determinant $\begin{vmatrix} k & 8 \\ 2 & 4 \end{vmatrix} = 4$, then find the value of k.

Solution : Given $\begin{vmatrix} k & 8 \\ 2 & 4 \end{vmatrix} = 4$

$\Rightarrow \quad 4k - 16 = 4$

$\Rightarrow \quad k = 5.$

Example 5. If determinant $\begin{vmatrix} k & 3 \\ -1 & k \end{vmatrix} = 7$ then find the vlaue of k.

Solution : Given $\begin{vmatrix} k & 3 \\ -1 & k \end{vmatrix} = 7$

$\Rightarrow \quad k^2 - (-3) = 7 \quad \Rightarrow \quad k^2 + 3 = 7$

$\Rightarrow \quad k^2 = 4 \quad \Rightarrow \quad k = \pm 2.$

Example 6. Evaluate the determinant $A = \begin{vmatrix} 2 & 4 & 1 \\ 8 & 5 & 2 \\ -1 & 3 & 7 \end{vmatrix}$ and write the cofactors and minors of elements of second row.

Solution: Minors : $A_{21} = \begin{vmatrix} 4 & 1 \\ 3 & 7 \end{vmatrix} = 28 - 3 = 25$, $A_{22} = \begin{vmatrix} 2 & 1 \\ -1 & 7 \end{vmatrix} = 14 - (-1) = 15$, $A_{23} = \begin{vmatrix} 2 & 4 \\ -1 & 3 \end{vmatrix} = 6 - (-4) = 10$

$\therefore$ Cofactors $\quad F_{21} = -A_{21} = -25, \quad F_{22} = A_{22} = 15, \quad F_{23} = -A_{23} = -10$

Thus the vlaue of determinant A is $= 8 \cdot F_{21} + 5 \cdot F_{22} + 2 \cdot F_{23}$

$$= 8(-25) + 5(15) + 2(-10)$$

$$= -200 + 75 - 20 = -145.$$

Example 7. Evaluate the determinant $\begin{vmatrix} 3 & -7 & 13 \\ 5 & 0 & 0 \\ 0 & 11 & 2 \end{vmatrix}$

Solution : Expanding along second row as it has two zeroes

$$\begin{vmatrix} 3 & -7 & 13 \\ 5 & 0 & 0 \\ 0 & 11 & 2 \end{vmatrix} = 5\times(-1)\begin{vmatrix} -7 & 13 \\ 11 & 2 \end{vmatrix} + 0 - 0$$

$$= -5[-14-143] = 785.$$

Exercise 4.1

1. For what value of k is the value of the determinant $\begin{vmatrix} k & 2 \\ 4 & -3 \end{vmatrix}$ zero?

2. If $\begin{vmatrix} x & y \\ 2 & 4 \end{vmatrix} = 0$ then find the ratio $x : y$.

3. If $\begin{vmatrix} 2 & 3 \\ y & x \end{vmatrix} = 4$ and $\begin{vmatrix} x & y \\ 4 & 2 \end{vmatrix} = 7$ then evaluate x and y.

4. If $\begin{vmatrix} x-1 & x-2 \\ x & x-3 \end{vmatrix} = 0$ then find the value of x.

5. Evaluate the determinant and also find the minors and cofactors of elements of first row

   (i) $\begin{vmatrix} 1 & -3 & 2 \\ 4 & -1 & 2 \\ 3 & 5 & 2 \end{vmatrix}$ (ii) $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$

6. Evaluate the determinant $\begin{vmatrix} 3 & -11 & 1 \\ 5 & 0 & 0 \\ -10 & 3 & 0 \end{vmatrix}$

7. Prove that $\begin{vmatrix} 1 & a & b \\ -a & 1 & c \\ -b & -c & 1 \end{vmatrix} = 1 + a^2 + b^2 + c^2$.

4.09 Properties of Determinants

(i) The value of the determinant remains unchanged if its rows and columns are interchanged.

Proof : Let
$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},$$

and
$$\Delta_1 = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix},$$

$\therefore$
$$\Delta = \left| \begin{array}{ccc|cc} a_1 & b_1 & c_1 & a_1 & b_1 \\ a_2 & b_2 & c_2 & a_2 & b_2 \\ a_3 & b_3 & c_3 & a_3 & b_3 \end{array} \right. \qquad \text{(by Sarrus figure)}$$

$$\Delta = \left(a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3\right) - \left(a_3b_2c_1 + b_3c_2a_1 + c_3a_2b_1\right) \qquad (1)$$

and
$$\Delta_1 = \left| \begin{array}{ccc|cc} a_1 & a_2 & a_3 & a_1 & a_2 \\ b_1 & b_2 & b_3 & b_1 & b_2 \\ c_1 & c_2 & c_3 & c_1 & c_2 \end{array} \right. \qquad \text{(by Sarrus figure)}$$

$$\Delta_1 = \left(a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2\right) - \left(c_1b_2a_3 + c_2b_3a_1 + c_3b_1a_2\right) \qquad (2)$$

$\therefore$ from (1) and (2) $\Delta = \Delta_1$

$\therefore\ |A^T| = |A|$, where A^T, is a transpose of square matrix A.

(ii) If any two rows (or columns) of a determinant are interchanged, then sign of determiant changes, but value remains unchanged.

Proof : Let
$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

and
$$\Delta_1 = \begin{vmatrix} b_1 & a_1 & c_1 \\ b_2 & a_2 & c_2 \\ b_3 & a_3 & c_3 \end{vmatrix},$$

(by interchanging the first and second columns of the determinant)

$\therefore$
$$\Delta = \left| \begin{array}{ccc|cc} a_1 & b_1 & c_1 & a_1 & b_1 \\ a_2 & b_2 & c_2 & a_2 & b_2 \\ a_3 & b_3 & c_3 & a_3 & b_3 \end{array} \right. \qquad \text{(by Sarrus figure)}$$

$$\Delta = \left(a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3\right) - \left(a_3b_2c_1 + b_3c_2a_1 + c_3a_2b_1\right) \qquad (1)$$

and $$\Delta_1 = \begin{vmatrix} b_1 & a_1 & c_1 \\ b_2 & a_2 & c_2 \\ b_3 & a_3 & c_3 \end{vmatrix} \begin{matrix} b_1 & a_1 \\ b_2 & a_2 \\ b_3 & a_3 \end{matrix}$$ (by Sarrus figure)

$$\Delta_1 = \left(b_1a_2c_3 + a_1c_2b_3 + c_1b_2a_3\right) - \left(b_3a_2c_1 + a_3c_2b_1 + c_3b_2a_1\right) \quad (2)$$

