

DEFINITE INTEGRATION [JEE ADVANCED PREVIOUS YEAR SOLVED PAPER]

JEE ADVANCED

Single Correct Answer Type

1. The value of the definite integral $\int_0^1 (1 + e^{-x^2}) dx$ is

a. -1 b. 2
c. $1 + e^{-1}$ d. none of these

(IIT-JEE 1981)

2. Let a, b, c be nonzero real numbers such that

$$\int_0^1 (1 + \cos^8 x) (ax^2 + bx + c) dx = \int_0^2 (1 + \cos^8 x) (ax^2 + bx + c) dx.$$

Then, the quadratic equation $ax^2 + bx + c = 0$ has

a. no root in $(0, 2)$
b. at least one root in $(0, 2)$
c. a double root in $(0, 2)$
d. two imaginary roots

(IIT-JEE 1981)

3. The value of the integral $\int_0^{\pi/2} \frac{\sqrt{\cot x}}{\sqrt{\cot x} + \sqrt{\tan x}} dx$ is

a. $\pi/4$ b. $\pi/2$
c. π d. none of these

(IIT-JEE 1983)

4. For any integer n , the integral $\int_0^\pi e^{\cos^2 x} \cos^3 (2n+1)x dx$ has the value

a. π b. 1
c. 0 d. none of these

(IIT-JEE 1985)

5. Let $f: R \rightarrow R$ and $g: R \rightarrow R$ be continuous functions. Then the value of the integral

$$\int_{-\pi/2}^{\pi/2} [f(x) + f(-x)] [g(x) - g(-x)] dx$$
 is

a. π b. 1
c. -1 d. 0 (IIT-JEE 1990)

6. The value of $\int_0^{\pi/2} \frac{dx}{1 + \tan^3 x}$ is

a. 0 b. 1
c. $\pi/2$ d. $\pi/4$ (IIT-JEE 1993)

7. If $f(x) = A \sin\left(\frac{\pi x}{2}\right) + B$, $f'\left(\frac{1}{2}\right) = \sqrt{2}$, and

$$\int_0^1 f(x) dx = \frac{2A}{\pi},$$
 then constants A and B are

a. $\frac{\pi}{2}$ and $\frac{\pi}{2}$ b. $\frac{2}{\pi}$ and $\frac{3}{\pi}$
c. 0 and $\frac{-4}{\pi}$ d. $\frac{4}{\pi}$ and 0

(IIT-JEE 1995)

8. The value of $\int_0^{2\pi} [2 \sin x] dx$, where $[.]$ represents the greatest integral function, is

a. $\frac{-5\pi}{3}$ b. $-\pi$ c. $\frac{5\pi}{3}$ d. -2π

(IIT-JEE 1995)

9. Let f be a positive function. Let $I_1 = \int_{1-k}^k x f[x(1-x)] dx$, $I_2 = \int_{1-k}^k f[x(1-x)] dx$, where $2k-1 > 0$. Then $\frac{I_1}{I_2}$ is
 a. 2 b. k c. $\frac{1}{2}$ d. 1
 (IIT-JEE 1997)
10. If $g(x) = \int_0^x \cos^4 t \, dt$, then $g(x + \pi)$ equals
 a. $g(x) + g(\pi)$ b. $g(x) - g(\pi)$
 c. $g(x) g(\pi)$ d. $\frac{g(x)}{g(\pi)}$ (IIT-JEE 1997)
11. $\int_{\pi/4}^{3\pi/4} \frac{dx}{1+\cos x}$ is equal to
 a. 2 b. -2 c. $1/2$ d. $-1/2$
 (IIT-JEE 1999)
12. If for a real number y , $[y]$ is the greatest integral function less than or equal to y , then the value of the integral $\int_{\pi/2}^{3\pi/2} [2\sin x] dx$ is
 a. $-\pi$ b. 0 c. $-\pi/2$ d. $\pi/2$
 (IIT-JEE 1999)
13. Let $g(x) = \int_0^x f(t) dt$, where f is such that $\frac{1}{2} \leq f(t) \leq 1$, for $t \in [0, 1]$, and $0 \leq f(t) \leq \frac{1}{2}$, for $t \in [1, 2]$. Then $g(2)$ satisfies the inequality
 a. $-\frac{3}{2} \leq g(2) < \frac{1}{2}$ b. $\frac{1}{2} \leq g(2) \leq \frac{3}{2}$
 c. $\frac{3}{2} < g(2) \leq \frac{5}{2}$ d. $2 < g(2) < 4$
 (IIT-JEE 2000)
14. If $f(x) = \begin{cases} e^{\cos x} \sin x, & \text{for } |x| \leq 2 \\ 2, & \text{otherwise} \end{cases}$, then $\int_{-2}^3 f(x) dx =$
 a. 0 b. 1 c. 2 d. 3
 (IIT-JEE 2000)
15. The value of the integral $\int_{e^{-1}}^{e^2} \left| \frac{\log_e x}{x} \right| dx$ is
 a. $3/2$ b. $5/2$ c. 3 d. 5
 (IIT-JEE 2000)
16. The value of $\int_{-\pi}^{\pi} \frac{\cos^2 x}{1+a^x} dx$, where $a > 0$, is
 a. π b. $a\pi$
 c. $\pi/2$ d. 2π (IIT-JEE 2001)
17. Let $f(x) = \int_1^x \sqrt{2-t^2} \, dt$. Then the real roots of the equation $x^2 - f'(x) = 0$ are
 a. ± 1 b. $\pm \frac{1}{\sqrt{2}}$
 c. $\pm \frac{1}{2}$ d. 0 and 1
 (IIT-JEE 2002)
18. Let $T > 0$ be a fixed real number. Suppose f is continuous function such that for all $x \in R$, $f(x+T) = f(x)$. If $I = \int_0^T f(x) dx$, then the value of $\int_3^{3+3T} f(2x) dx$ is
 a. $3/2 I$ b. $2I$
 c. $3I$ d. $6I$ (IIT-JEE 2002)
19. The integral $\int_{-1/2}^{1/2} \left([x] + \ln \left(\frac{1+x}{1-x} \right) \right) dx$ is equal to (where $[\cdot]$ represents the greatest integer function)
 a. $-\frac{1}{2}$ b. 0 c. 1 d. $2 \ln \left(\frac{1}{2} \right)$
 (IIT-JEE 2002)
20. If $L(m, n) = \int_0^1 t^m (1+t)^n dt$, then the expression for $L(m, n)$ in terms of $L(m+1, n-1)$ is, ($m, n \in N$),
 a. $\frac{2^n}{m+1} - \frac{n}{m+1} L(m+1, n-1)$
 b. $\frac{n}{m+1} L(m+1, n-1)$
 c. $\frac{2^n}{m+1} + \frac{n}{m+1} L(m+1, n-1)$
 d. $\frac{m}{n+1} L(m+1, n-1)$ (IIT-JEE 2003)
21. If $f(x) = \int_{x^2}^{x^2+1} e^{-t^2} dt$, then $f(x)$ increases in
 a. (0, 2) b. no value of x
 c. (0, ∞) d. $(-\infty, 0)$ (IIT-JEE 2003)
22. If $f(x)$ is differentiable and $\int_0^{t^2} x f(x) dx = \frac{2}{5} t^5$, then $f\left(\frac{4}{25}\right)$ equals
 a. $2/5$ b. $-5/2$
 c. 1 d. $5/2$ (IIT-JEE 2004)
23. The value of the integral $\int_0^1 \sqrt{\frac{1-x}{1+x}} dx$ is
 a. $\frac{\pi}{2} + 1$ b. $\frac{\pi}{2} - 1$
 c. -1 d. 1 (IIT-JEE 2004)
24. $\int_{-2}^0 \{x^3 + 3x^2 + 3x + 3 + (x+1)\cos(x+1)\} dx$ is equal to
 a. -4 b. 0
 c. 4 d. 6 (IIT-JEE 2005)
25. $\lim_{x \rightarrow \frac{\pi}{4}} \frac{\int_{\sec^2 x}^{\sec^2 x} f(t) dt}{x^2 - \frac{\pi^2}{16}}$ equals
 a. $\frac{8}{\pi} f(2)$ b. $\frac{2}{\pi} f(2)$

