

## प्रतिलोम वृत्तीय फलन

### Ex 2.1

प्रश्न 1. निम्नलिखित कोणों के मुख्य मान ज्ञात कीजिए:

(i)  $\sin^{-1}(1)$                       (ii)  $\cos^{-1}\left(-\frac{1}{2}\right)$

(iii)  $\sec^{-1}(-\sqrt{2})$                       (iv)  $\operatorname{cosec}^{-1}(1)$

(v)  $\cot^{-1}\left(-\sqrt{\frac{1}{3}}\right)$                       (vi)  $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

हल : (i)  $\sin^{-1}(1)$

$\sin^{-1}$  की मुख्य मान शाखा  $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$  है।

माना  $\sin^{-1}(1) = x$

$$\Rightarrow \sin x = 1$$

$$\Rightarrow \sin x = \sin \frac{\pi}{2}$$

$$\Rightarrow x = \frac{\pi}{2}$$

$$\therefore \sin^{-1}(1) \text{ का मुख्य मान} = \frac{\pi}{2}$$

(ii)  $\cos^{-1}\left(-\frac{1}{2}\right)$

$\cos^{-1}$  की मुख्य मान शाखा  $[0, \pi]$  है।

माना  $\cos^{-1}\left(-\frac{1}{2}\right) = x$

$$\Rightarrow \cos x = -\frac{1}{2} = -\cos \frac{\pi}{3}$$

$$\Rightarrow \cos x = \cos \left[ \pi - \frac{\pi}{3} \right] = \cos \frac{2\pi}{3}$$

$$\text{जहाँ } \frac{2\pi}{3} \in [0, \pi]$$

$$\Rightarrow x = \frac{2\pi}{3}$$

$$\therefore \cos^{-1}\left(-\frac{1}{2}\right) \text{ का मुख्य मान} = \frac{2\pi}{3}$$

(iii)  $\sec^{-1}(-\sqrt{2})$

$\sec^{-1}$  का मुख्य मान शाखा  $[0, \pi] - \left\{\frac{\pi}{2}\right\}$  है।

$$\text{माना } \sec^{-1}(-\sqrt{2}) = x$$

$$\Rightarrow \sec x = -\sqrt{2} = -\sec \frac{\pi}{4}$$

$$\Rightarrow \sec x = \sec\left(\pi - \frac{\pi}{4}\right) = \sec \frac{3\pi}{4}$$

$$\text{जहाँ } \frac{3\pi}{4} \in [0, \pi] - \left\{\frac{\pi}{2}\right\}$$

$$\Rightarrow x = \frac{3\pi}{4}$$

$$\therefore \sec^{-1}(-\sqrt{2}) \text{ का मुख्य मान } = \frac{3\pi}{4}$$

(iv)  $\operatorname{cosec}^{-1}(-1)$

$\operatorname{cosec}^{-1}$  की मुख्य मान शाखा  $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$  है।

माना  $\operatorname{cosec}^{-1}(-1) = x$

$$\Rightarrow \operatorname{cosec} x = -1 = -\operatorname{cosec} \frac{\pi}{2}$$

$$\Rightarrow x = -\frac{\pi}{2}$$

$$\therefore \operatorname{cosec}^{-1}(-1) \text{ का मुख्य मान } = -\frac{\pi}{2}$$

(v)  $\cot^{-1}\left(-\frac{1}{\sqrt{3}}\right)$

$\cot^{-1}$  का मुख्य मान शाखा  $[0, \pi]$  है।

$$\text{माना } \cot^{-1}\left(-\frac{1}{\sqrt{3}}\right) = x$$

$$\Rightarrow \cot x = -\frac{1}{\sqrt{3}} = \cot\left(\pi - \frac{\pi}{3}\right)$$

$$\Rightarrow x = \frac{2\pi}{3} \quad \text{जहाँ } \frac{2\pi}{3} \in [0, \pi]$$

$$\therefore \cot^{-1}\left(-\frac{1}{\sqrt{3}}\right) \text{ का मुख्य मान } = \frac{2\pi}{3}$$

(vi)  $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

$\tan^{-1}$  की मुख्य मान शाखा  $[-\frac{\pi}{2}, \frac{\pi}{2}]$  है।

$$\text{माना } \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = x$$

$$\Rightarrow \tan x = \frac{1}{\sqrt{3}} = \tan \frac{\pi}{6}$$

$$\Rightarrow x = \frac{\pi}{6} \quad \text{जहाँ } \frac{\pi}{6} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$\therefore \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) \text{ का मुख्य मान } = \frac{\pi}{6}$$

प्रश्न 2.

$$2 \tan^{-1} \frac{1}{2} - \tan^{-1} \frac{1}{7} = \frac{\pi}{4}$$

हल : LHS

$$= 2 \tan^{-1} \frac{1}{2} - \tan^{-1} \frac{1}{7}$$

$$= \tan^{-1} \frac{2 \times \frac{1}{2}}{1 - \left(\frac{1}{2}\right)^2} - \tan^{-1} \frac{1}{7}$$

$$\left[ \because 2 \tan^{-1} x = \tan^{-1} \frac{2x}{1-x^2} \right]$$

$$= \tan^{-1} \frac{1}{1 - \frac{1}{4}} - \tan^{-1} \frac{1}{7}$$

$$= \tan^{-1} \frac{4}{3} - \tan^{-1} \frac{1}{7} = \tan^{-1} \frac{\frac{4}{3} - \frac{1}{7}}{1 + \frac{4}{3} \times \frac{1}{7}}$$

$$\left[ \because \tan^{-1} x - \tan^{-1} y = \tan^{-1} \frac{x-y}{1+xy} \right]$$

$$= \tan^{-1} \frac{28-3}{21+4} = \tan^{-1} \frac{25}{25}$$

$$= 2 \tan^{-1} \frac{1}{2} - \tan^{-1} \frac{1}{7}$$

$$= \tan^{-1} \frac{2 \times \frac{1}{2}}{1 - \left(\frac{1}{2}\right)^2} - \tan^{-1} \frac{1}{7}$$

$$\left[ \because 2 \tan^{-1} x = \tan^{-1} \frac{2x}{1-x^2} \right]$$

$$= \tan^{-1} \frac{1}{1 - \frac{1}{4}} - \tan^{-1} \frac{1}{7}$$

$$= \tan^{-1} \frac{4}{3} - \tan^{-1} \frac{1}{7} = \tan^{-1} \frac{\frac{4}{3} - \frac{1}{7}}{1 + \frac{4}{3} \times \frac{1}{7}}$$

$$\left[ \because \tan^{-1} x - \tan^{-1} y = \tan^{-1} \frac{x-y}{1+xy} \right]$$

$$= \tan^{-1} \frac{28-3}{21+4} = \tan^{-1} \frac{25}{25}$$

इति सिद्धम्।

प्रश्न 3.

$$\tan^{-1} \frac{17}{19} - \tan^{-1} \frac{2}{3} = \tan^{-1} \frac{1}{7}$$

हल : LHS

$$= \tan^{-1} \frac{17}{19} - \tan^{-1} \frac{2}{3}$$

$$= \tan^{-1} \frac{\frac{17}{19} - \frac{2}{3}}{1 + \frac{17}{19} \times \frac{2}{3}}$$

$$\left[ \because \tan^{-1} x - \tan^{-1} y = \tan^{-1} \left( \frac{x-y}{1+xy} \right) \right]$$

$$\begin{aligned}
&= \tan^{-1} \frac{51-38}{\frac{19 \times 3}{57+34}} \\
&= \tan^{-1} \frac{13}{\frac{19 \times 3}{19 \times 3}} \\
&= \tan^{-1} \frac{13}{91} = \tan^{-1} \frac{1}{7} \\
&= \text{R.H.S.}
\end{aligned}$$

इति सिद्धम्।

प्रश्न 4.

$$\cos^{-1} \frac{63}{65} + 2 \tan^{-1} \frac{1}{5} = \sin^{-1} \frac{3}{5}$$

हल : LHS

$$\begin{aligned}
&= \cos^{-1} \frac{63}{65} + 2 \tan^{-1} \frac{1}{5} \\
&= \cos^{-1} \frac{63}{65} + \cos^{-1} \left( \frac{1 - \left(\frac{1}{5}\right)^2}{1 + \left(\frac{1}{5}\right)^2} \right) \\
&\quad \left[ \because 2 \tan^{-1} x = \cos^{-1} \left( \frac{1 - x^2}{1 + x^2} \right) \right] \\
&= \cos^{-1} \frac{63}{65} + \cos^{-1} \left( \frac{\frac{24}{25}}{\frac{26}{25}} \right) \\
&= \cos^{-1} \frac{63}{65} + \cos^{-1} \frac{12}{13} \\
&= \cos^{-1} \left[ \frac{63 \times 12}{65 \times 13} - \sqrt{\left\{ 1 - \left(\frac{63}{65}\right)^2 \right\} \left\{ 1 - \left(\frac{12}{13}\right)^2 \right\}} \right]
\end{aligned}$$

$$\begin{aligned}
& \left[ \because \cos^{-1} x + \cos^{-1} y = \left[ xy - \sqrt{(1-x^2)(1-y^2)} \right] \right] \\
& = \cos^{-1} \left[ \frac{63 \times 12}{65 \times 13} - \frac{\sqrt{\{(65)^2 - (63)^2\} \{169 - 144\}}}{65 \times 13} \right] \\
& = \cos^{-1} \left[ \frac{63 \times 12}{65 \times 13} - \frac{\sqrt{128 \times 2 \times 25}}{65 \times 13} \right] \\
& = \cos^{-1} \left[ \frac{63 \times 12 - 2 \times 8 \times 5}{65 \times 13} \right] \\
& = \cos^{-1} \left[ \frac{756 - 80}{65 \times 13} \right] \\
& = \cos^{-1} \left[ \frac{676}{65 \times 13} \right] \\
& = \cos^{-1} \frac{4}{5} \\
& = \sin^{-1} \sqrt{1 - \left(\frac{4}{5}\right)^2} \quad \left[ \because \cos^{-1} x = \sin^{-1} \sqrt{1 - x^2} \right] \\
& = \sin^{-1} \sqrt{\frac{25 - 16}{25}} \\
& = \sin^{-1} \sqrt{\frac{9}{25}} \\
& = \sin^{-1} \frac{3}{5} \\
& = \text{R.H.S.}
\end{aligned}$$