$\therefore$ from (1) and (2) $\Delta_1 = -\Delta$

(iii) If any two rows (or columns) of a determinant are identical (all corresponding elements are same), then value of determinant is zero

Proof : $$\begin{vmatrix} a & b & c \\ a & b & c \\ x & y & z \end{vmatrix} = \begin{vmatrix} a & b & c \\ a & b & c \\ x & y & z \end{vmatrix} \begin{matrix} a & b \\ a & b \\ x & y \end{matrix}$$ (by Sarrus figure)

$$= \left(abz + bcx + cay\right) - \left(xbc + yca + zab\right)$$

$$= 0.$$

and $$\begin{vmatrix} x & a & x \\ y & b & y \\ z & c & z \end{vmatrix} = \begin{vmatrix} x & a & x \\ y & b & y \\ z & c & z \end{vmatrix} \begin{matrix} x & a \\ y & b \\ z & c \end{matrix}$$ (by Sarrus figure)

$$= \left(xbz + ayz + xyc\right) - \left(zbx + cyx + zya\right)$$

$$= 0.$$

(iv) If each element of a row (or a column) of a determinant is multiplied by a constant *k*, then its value gets multiplied by *k*.

Proof : Let $$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

and $$\Delta_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ ka_3 & kb_3 & kc_3 \end{vmatrix},$$

$\therefore$ By Sarrus figure (multiplying the third row by k)

$$\Delta = \left(a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3\right) - \left(a_3b_2c_1 + b_3c_2a_1 + c_3a_2b_1\right) \quad (1)$$

and $$\Delta_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ ka_3 & kb_3 & kc_3 \end{vmatrix} \begin{matrix} a_1 & b_1 \\ a_2 & b_2 \\ ka_3 & kb_3 \end{matrix}$$ (by Sarrus figure)

$$\Delta_1 = \left(a_1b_2kc_3 + b_1c_2ka_3 + c_1a_2kb_3\right) - \left(ka_3b_2c_1 + kb_3c_2a_1 + kc_3a_2b_1\right)$$

$$= k\left\{\left(a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3\right) - \left(a_3b_2c_1 + b_3c_2a_1 + c_3a_2b_1\right)\right\}$$

$$= k\Delta$$

$\therefore \quad \Delta_1 = k\Delta$

Corollory :: Let Δ_1 be the determinant obtained by multiplying the each elements of Δ by k then

$\Delta_1 = k\Delta,$ when the order of Δ is one

$\Delta_1 = k^2\Delta,$ when the order of Δ is two

$\Delta_1 = k^3\Delta,$ when the order of Δ is three

$\Delta_1 = k^4\Delta,$ when the order of Δ is four

i.e. $\quad \Delta_1 = k^n\Delta$ when the order of Δ is n

(v) If each elements of a row or column of a determinant are expressed as sum of two (or more) terms the determinant can be expressed as sum of two (or more) determinants.

Proof : Let $\quad \Delta = \begin{vmatrix} a_1+d_1 & b_1 & c_1 \\ a_2+d_2 & b_2 & c_2 \\ a_3+d_3 & b_3 & c_3 \end{vmatrix}$

Expanding along first row

$$\Delta = \left(a_1+d_1\right)\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - \left(a_2+d_2\right)\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + \left(a_3+d_3\right)\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

$$= \left\{a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}\right\} + \left\{d_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - d_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + d_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}\right\}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

(vi) If the elements of any row or column of a determinant is added or substracted with any of other row (or column) with a multiple of constant, then the value of the determinant does not changes.

Proof : Let $\quad \Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

and $\quad \Delta_1 = \begin{vmatrix} a_1+kc_1 & b_1 & c_1 \\ a_2+kc_2 & b_2 & c_2 \\ a_3+kc_3 & b_3 & c_3 \end{vmatrix},$

(by adding first column with k times the third column)

$$\therefore \quad \Delta_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} kc_1 & b_1 & c_1 \\ kc_2 & b_2 & c_2 \\ kc_3 & b_3 & c_3 \end{vmatrix} \qquad [\text{Property } (v)]$$

$$= \Delta + k\begin{vmatrix} c_1 & b_1 & c_1 \\ c_2 & b_2 & c_2 \\ c_3 & b_3 & c_3 \end{vmatrix} \qquad [\text{Property (iv)}]$$

$$= \Delta + k \times 0$$

$$= \Delta. \qquad [\text{Property (iii)}]$$

(vii) If elements of one row (or column) are multiplied with cofactors of elements of any other row (or column), then their sum is zero.

Proof : Let

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \qquad (1)$$

$$\Rightarrow \quad \Delta = a_{11}F_{11} + a_{12}F_{12} + a_{13}F_{13} \quad \text{(Expanding along first rows)} \qquad (2)$$

substituting in (1) of a_{11}, a_{12} and a_{13} by a_{21}, a_{22} and a_{23}

$$\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 0 \qquad [\text{Property (iii)}] \quad (3)$$

thsu from (1) and (3) $\qquad 0 = a_{21}F_{11} + a_{22}F_{12} + a_{23}F_{13}$

similarly $\qquad 0 = a_{31}F_{11} + a_{32}F_{12} + a_{33}F_{13}$ $\qquad$ etc.

(viii) If the elements of any row or column of a determinant are zeroes then the vlaue of the determinant is zero.

Proof :

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ 0 & 0 & 0 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad \text{expanding along second row}$$

$$= -0 \times \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + 0 \times \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} - 0 \times \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}$$

$$= 0$$

(ix) In a Triangular matrix the value of the determinant is the product of the elements of the diagonals.