c. $\frac{2}{\pi} f\left(\frac{1}{2}\right)$ d. $4f(2)$ (IIT-JEE 2007)

26. Let f be a non-negative function defined on the interval $[0, 1]$. If $\int_0^x \sqrt{1 - (f'(t))^2} dt = \int_0^x f(t) dt$, $0 \leq x \leq 1$, and $f(0) = 0$, then

a. $f\left(\frac{1}{2}\right) < \frac{1}{2}$ and $f\left(\frac{1}{3}\right) > \frac{1}{3}$

b. $f\left(\frac{1}{2}\right) > \frac{1}{2}$ and $f\left(\frac{1}{3}\right) > \frac{1}{3}$

c. $f\left(\frac{1}{2}\right) < \frac{1}{2}$ and $f\left(\frac{1}{3}\right) < \frac{1}{3}$

d. $f\left(\frac{1}{2}\right) > \frac{1}{2}$ and $f\left(\frac{1}{3}\right) < \frac{1}{3}$ (IIT-JEE 2009)

27. The value of $\int_0^1 \frac{x^4(1-x)^4}{1+x^2} dx$ is (are)

a. $\frac{22}{7} - \pi$ b. $\frac{2}{105}$ c. 0 d. $\frac{71}{15} - \frac{3\pi}{2}$ (IIT-JEE 2010)

28. The value of $\lim_{x \rightarrow 0} \frac{1}{x^3} \int_0^x \frac{t \ln(1+t)}{t^4+4} dt$ is

a. 0 b. $\frac{1}{12}$ c. $\frac{1}{24}$ d. $\frac{1}{64}$ (IIT-JEE 2010)

29. Let f be a real-valued function defined on the interval $(-1, 1)$ such that $e^{-x} f(x) = 2 + \int_0^x \sqrt{t^4+1} dt$, for all $x \in (-1, 1)$ and let f^{-1} be the inverse function of f . Then $(f^{-1})'(2)$ is equal to

a. 1 b. $1/3$ c. $1/2$ d. $1/e$ (IIT-JEE 2010)

30. The value of $\int_{\sqrt{\ln 2}}^{\sqrt{\ln 3}} \frac{x \sin x^2}{\sin x^2 + \sin(\ln 6 - x^2)} dx$ is

a. $\frac{1}{4} \ln \frac{3}{2}$ b. $\frac{1}{2} \ln \frac{3}{2}$

c. $\ln \frac{3}{2}$ d. $\frac{1}{6} \ln \frac{3}{2}$ (IIT-JEE 2011)

31. Let $f: [-1, 2] \rightarrow [0, \infty)$ be a continuous function such that $f(x) = f(1-x)$ for all $x \in [-1, 2]$. Let $R_1 = \int_{-1}^2 xf(x) dx$, and

R_2 be the area of the region bounded by $y = f(x)$, $x = -1$, $x = 2$, and the x -axis. Then

a. $R_1 = 2R_2$ b. $R_1 = 3R_2$ c. $2R_1 = R_2$ d. $3R_1 = R_2$ (IIT-JEE 2011)

32. The value of the integral $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(x^2 + \log_e \frac{\pi+x}{\pi-x} \right) \cos x dx$ is

a. 0 b. $\frac{\pi^2}{2} - 4$ c. $\frac{\pi^2}{2} + 4$ d. $\frac{\pi^2}{2}$ (IIT-JEE 2012)

33. Let $f: \left[\frac{1}{2}, 1\right] \rightarrow R$ (the set of all real numbers) be a positive, non-constant, and differentiable function such that $f'(x) < 2f(x)$ and $f(1/2) = 1$. Then the value of $\int_{1/2}^1 f(x) dx$ lies in the interval

a. $(2e-1, 2e)$ b. $(e-1, 2e-1)$ c. $\left(\frac{e-1}{2}, e-1\right)$ d. $\left(0, \frac{e-1}{2}\right)$ (JEE Advanced 2013)

34. Let $f: [0, 2] \rightarrow R$ be a function which is continuous on $[0, 2]$ and is differentiable on $(0, 2)$ with $f(0) = 1$.