इति सिद्धम्।

**प्रश्न 5.**  $\sec^2(\tan^{-1}2) + \operatorname{cosec}^2(\cot^{-1}3) = 15$

**हल :** माना  $\tan^{-1}2 = \theta \Rightarrow \tan \theta = 2$

$$\therefore \sec^2 \theta = 1 + \tan^2 \theta$$

$$= 1 + (2)^2 = 1 + 4 = 5$$

$$\therefore \sec^2(\tan^{-1}2) = 5 \dots(i)$$

माना  $\cot^{-1}3 = \Phi$

$$\Rightarrow \cot \Phi = 3$$

$$\therefore \operatorname{cosec}^2 \Phi = 1 + \cot^2 \Phi$$

$$= 1 + (3)^2 = 1 + 9 = 10$$

$$\therefore \operatorname{cosec}^2(\cot^{-1}3) = 10 \dots(ii)$$

(i) और (ii) को जोड़ने पर

$$\sec^2(\tan^{-1}2) + \operatorname{cosec}^2(\cot^{-1}3) = 5 + 10$$

$$\sec^2(\tan^{-1}2) + \operatorname{cosec}^2(\cot^{-1}3) = 15.$$

इति सिद्धम्

**प्रश्न 6.**

$$2 \tan^{-1} x = \sin^{-1} \frac{2x}{1+x^2} = \cos^{-1} \frac{1-x^2}{1+x^2}$$

हल : (i)  $2 \tan^{-1} x = \sin^{-1} \frac{2x}{1+x^2}$

माना  $\tan^{-1} x = \theta$

$$\therefore x = \tan \theta$$

$$\begin{aligned} \text{R.H.S.} &= \sin^{-1} \frac{2x}{1+x^2} \\ &= \sin^{-1} \left( \frac{2 \tan \theta}{1 + \tan^2 \theta} \right) \end{aligned}$$

$$\begin{aligned}
&= \sin^{-1} \left( \frac{2 \tan \theta}{\sec^2 \theta} \right) \\
&= \sin^{-1} (2 \sin \theta \cos \theta) \\
&= \sin^{-1} (\sin 2\theta) \\
&= 2\theta \\
&= 2 \tan^{-1} x \\
&= \text{L.H.S.}
\end{aligned}$$

$$\begin{aligned}
\text{(ii) R.H.S.} &= \cos^{-1} \left( \frac{1-x^2}{1+x^2} \right) \\
&= \cos^{-1} \left( \frac{1-\tan^2 \theta}{1+\tan^2 \theta} \right) \\
&= \cos^{-1} \left( \frac{\cos^2 \theta - \sin^2 \theta}{\cos^2 \theta + \sin^2 \theta} \right) \\
&= \cos^{-1} (\cos 2\theta) \\
&= 2\theta \\
&= 2 \tan^{-1} x \\
&= \text{L.H.S.}
\end{aligned}$$

$$\begin{aligned}
\text{अतः } 2 \tan^{-1} x &= \sin^{-1} \frac{2x}{1+x^2} \\
&= \cos^{-1} \frac{1-x^2}{1+x^2}
\end{aligned}$$

इति सिद्धम्।

प्रश्न 7.

$$\tan^{-1} \sqrt{\frac{ax}{bc}} + \tan^{-1} \sqrt{\frac{bx}{ca}} + \tan^{-1} \sqrt{\frac{cx}{ab}} = \pi$$

जहाँ  $a + b + c = x$

हल : LHS

$$\begin{aligned}
&= \tan^{-1}\left(\frac{ax}{bc}\right) + \tan^{-1}\left(\frac{bx}{ca}\right) + \tan^{-1}\left(\frac{cx}{ab}\right) \\
&= \tan^{-1}\left[\frac{\sqrt{\frac{ax}{bc}} + \sqrt{\frac{bx}{ca}} + \sqrt{\frac{ca}{ab}} - \sqrt{\frac{ax}{bc}} \cdot \sqrt{\frac{bx}{ca}} \cdot \sqrt{\frac{cx}{ab}}}{1 - \sqrt{\frac{ax}{bc}} \sqrt{\frac{bx}{ca}} - \sqrt{\frac{bx}{ca}} \sqrt{\frac{cx}{ab}} - \sqrt{\frac{ca}{ab}} \sqrt{\frac{ax}{bc}}}\right] \\
&\left(\because \tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \tan^{-1} \frac{x + y + z - xyz}{1 - xy - yz - zx}\right) \\
&= \tan^{-1}\left[\frac{\frac{a\sqrt{x} + b\sqrt{x} + c\sqrt{x}}{\sqrt{abc}} - \frac{\sqrt{x}\sqrt{x}\sqrt{x}}{\sqrt{abc}}}{1 - \frac{x}{c} - \frac{x}{a} - \frac{x}{b}}\right] \\
&= \left[\frac{\frac{\sqrt{x}(a+b+c) - x\sqrt{x}}{\sqrt{abc}}}{\left(1 - \frac{x}{a} - \frac{x}{b} - \frac{x}{c}\right)}\right] \\
&= \left[\frac{\frac{x\sqrt{x} - x\sqrt{x}}{\sqrt{abc}}}{\left(1 - \frac{x}{a} - \frac{x}{b} - \frac{x}{c}\right)}\right] \quad (\because a + b + c = x)
\end{aligned}$$

$$= \tan^{-1}(0)$$

$$= \tan^{-1}(\pi)$$

$$= \pi$$

$$= \text{RHS}$$

इति सिद्धम्।

प्रश्न 8.

$$\frac{1}{2} \tan^{-1} x = \cos^{-1} \left\{ \frac{1 + \sqrt{1 + x^2}}{2\sqrt{1 + x^2}} \right\}^{\frac{1}{2}}$$

हल : माना  $\tan^{-1} x = \theta$

$$x = \tan \theta$$

$$\begin{aligned} \text{R.H.S.} &= \cos^{-1} \left\{ \frac{1 + \sqrt{1 + \tan^2 \theta}}{2\sqrt{1 + \tan^2 \theta}} \right\}^{\frac{1}{2}} \\ &= \cos^{-1} \left\{ \frac{1 + \sec \theta}{2\sec \theta} \right\}^{\frac{1}{2}} = \cos^{-1} \left( \frac{\cos \theta + 1}{2} \right)^{\frac{1}{2}} \\ &= \cos^{-1} \left( \frac{2\cos^2 \frac{\theta}{2}}{2} \right)^{\frac{1}{2}} = \cos^{-1} \cos \frac{\theta}{2} = \frac{\theta}{2} \\ &= \frac{1}{2} \tan^{-1} x = \text{L.H.S.} \end{aligned}$$

इति सिद्धम्।

**प्रश्न 9.** यदि  $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = \pi$ , तो सिद्ध कीजिए कि  $x^2 + y^2 + z^2 + 2xyz = 1$ .

**हल :**  $\because \cos^{-1}x + \cos^{-1}y + \cos^{-1}z = \pi$

$$\Rightarrow \cos^{-1}x + \cos^{-1}y = \pi - \cos^{-1}z$$

$$\Rightarrow \cos^{-1} [xy - \sqrt{1-x^2}\sqrt{1-y^2}] = \cos^{-1}(-z)$$

$$\Rightarrow [\because \cos^{-1}x + \cos^{-1}y = \cos^{-1} [xy - \sqrt{1-x^2}\sqrt{1-y^2}] \text{ तथा } (\cos^{-1}(-x) = \pi - \cos^{-1}x)]$$

$$\Rightarrow xy - \sqrt{1-x^2}\sqrt{1-y^2} = (-z)$$

$$\Rightarrow xy + z = \sqrt{1-x^2}\sqrt{1-y^2}$$

$$\Rightarrow (xy + z)^2 = (1-x^2)(1-y^2)$$

$$\Rightarrow x^2y^2 + z^2 + 2xyz = 1 - y^2 - x^2 + x^2y^2$$

$$\Rightarrow z^2 + 2xyz = 1 - y^2 - x^2$$

$$\Rightarrow x^2 + y^2 + z^2 + 2xyz = 1$$

इति सिद्धम्।

प्रश्न 10. यदि  $\sin^{-1} x + \sin^{-1} y + \sin^{-1} z = \pi$ , तो सिद्ध कीजिए

कि  $x\sqrt{1-x^2} + y\sqrt{1-y^2} + z\sqrt{1-z^2} = 2xyz$ .