For example: (i) $\begin{vmatrix} a & b \\ 0 & c \end{vmatrix} = ac - 0 = ac$

(ii) $\begin{vmatrix} a & 0 \\ b & c \end{vmatrix} = ac - 0 = ac$

(iii) $\begin{vmatrix} a & b & c \\ 0 & x & y \\ 0 & 0 & \ell \end{vmatrix} = \ell \begin{vmatrix} a & b \\ 0 & x \end{vmatrix} = \ell(ax) = a\ell x$

(iv) $\begin{vmatrix} a & 0 & 0 \\ b & x & 0 \\ c & y & \ell \end{vmatrix} = a \begin{vmatrix} x & 0 \\ y & \ell \end{vmatrix} = a(x\ell - 0) = a\ell x$

Corollory : $|I_n| = 1,$ where I_n, n is the identity matrix of order n

$$\Rightarrow \qquad \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

(x) If a determinant has polynomial with variable x and if by substituting a in place of x the value of determinant is zero then $x–a$ will be a factor of the determinant.

For example : In $\Delta = \begin{vmatrix} 1 & x & x^2 \\ 1 & a & a^2 \\ 1 & b & b^2 \end{vmatrix}$ if by substituting $x = a$ and $x = b$ the value of Δ becomes zero

then $(x-a)$ and $(x-b)$ will be the two factors of the determinant.

$\therefore$ To solve for Δ subtracting second row from first and third row from first we have

$$\Delta = \begin{vmatrix} 1 & x & x^2 \\ 0 & a-x & a^2-x^2 \\ 0 & b-x & b^2-x^2 \end{vmatrix} = \begin{vmatrix} a-x & a^2-x^2 \\ b-x & b^2-x^2 \end{vmatrix}$$

$$= (a-x)(b-x)\begin{vmatrix} 1 & a+x \\ 1 & b+x \end{vmatrix}$$

$$= (a-x)(b-x)(b+x-a-x)$$

$$= (a-x)(b-x)(b-a)$$

$$= (x-a)(x-b)(b-a)$$

4.10 Elementary operations

If the order of Δ is $n \geq 2$ then $R_1, R_2, R_3, \ldots$ represents first row, second row, third row. . . and $C_1, C_2, C_3, \ldots$ represents first columns, second column, third column . . . etc.

(i) Operation $R_i \leftrightarrow R_j$ means *i*th and *j*th rows are mutually interchanged and $C_i \leftrightarrow C_j$ means that *i*th and *j*th columns are mutually interchanged.

(ii) Operation $R_i \to kR_i$ means that every element of *i*th row is multiplied by k whereas $C_i \to kC_i$ means that every element of *i*th column is multiplied by *k*.

(iii) Operation $R_i = R_i + kR_j$ refers that every element of *i*th row is added to *k* times the elements in *j*th row similarly $C_i = C_i + kC_j$ refers that every element of *i*th row is added to *k* times the elements in *j*th column

4.11 Product of determinants

I. The product of second order determinant can be done as given below:

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \times \begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{vmatrix} = \begin{vmatrix} a_1\alpha_1 + b_1\alpha_2 & a_1\beta_1 + b_1\beta_2 \\ a_2\alpha_1 + b_2\alpha_2 & a_2\beta_1 + b_2\beta_2 \end{vmatrix} \quad \text{(Row multiply by column)}$$

and

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \times \begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{vmatrix} = \begin{vmatrix} a_1\alpha_1 + b_1\beta_1 & a_1\alpha_2 + b_1\beta_2 \\ a_2\alpha_1 + b_2\beta_1 & a_2\alpha_2 + b_2\beta_2 \end{vmatrix} \quad \text{(Row multiply by Row)}$$

$\because \quad |A^T| = |A|$

II. The product of third order determinant can be done as given below:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix} = \begin{vmatrix} a_1\alpha_1 + b_1\alpha_2 + c_1\alpha_3 & a_1\beta_1 + b_1\beta_2 + c_1\beta_3 & a_1\gamma_1 + b_1\gamma_2 + c_1\gamma_3 \\ a_2\alpha_1 + b_2\alpha_2 + c_2\alpha_3 & a_2\beta_1 + b_2\beta_2 + c_2\beta_3 & a_2\gamma_1 + b_2\gamma_2 + c_2\gamma_3 \\ a_3\alpha_1 + b_3\alpha_2 + c_3\alpha_3 & a_3\beta_1 + b_3\beta_2 + c_3\beta_2 & a_3\gamma_1 + b_3\gamma_2 + c_3\gamma_3 \end{vmatrix}$$

and

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix} = \begin{vmatrix} a_1\alpha_1 + b_1\beta_1 + c_1\gamma_1 & a_1\alpha_2 + b_1\beta_2 + c_1\gamma_2 & a_1\alpha_3 + b_1\beta_3 + c_1\gamma_3 \\ a_2\alpha_1 + b_2\beta_1 + c_2\gamma_1 & a_2\alpha_2 + b_2\beta_2 + c_2\gamma_2 & a_2\alpha_3 + b_2\beta_3 + c_2\gamma_3 \\ a_3\alpha_1 + b_3\beta_1 + c_3\gamma_1 & a_3\alpha_2 + b_3\beta_2 + c_3\gamma_2 & a_3\alpha_3 + b_3\beta_3 + c_3\gamma_3 \end{vmatrix}$$

Note : The product of two different order determinants is also possible.

For example : $\Delta_1 = \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix}$ and $\Delta_2 = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 1 & 2 & 4 \end{vmatrix}$

$$\Delta_1 \cdot \Delta_2 = \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} \times \begin{vmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 1 & 2 & 4 \end{vmatrix}$$

$$\therefore \quad = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{vmatrix} \times \begin{vmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 1 & 2 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 11 \\ 5 & 4 & 10 \end{vmatrix}$$

$$= 1(50-44) - 2(40-55) + 3(16-25)$$

$$= 6 + 30 - 27 = 9. \qquad (1)$$

Now $\Delta_1 = \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = 1 - 4 = -3. \qquad (2)$

and $\Delta_2 = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 1 & 2 & 4 \end{vmatrix} = 1(4-6) - 2(8-3) + 3(4-1)$

$$= -2 - 10 + 9 = -3. \qquad (3)$$

from (1), (2) and (3)

$$\Delta_1 \cdot \Delta_2 = 9.$$

Illustrative Examples

Example 8. Evaluate the determinant $\begin{vmatrix} 49 & 1 & 6 \\ 39 & 7 & 4 \\ 10 & 2 & 1 \end{vmatrix}$ without cxpansion.

Solution : Using operation $C_1 \to C_1 - 8C_3$

$$\begin{vmatrix} 1 & 1 & 6 \\ 7 & 7 & 4 \\ 2 & 2 & 1 \end{vmatrix} = 0 \qquad [\because C_1 = C_2 \quad \text{Property (iii)}]$$

Example 9. Evaluate the determinant $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$ without expansion.