Let $F(x) = \int_0^{x^2} f(\sqrt{t}) dt$ for $x \in [0, 2]$. If $F'(x) = f'(x)$ for all $x \in (0, 2)$, then $F(2)$ equals

a. $e^2 - 1$ b. $e^4 - 1$ c. $e - 1$ d. e^4 (JEE Advanced 2014)

35. The following integral $\int_{\pi/4}^{\pi/2} (2 \operatorname{cosec} x)^{17} dx$ is equal to

a. $\int_0^{\log(1+\sqrt{2})} 2(e^u + e^{-u})^{16} du$ b. $\int_0^{\log(1+\sqrt{2})} 2(e^u + e^{-u})^{17} du$ c. $\int_0^{\log(1+\sqrt{2})} 2(e^u - e^{-u})^{17} du$ d. $\int_0^{\log(1+\sqrt{2})} 2(e^u - e^{-u})^{16} du$ (JEE Advanced 2014)

36. Let $f'(x) = \frac{192x^3}{2 + \sin^4 \pi x}$ for all $x \in R$ with $f\left(\frac{1}{2}\right) = 0$. If

$m \leq \int_{1/2}^1 f(x) dx \leq M$, then the possible values of m and M are

- a. $m = 13, M = 24$ b. $m = \frac{1}{4}, M = \frac{1}{2}$
 c. $m = -11, M = 0$ d. $m = 1, M = 12$
 (JEE Advanced 2015)

Multiple Correct Answer Type

- If $\int_0^x f(t)dt = x + \int_x^1 t f(t)dt$, then the value of $f(1)$ is
 a. $1/2$ b. 0 c. 1 d. $-1/2$
 (IIT-JEE 1998)
- Let $f(x) = x - [x]$, for every real number x , where $[x]$ is the integral part of x . Then $\int_{-1}^1 f(x) dx$ is
 a. 1 b. 2 c. 0 d. $1/2$
 (IIT-JEE 1998)
- Let $S_n = \sum_{k=1}^n \frac{n}{n^2 + kn + k^2}$ and $T_n = \sum_{k=0}^{n-1} \frac{n}{n^2 + kn + k^2}$ for $n = 1, 2, 3, \dots$. Then
 a. $S_n < \frac{\pi}{3\sqrt{3}}$ b. $S_n > \frac{\pi}{3\sqrt{3}}$
 c. $T_n < \frac{\pi}{3\sqrt{3}}$ d. $T_n > \frac{\pi}{3\sqrt{3}}$
 (IIT-JEE 2008)
- Let $f(x)$ be a non-constant twice differentiable function defined on $(-\infty, \infty)$ such that $f(x) = f(1-x)$ and $f'\left(\frac{1}{4}\right) = 0$. Then
 a. $f'(x)$ vanishes at least twice on $[0, 1]$
 b. $f'\left(\frac{1}{2}\right) = 0$
 c. $\int_{-1/2}^{1/2} f\left(x + \frac{1}{2}\right) \sin x dx = 0$
 d. $\int_0^{1/2} f(t) e^{\sin \pi t} dt = \int_{1/2}^1 f(1-t) e^{\sin \pi t} dt$ (IIT-JEE 2008)
- If $I_n = \int_{-\pi}^{\pi} \frac{\sin nx}{(1 + \pi^x) \sin x} dx$, $n = 0, 1, 2, \dots$, then
 a. $I_n = I_{n+2}$ b. $\sum_{m=1}^{10} I_{2m+1} = 10\pi$
 c. $\sum_{m=1}^{10} I_{2m} = 0$ d. $I_n = I_{n+1}$
 (IIT-JEE 2009)
- Let f be a real-valued function defined on the interval $(0, \infty)$ by $f(x) = \ln x + \int_0^x \sqrt{1 + \sin t} dt$. Then which of the following statement(s) is (are) true?
 a. $f''(x)$ exists for all $x \in (0, \infty)$.
 b. $f(x)$ exists for all $x \in (0, \infty)$ and f is continuous on $(0, \infty)$ but not differentiable on $(0, \infty)$.
 c. There exists $\alpha > 1$ such that $|f'(x)| < |f(x)|$ for all $x \in (\alpha, \infty)$.
 d. There exists $\beta > 0$ such that $|f(x)| + |f'(x)| \leq \beta$ for all $x \in (0, \infty)$.
 (IIT-JEE 2010)
- Let S be the area of the region enclosed by $y = e^{-x^2}$, $y = 0$, $x = 0$, and $x = 1$. Then
 a. $S \geq \frac{1}{e}$ b. $S \geq 1 - \frac{1}{e}$
 c. $S \leq \frac{1}{4} \left(1 + \frac{1}{\sqrt{e}}\right)$ d. $S \leq \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{e}} \left(1 - \frac{1}{\sqrt{2}}\right)$
 (IIT-JEE 2012)
- If $f(x) = \int_0^x e^{t^2} (t-2)(t-3) dt$ for all $x \in (0, \infty)$, then
 a. f has a local maximum at $x = 2$
 b. f is decreasing on $(2, 3)$
 c. there exists some $c \in (0, \infty)$ such that $f'(c) = 0$
 d. f has a local minimum at $x = 3$ (IIT-JEE 2012)
- For $a \in \mathbb{R}$ (the set of all real numbers), $a \neq -1$,

$$\lim_{n \rightarrow \infty} \frac{(1^a + 2^a + \dots + n^a)}{(n+1)^{a-1} [(na+1) + (na+2) + \dots + (na+n)]} = \frac{1}{60}$$