हल : माना

$$\sin^{-1} x = A \therefore x = \sin A$$

$$\sin^{-1} y = B \therefore y = \sin B$$

$$\sin^{-1} z = C \therefore z = \sin C$$

$$\therefore \sin^{-1} x + \sin^{-1} y + \sin^{-1} z = \pi$$

$$A + B + C = \pi$$

$\therefore$  सिद्ध करना है कि

$$x\sqrt{1-x^2} + y\sqrt{1-y^2} + z\sqrt{1-z^2} = 2xyz$$

L.H.S.

$$= \sin A \sqrt{1 - \sin^2 A} + \sin B \sqrt{1 - \sin^2 B} + \sin C \sqrt{1 - \sin^2 C}$$

$$= \sin A \cos A + \sin B \cos B + \sin C \cos C$$

$$= \frac{1}{2} [2 \sin A \cos A + 2 \sin B \cos B + 2 \sin C \cos C]$$

$$= \frac{1}{2} [\sin 2A + \sin 2B + \sin 2C]$$

$$= \frac{1}{2} \left[ 2 \sin \left( \frac{2A+2B}{2} \right) \cos \left( \frac{2A-2B}{2} \right) + \sin 2C \right]$$

$$\left[ \because \sin C + \sin D = 2 \sin \frac{C+D}{2} \cos \frac{C-D}{2} \right]$$

$$\Rightarrow \frac{1}{2} [2 \sin (A+B) \cos (A-B) + 2 \sin C \cos C]$$

$$\Rightarrow \sin (A+B) \cos (A-B) + \sin C \cos C$$

$$\Rightarrow \sin (\pi - C) \cos (A-B) + \sin C \cos C$$

$$\Rightarrow \sin C \cos (A-B) + \sin C \cos C$$

$$\Rightarrow \sin ( [\cos (A-B) + \cos \{\pi - (A+B)\}] )$$

$$\Rightarrow \sin ( [\cos (A-B) - \cos (A+B)] )$$

$$\Rightarrow \sin ( [2 \sin A \sin B] )$$

$$[\because \cos C - \cos D = 2 \sin \frac{C+D}{2} \sin \frac{D-C}{2}]$$

$$= 2 \sin A \sin B \sin C$$

$$= 2xyz$$

$$= \text{R.H.S.}$$

इति सिद्धम्

प्रश्न 11. यदि  $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \frac{\pi}{2}$ , तो सिद्ध कीजिए कि  $xy + yz + zx = 1$ .

हल : प्रश्नानुसार

$$\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \frac{\pi}{2}$$

$$\Rightarrow \tan^{-1} \left( \frac{x+y+z-xyz}{1-xy-yz-xz} \right) = \frac{\pi}{2}$$

$$\Rightarrow \frac{x+y+z-xyz}{1-xy-yz-xz} = \tan \frac{\pi}{2}$$

$$\Rightarrow \frac{x+y+z-xyz}{1-xy-yz-xz} = \infty$$

$$\Rightarrow 1 - xy - yz - zx = 0$$

$$\Rightarrow xy + yz + zx = 1$$

इति सिद्धम्

प्रश्न 12. यदि

$$\frac{1}{2} \sin^{-1} \frac{2x}{1-x^2} + \frac{1}{2} \cos^{-1} \frac{1-y^2}{1+y^2} + \frac{1}{3} \tan^{-1} \frac{3z-z^3}{1-3z^2} = 5\pi,$$

तो सिद्ध कीजिए कि  $x + y + z = xyz$ .

हल : माना  $x = \tan A, y = \tan B, z = \tan C$

$$\frac{1}{2} \sin^{-1} \left( \frac{2x}{1+x^2} \right) + \frac{1}{2} \cos^{-1} \left( \frac{1-y^2}{1+y^2} \right) + \frac{1}{3} \tan^{-1} \left( \frac{3z-z^3}{1-3z^2} \right) = 5\pi$$

$$\Rightarrow \frac{1}{2} \sin^{-1} \left( \frac{2 \tan A}{1 + \tan^2 A} \right) + \frac{1}{2} \cos^{-1} \left( \frac{1 - \tan^2 B}{1 + \tan^2 B} \right) + \frac{1}{3} \tan^{-1} \left( \frac{3 \tan C - \tan^3 C}{1 - 3 \tan^2 C} \right) = 5\pi$$

$$\Rightarrow \frac{1}{2} \sin^{-1} (\sin 2A) + \frac{1}{2} \cos^{-1} (\cos 2B) + \frac{1}{3} \tan^{-1} (\tan 3C) = 5\pi$$

$$\Rightarrow \frac{1}{2}(2A) + \frac{1}{2}(2B) + \frac{1}{2}(3C) = 5\pi$$

$$\Rightarrow A + B + C = 5\pi$$

$$\Rightarrow A + B = 5\pi - C$$

$$\tan(A + B) = \tan(5\pi - C)$$

$$\Rightarrow \frac{\tan A + \tan B}{1 - \tan A \tan B} = -\tan C$$

$$\Rightarrow \frac{x + y}{1 - xy} = -x$$

( $\tan A$ ,  $\tan B$  व  $\tan C$  के मान रखने पर)

$$\Rightarrow x + y = -2(1 - xy) = -2 + 2xy$$

$$x + y + z = xyz.$$

इति सिद्धम्

प्रश्न 13. यदि

$$\sec^{-1}(\sqrt{1+x^2}) + \operatorname{cosec}^{-1}\left(\frac{\sqrt{1+y^2}}{y}\right) + \cot^{-1}\left(\frac{1}{z}\right) = 3\pi,$$

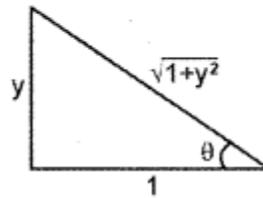
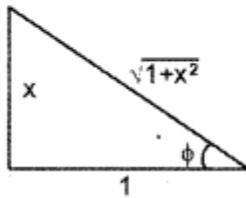
तो सिद्ध कीजिए कि  $x + y + z = xyz$ .

हल : प्रश्नानुसार

$$\sec^{-1}(\sqrt{1+x^2}) + \operatorname{cosec}^{-1}\left(\frac{\sqrt{1+y^2}}{y}\right) + \cot^{-1}\left(\frac{1}{z}\right) = 3\pi$$

$$\Rightarrow \cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right) + \tan^{-1}\frac{y}{\sqrt{1+y^2}} \quad (2) = 3\pi$$

पाइथागोरस प्रमेय तथा त्रिकोणमितीय अनुपातों से,



जहाँ  $\cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right) = \theta = \tan^{-1}\left(\frac{x}{1}\right)$

$$\text{तथा} \quad \sin^{-1}\left(\frac{y}{\sqrt{1+y^2}}\right) = \phi = \tan^{-1}\left(\frac{y}{1}\right)$$

$$\Rightarrow \tan^{-1}(x) + \tan^{-1}(y) + \tan^{-1}(z) = 3\pi$$

$$\Rightarrow \tan^{-1}\left(\frac{x+y+z-xyz}{1-xy-yz-zx}\right) = 3\pi$$

$$\Rightarrow \frac{x+y+z-xyz}{1-xy-yz-zx} = \tan(3\pi) = 0$$

$$x+y+z-xyz=0$$

$$x+y+z=xyz$$

इति सिद्धम्॥

**प्रश्न 14. सिद्ध कीजिए कि**

$$\tan^{-1}x + \cot^{-1}(x+1) = \tan^{-1}(x^2+x+1)$$

**हल :** LHS

$$= \tan^{-1}x + \cot^{-1}(x+1)$$

$$= \tan^{-1}x + \tan^{-1}\left(\frac{1}{x+1}\right)$$

$$= \tan^{-1}\left(\frac{x + \frac{1}{x+1}}{1 - x \times \frac{1}{x+1}}\right)$$

$$= \tan^{-1}\left(\frac{x^2+x+1}{x+1-x}\right)$$

$$= \tan^{-1}(x^2+x+1)$$

= RHS

इति सिद्धम्॥

**प्रश्न 15. यदि  $\tan^{-1}x, \tan^{-1}y, \tan^{-1}z$ , समान्तर श्रेणी में हो, तो सिद्ध कीजिए कि  $y^2(x+z) + 2y(1-xz) - x-z = 0$**

**हल :**  $\tan^{-1}x, \tan^{-1}y, \tan^{-1}z$ , समान्तर श्रेणी में हैं, अतः

$$\therefore \tan^{-1} z + \tan^{-1} x = 2 \tan^{-1} y$$

$$\Rightarrow \tan^{-1} \left( \frac{z+x}{1-zx} \right) = 2 \tan^{-1} y$$

$$\Rightarrow \tan^{-1} \left( \frac{z+x}{1-zx} \right) = \tan^{-1} \left( \frac{2y}{1-y^2} \right)$$

$$\Rightarrow \frac{z+x}{1-zx} = \frac{2y}{1-y^2}$$

$$\Rightarrow (z+x)(1-y^2) = 2y(1-zx)$$

$$\Rightarrow z+x-y^2(x+z) = 2y(1-zx)$$

$$\Rightarrow y^2(x+z) + 2y(1-zx) - x - z = 0$$

इति सिद्धम्॥

**प्रश्न 16.** यदि  $x^3 + px^2 + qx + p = 0$  के मूल  $\alpha, \beta, \gamma$  हो, तो सिद्ध कीजिए कि एक विशेष परिस्थिति के अलावा  $\tan^{-1} \alpha + \tan^{-1} \beta + \tan^{-1} \gamma = n\pi$  और वह विशेष स्थिति भी ज्ञात कीजिए जब ऐसा नहीं होता है।