Solution :

$$\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix} = \begin{vmatrix} 1 & a & a+b+c \\ 1 & b & c+a+b \\ 1 & c & a+b+c \end{vmatrix} \qquad (C_3 \to C_3 + C_2)$$

$$= (a+b+c)\begin{vmatrix} 1 & a & 1 \\ 1 & b & 1 \\ 1 & c & 1 \end{vmatrix} \qquad [\text{Property (iv)}]$$

$$= (a+b+c)(0)$$

$$= 0 \qquad [\because C_1 = C_3 \quad \text{Property (iii)}]$$

Example 10. Evaluate the determinant $\begin{vmatrix} a-b & m-n & x-y \\ b-c & n-p & y-z \\ c-a & p-m & z-x \end{vmatrix}$ without expansion.

Solution : $\begin{vmatrix} a-b & m-n & x-y \\ b-c & n-p & y-z \\ c-a & p-m & z-x \end{vmatrix}$

Using operation $R_1 \to R_1 + R_2 + R_3$

$$= \begin{vmatrix} 0 & 0 & 0 \\ b-c & n-p & y-z \\ c-a & p-m & z-x \end{vmatrix} = 0$$ [Using Property (viii)]

Example 11. Prove that

$$\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = (x-y)(y-z)(z-x).$$

Solution : L.H.S. $= \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$

Usign $R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3$

$$= \begin{vmatrix} 0 & x-y & x^2-y^2 \\ 0 & y-z & y^2-z^2 \\ 1 & z & z^2 \end{vmatrix}$$

$$= (x-y)(y-z)\begin{vmatrix} 0 & 1 & x+y \\ 0 & 1 & y+z \\ 1 & z & z^2 \end{vmatrix}$$ [property (iv)]

Expanding along first column

$$= (x-y)(y-z)\left\{0 - 0 + 1\begin{vmatrix} 1 & x+y \\ 1 & y+z \end{vmatrix}\right\}$$

$$= (x-y)(y-z)(y+z-x-y)$$

$$= (x-y)(y-z)(z-x).$$

$= \text{R.H.S.}$

Example 12. Without expanding, prove that

$$\Delta = \begin{vmatrix} b+c & c+a & a+b \\ q+r & r+p & p+q \\ y+z & z+x & x+y \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}.$$

Solution :

$$\Delta = \begin{vmatrix} b+c & c+a & a+b \\ q+r & r+p & p+q \\ y+z & z+x & x+y \end{vmatrix}$$

$$= \begin{vmatrix} 2c & c+a & a+b \\ 2r & r+p & p+q \\ 2z & z+x & x+y \end{vmatrix}$$ (Property $C_1 \to C_1 + C_2 - C_3$)

$$= 2\begin{vmatrix} c & c+a & a+b \\ r & r+p & p+q \\ z & z+x & x+y \end{vmatrix}$$ [Property (iv)]

$$= 2\begin{vmatrix} c & a & a+b \\ r & p & p+q \\ z & x & x+y \end{vmatrix}$$ (operation $C_2 \to C_2 - C_1$)

$$= 2\begin{vmatrix} c & a & b \\ r & p & q \\ z & x & y \end{vmatrix}$$ (operation $C_3 \to C_3 - C_2$)

$$= -2\begin{vmatrix} a & c & b \\ p & r & q \\ x & z & y \end{vmatrix}$$ (operation $C_1 \leftrightarrow C_2$)

$$= 2\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}$$ (operation $C_2 \leftrightarrow C_3$)

Example 13. If x, y, z are different and real,

$$\begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix} = 0$$

then Prove that $xyz = -1$.

Solution :

given
$$\begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix} = 0$$

$$\Rightarrow \quad \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + \begin{vmatrix} x & x^2 & x^3 \\ y & y^2 & y^3 \\ z & z^2 & z^3 \end{vmatrix} = 0 \qquad \text{[property (v)]}$$

$$\Rightarrow \quad -\begin{vmatrix} x & 1 & x^2 \\ y & 1 & y^2 \\ z & 1 & z^2 \end{vmatrix} + xyz\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = 0 \qquad \text{[property (ii) and (iv)]}$$

$$\Rightarrow \quad \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} + xyz\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = 0 \qquad \text{[property (ii)]}$$

$$\Rightarrow \quad (1+xyz)\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = 0 \qquad \text{[from example (11)]}$$

$\Rightarrow \quad (1+xyz)(x-y)(y-z)(z-x) = 0$

$\because \quad x \neq y \neq z \Rightarrow x-y \neq 0,\ y-z \neq 0$ तथा $z-x \neq 0$

$\Rightarrow \quad 1+xyz = 0 \Rightarrow xyz = -1\cdot$

Example 14. Evaluate the determinant $\begin{vmatrix} 1/a & a^2 & bc \\ 1/b & b^2 & ca \\ 1/c & c^2 & ab \end{vmatrix}$

Solution :
$$\begin{vmatrix} 1/a & a^2 & bc \\ 1/b & b^2 & ca \\ 1/c & c^2 & ab \end{vmatrix} = \frac{1}{abc}\begin{vmatrix} 1 & a^3 & abc \\ 1 & b^3 & abc \\ 1 & c^3 & abc \end{vmatrix} \qquad \text{(Operation } R_1 \to aR_1,\ R_2 \to bR_2 \text{ and } R_3 \to cR_3\text{)}$$

$$= \frac{abc}{abc}\begin{vmatrix} 1 & a^3 & 1 \\ 1 & b^3 & 1 \\ 1 & c^3 & 1 \end{vmatrix} = 0 \qquad [\because C_1 = C_3, \text{ property (iii)}]$$