 Then $a =$
 a. 5 b. 7 c. $\frac{-15}{2}$ d. $\frac{-17}{2}$
 (JEE Advanced 2013)
- Let $f: [a, b] \rightarrow [1, \infty)$ be a continuous function and let $g: \mathbb{R} \rightarrow \mathbb{R}$ be defined as

$$g(x) = \begin{cases} 0 & \text{if } x < a \\ \int_a^x f(t) dt & \text{if } a \leq x \leq b, \\ \int_b^x f(t) dt & \text{if } x > b \end{cases}$$

 Then
 a. $g(x)$ is continuous but not differentiable at a
 b. $g(x)$ is differentiable on $\mathbb{R}$
 c. $g(x)$ is continuous but not differentiable at b
 d. $g(x)$ is continuous and differentiable at either a or b but not both
 (JEE Advanced 2014)
- Let $f: (0, \infty) \rightarrow \mathbb{R}$ be given by $f(x) = \int_{1/x}^x e^{-\left(t + \frac{1}{t}\right)} \frac{dt}{t}$, then
 a. $f(x)$ is monotonically increasing on $[1, \infty)$
 b. $f(x)$ is monotonically decreasing on $(0, 1)$
 c. $f(x) + f\left(\frac{1}{x}\right) = 0$, for all $x \in (0, \infty)$
 d. $f(2^x)$ is an odd function of x on $\mathbb{R}$
 (JEE Advanced 2014)

12. The option(s) with the values of a and L that satisfy the following equation is (are)

$$\frac{\int_0^{4\pi} e^t (\sin^6 at + \cos^4 at) dt}{\int_0^{\pi} e^t (\sin^6 at + \cos^4 at) dt} = L?$$

- a. $a = 2, L = \frac{e^{4\pi} - 1}{e^{\pi} - 1}$ b. $a = 2, L = \frac{e^{4\pi} + 1}{e^{\pi} + 1}$
c. $a = 4, L = \frac{e^{4\pi} - 1}{e^{\pi} - 1}$ d. $a = 4, L = \frac{e^{4\pi} + 1}{e^{\pi} + 1}$

(JEE Advanced 2015)

13. Let $f(x) = 7 \tan^8 x + 7 \tan^6 x - 3 \tan^4 x - 3 \tan^2 x$ for all $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Then the correct expression(s) is (are)

- a. $\int_0^{\pi/4} xf(x)dx = \frac{1}{12}$ b. $\int_0^{\pi/4} f(x)dx = 0$
c. $\int_0^{\pi/4} xf(x)dx = \frac{1}{6}$ d. $\int_0^{\pi/4} f(x)dx = 1$

(JEE Advanced 2015)

Linked Comprehension Type

For Problem 1-3

Let the definite integral be defined by the formula $\int_a^b f(x)dx = \frac{b-a}{2} (f(a) + f(b))$. For more accurate result, for $c \in (a, b)$, we can use $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx = F(c)$ so that for $c = \frac{a+b}{2}$, we get $\int_a^b f(x)dx = \frac{b-a}{4} (f(a) + f(b) + 2f(c))$.

(IIT-JEE 2008)

1. $\int_0^{\pi/2} \sin x dx$ is equal to

- a. $\frac{\pi}{8}(1 + \sqrt{2})$ b. $\frac{\pi}{4}(1 + \sqrt{2})$
c. $\frac{\pi}{8\sqrt{2}}$ d. $\frac{\pi}{4\sqrt{2}}$

2. If $\lim_{x \rightarrow a} \frac{\int_a^x f(x)dx - \left(\frac{x-a}{2}\right)(f(x) + f(a))}{(x-a)^3} = 0$, then $f(x)$ is of maximum degree

- a. 4 b. 3 c. 2 d. 1

3. If $f''(x) < 0 \forall x \in (a, b)$ and c is a point such that $a < c < b$, and $(c, f(c))$ is the point lying on the curve for which $F(c)$ is maximum, then $f'(c)$ is equal to

- a. $\frac{f(b) - f(a)}{b - a}$ b. $\frac{2(f(b) - f(a))}{b - a}$
c. $\frac{2f(b) - f(a)}{2b - a}$ d. 0

For Problem 4-5

Consider the function $f: (-\infty, \infty) \rightarrow (-\infty, \infty)$ defined by

$$f(x) = \frac{x^2 - ax + 1}{x^2 + ax + 1}, 0 < a < 2.$$

4. Let $g(x) = \int_0^x \frac{f'(t)}{1+t^2} dt$ which of the following is true?
a. $g'(x)$ is positive on $(-\infty, 0)$ and negative on $(0, \infty)$
b. $g'(x)$ is negative on $(-\infty, 0)$ and positive on $(0, \infty)$
c. $g'(x)$ changes sign on both $(-\infty, 0)$ and $(0, \infty)$
d. $g'(x)$ does not change sign on $(-\infty, \infty)$

(IIT-JEE 2008)

5. Let $g(x) = \int_1^x \left(\frac{2(t-1)}{t+1} - \log_e t \right) f(t) dt$ for all $x \in (1, \infty)$.

Then which of the following is true?

- a. g is increasing on $(1, \infty)$
b. g is decreasing on $(1, \infty)$
c. g is increasing on $(1, 2)$ and decreasing on $(2, \infty)$
d. g is decreasing on $(1, 2)$ and increasing on $(2, \infty)$

(IIT-JEE 2012)

For Problems 6 and 7

Given that for each $a \in (0, 1)$, $\lim_{h \rightarrow 0^+} \int_h^{1-h} t^{-a} (1-t)^{a-1} dt$ exists.