हल : दिया है :

$\alpha, \beta, \gamma$  समीकरण :  $x^3 + px^2 + qx + p = 0$  के मूल हैं; अतः

$$\alpha + \beta + \gamma = - \left( \frac{x^2 \text{ का गुणांक}}{x^3 \text{ का गुणांक}} \right) = - \frac{p}{1} = -p$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = \left( \frac{x^2 \text{ का गुणांक}}{x^3 \text{ का गुणांक}} \right) = \frac{q}{1} = q$$

$$\text{तथा } \alpha\beta\gamma = - \left( \frac{\text{अचर पद}}{x^3 \text{ का गुणांक}} \right) = -p$$

$$\text{L.H.S.} = \tan^{-1} \alpha + \tan^{-1} \beta + \tan^{-1} \gamma$$

$$= \tan^{-1} \left[ \frac{\alpha + \beta + \gamma - \alpha\beta\gamma}{1 - (\alpha\beta + \beta\gamma + \gamma\alpha)} \right]$$

$$= \tan^{-1} (0) [\because \alpha + \beta + \gamma = \alpha\beta\gamma = -p]$$

$$= n\pi$$

$$= \text{RHS}$$

प्रश्न 17.

$$\sec^{-1}\left(\frac{x}{a}\right) - \sec^{-1}\left(\frac{x}{b}\right) = \sec^{-1}b - \sec^{-1}a.$$

हल :

$$\sec^{-1}\left(\frac{x}{a}\right) - \sec^{-1}\left(\frac{x}{b}\right) = \sec^{-1}(b) - \sec^{-1}(a)$$

$$\Rightarrow \sec^{-1}\left(\frac{x}{a}\right) + \sec^{-1}(a) = \sec^{-1}\left(\frac{x}{b}\right) + \sec^{-1}(b)$$

$$\Rightarrow \cos^{-1}\left(\frac{a}{x}\right) + \cos^{-1}\left(\frac{1}{a}\right) = \cos^{-1}\left(\frac{b}{x}\right) + \cos^{-1}\left(\frac{1}{b}\right)$$

$$\Rightarrow \cos^{-1}\left[\frac{a}{x} \cdot \frac{1}{a} - \sqrt{1 - \left(\frac{a}{x}\right)^2} \sqrt{1 - \left(\frac{1}{a}\right)^2}\right]$$

$$= \cos^{-1}\left[\frac{b}{x} \cdot \frac{1}{b} - \sqrt{1 - \left(\frac{b}{x}\right)^2} \sqrt{1 - \left(\frac{1}{b}\right)^2}\right]$$

$$\Rightarrow \frac{1}{x} - \sqrt{1 - \frac{a^2}{x^2} - \frac{1}{a^2} + \frac{1}{x^2}} = \frac{1}{x} - \sqrt{1 - \frac{b^2}{x^2} - \frac{1}{b^2} + \frac{1}{x^2}}$$

$$1 - \frac{a^2}{x^2} - \frac{1}{a^2} + \frac{1}{x^2} = 1 - \frac{b^2}{x^2} - \frac{1}{b^2} + \frac{1}{x^2}$$

$$\Rightarrow \frac{b^2}{x^2} + \frac{1}{b^2} = \frac{a^2}{x^2} + \frac{1}{a^2}$$

$$\Rightarrow \frac{b^2}{x^2} - \frac{a^2}{x^2} = \frac{1}{a^2} - \frac{1}{b^2}$$

$$\Rightarrow (b^2 - a^2) = x^2 \left(\frac{b^2 - a^2}{a^2 b^2}\right)$$

$$\Rightarrow x^2 = a^2 b^2$$

$$\Rightarrow x = \pm ab.$$

प्रश्न 18.

$$\cos^{-1} \frac{x^2-1}{x^2+1} + \tan^{-1} \frac{2x}{x^2-1} = \frac{2\pi}{3}$$

हल :

$$\cos^{-1} \left( \frac{x^2-1}{x^2+1} \right) + \tan^{-1} \left( \frac{2x}{x^2-1} \right) = \frac{2\pi}{3}$$

$$\Rightarrow \pi - \cos^{-1} \left( \frac{1-x^2}{1+x^2} \right) - \tan^{-1} \left( \frac{2x}{1-x^2} \right) = \frac{2\pi}{3}$$

$$[\because \cos^{-1}(-x) = \pi - \cos^{-1} x \\ \tan^{-1}(-x) = -\tan^{-1} x]$$

$$\Rightarrow \pi - 2 \tan^{-1} x - 2 \tan^{-1} x = \frac{2\pi}{3}$$

$$\Rightarrow 4 \tan^{-1} x = \pi - 2 \frac{\pi}{3}$$

$$\Rightarrow 4 \tan^{-1} x = \frac{\pi}{3}$$

$$\Rightarrow \tan^{-1} x = \frac{\pi}{12} \Rightarrow x = \tan \frac{\pi}{12}$$

अतः  $x = \tan \left( \frac{\pi}{12} \right)$

प्रश्न 19.

$$\tan^{-1} \frac{1}{1+2x} + \tan^{-1} \frac{1}{4x+1} = \tan^{-1} \frac{2}{x^2}$$

हल :

$$\therefore \tan^{-1} \frac{1}{2x} + \tan^{-1} \frac{1}{4x+1} = \tan^{-1} \frac{2}{x^2}$$

$$\therefore \tan^{-1} A + \tan^{-1} B = \tan^{-1} \left( \frac{A+B}{1-AB} \right)$$

$$\therefore \tan^{-1} \left[ \frac{\frac{1}{1+2x} + \frac{1}{4x+1}}{1 - \left(\frac{1}{1+2x}\right) \left(\frac{1}{4x+1}\right)} \right] = \tan^{-1} \frac{2}{x^2}$$

$$\Rightarrow \tan^{-1} \left[ \frac{\frac{4x+1+2x+1}{(1+2x)(1+4x)}}{\frac{(1+2x)(1+4x)-1}{(1+2x)(1+4x)}} \right] = \tan^{-1} \frac{2}{x^2}$$

$$\Rightarrow \tan^{-1} \left[ \frac{6x+2}{1+2x+4x+8x^2-1} \right] = \tan^{-1} \frac{2}{x^2}$$

$$\Rightarrow \tan^{-1} \left( \frac{6x+2}{8x^2+6x} \right) = \tan^{-1} \frac{2}{x^2}$$

$$\Rightarrow \frac{3x+1}{4x^2+3x} = \frac{2}{x^2}$$

$$\Rightarrow 3x^3 + x^2 = 8x^2 + 6x$$

$$\Rightarrow x(3x^2 - 7x - 6) = 0$$

$$\Rightarrow x(3x^2 - 9x + 2x - 6) = 0$$

$$\Rightarrow x[3x(x-3) + 2(x-3)] = 0$$

$$\Rightarrow x(x-3)(3x+2) = 0$$

$$\Rightarrow x = 0, x = 3, x = -\frac{2}{3}$$

प्रश्न 20.

$$\tan^{-1} \frac{x+7}{x-1} + \tan^{-1} \frac{x-1}{x} = \pi - \tan^{-1} 7$$

हल :

$$\tan^{-1} \frac{x+7}{x-1} + \tan^{-1} \frac{x-1}{x} = \pi - \tan^{-1} 7$$

$$\tan^{-1} \left[ \frac{\frac{x+7}{x-1} + \frac{x-1}{x}}{1 - \left(\frac{x+7}{x-1}\right) \left(\frac{x-1}{x}\right)} \right] = \pi - \tan^{-1} 7$$

$$\Rightarrow \tan^{-1} \left[ \frac{\frac{x(x+7) + (x-1)^2}{x(x-1)}}{\frac{x(x-1) - (x+7)(x-1)}{x(x-1)}} \right] = \pi - \tan^{-1} 7$$

$$\Rightarrow \tan^{-1} \left[ \frac{x^2 + 7x + x^2 - 2x + 1}{x^2 - x - x^2 - 7x + x + 7} \right] = \pi - \tan^{-1} 7$$

$$\Rightarrow \tan^{-1} \left( \frac{2x^2 + 5x + 1}{-7x + 7} \right) + \tan^{-1} 7 = \pi$$

$$\Rightarrow \tan^{-1} \left[ \frac{\frac{2x^2 + 5x + 1}{-7x + 7} + 7}{1 - \left( \frac{2x^2 + 5x + 1}{-7x + 7} \right) \times 7} \right] = \pi$$

$$\Rightarrow \tan^{-1} \left[ \frac{2x^2 + 5x + 1 - 49x + 49}{-7x + 7 - 14x^2 - 35x - 7} \right] = \pi$$

$$\Rightarrow \frac{2x^2 - 44x + 50}{-14x^2 - 42x} = \tan \pi$$

$$\Rightarrow 2x^2 - 44x + 50 = 0$$

$$\Rightarrow x^2 - 22x + 25 = 0$$

$$\Rightarrow x = \frac{22 \pm \sqrt{484 - 4 \times 25}}{2 \times 1}$$

$$\Rightarrow x = \frac{22 \pm \sqrt{484 - 100}}{2}$$

$$\Rightarrow x = \frac{22 \pm 8\sqrt{6}}{2}$$

$$\Rightarrow x = 11 \pm 4\sqrt{6}$$

प्रश्न 21.