Example 15. Prove that

$$\begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix} = 2(a+b+c)^3$$

Solution : L.H.S. $= \begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix}$

$$= \begin{vmatrix} 2(a+b+c) & a & b \\ 2(a+b+c) & b+c+2a & b \\ 2(a+b+c) & a & c+a+2b \end{vmatrix} \qquad \text{(opration } C_1 \to C_1 + C_2 + C_3\text{)}$$

$$= 2(a+b+c)\begin{vmatrix} 1 & a & b \\ 1 & b+c+2a & b \\ 1 & a & c+a+2b \end{vmatrix} \qquad \text{[property (iv)]}$$

$$= 2(a+b+c)\begin{vmatrix} 1 & a & b \\ 0 & b+c+a & 0 \\ 0 & 0 & c+a+b \end{vmatrix} \qquad \text{(operation } R_2 \to R_2 - R_1 \text{ and } R_3 \to R_3 - R_1\text{)}$$

$$= 2(a+b+c)\left\{1 \cdot \begin{vmatrix} b+c+a & 0 \\ 0 & c+a+b \end{vmatrix}\right\}$$

$$= 2(a+b+c)\begin{vmatrix} a+b+c & 0 \\ 0 & a+b+c \end{vmatrix}$$

$$= 2(a+b+c)(a+b+c)^2$$

$$= 2(a+b+c)^3$$

= RHS

Example 16. Prove that

$$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right).$$

Solution : L.H.S. $= \begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix}$

$$= abc \begin{vmatrix} \frac{1}{a}+1 & \frac{1}{a} & \frac{1}{a} \\ \frac{1}{b} & \frac{1}{b}+1 & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & 1+\frac{1}{c} \end{vmatrix}$$ (taking a, b and c from first, second and third row)

$$= abc \begin{vmatrix} 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \\ \frac{1}{b} & 1+\frac{1}{b} & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & 1+\frac{1}{c} \end{vmatrix}$$ (operation $R_1 \to R_1 + R_2 + R_3$)

$$= abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \begin{vmatrix} 1 & 1 & 1 \\ \frac{1}{b} & 1+\frac{1}{b} & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & 1+\frac{1}{c} \end{vmatrix}$$ [property (iv)]

$$= abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \begin{vmatrix} 0 & 0 & 1 \\ -1 & 1 & \frac{1}{b} \\ 0 & -1 & 1+\frac{1}{c} \end{vmatrix}$$ (Using operation $C_1 \to C_1 - C_2$ and $C_2 \to C_2 - C_3$)

$$= abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\left\{0+0+1\begin{vmatrix} -1 & 1 \\ 0 & -1 \end{vmatrix}\right\}$$

$$= abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)(1-0)$$

$$= abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

= R.H.S.

Example 17. Solve the equation $\begin{vmatrix} x+a & b & c \\ c & x+b & a \\ a & b & x+c \end{vmatrix} = 0$

Solution :

$$\begin{vmatrix} x+a & b & c \\ c & x+b & a \\ a & b & x+c \end{vmatrix} = 0$$

$$\begin{vmatrix} x+a+b+c & b & c \\ x+a+b+c & x+b & a \\ x+a+b+c & b & x+c \end{vmatrix} = 0 \qquad \text{(operation } C_1 \to C_1 + C_2 + C_3\text{)}$$

or $$(x+a+b+c)\begin{vmatrix} 1 & b & c \\ 1 & x+b & a \\ 1 & b & x+c \end{vmatrix} = 0$$

or $$(x+a+b+c)\begin{vmatrix} 0 & -x & c-a \\ 0 & x & a-x-c \\ 1 & b & x+c \end{vmatrix} = 0 \qquad \text{(using operation } R_1 \to R_1 - R_2 \text{ and } R_2 \to R_2 - R_3\text{)}$$

or $$(x+a+b+c)\begin{vmatrix} -x & c-a \\ x & a-x-c \end{vmatrix} = 0 \qquad \text{(expanding } C_1 \text{)}$$

or $$(x+a+b+c)\begin{vmatrix} 0 & -x \\ x & a-x-c \end{vmatrix} = 0 \qquad \text{(operation } R_1 \to R_1 + R_2\text{)}$$

$\Rightarrow$ $$(x+a+b+c)(0+x^2) = 0$$

$\Rightarrow$ $$x^2(x+a+b+c) = 0$$

$\Rightarrow$ $x^2 = 0$ or $x+a+b+c = 0$

$\Rightarrow$ $x = 0$ or $x = -(a+b+c)$

Example 18. Prove that

$$\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ yz & zx & xy \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix} = (y-z)(z-x)(x-y)(yz+zx+xy)\cdot$$

Solution : L.H.S. $= \begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ yz & zx & xy \end{vmatrix}$

$$= \frac{1}{xyz}\begin{vmatrix} x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \\ xyz & xyz & xyz \end{vmatrix}$$ (operation $C_1 \to xC_1, C_2 \to yC_2, C_3 \to zC_3$)

$$= \frac{xyz}{xyz}\begin{vmatrix} x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \\ 1 & 1 & 1 \end{vmatrix}$$ (taking out xyz from the operation R_3)

$$= -\begin{vmatrix} x^2 & y^2 & z^2 \\ 1 & 1 & 1 \\ x^3 & y^3 & z^3 \end{vmatrix}$$ (operation $R_2 \leftrightarrow R_3$)

$$= \begin{vmatrix} 1 & 1 & 1 \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix}$$ (operation $R_1 \leftrightarrow R_2$)

$$= \begin{vmatrix} 0 & 0 & 1 \\ x^2-y^2 & y^2-z^2 & z^2 \\ x^3-y^3 & y^3-z^3 & z^3 \end{vmatrix}$$ (operation $C_1 \to C_1 - C_2$ and $C_2 \to C_2 - C_3$)

$$= \begin{vmatrix} x^2-y^2 & y^2-z^2 \\ x^3-y^3 & y^3-z^3 \end{vmatrix}$$ (Epanding R_1)

$$= \begin{vmatrix} (x-y)(x+y) & (y+z)(y-z) \\ (x-y)(x^2+xy+y^2) & (y-z)(y^2+yz+z^2) \end{vmatrix}$$

$$= (x-y)(y-z)\begin{vmatrix} x+y & y+z \\ x^2+xy+y^2 & y^2+yz+z^2 \end{vmatrix}$$

$$=(x-y)(y-z)\begin{vmatrix} x+y & z-x \\ x^2+xy+y^2 & yz+z^2-x^2-xy \end{vmatrix} \quad \text{(operation } C_2 \to C_2 - C_1 \text{)}$$

$$=(x-y)(y-z)\begin{vmatrix} x+y & z-x \\ x^2+xy+y^2 & (z-x)(z+x)+y(z-x) \end{vmatrix}$$

$$=(x-y)(y-z)\begin{vmatrix} x+y & z-x \\ x^2+xy+y^2 & (z-x)(z+x+y) \end{vmatrix}$$

$$=(x-y)(y-z)(z-x)\begin{vmatrix} x+y & 1 \\ x^2+xy+y^2 & z+x+y \end{vmatrix}$$

$$=(x-y)(y-z)(z-x)\{(x+y)(z+x+y)-(x^2+xy+y^2)\}$$

$$=(x-y)(y-z)(z-x)\cdot(zx+x^2+xy+yz+xy+y^2-x^2-xy-y^2)$$

$$=(x-y)(y-z)(z-x)(xy+yz+zx)$$

= R.H.S.