Let this limit be $g(a)$. In addition, it is given that the function $g(a)$ is differentiable on $(0, 1)$. (JEE Advanced 2014)

6. The value of $g\left(\frac{1}{2}\right)$ is
a. π b. 2π c. $\frac{\pi}{2}$ d. $\frac{\pi}{4}$
7. The value of $g'\left(\frac{1}{2}\right)$ is
a. $\frac{\pi}{2}$ b. π c. $-\frac{\pi}{2}$ d. 0

For Problem 8 and 9

Let $F: \mathbb{R} \rightarrow \mathbb{R}$ be a thrice differentiable function. Suppose that $F(1) = 0$, $F(3) = -4$ and $F'(x) < 0$ for all $x \in (1/2, 3)$. Let $f(x) = xF(x)$ for all $x \in \mathbb{R}$. (JEE Advanced 2015)

8. The correct statement(s) is (are)
a. $f'(1) < 0$
b. $f(2) < 0$
c. $f''(x) \neq 0$ for any $x \in (1, 3)$
d. $f''(x) = 0$ for some $x \in (1, 3)$
9. If $\int_1^3 x^2 F'(x) dx = -12$ and $\int_1^3 x^3 F''(x) dx = 40$, then the correct expression(s) is (are)
a. $9f'(3) + f'(1) - 32 = 0$ b. $\int_1^3 f(x) dx = 12$
c. $9f'(3) - f'(1) + 32 = 0$ d. $\int_1^3 f(x) dx = -12$

Matching Column Type

1. Match the statements/expressions given in Column I with the values given in Column II.

Column I	Column II
(i) Two rays in the first quadrant $x + y = a $ and $ax - y = 1$ intersects each other in the interval $a \in (a_0, \infty)$, the value of a_0 is	(a) 2
(ii) Point (α, β, γ) lies on the plane $x + y + z = 2$. Let $\vec{a} = \alpha\hat{i} + \beta\hat{j} + \gamma\hat{k}$, $\hat{k} \times (\hat{k} \times \vec{a}) = 0$, then $\gamma =$	(b) $4/3$
(iii) $\left \int_0^1 (1-y^2)dy \right + \left \int_1^0 (y^2-1)dy \right $	(c) $\left \int_0^1 \sqrt{1-x}dx \right + \left \int_{-1}^0 \sqrt{1+x}dx \right $
(iv) If $\sin A \sin B \sin C + \cos A \cos B = 1$, then the value of $\sin C =$	(d) 1

(IITJEE 2006)

2. Match the statements/expressions given in Column I with the values given in Column II.

Column I	Column II
(i) $\int_0^{\pi/2} (\sin x)^{\cos x} (\cos x \cot x - \log(\sin x)^{\sin x}) dx$	(a) 1
(ii) Area bounded by $-4y^2 = x$ and $x - 1 = 5y^2$	(b) 0
(iii) Cosine of the angle of intersection of curves $y = 3^{x-1} \log x$ and $y = x^x - 1$ is	(c) $6 \ln 2$
	(d) $4/3$

(IITJEE 2006)

3. Match the statements/expressions given in Column I with the values given in Column II.

Column I	Column II
(a) $\int_{-1}^1 \frac{dx}{1+x^2}$	(p) $\frac{1}{2} \log\left(\frac{2}{3}\right)$
(b) $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$	(q) $2 \log\left(\frac{2}{3}\right)$
(c) $\int_2^3 \frac{dx}{1-x^2}$	(r) $\frac{\pi}{3}$
(d) $\int_1^2 \frac{dx}{x\sqrt{x^2-1}}$	(s) $\frac{\pi}{2}$

(IIT-JEE 2007)

4. Match the statements/expressions in Column I with the open intervals in Column II.

Column I	Column II
(a) Interval contained in the domain of definition of non-zero solutions of the differential equation $(x-3)^2 y' + y = 0$	(p) $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
(b) Interval containing the value of the integral $\int_1^5 (x-1)(x-2)(x-3)(x-4)(x-5) dx$	(q) $\left(0, \frac{\pi}{2}\right)$
(c) Interval in which at least one of the points locus maximum of $\cos^2 x + \sin x$ lies	(r) $\left(\frac{\pi}{8}, \frac{5\pi}{4}\right)$
(d) Interval in which $\tan^{-1}(\sin x + \cos x)$ is increasing	(s) $\left(0, \frac{\pi}{8}\right)$
	(t) $(-\pi, \pi)$

(IIT-JEE 2009)

5. Match the statements given in Column I with the values given in Column II.

Column I	Column II
(a) If $\vec{a} = \hat{j} + \sqrt{3}\hat{k}$, $\vec{b} = -\hat{j} + \sqrt{3}\hat{k}$ and $\vec{c} = 2\sqrt{3}\hat{k}$ form a triangle, then the internal angle of the triangle between $\vec{a}$ and $\vec{b}$ is	(p) $\frac{\pi}{6}$
(b) If $\int_a^b (f(x) - 3x) dx = a^2 - b^2$, then the value of $f\left(\frac{\pi}{6}\right)$ is	(q) $\frac{2\pi}{3}$
(c) The value of $\frac{\pi^2}{\log_e 3} \int_{7/6}^{5/6} \sec(\pi x) dx$ is	(r) $\frac{\pi}{3}$
(d) The maximum value of $\left \operatorname{Arg} \left(\frac{1}{1-z} \right) \right $ for $ z = 1, z \neq 1$ is given by	(s) π
	(t) $\frac{\pi}{2}$

(IITJEE 2011)

6. Match the statements/expressions given in Column I with the values given in Column II.

Column I	Column II
(p) The number of polynomials $f(x)$ with non-negative integer coefficients of degree ≤ 2 , satisfying $f(0) = 0$ and $\int_0^1 f(x) dx = 1$, is	(1) 8