$$\sin^{-1} \left( \frac{1}{\sqrt{5}} \right) + \cot^{-1} x = \frac{\pi}{4}$$

हल :

$$\sin^{-1} \left( \frac{1}{\sqrt{5}} \right) + \cot^{-1} x = \frac{\pi}{4}$$

$$\therefore \sin^{-1} \left( \frac{1}{x} \right) = y = \operatorname{cosec}^{-1} x$$

$$\therefore \sin^{-1} \left( \frac{1}{\sqrt{5}} \right) = \operatorname{cosec}^{-1} (\sqrt{5})$$

माना  $\operatorname{cosec}^{-1} \sqrt{5} = y$

$$\therefore \operatorname{cosec} = \sqrt{5}$$

$$\therefore \operatorname{cosec}^2 y = 5$$

$$1 + \cot^2 y = 5$$

$$\cot^2 y = 4$$

$$\cot y = \pm 2$$

$$y = \cot^{-1} (2)$$

(+ चिह्न लेने पर)

$$\therefore y + \cot^{-1} x = \frac{\pi}{4}$$

$$\Rightarrow \cot^{-1} (2) + \cot^{-1} x = \frac{\pi}{4}$$

$$\Rightarrow \cot^{-1} \left( \frac{2x-1}{2+x} \right) = \frac{\pi}{4}$$

$$\left[ \because \cot^{-1} x + \cot^{-1} y = \cot^{-1} \left( \frac{xy-1}{x+y} \right) \right]$$

$$\Rightarrow \frac{2x-1}{2+x} = \cot \frac{\pi}{4} \Rightarrow \frac{2x-1}{2+x} = 1$$

$$\Rightarrow 2x - 1 = 2 + x$$

$$\Rightarrow 2x - x = 2 + 1$$

$$\Rightarrow x = 3$$

प्रश्न 22.

$$3 \tan^{-1} \frac{1}{2+\sqrt{3}} - \tan^{-1} \frac{1}{x} = \tan^{-1} \frac{1}{3}$$

हल :

$$\begin{aligned}
& 3 \tan^{-1} \left( \frac{1}{2+\sqrt{3}} \right) - \tan^{-1} \left( \frac{1}{x} \right) = \tan^{-1} \left( \frac{1}{3} \right) \\
\Rightarrow & 3 \tan^{-1} \left[ \frac{2-\sqrt{3}}{(2+\sqrt{3})(2-\sqrt{3})} \right] = \tan^{-1} \left( \frac{1}{x} \right) + \tan^{-1} \left( \frac{1}{3} \right) \\
\Rightarrow & 3 \tan^{-1} \left[ \frac{2-\sqrt{3}}{4-(\sqrt{3})^2} \right] = \tan^{-1} \left[ \frac{\frac{1}{x} + \frac{1}{3}}{1 - \frac{1}{x} \times \frac{1}{3}} \right] \\
\Rightarrow & 3 \tan^{-1} (2-\sqrt{3}) = \tan^{-1} \left( \frac{3+x}{3x-1} \right) \\
\Rightarrow & \tan^{-1} \left\{ \frac{3(2-\sqrt{3}) - (2-\sqrt{3})^3}{1-3(2-\sqrt{3})^2} \right\} = \frac{3+x}{3-1} \\
& \left[ \because 3 \tan^{-1} x = \tan^{-1} \left( \frac{3x-x^3}{1-3x^2} \right) \right] \\
\Rightarrow & \frac{3+x}{3x-1} = \left[ \frac{3(2-\sqrt{3}) - (8-3\sqrt{3}-12\sqrt{3}+18)}{1-3(4+3-4\sqrt{3})} \right] \\
\Rightarrow & \frac{3+x}{3x-1} = \frac{6-3\sqrt{3}-8+3\sqrt{3}-18+12\sqrt{3}}{1-3(7-4\sqrt{3})} \\
\Rightarrow & \frac{3+x}{3x-1} = \frac{-20+12\sqrt{3}}{-20+12\sqrt{3}} \\
\Rightarrow & \frac{3+x}{3x-1} = \frac{20+12\sqrt{3}}{20+12\sqrt{3}} \Rightarrow \frac{3+x}{3x-1} = 1 \\
\Rightarrow & 3+x = 3x-1 \\
\Rightarrow & 2x = 4 \\
\Rightarrow & x = 2
\end{aligned}$$

**प्रश्न 23.**  $\sin 2 (\cos^{-1} \{ \cot (2 \tan^{-1} x) \}) = 0$

**हल :** दी गई समीकरण है

$$\sin 2 [\cos^{-1} \{ \cot (2 \tan^{-1} x) \}] = 0$$

$$\Rightarrow \sin 2 \left[ \cos^{-1} \left\{ \cot \left( \tan^{-1} \frac{2x}{1-x^2} \right) \right\} \right] = 0$$

$$\left[ \because 2 \tan^{-1} x = \tan^{-1} \left( \frac{2x}{1-x^2} \right) \right]$$

$$\Rightarrow \sin 2 \left[ \cos^{-1} \left\{ \cot \left( \cot^{-1} \frac{1-x^2}{2x} \right) \right\} \right] = 0$$

$$\left[ \because \cot^{-1} x = \tan^{-1} \frac{1}{x} \right]$$

$$\Rightarrow \sin 2 \left[ \cos^{-1} \left( \frac{1-x^2}{2x} \right) \right] = 0$$

$$\Rightarrow \sin 2 \left[ \sin^{-1} \left\{ \sqrt{\frac{6x^2-1-x^4}{4x^2}} \right\} \right] = 0$$

$$\left[ \because \cos^{-1} y = \sin^{-1} \sqrt{1-y^2} \right]$$

$$\Rightarrow \sin \left[ \sin^{-1} \left\{ 2 \left( \frac{\sqrt{6x^2-1-x^4}}{2x} \right) \left( \sqrt{1-\frac{6x^2-1-x^4}{4x^2}} \right) \right\} \right] = 0$$

$$\left[ \because 2 \sin^{-1} x = \sin^{-1} (2x\sqrt{1-x^2}) \right]$$

$$2 \left( \frac{\sqrt{6x^2-1-x^4}}{2x} \right) \left( \sqrt{\frac{4x^2-6x^2+1+x^4}{4x^2}} \right) = 0$$

$$\left[ \because \sin (\sin^{-1} x) = x \right]$$

$$\Rightarrow \sqrt{6x^2-1-x^4} \sqrt{-2x^2+x^4+1} = 0$$

$$\Rightarrow (6x^2-1-x^4)(x^4-2x^2+1) = 0$$

$$\Rightarrow 6x^2-1-x^4 = 0 \dots (1)$$

$$\Rightarrow x^4-2x^2+1 = 0 \dots (2)$$

समीकरण (1) से,

$$\Rightarrow 6x^2-1-x^4 = 0$$

$$\Rightarrow x^4-6x^2+1 = 0$$

$$\Rightarrow x^4-2 \times 3x^2 + (3)^2 - 8 = 0$$

$$\Rightarrow (x^2-3)^2 = 8$$

$$\Rightarrow x^2-3 = \pm 2\sqrt{2}$$

$$\Rightarrow x^2 = 3 \pm 2\sqrt{2}$$

$$\Rightarrow x^2 = 1 + 2 \pm 2\sqrt{2}$$

$$\Rightarrow x^2 = (1)^2 + (\sqrt{2})^2 \pm 2\sqrt{2}$$

$$\Rightarrow x^2 = (1 \pm \sqrt{2})^2$$

$$\Rightarrow x = \pm (1 + \sqrt{2})$$

समीकरण (2) से

$$\Rightarrow x^4 - 2x^2 + 1 = 0$$

$$\Rightarrow (x^2)^2 - 2x^2 + (1)^2 = 0$$

$$\Rightarrow (x^2 - 1)^2 = 0$$

$$\Rightarrow x^2 = 1$$

$$\Rightarrow x = \pm 1$$

प्रश्न 24.

$$\begin{aligned} \tan^{-1} \left( \frac{1}{4} \right) + 2 \tan^{-1} \left( \frac{1}{5} \right) + \tan^{-1} \left( \frac{1}{6} \right) \\ + \tan^{-1} \left( \frac{1}{x} \right) = \frac{\pi}{4} \end{aligned}$$