Example 19. Evaluate the following $\begin{vmatrix} 1 & \log_x y & \log_x z \\ \log_y x & 1 & \log_y z \\ \log_z x & \log_z y & 1 \end{vmatrix}$ without expansion.

Solution : We know that $\log_n m = \dfrac{\log m}{\log n}$

$$\therefore \quad \begin{vmatrix} 1 & \log_x y & \log_x z \\ \log_y x & 1 & \log_y z \\ \log_z x & \log_z y & 1 \end{vmatrix} = \begin{vmatrix} 1 & \dfrac{\log y}{\log x} & \dfrac{\log z}{\log x} \\ \dfrac{\log x}{\log y} & 1 & \dfrac{\log z}{\log y} \\ \dfrac{\log x}{\log z} & \dfrac{\log y}{\log z} & 1 \end{vmatrix}$$

$$= \frac{1}{\log x \cdot \log y \cdot \log z}\begin{vmatrix} \log x & \log y & \log z \\ \log x & \log y & \log z \\ \log x & \log y & \log z \end{vmatrix}$$

(operation $R_1 \to \log x \cdot R_1; R_2 \to \log y \cdot R_2; R_3 \to \log z \cdot R_3$)

$$= \frac{1}{\log x \cdot \log y \cdot \log z} \times 0 \qquad (\because \quad R_1 = R_2 = R_3 \ )$$

= 0

Example 20. Prove that

$$\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix} = 2abc(a+b+c)^3.$$

Solution : L.H.S. $= \begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix}$

$$= \begin{vmatrix} (b+c)^2-a^2 & 0 & a^2 \\ 0 & (c+a)^2-b^2 & b^2 \\ c^2-(a+b)^2 & c^2-(a+b)^2 & (a+b)^2 \end{vmatrix}$$ (operation $C_1 \to C_1 - C_3$ and $C_2 \to C_2 - C_3$)

$$= \begin{vmatrix} (b+c+a)(b+c-a) & 0 & a^2 \\ 0 & (c+a+b)(c+a-b) & b^2 \\ (c+a+b)(c-a-b) & (c+a+b)(c-a-b) & (a+b)^2 \end{vmatrix}$$

$$= (a+b+c)^2 \begin{vmatrix} b+c-a & 0 & a^2 \\ 0 & c+a-b & b^2 \\ c-a-b & c-a-b & (a+b)^2 \end{vmatrix}$$ (Taking out $(a+b+c)$ from C_1 and C_2)

$$= (a+b+c)^2 \begin{vmatrix} b+c-a & 0 & a^2 \\ 0 & c+a-b & b^2 \\ -2b & -2a & 2ab \end{vmatrix}$$ (operation $R_3 \to R_3 - R_1 - R_2$)

$$= (a+b+c)^2 \begin{vmatrix} b+c & \frac{a^2}{b} & a^2 \\ \frac{b^2}{a} & c+a & b^2 \\ 0 & 0 & 2ab \end{vmatrix}$$ (operation $C_1 \to C_1 + \frac{C_3}{a}$ and $C_2 \to C_2 + \frac{C_3}{b}$)

$$=(a+b+c)^2\left\{0+0+2ab\begin{vmatrix} b+c & \frac{a^2}{b} \\ \frac{b^2}{a} & (c+a)\end{vmatrix}\right\}$$ (Expanding along R_3)

$$=(a+b+c)^2\cdot 2ab\{(b+c)(c+a)-ab\}$$

$$=(a+b+c)^2\cdot 2ab\left(bc+ab+c^2+ca-ab\right)$$

$$=(a+b+c)^2\cdot 2ab\left(bc+c^2+ca\right)$$

$$=(a+b+c)^2\cdot 2abc(b+c+a)$$

$$=2abc(a+b+c)^3=\text{R.H.S.}$$

Example 21. Prove that

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b\end{vmatrix}^2=\begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ac-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2\end{vmatrix}.$$

Solution : L.H.S. $=\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b\end{vmatrix}^2=\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b\end{vmatrix}\times\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b\end{vmatrix}$

$$=\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b\end{vmatrix}\times(-1)\begin{vmatrix} a & c & b \\ b & a & c \\ c & b & a\end{vmatrix}$$ $(C_2 \leftrightarrow C_3)$

$$=\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b\end{vmatrix}\times\begin{vmatrix} -a & c & b \\ -b & a & c \\ -c & b & a\end{vmatrix}$$

$$=\begin{vmatrix} -a^2+bc+bc & -ab+ab+c^2 & -ac+b^2+ac \\ -ab+c^2+ab & -b^2+ac+ac & -bc+bc+a^2 \\ -ac+ac+b^2 & -bc+a^2+bc & -c^2+ab+ab\end{vmatrix}$$ (multiply row by row)

$$=\begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ac-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2\end{vmatrix}$$

= R.H.S.