(q) The number of points in the interval $[-\sqrt{13}, \sqrt{13}]$ at which $f(x) = \sin(x^2) + \cos(x^2)$ attains its maximum value, is	(2) 2
(r) $\int_{-2}^2 \frac{3x^2}{1+e^x} dx$ equals	(3) 4
(s) $\frac{\int_{-1/2}^{1/2} \cos 2x \cdot \log\left(\frac{1+x}{1-x}\right) dx}{\int_0^{1/2} \cos 2x \cdot \log\left(\frac{1+x}{1-x}\right) dx}$ equals	(4) 0

(JEE Advanced 2014)

Codes:

- | | | | | |
|----|-----|-----|-----|-----|
| | (p) | (q) | (r) | (s) |
| a. | (3) | (2) | (4) | (1) |
| b. | (2) | (3) | (4) | (1) |
| c. | (3) | (2) | (1) | (4) |
| d. | (2) | (3) | (1) | (4) |

Integer Answer Type

1. Let $f: R \rightarrow R$ be a continuous function which satisfies $f(x)$

$$= \int_0^x f(t) dt. \text{ Then the value of } f(\ln 5) \text{ is}$$

(IIT-JEE 2009)

2. For any real number x , let $[x]$ denote the largest integer less than or equal to x . Let f be a real-valued function defined on the interval $[-10, 10]$ by

$$f(x) = \begin{cases} x - [x], & \text{if } [x] \text{ is odd} \\ 1 + [x] - x, & \text{if } [x] \text{ is even} \end{cases}$$

Then the value of $\frac{\pi^2}{10} \int_{-10}^{10} f(x) \cos \pi x dx$ is

(IIT-JEE 2010)

3. The value of $\int_0^1 4x^3 \left\{ \frac{d^2}{dx^2} (1-x^2)^5 \right\} dx$ is
- (JEE Advanced 2014)

4. Let $f: R \rightarrow R$ be a continuous odd function, which vanishes exactly at one point and $f(1) = \frac{1}{2}$. Suppose that

$$F(x) = \int_{-1}^x f(t) dt \text{ for all } x \in [-1, 2] \text{ and } G(x) = \int_{-1}^x t |f(f(t))| dt$$

for all $x \in [-1, 2]$. If $\lim_{x \rightarrow 1} \frac{F(x)}{G(x)} = \frac{1}{14}$, then the value of

$$f\left(\frac{1}{2}\right) \text{ is}$$

(JEE Advanced 2015)

5. If $\alpha = \int_0^1 (e^{9x+3\tan^{-1}x}) \left(\frac{12+9x^2}{1+x^2} \right) dx$ where $\tan^{-1}x$ takes only principal values, then the value of $\left(\log_e |1+\alpha| - \frac{3\pi}{4} \right)$ is
- (JEE Advanced 2015)

6. Let $F(x) = \int_x^{x^2+\frac{\pi}{6}} 2 \cos^2 t dt$ for all $x \in R$ and $f: \left[0, \frac{1}{2}\right] \rightarrow [0, \infty)$ be a continuous function. For $a \in \left[0, \frac{1}{2}\right]$, if $F'(a) + 2$ is the area of the region bounded by $x = 0$, $y = 0$, $y = f(x)$ and $x = a$, then $f(0)$ is
- (JEE Advanced 2015)

7. Let $f: R \rightarrow R$ be a function defined by $f(x) = \begin{cases} [x], & x \leq 2 \\ 0, & x > 2 \end{cases}$ where $[x]$ is the greatest integer less than or equal to x . If $I = \int_{-1}^2 \frac{xf(x^2)}{2+f(x+1)} dx$, then the value of $(4I-1)$ is
- (JEE Advanced 2015)

Fill in the Blanks Type

$$1. f(x) = \begin{vmatrix} \sec x & \cos x & \sec^2 x + \cot x \operatorname{cosec} x \\ \cos^2 x & \cos^2 x & \operatorname{cosec}^2 x \\ 1 & \cos^2 x & \cos^2 x \end{vmatrix}.$$

Then $\int_0^{\pi/2} f(x) dx = \underline{\hspace{2cm}}$. (IIT-JEE 1987)

2. The integral $\int_0^{1.5} [x^2] dx$, where $[\cdot]$ denotes the greatest integer function, equals $\underline{\hspace{2cm}}$. (IIT-JEE 1988)

3. The value of $\int_{-2}^2 |1-x^2| dx$ is $\underline{\hspace{2cm}}$. (IIT-JEE 1989)

4. The value of $\int_{\pi/4}^{3\pi/4} \frac{\phi}{1+\sin \phi} d\phi$ is $\underline{\hspace{2cm}}$. (IIT-JEE 1993)

5. The value of $\int_2^3 \frac{\sqrt{x}}{\sqrt{5-x} + \sqrt{x}} dx$ is $\underline{\hspace{2cm}}$. (IIT-JEE 1994)

6. If for non-zero x , $af(x) + bf\left(\frac{1}{x}\right) = \frac{1}{x} - 5$, where $a \neq b$, then $\int_1^2 f(x) dx = \underline{\hspace{2cm}}$. (IIT-JEE 1996)

7. For $n \in N$, $\int_0^{2\pi} \frac{x \sin^{2n} x}{\sin^{2n} x + \cos^{2n} x} dx = \underline{\hspace{2cm}}$. (IIT-JEE 1996)

8. The value of $\int_1^{e^{37}} \frac{\pi \sin(\pi \ln x)}{x} dx$ is $\underline{\hspace{2cm}}$. (IIT-JEE 1997)

9. Let $\frac{d}{dx} F(x) = \frac{e^{\sin x}}{x}$, $x > 0$. If $\int_1^4 \frac{2e^{\sin x^2}}{x} dx = F(k) - F(1)$, then one of the possible values of k is _____.
(IIT-JEE 1997)