हल :

$$\begin{aligned} \tan \left( \frac{\frac{1}{x} + \frac{1}{4}}{1 - \frac{1}{x} \times \frac{1}{4}} \right) + \tan^{-1} \left\{ \frac{2 \times \frac{1}{5}}{1 - \left( \frac{1}{5} \right)^2} \right\} + \tan^{-1} \frac{1}{6} \\ = \frac{\pi}{4} \end{aligned}$$

$$\Rightarrow \tan^{-1} \left( \frac{4+x}{4x-1} \right) + \tan^{-1} \left( \frac{5}{12} \right) + \tan^{-1} \frac{1}{6} = \frac{\pi}{4}$$

$$\Rightarrow \tan^{-1} \left( \frac{4+x}{4x-1} \right) + \tan^{-1} \left\{ \frac{\frac{5}{12} + \frac{1}{6}}{1 - \frac{5}{12} \times \frac{1}{6}} \right\} = \frac{\pi}{4}$$

$$\Rightarrow \tan^{-1} \left( \frac{4+x}{4x-1} \right) + \tan^{-1} \left( \frac{30+12}{72-5} \right) = \frac{\pi}{4}$$

$$\Rightarrow \tan^{-1} \left( \frac{4+x}{4x-1} \right) + \tan^{-1} \left( \frac{42}{67} \right) = \frac{\pi}{4}$$

$$\Rightarrow \tan^{-1} \left\{ \frac{\frac{4+x}{4x-1} + \frac{42}{67}}{1 - \left( \frac{4+x}{4x-1} \right) \left( \frac{42}{67} \right)} \right\} = \frac{\pi}{4}$$

$$\Rightarrow \frac{67(4+x) + 42(4x-1)}{(4x-1)67 - 42(4+x)} = \tan \frac{\pi}{4}$$

$$\Rightarrow \frac{268 + 67x + 168x - 42}{268x - 67 - 168 - 42x} = 1$$

$$\Rightarrow 268 + 67x + 168x - 42 = 268x - 67 - 168 - 42x$$

$$\Rightarrow 277x - 268x = -503 + 42$$

$$\Rightarrow 9x = -461$$

$$\Rightarrow x = \frac{-461}{9}$$

प्रश्न 25.

$$\sin^{-1}x - \sin^{-1}y = \frac{2\pi}{3}; \cos^{-1}x - \cos^{-1}y = \frac{\pi}{3}$$

हल : प्रश्नानुसार,

$$\sin^{-1}x - \sin^{-1}y = \frac{2\pi}{3} \dots (i)$$

$$\cos^{-1}x - \cos^{-1}y = \frac{\pi}{3} \dots (ii)$$

$$\left( \frac{\pi}{2} - \sin^{-1}x \right) - \left( \frac{\pi}{2} - \sin^{-1}y \right) = \frac{\pi}{3}$$

समीकरण (ii) में  $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$  सूत्र के प्रयोग से

$$\Rightarrow \frac{\pi}{2} - \sin^{-1}x - \frac{\pi}{2} + \sin^{-1}y = \frac{\pi}{3}$$

$$\Rightarrow -\sin^{-1}x + \sin^{-1}y = \frac{\pi}{3} \dots (iii)$$

समीकरण (i) व (iii) से

$$2 \sin^{-1}y = \frac{2\pi}{3} + \frac{\pi}{3} = \pi$$

$$\Rightarrow \sin^{-1} y = \frac{\pi}{2}$$

$$\Rightarrow y = \sin \frac{\pi}{2} = 1 \quad \dots(\text{iv})$$

समी. (i) व (iv) से,

$$\sin^{-1} x + \tan^{-1}(1) = \frac{2\pi}{3}$$

$$\Rightarrow \sin^{-1} x + \frac{\pi}{2} = \frac{2\pi}{3}$$

$$\Rightarrow \sin^{-1} x = \frac{2\pi}{3} - \frac{\pi}{2} = \frac{\pi}{6}$$

$$\Rightarrow x = \sin \frac{\pi}{6} = \frac{1}{2}$$

अतः  $x = \frac{1}{2}, y = 1$ .

## Miscellaneous Exercise

प्रश्न 1.  $\tan^{-1}(-1)$  का मुख्य मान है

- (a)  $45^\circ$
- (b)  $135^\circ$
- (c)  $-45^\circ$
- (d)  $-60^\circ$

हल :

$$\begin{aligned}\because \tan^{-1}(-x) &= -\tan^{-1} x \\ \therefore \tan^{-1}(-1) &= -\tan^{-1}(1)\end{aligned}$$

माना  $\tan^{-1} 1 = \theta$

$$\begin{aligned}\therefore \tan \theta &= 1 \\ \tan \theta &= \tan 45^\circ \\ \therefore \theta &= 45^\circ \\ \therefore \tan^{-1}(-1) &= -45^\circ\end{aligned}$$

अतः सही विकल्प (c) है।

प्रश्न 2.

$2 \tan^{-1}(1/2)$  बराबर है

- (a)  $\cos^{-1}\left(\frac{3}{5}\right)$
- (b)  $\cos^{-1}\left(\frac{3}{4}\right)$
- (c)  $\cos^{-1}\left(\frac{5}{3}\right)$
- (d)  $\cos^{-1}\left(\frac{1}{2}\right)$

हल :

$$\begin{aligned}2 \tan^{-1}\left(\frac{1}{2}\right) & \\ \left\{ \text{सूत्र : } 2 \tan^{-1} x &= \tan^{-1}\left(\frac{2x}{1-x^2}\right) \text{ से} \right\} \\ &= \tan^{-1}\left(\frac{2 \times \frac{1}{2}}{1 - \frac{1}{4}}\right) \\ &= \tan^{-1}\left(\frac{4}{3}\right) = \tan^{-1}\left(\frac{3}{5}\right)\end{aligned}$$

अतः सही विकल्प (a) है।

प्रश्न 3. यदि  $\tan^{-1} (3/4) = \theta$ , तो  $\sin \theta$  का मान है

- (a)  $\frac{5}{3}$       (b)  $\frac{3}{5}$       (c)  $\frac{4}{3}$       (d)  $\frac{1}{4}$

हल :

प्रश्नानुसार,

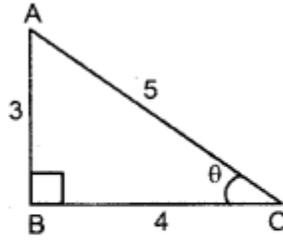
$$\tan^{-1} \left( \frac{3}{4} \right) = \theta$$

$$\Rightarrow \tan \theta = \frac{3}{4}$$

समकोण त्रिभुज बनाने पर

$$\sin \theta = \frac{3}{5}$$

$$\therefore \sin^{-1} = \left( \frac{3}{5} \right)$$



अतः सही उत्तर का विकल्प (b) है।

प्रश्न 4.  $\cot [\tan^{-1} a + \cot^{-1} a]$  का मान है

- (a) 1  
(b)  $\infty$   
(c) 0  
(d) इनमें से कोई नहीं

हल :  $\cos (\tan^{-1} a + \cot^{-1} a)$ .

$$= \cot \frac{\pi}{2} (\because \tan^{-1} x + \cot^{-1} x = \frac{\pi}{2})$$

$$= 0$$

अतः सही उत्तर का विकल्प (c) है।

प्रश्न 5. यदि  $\sin^{-1} \frac{1}{2} = x$  तो का व्यापक मान है

- (a)  $2n\pi \pm \frac{\pi}{6}$       (b)  $\frac{\pi}{6}$   
(c)  $n\pi \pm \frac{\pi}{6}$       (d)  $n\pi(-1)^n \frac{\pi}{6}$

हल : दिया है,

$$\sin^{-1} \left( \frac{1}{2} \right) = x \Rightarrow \sin x = \frac{1}{2} = \sin^{-1} \frac{\pi}{6}$$

$$\Rightarrow x = \frac{\pi}{6}$$

$$\therefore x \text{ का व्यापक मान } \theta = n\pi + (-1)^n \frac{\pi}{6}$$

अतः सही उत्तर का विकल्प (d) है।

**प्रश्न 6.  $2 \tan (\tan^{-1} x + \tan^{-1} x^3)$  का मान है**

- (a)  $\frac{2x}{1-x^2}$   
 (b)  $1+x^2$   
 (c)  $2x$   
 (d) इनमें से कोई नहीं

**हल :**

$$\begin{aligned} &= 2 \tan \left\{ \tan^{-1} \left( \frac{x+x^3}{1-x \times x^3} \right) \right\} \\ &= 2 \tan \left\{ \tan^{-1} \left( \frac{x(1+x^2)}{1-x^4} \right) \right\} \\ &= 2 \tan \left\{ \tan^{-1} \frac{x(1+x^2)}{(1-x^2)(1+x^2)} \right\} \\ &= 2 \frac{x}{1-x^2} = \frac{2x}{1-x^2} \end{aligned}$$

अतः सही विकल्प (a) है।

**प्रश्न 7. यदि  $\tan^{-1} (3x) + \tan^{-1} (2x) = \frac{\pi}{2}$ . तो  $x$  का मान**

- (a)  $\frac{1}{6}$       (b)  $\frac{1}{3}$       (c)  $\frac{1}{10}$       (d)  $\frac{1}{2}$

$$\text{हल : } \tan^{-1} (3x) + \tan^{-1} (2x) = \frac{\pi}{2}$$

$$\Rightarrow \tan^{-1} \left\{ \frac{3x+2x}{1-3x \times 2x} \right\} = \frac{\pi}{4}$$

$$\Rightarrow \left( \frac{5x}{1-6x^2} \right) = \tan \frac{\pi}{4}$$

$$\Rightarrow \frac{5x}{1-6x^2} = 1$$

$$\Rightarrow 1 - 6x^2 = 5x$$

$$\Rightarrow 6x^2 + 5x - 1 = 0$$

$$\Rightarrow 6x^2 + 6x - x - 1 = 0$$

$$\Rightarrow 6x(x+1) - 1(x+1) = 0$$

$$\Rightarrow (x+1)(6x-1) = 0$$

$$\Rightarrow x = -1, x = \frac{1}{6}$$

अतः सही विकल्प (a) है।

**प्रश्न 8.**

$$\sin^{-1} \left( \frac{\sqrt{3}}{2} \right) + 2 \cos^{-1} \left( \frac{\sqrt{3}}{2} \right)$$