Exercise 4.2

1. If $\begin{vmatrix} \ell & m \\ 2 & 3 \end{vmatrix} = 0$ then find the ratio $\ell : m$

2. Find the minor of the elements of second row of determinant $\begin{vmatrix} 2 & 3 & 4 \\ 3 & 6 & 5 \\ 1 & 8 & 9 \end{vmatrix}$

3. Evaluate the determinant $\begin{vmatrix} 13 & 16 & 19 \\ 14 & 17 & 20 \\ 15 & 18 & 21 \end{vmatrix}$

4. If the first and the third columns of the determinant are interchanged then write the change in the determinant?

5. Prove that

$$\begin{vmatrix} 1 & yz & y+z \\ 1 & zx & z+x \\ 1 & xy & x+y \end{vmatrix} = (x-y)(y-z)(z-x).$$

6. Evaluate the determinant $\begin{vmatrix} 0 & b^2a & c^2a \\ a^2b & 0 & c^2b \\ a^2c & b^2c & 0 \end{vmatrix}$

7. Solve the following determinant:

$$\begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ x-4 & 2x-9 & 3x-16 \\ x-8 & 2x-27 & 3x-64 \end{vmatrix} = 0.$$

8. Without expanding evaluate the determinant

$$\begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix} = \begin{vmatrix} x & y & z \\ p & q & r \\ a & b & c \end{vmatrix} = \begin{vmatrix} y & b & q \\ x & a & p \\ z & c & r \end{vmatrix}.$$

9. Prove that

$$\begin{vmatrix} b+c & a+b & a \\ c+a & b+c & b \\ a+b & c+a & c \end{vmatrix} = a^3+b^3+c^3-3abc.$$

10. Evaluate the determinant $\begin{vmatrix} 1^2 & 2^2 & 3^2 \\ 2^2 & 3^2 & 4^2 \\ 3^2 & 4^2 & 5^2 \end{vmatrix}$

11. If ω is the cube root of unity then find the value of the determinant $\begin{vmatrix} 1 & \omega^3 & \omega^2 \\ \omega^3 & 1 & \omega \\ \omega^2 & \omega & 1 \end{vmatrix}$.

12. Prove that :

$$\begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix} = 4a^2b^2c^2 \cdot$$

13. If in the determinant $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ $A_1, B_1, C_1, \ldots$ are the cofactors of elements $a_1, b_1, c_1, \ldots$ then

Prove that

$$\Delta^2 = \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} \cdot$$

[HINT : $\Delta \cdot \Delta' = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}$,

$$= \begin{vmatrix} \Delta & 0 & 0 \\ 0 & \Delta & 0 \\ 0 & 0 & \Delta \end{vmatrix} = \Delta^3$$

$\therefore$ $\Delta\Delta' = \Delta^3$ or $\Delta' = \Delta^2$

Miscellaneous Exericse – 4

1. The vlaue of the determinant $\begin{vmatrix} \cos 80° & -\cos 10° \\ \sin 80° & \sin 10° \end{vmatrix}$ is

(a) 0 (b) 1 (c) –1 (d) none of these.

2. The cofactors of first column in the determinant $\begin{vmatrix} 5 & 20 \\ 3 & -1 \end{vmatrix}$ are

(a) –1, 3 (b) –1, –3 (c) –1, 20 (d) –1, –20.

3. If $\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 2 & 4 \end{vmatrix}$ then the value of the determinant $\begin{vmatrix} -2 & -4 & -6 \\ -8 & -10 & -12 \\ -2 & -4 & -8 \end{vmatrix}$ is

(a) -2Δ (b) 8Δ (c) -8Δ (d) -6Δ.

4. Which among the below given determinants is same as determinant $\begin{vmatrix} 1 & 0 & 2 \\ 3 & -2 & -1 \\ 2 & 5 & 4 \end{vmatrix}$?

(a) $\begin{vmatrix} 2 & 5 & 4 \\ 3 & -2 & -1 \\ 1 & 0 & 2 \end{vmatrix}$ (b) $\begin{vmatrix} 1 & 3 & 2 \\ 2 & -1 & 4 \\ 0 & -2 & 5 \end{vmatrix}$ (c) $-\begin{vmatrix} 2 & -1 & 4 \\ 0 & -2 & 5 \\ 1 & 3 & 2 \end{vmatrix}$ (d) $\begin{vmatrix} 2 & 0 & 1 \\ -1 & -2 & 3 \\ 4 & 5 & 2 \end{vmatrix}$.

5. The value of the determinant $\begin{vmatrix} \cos 50^\circ & \sin 10^\circ \\ \sin 50^\circ & \cos 10^\circ \end{vmatrix}$ is

(a) 0 (b) 1 (c) 1 / 2 (d) –1 / 2.

6. The value of the determinant $\begin{vmatrix} 1 & bc & a(b+c) \\ 1 & ca & b(c+a) \\ 1 & ab & c(a+b) \end{vmatrix}$ is

(a) $ab+bc+ca$ (b) 0 (c) 1 (d) abc.

7. If ω is the root of unity then the value of the determinant $\begin{vmatrix} 1 & \omega^4 & \omega^8 \\ \omega^4 & \omega^8 & 1 \\ \omega^8 & 1 & \omega^4 \end{vmatrix}$ is

(a) ω^2 (b) ω (c) 1 (d) 0.

8. If $\begin{vmatrix} 4 & 1 \\ 2 & 1 \end{vmatrix}^2 = \begin{vmatrix} 3 & 2 \\ 1 & x \end{vmatrix} - \begin{vmatrix} x & 3 \\ -2 & 1 \end{vmatrix}$ then the value of x is

(a) 6 (b) 7 (c) 8 (d) 0.

9. If $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ and cofactors corresponding to elements $a_{11}, a_{12}, a_{13}, \ldots$ are $F_{11}, F_{12}, F_{13}, \ldots$

then the correct statement is

(a) $a_{12}F_{12} + a_{22}F_{22} + a_{32}F_{32} = 0$ (b) $a_{12}F_{12} + a_{22}F_{22} + a_{32}F_{32} \neq \Delta$

(c) $a_{12}F_{12} + a_{22}F_{22} + a_{32}F_{32} = \Delta$ (d) $a_{12}F_{12} + a_{22}F_{22} + a_{32}F_{32} = -\Delta$.

10. The value of the determinant $\begin{vmatrix} x+y & y+z & z+x \\ z & x & y \\ 2 & 2 & 2 \end{vmatrix}$ is

(a) $x+y+z$ (b) $2(x+y+z)$ (c) 1 (d) 0.