True/False Type

1. The value of the integral $\int_0^{2a} \frac{f(x)}{f(x) + f(2a-x)} dx$ is equal to a .
(IIT-JEE 1988)

Subjective Type

- Show that $\lim_{n \rightarrow \infty} \left(\frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{6n} \right) = \log 6$.
(IIT-JEE 1981)
- Evaluate $\int_0^1 (tx + 1 - x)^n dx$, where n is a positive integer and t is a parameter independent of x . Hence, show that $\int_0^1 x^k (1-x)^{n-k} dx = [{}^nC_k(n+1)]^{-1}$ for $k = 0, 1, \dots, n$.
(IIT-JEE 1981)
- Show that $\int_0^\pi x f(\sin x) dx = \frac{\pi}{2} \int_0^\pi f(\sin x) dx$.
(IIT-JEE 1982)
- Find the value of $\int_{-1}^{3/2} |x \sin \pi x| dx$.
(IIT-JEE 1982)
- Evaluate $\int_0^{\pi/4} \frac{\sin x + \cos x}{9 + 16 \sin 2x} dx$.
(IIT-JEE 1983)
- Evaluate $\int_0^{1/2} \frac{x \sin^{-1} x}{\sqrt{1-x^2}} dx$.
(IIT-JEE 1984)
- Given a function $f(x)$ such that
 - it is integrable over every interval on the real line, and
 - $f(t+x) = f(x)$, for every x and a real t .
 Then show that the integral $\int_a^{a+t} f(x) dx$ is independent of a .
- Evaluate $\int_0^{\pi/2} \frac{x \sin x \cos x}{\cos^4 x + \sin^4 x} dx$.
(IIT-JEE 1985)
- Evaluate $\int_0^\pi \frac{x dx}{1 + \cos \alpha \sin x}$, where $0 < \alpha < \pi$.
(IIT-JEE 1986)
- If f and g are continuous functions on $[0, a]$ satisfying $f(x) = f(a-x)$ and $g(x) + g(a-x) = 2$, then show that $\int_0^a f(x) g(x) dx = \int_0^a f(x) dx$.
(IIT-JEE 1989)
- Show that $\int_0^{\pi/2} f(\sin 2x) \sin x dx$

$$= \sqrt{2} \int_0^{\pi/4} f(\cos 2x) \cos x dx. \quad (\text{IIT-JEE 1990})$$

12. Prove that for any positive integer k ,

$$\frac{\sin 2kx}{\sin x} = 2[\cos x + \cos 3x + \dots + \cos (2k-1)x].$$

$$\text{Hence, prove that } \int_0^{\pi/2} \sin 2kx \cot x dx = \frac{\pi}{2}.$$

(IIT-JEE 1990)

13. If f is a continuous function with $\int_0^x f(t) dt \rightarrow \infty$ as $|x| \rightarrow \infty$, then show that every line $y = mx$ intersects the curve $y^2 + \int_0^x f(t) dt = 2$.
(IIT-JEE 1991)

14. Evaluate $\int_0^\pi \frac{x \sin 2x \sin \left(\frac{\pi}{2} \cos x \right)}{2x - \pi} dx$.
(IIT-JEE 1991)

15. Determine a positive integer $n \leq 5$ such that

$$\int_0^1 e^x (x-1)^n = 16 - 6e. \quad (\text{IIT-JEE 1992})$$

16. Evaluate $\int_2^3 \frac{2x^5 + x^4 - 2x^3 + 2x^2 + 1}{(x^2 + 1)(x^4 - 1)} dx$.
(IIT-JEE 1993)

17. Show that $\int_0^{n\pi+v} |\sin x| dx = 2n + 1 - \cos v$, where n is a positive integer and $0 \leq v < \pi$.
(IIT-JEE 1994)

18. If $U_n = \int_0^\pi \frac{1 - \cos nx}{1 - \cos x} dx$, where n is positive integer or zero, then show that $U_{n+2} + U_n = 2U_{n+1}$. Hence, deduce that $\int_0^{\pi/2} \frac{\sin^2 n\theta}{\sin^2 \theta} = \frac{1}{2} n\pi$.
(IIT-JEE 1995)

19. Evaluate the definite integral

$$\int_{-1/\sqrt{3}}^{1/\sqrt{3}} \left(\frac{x^4}{1-x^4} \right) \cos^{-1} \left(\frac{2x}{1+x^2} \right) dx. \quad (\text{IIT-JEE 1995, 1996})$$

20. Evaluate $\int_0^{\pi/4} \ln(1 + \tan x) dx$.
(IIT-JEE 1995, 1996)

21. Let $a + b = 4$, where $a < 2$, and let $g(x)$ be a differentiable function. If $\frac{dg}{dx} > 0$ for all x , prove that

$$\int_0^a g(x) dx + \int_0^b g(x) dx \text{ increases as } (b-a) \text{ increases.} \quad (\text{IIT-JEE 1997})$$

22. Determine the value of $\int_{-\pi}^\pi \frac{2x(1 + \sin x)}{1 + \cos^2 x} dx$.
(IIT-JEE 1997)

23. Prove that $\int_0^1 \tan^{-1}\left(\frac{1}{1-x+x^2}\right) dx = 2 \int_0^1 \tan^{-1} x dx$.

Hence or otherwise, evaluate the integral

$\int_0^1 \tan^{-1}(1-x+x^2) dx$. (IIT-JEE 1998)

24. For $x > 0$, let $f(x) = \int_1^x \frac{\log t}{1+t} dt$. Find the function

$f(x) + f\left(\frac{1}{x}\right)$ and find the value of $f(e) + f\left(\frac{1}{e}\right)$. (IIT-JEE 2000)

25. If $y(x) = \int_{\pi^2/16}^{x^2} \frac{\cos x \cos \sqrt{\theta}}{1 + \sin^2 \sqrt{\theta}} d\theta$, then find $\frac{dy}{dx}$ at $x = \pi$.