का मान है

(a)  $\frac{\pi}{2}$       (b)  $\frac{\pi}{3}$       (c)  $\frac{2\pi}{3}$       (d)  $\pi$

हल :

$$\begin{aligned} & \sin^{-1} \frac{\sqrt{3}}{2} + 2 \cos^{-1} \left( \frac{\sqrt{3}}{2} \right) \\ &= \left[ \sin^{-1} \left( \frac{\sqrt{3}}{2} \right) + \cos^{-1} \left( \frac{\sqrt{3}}{2} \right) \right] + \cos^{-1} \left( \frac{\sqrt{3}}{2} \right) \\ &= \frac{\pi}{2} + \frac{\pi}{6} = \frac{2\pi}{3} \end{aligned}$$

अतः सही विकल्प (c) है।

**प्रश्न 9.** यदि  $\tan^{-1}(1) + \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \sin^{-1} x$ , तो  $x$  का मान है

(a) -1

(b) 0

(c) 1

(d)  $-\frac{1}{2}$

हल :

$$\tan^{-1}(1) + \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \sin^{-1} x$$

$$\sin^{-1} x = \frac{\pi}{4} + \frac{\pi}{4}$$

$$\sin^{-1} x = \frac{\pi}{2}$$

$$x = \sin \frac{\pi}{2}$$

$$x = 1$$

अतः सही विकल्प (c) है।

प्रश्न 10. यदि  $\cot^{-1} x + \tan^{-1}\left(\frac{1}{3}\right) = \frac{\pi}{2}$  तो  $x$  का मान है

(a) 1

(b) 3

(c)  $\frac{1}{3}$

(d) इनमें से कोई नहीं

हल :

$$\cot^{-1}(x) + \tan^{-1}\left(\frac{1}{3}\right) = \frac{\pi}{2}$$

$$\Rightarrow \cot^{-1} x = \frac{\pi}{2} - \tan^{-1}\left(\frac{1}{3}\right)$$

$$\Rightarrow \cot^{-1} x = \cot^{-1} \frac{1}{3}$$

अतः तुलना से,  $x = \frac{1}{3}$

अतः सही विकल्प (c) है।

प्रश्न 11. यदि  $4 \sin^{-1} x + \cos^{-1} x = \pi$ , तो  $x$  का मान कीजिए।

हल :

$$4 \sin^{-1} x + \cos^{-1} x = \pi$$

$$\Rightarrow 4 \sin^{-1} x + \frac{\pi}{2} - \sin^{-1} x = \pi$$

$$\left[ \because \sin^{-1} x + \cos^{-1} x = \frac{\pi}{2} \right]$$

$$\Rightarrow 3 \sin^{-1} x = \frac{\pi}{2} \Rightarrow \sin^{-1} x = \frac{\pi}{6}$$

$$\Rightarrow x = \sin \frac{\pi}{6} \Rightarrow x = \frac{1}{2}$$

प्रश्न 12.

$$\cos \left[ \left( \frac{\pi}{2} \right) + \sin^{-1} \left( \frac{1}{3} \right) \right]$$

का मान कीजिए।

हल :

$$\cos \left[ \left( \frac{\pi}{2} \right) + \sin^{-1} \left( \frac{1}{3} \right) \right]$$

$$= -\sin \left( \sin^{-1} \frac{1}{3} \right) \left[ \because \cos \left( \frac{\pi}{2} + \theta \right) = -\sin \theta \right]$$

$$= -\frac{1}{3}$$

प्रश्न 13. यदि

$$\sin^{-1} \left( \frac{3}{4} \right) + \sec^{-1} \left( \frac{4}{3} \right) = x$$

तो  $x$  का मान कीजिए।

हल :

$$\sin^{-1} \left( \frac{3}{4} \right) + \sec^{-1} \left( \frac{4}{3} \right) = x$$

$$\sin^{-1} \left( \frac{3}{4} \right) + \cos^{-1} \left( \frac{3}{4} \right) = x$$

$$\frac{\pi}{2} = x \quad \left[ \because \sin^{-1} x + \cos^{-1} x = \frac{\pi}{2} \right]$$

$$x = \frac{\pi}{2}$$

प्रश्न 14.

$$\sin^{-1} \left( \frac{4}{5} \right) + 2 \tan^{-1} \left( \frac{1}{3} \right)$$

का मान कीजिए।

हल :

$$\sin^{-1} \left( \frac{4}{5} \right) + 2 \tan^{-1} \left( \frac{1}{3} \right)$$

$$= \sin^{-1} \left( \frac{4}{5} \right) + \cos^{-1} \left\{ \frac{1 - \left( \frac{1}{3} \right)^2}{1 + \left( \frac{1}{3} \right)^2} \right\}$$

$$\because 2 \tan^{-1} x = \cos^{-1} \left\{ \frac{1 - x^2}{1 + x^2} \right\}$$

$$= \sin^{-1} \left( \frac{4}{5} \right) + \cos^{-1} \left( \frac{\frac{8}{9}}{\frac{10}{9}} \right)$$

$$= \sin^{-1} \left( \frac{4}{5} \right) + \cos^{-1} \left( \frac{4}{5} \right)$$

$$= \frac{\pi}{2}$$

प्रश्न 15.

यदि

$$\sin^{-1} \left( \frac{5}{13} \right) + \sin^{-1} \left( \frac{12}{x} \right) = 90^\circ,$$

तो  $x$  का मान कीजिए।

हल :

$$\sin^{-1}\left(\frac{5}{13}\right) + \sin^{-1}\left(\frac{12}{x}\right) = 90^\circ$$

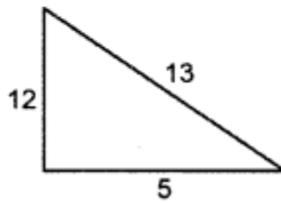
$$\Rightarrow \sin^{-1}\left(\frac{5}{13}\right) + \sin^{-1}\left(\frac{12}{x}\right) = \sin^{-1}\left(\frac{5}{13}\right) + \cos^{-1}\left(\frac{5}{13}\right)$$

$$\left[ \because \sin^{-1}\left(\frac{5}{13}\right) + \cos^{-1}\left(\frac{5}{13}\right) = \frac{\pi}{2} \right]$$

$$\Rightarrow \sin^{-1}\frac{12}{x} = \cos^{-1}\frac{5}{13}$$

$$\Rightarrow \sin^{-1}\frac{12}{x} = \sin^{-1}\frac{12}{13}$$

$\cos^{-1}\left(\frac{5}{13}\right)$  के लिए



तुलना से,

$$x = 13$$

प्रश्न 16.

सिद्ध कीजिए कि :

$$\sin^{-1}\frac{3}{5} - \cos^{-1}\frac{12}{13} = \sin^{-1}\frac{16}{65}$$

हल :

$$\text{L.H.S.} = \sin^{-1}\left(\frac{3}{5}\right) - \cos^{-1}\left(\frac{12}{13}\right)$$

$$= \sin^{-1}\left(\frac{3}{5}\right) - \sin^{-1}\left(\sqrt{1 - \left(\frac{12}{13}\right)^2}\right)$$

$$\left[ \because \cos^{-1} x = \sin^{-1} \sqrt{1 - x^2} \right]$$

$$= \sin^{-1}\left(\frac{3}{5}\right) - \sin^{-1}\left(\frac{3}{13}\right)$$

$$= \sin^{-1}\left[\frac{3}{5}\sqrt{1 - \frac{25}{169}} - \frac{5}{13}\sqrt{1 - \frac{9}{25}}\right]$$

$$\begin{aligned} & \left[ \because \sin^{-1} x - \sin^{-1} y = \sin^{-1} (x\sqrt{1-y^2} - y\sqrt{1-x^2}) \right] \\ & = \left[ \frac{3}{5} \times \frac{12}{13} - \frac{5}{12} \cdot \frac{4}{5} \right] \\ & = \sin^{-1} \left( \frac{16}{65} \right) = \text{R.H.S.} \quad \text{इति सिद्धम्।} \end{aligned}$$

प्रश्न 17. यदि  $\tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \pi$ , तो सिद्ध कीजिए :  $x + y + z = xyz$

हल :  $\tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \pi$

$$\begin{aligned} \Rightarrow & \left( \frac{x+y+z-xyz}{1-xy-yz-zx} \right) \tan^{-1} = \pi \\ \Rightarrow & \frac{x+y+z-xyz}{1-xy-yz-zx} = \tan \pi \end{aligned}$$

$$x + y + z - xyz = 0 \times (1 - xy - yz + xyz)$$

$$x + y + z - xyz = 0$$

$$\text{अतः } x + y + z = xyz$$

इति सिद्धम्।

प्रश्न 18. सिद्ध कीजिए कि :

$$\tan^{-1} \left[ \frac{1}{2} \tan 2A \right] + \tan^{-1} (\cot A) + \tan^{-1} (\cot A) = 0.$$