11. Solve the following equation $\begin{vmatrix} 1 & 2 & 3 \\ 4 & x & 6 \\ 7 & 8 & 9 \end{vmatrix} = 0.$

12. Evaluate the determinant $\begin{vmatrix} 1 & 3 & 9 \\ 3 & 9 & 1 \\ 9 & 1 & 3 \end{vmatrix}.$

13. Evaluate the determinant $\begin{vmatrix} 1+a & b & c \\ a & 1+b & c \\ a & b & 1+c \end{vmatrix}$.

14. Prove that

$$\begin{vmatrix} -a^2 & ab & ac \\ ab & -b^2 & bc \\ ca & cb & -c^2 \end{vmatrix} = 4a^2b^2c^2.$$

15. Prove that one root of the equation is $x = 2$ and hence find the remaining roots

$$\begin{vmatrix} x & -6 & -1 \\ 2 & -3x & x-3 \\ -3 & 2x & x+2 \end{vmatrix} = 0.$$

Prove that [Q 16 to 20]

16. $\begin{vmatrix} a+b+c & -c & -b \\ -c & a+b+c & -a \\ -b & -a & c+a+b \end{vmatrix} = 2(a+b)(b+c)(c+a).$

17. $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3.$

18. $\begin{vmatrix} y+z & x & y \\ z+x & z & x \\ x+y & y & z \end{vmatrix} = (x+y+z)(x-z)^2.$

19. $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (b-c)(c-a)(a-b)(a+b+c).$

20. $\begin{vmatrix} \frac{a^2+b^2}{c} & c & c \\ a & \frac{b^2+c^2}{a} & a \\ b & b & \frac{c^2+a^2}{b} \end{vmatrix} = 4abc$ (Hint: using operation $R_1 \to cR_1, R_2 \to aR_2$ and $R_3 \to bR_3$)

21. If $a+b+c=0$ then solve the following equation

$$\begin{vmatrix} a-x & c & b \\ c & b-x & a \\ b & a & c-x \end{vmatrix} = 0.$$

22. Prove that

$$\begin{vmatrix} a & a+b & a+2b \\ a+2b & a & a+b \\ a+b & a+2b & a \end{vmatrix} = 9(a+b)b^2$$

23. If $p+q+r=0$ then prove that

$$\begin{vmatrix} pa & qb & rc \\ qc & ra & pb \\ rb & pc & qa \end{vmatrix} = pqr\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$$

(Hint : L.H.S. $= pqr(a^3+b^3+c^3) - abc(p^3+q^3+r^3)$ $\because$ $p+q+r=0 \Rightarrow p^3+q^3+r^3 = 3pqr$

$\therefore$ L.H.S. $= pqr(a^3+b^3+c^3-3abc) =$ R.H.S.

24. Prove that

$$\begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix} = (5x+4)(x-4)^2$$

IMPORTANT POINTS

1. Second order determinant $\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1 \cdot$

2. Third order determinant =

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$\Delta = \left|\begin{array}{ccc|cc} a_1 & b_1 & c_1 & a_1 & b_1 \\ a_2 & b_2 & c_2 & a_2 & b_2 \\ a_3 & b_3 & c_3 & a_3 & b_3 \end{array}\right.$$

(From Sarrus diagram)

$$= \left(a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3\right) - \left(a_3b_2c_1 + b_3c_2a_1 + c_3a_2b_1\right)$$

3. Difference between matrix and determinant.
   - (i) There is no value of matrix whereas determinant has a unique value
   - (ii) Matrix can be of any order while determinant is always of order $n \times n$.
   - (iii) In determinant $|A| = |A^T|$ whereas in matrix $[A] \neq [A^T] \cdot$

4. Minor of an element a_{ij} of the determinant of matrix A is the determinant obtained by deleting *i*th row and *j*th column and denoted by A*ij*.

5. Cofactor of element $a_{ij} = (-1)^{i+j}$ Minor

   $\Rightarrow$ Cofactor of $a_{ij} = a_{ij}$, when $i + j$ is even

   $= -(a_{ij}$ Minor of), when $i + j$ is odd

6. Expansion of determinant $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$
   - (i) in terms of minors $\Delta = a_{11}A_{11} - a_{12}A_{12} + a_{13}A_{13}$
   - (ii) in terms of co-factors $\Delta = a_{11}F_{11} + a_{12}F_{12} + a_{13}F_{13}$

7. For any square matrix A, the |A| satisfies following properties.
   - (i) If we interchange any two rows (or columns), then sign of determinant changes, but value remains unchanged.
   - (ii) If any two rows or any two columns are identical or propertional, then value of determinant is zero.
   - (iii) If we multiply each element of a row or a column of a determinant by constant *k*, then vlaue of determinant is multiplied by *k*.

(iv) Multiplying a determinant by k means multiply elements of only one row (or one column) by k.

(v) If elements of a row or a column in a determinant can be expressed as sum of two or more elements, then the given determinant can be expressed as sum of two or more determinants.

(vi) If each element of a row or a column of a determinant the equimultiples of corresponding elements of other rows or columns are added or subtracted, then value of determinant remains same.

(vii) If all rows are converted into columns or all columns converted in rows in any determinant the value of determinant remains same.

(viii) If any row or column contains all its element as zero then the value of determinant will be zero.

(ix) Value of Determinant of triangular matrices is equal to product of element of principal diagonal.

(x) Multiplication of determinant is done by row to column and row to row law.

Answers

Exericse 4.1

1. $\frac{-8}{3}$ 2. 1 : 2 3. $x = \frac{-5}{2}, y = -3$ 4. $\frac{3}{2}$

5. (i) $A_{11} = -12, A_{21} = -16, A_{31} = -4$

$F_{11} = -12, F_{21} = 16, F_{31} = -4, \quad 40$

(ii) $A_{11} = bc - f^2, \; A_{21} = hc - fg, \; A_{31} = hf - bg$

$F_{11} = bc - f^2, \; F_{21} = fg - hc, \; F_{31} = hf - bg;$

$abc + 2fgh - af^2 - bg^2 - ch^2$

6. 15

Exercise 4.2

1. 2 : 3 2. Minor of $3 = \begin{vmatrix} 3 & 4 \\ 8 & 9 \end{vmatrix}$, Minor of $6 = \begin{vmatrix} 2 & 4 \\ 1 & 9 \end{vmatrix}$ and Minor of $5 = \begin{vmatrix} 2 & 3 \\ 1 & 8 \end{vmatrix}$.

3. 0 4. The sign of the determinant changes 6. $2a^3b^3c^3$

7. x = 4 10. = –8 11. 3

Miscellaneous Exercise . 4

1. (b) 2. (d) 3. (c) 4. (c) 5. (c) 6. (b) 7. (d)

8. (a) 9. (c) 10. (d) 11. 5 12. –676 13. $1 + a + b + c$

15. 1, –3 21. $0, \pm\sqrt{\frac{3}{2}\left(a^2 + b^2 + c^2\right)}$

❑