हल :

$$\tan^{-1} \left[ \frac{1}{2} \tan 2A \right] + \tan^{-1} (\cot A) + \tan^{-1} (\cot A) = 0.$$

$$\begin{aligned} & = \tan^{-1} \left( \frac{1}{2} \times \frac{2 \tan A}{1 - \tan^2 A} \right) + \tan^{-1} \left( \frac{1}{\tan A} \right) \\ & \quad + \tan^{-1} \left( \frac{1}{\tan^2 A} \right) \end{aligned}$$

$$\text{माना } \tan A = x$$

$$\begin{aligned}
\text{तब L.H.S.} &= \tan^{-1} \left( \frac{x}{1-x^2} \right) + \tan^{-1} \left( \frac{1}{x} \right) \\
&\quad + \tan^{-1} \left( \frac{1}{x^2} \right) \\
&= \tan^{-1} \left\{ \frac{\frac{x}{1-x^2} + \frac{1}{x}}{1 - \left( \frac{x}{1-x^2} \right) \frac{1}{x}} \right\} + \tan^{-1} \left( \frac{1}{x^2} \right) \\
&= \tan^{-1} \left\{ \frac{\frac{x^2 + 1 - x^2}{x(1-x^2)}}{\frac{1-x^2-1}{x(1-x^2)}} \right\} + \tan^{-1} \left( \frac{1}{x^2} \right) \\
&= \tan^{-1} \left( \frac{1}{-x^2} \right) + \tan^{-1} \left( \frac{1}{x^2} \right) \\
&= -\tan^{-1} \left( \frac{1}{x^2} \right) + \tan^{-1} \left( \frac{1}{x^2} \right) \\
&\quad [\because \tan^{-1}(-\theta) = -\tan^{-1} \theta] \\
&= 0 \\
&= \text{R.H.S.} \qquad \qquad \qquad \text{इति सिद्धम्।}
\end{aligned}$$

प्रश्न 19. सिद्ध कीजिए कि :

$$\tan^{-1} x = 2 \tan^{-1} [\operatorname{cosec}(\tan^{-1} x) - \tan(\cot^{-1} x)]$$

हल : माना  $\tan^{-1} \theta$

$$\Rightarrow x = \tan \theta = \cot \left( \frac{\pi}{2} - \theta \right)$$

$$\Rightarrow \cot^{-1} x = \frac{\pi}{2} - \theta$$

$$\text{R.H.S.} = 2 \tan^{-1} \left[ \operatorname{cosec} \theta - \tan \left( \frac{\pi}{2} - \theta \right) \right]$$

[समी. (i) से]

$$= 2 \tan^{-1} [\operatorname{cosec} \theta - \cot \theta]$$

$$\begin{aligned}
&= 2 \tan^{-1} \left( \frac{1 - \cos \theta}{\sin \theta} \right) \\
&= 2 \tan^{-1} \left( \frac{1 - \cos \theta}{\sin \theta} \right) \\
&= 2 \tan^{-1} \left( \frac{2 \sin^2 \frac{\theta}{2}}{2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}} \right) \\
&= 2 \tan^{-1} \left( \frac{\sin \frac{\theta}{2}}{\cos \frac{\theta}{2}} \right) = 2 \tan^{-1} \left( \tan \frac{\theta}{2} \right) \\
&= 2 \times \frac{\theta}{2} = \theta = \tan^{-1} x = \text{L.H.S.}
\end{aligned}$$

=  $\tan^{-1} x$

= RHS

इति सिद्धम्।

प्रश्न 20. यदि  $\Phi = \tan^{-1} \frac{x\sqrt{3}}{2K-x}$  और  $\theta = \tan^{-1} \frac{2x-K}{K\sqrt{3}}$  तो सिद्ध कीजिए  $\Phi - \theta$  का मान  $30^\circ$  है।

हल :

दिया है,  $\phi = \tan^{-1} \frac{x\sqrt{3}}{2K-x}$

$\therefore \tan \phi = \frac{x\sqrt{3}}{2K-x}$

तथा  $\theta = \tan^{-1} \frac{2x-K}{K\sqrt{3}}$

$\therefore \tan \theta = \frac{2x-K}{K\sqrt{3}}$

$\therefore \tan (\phi - \theta) = \frac{\tan \phi - \tan \theta}{1 + \tan \phi \tan \theta}$

$$\begin{aligned}
\tan (\phi - \theta) &= \frac{\frac{x\sqrt{3}}{2K-x} - \frac{2x-K}{K\sqrt{3}}}{1 + \left( \frac{x\sqrt{3}}{2K-x} \right) \left( \frac{2x-K}{K\sqrt{3}} \right)}
\end{aligned}$$

$$\Rightarrow \tan(\phi - \theta) = \frac{3Kx - (2K - x)(2x - K)}{(2K - x)K\sqrt{3}}$$

$$\Rightarrow \tan(\phi - \theta) = \frac{(2K - x)K\sqrt{3} + x\sqrt{3}(2x - K)}{(2K - x)K\sqrt{3}}$$

$$\Rightarrow \tan(\phi - \theta) = \frac{3Kx - (4xK - 2K^2 - 2x^2 + Kx)}{2\sqrt{3}K^2 - \sqrt{3}Kx + 2\sqrt{3}x^2 - \sqrt{3}Kx}$$

$$\Rightarrow \tan(\phi - \theta) = \frac{3Kx - 4Kx + 2K^2 + 2x^2 - Kx}{2\sqrt{3}K^2 - 2\sqrt{3}Kx + 2\sqrt{3}x^2}$$

$$\Rightarrow \tan(\phi - \theta) = \frac{2K^2 + 2x^2 - 2Kx}{2\sqrt{3}K^2 + 2\sqrt{3}x^2 - 2\sqrt{3}Kx}$$

$$\Rightarrow \tan(\phi - \theta) = \frac{2(K^2 + x^2 - Kx)}{2\sqrt{3}(K^2 + x^2 - Kx)}$$

$$\Rightarrow \tan(\phi - \theta) = \frac{1}{\sqrt{3}}$$

$$\tan(\phi - \theta) = \tan 30^\circ$$

$$\therefore \phi - \theta = 30^\circ$$

इति सिद्धम्।

प्रश्न 21.

सिद्ध कीजिए कि :

$$2 \tan^{-1} \left[ \tan(45^\circ - \alpha) \tan \frac{\beta}{2} \right] = \cos^{-1} \left( \frac{\sin 2\alpha + \cos \beta}{1 + \sin 2\alpha \cos \beta} \right)$$

हल : LHS

$$= 2 \tan^{-1} [\tan(45^\circ - \alpha) \tan \beta/2]$$

$$= 2 \tan^{-1} \left[ \left( \frac{\tan 45^\circ - \tan \alpha}{1 + \tan 45^\circ \tan \alpha} \right) \tan \frac{\beta}{2} \right]$$

$$= 2 \tan^{-1} \left[ \left( \frac{1 - \tan \alpha}{1 + \tan \alpha} \right) \tan \frac{\beta}{2} \right]$$

$$= 2 \tan^{-1} \left[ \left( \frac{\cos \alpha - \sin \alpha}{\cos \alpha + \sin \alpha} \right) \tan \frac{\beta}{2} \right]$$

$$= \cos^{-1} \left[ \frac{1 - \left\{ \left( \frac{\cos \alpha - \sin \alpha}{\cos \alpha + \sin \alpha} \right) \tan \frac{\beta}{2} \right\}^2}{1 + \left\{ \left( \frac{\cos \alpha - \sin \alpha}{\cos \alpha + \sin \alpha} \right) \tan \frac{\beta}{2} \right\}^2} \right]$$

$$\left[ \because 2 \tan^{-1} x = \cos^{-1} \left\{ \frac{1 - x^2}{1 + x^2} \right\} \right]$$

$$= \cos^{-1} \left[ \frac{(\cos \alpha + \sin \alpha)^2 \cos^2 \frac{\beta}{2} - (\cos \alpha - \sin \alpha)^2 \sin^2 \frac{\beta}{2}}{(\cos \alpha + \sin \alpha)^2 \cos^2 \frac{\beta}{2} + (\cos \alpha - \sin \alpha)^2 \sin^2 \frac{\beta}{2}} \right]$$

$$= \cos^{-1} \left[ \frac{(1 + \sin 2\alpha) \left( \frac{1 + \cos \beta}{2} \right) - (1 - \sin 2\alpha) \left( \frac{1 - \cos \beta}{2} \right)}{(1 + \sin 2\alpha) \left( \frac{1 + \cos \beta}{2} \right) + (1 - \sin 2\alpha) \left( \frac{1 - \cos \beta}{2} \right)} \right]$$

$$= \cos^{-1} \left[ \frac{\frac{1}{2} \{ (1 + \sin 2\alpha + \cos \beta + \sin 2\alpha \cos \beta) - (1 - \sin 2\alpha - \cos \beta + \sin 2\alpha \cos \beta) \}}{\frac{1}{2} \{ (1 + \sin 2\alpha + \cos \beta + \sin 2\alpha \sin \beta + 1 - \sin 2\alpha - \cos \beta + \sin 2\alpha \cos \beta) \}} \right]$$

$$= \cos^{-1} \left[ \frac{1 - \sin 2\alpha + \cos \beta + \sin 2\alpha \cos \beta - 1 + \sin 2\alpha + \cos \beta - \sin 2\alpha \cos \beta}{1 + \sin 2\alpha + \cos \beta + \sin 2\alpha \cos \beta + 1 - \sin 2\alpha - \cos \beta + \sin 2\alpha \cos \beta} \right]$$

$$= \cos^{-1} \left[ \frac{2(\sin 2\alpha + \cos \beta)}{2 + 2 \sin 2\alpha \cos \beta} \right]$$

$$= \cos^{-1} \left( \frac{\sin 2\alpha + \cos \beta}{1 + \sin 2\alpha \cos \beta} \right) = \text{R.H.S.}$$

इति सिद्धम्।