(IIT-JEE 2004)

26. Find the value of $\int_{-\pi/3}^{\pi/3} \frac{\pi + 4x^3}{2 - \cos\left(|x| + \frac{\pi}{3}\right)} dx$.

(IIT-JEE 2004)

27. Evaluate $\int_0^\pi e^{|\cos x|} \left(2 \sin\left(\frac{1}{2} \cos x\right) + 3 \cos\left(\frac{1}{2} \cos x\right) \right) \sin x dx$.

(IIT-JEE 2005)

28. Evaluate $5050 \frac{\int_0^1 (1-x^{50})^{100} dx}{\int_0^1 (1-x^{50})^{101} dx}$. (IIT-JEE 2006)

Answer Key

JEE Advanced

Single Correct Answer Type

- | | | | |
|--------|--------|--------|--------|
| 1. d. | 2. b. | 3. a. | 4. c. |
| 5. d. | 6. d. | 7. d. | 8. b. |
| 9. c. | 10. a. | 11. a. | 12. c. |
| 13. b. | 14. c. | 15. b. | 16. c. |
| 17. a. | 18. c. | 19. a. | 20. a. |
| 21. d. | 22. a. | 23. b. | 24. c. |
| 25. a. | 26. c. | 27. a. | 28. b. |
| 29. b. | 30. a. | 31. c. | 32. b. |
| 33. d. | 34. b. | 35. a. | 36. d. |

Multiple Correct Answers Type

- | | | | |
|---------------|------------|----------------|-------------------|
| 1. a. | 2. a. | 3. a., d. | 4. a., b., c., d. |
| 5. a., b., c. | 6. b., c. | 7. a., b., d. | 8. a., b., c., d. |
| 9. b., d. | 10. a., c. | 11. a., c., d. | 12. a., c. |
| 13. a., b. | | | |

Linked Comprehension Type

- | | | | |
|---------------|-----------|-------|-------|
| 1. a. | 2. d. | 3. b. | 4. b. |
| 5. b. | 6. a. | 7. d. | |
| 8. a., b., c. | 9. c., d. | | |

Matching Column Type

- (iii) - (b), (c)
- (i) - (a)
- (a) - (s); (b) - (s); (c) - (p); (d) - (r).
- (b) - (p), (t)
- (b) - (p); (c) - (s)
- d.

Integer Answer Type

- | | | | |
|------|------|------|------|
| 1. 0 | 2. 4 | 3. 2 | 4. 7 |
| 5. 9 | 6. 3 | 7. 0 | |

Fill in the Blanks Type

- | | |
|--|--|
| 1. $-\left(\frac{15\pi - 32}{60}\right)$ | 2. $2 - \sqrt{2}$ |
| 3. 4 | 4. $\pi(\sqrt{2} - 1)$ |
| 5. $\frac{1}{2}$ | 6. $\frac{1}{a^2 - b^2} \left[a \log 2 - 5a + \frac{7b}{2} \right]$ |
| 7. π^2 | 8. 2 |
| 9. 16 | |

True/False Type

- True

Subjective Type

- | | |
|-------------------------------------|--------------------------------------|
| 2. $\frac{t^{n+1} - 1}{(t-1)(n+1)}$ | 4. $\frac{3}{\pi} + \frac{1}{\pi^2}$ |
| 5. $\frac{1}{20} \log 3$ | 6. $\frac{6 - \pi\sqrt{3}}{12}$ |
| 8. $\frac{\pi^2}{16}$ | 9. $\frac{\pi\alpha}{\sin \alpha}$ |

$$14. \frac{8}{\pi^2}$$

$$15. n = 3$$

$$23. \log 2$$

$$24. \frac{1}{2} (\log x)^2$$

$$16. \frac{3}{2} \log 2 - \frac{1}{10}$$

$$17. 2n + 1 - \cos v$$

$$25. 2\pi$$

$$26. \frac{4\pi}{\sqrt{3}} \left[\tan^{-1} 3 - \frac{\pi}{4} \right]$$

$$19. -\frac{\pi}{\sqrt{3}} - \frac{\pi}{4} \log_e \left| \frac{\sqrt{3} + 1}{\sqrt{3} - 1} \right| + \frac{\pi^2}{12}$$

$$27. \frac{24}{5} \left[e \cos \left(\frac{1}{2} \right) + \frac{1}{2} e \sin \left(\frac{1}{2} \right) - 1 \right]$$

$$20. \frac{\pi}{8} \ln 2$$

$$22. \pi^2$$

$$28. 5051$$

Hints and Solutions

JEE Advanced

Single Correct Answer Type

1. d. $\int_0^1 (1 + e^{-x^2}) dx$

$$= \int_0^1 \left(1 + 1 - \frac{x^2}{1!} + \frac{x^4}{2!} - \frac{x^6}{3!} + \dots \right) dx$$

$$= \left[2x - \frac{x^3}{3 \cdot 1!} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \dots \right]_0^1$$

$$= \left[2 - \frac{1}{3 \cdot 1!} + \frac{1}{5 \cdot 2!} - \frac{1}{7 \cdot 3!} + \dots \right]$$

Clearly, d is the correct alternative.

2. b. Let $f(x) = \int (1 + \cos^8 x)(ax^2 + bx + c) dx$

$$\therefore f'(x) = (1 + \cos^8 x)(ax^2 + bx + c) \quad (1)$$

From the given conditions,

$$f(1) - f(0) = 0 \quad \text{or } f(0) = f(1) \quad (2)$$

$$\text{and } f(2) - f(0) = 0 \quad \text{or } f(0) = f(2) \quad (3)$$

From equations (2) and (3), we get $f(0) = f(1) = f(2)$.

By Rolle's theorem for $f(x)$ in $[0, 1]$: $f'(\alpha) = 0$, $\exists$ at least one α such that $0 < \alpha < 1$.

By Rolle's theorem for $f(x)$ in $[1, 2]$: $f'(\beta) = 0$, $\exists$ at least one β such that $1 < \beta < 2$.

Now, from equation (1), $f'(\alpha) = 0$

$$\text{or } (1 + \cos^8 \alpha)(a\alpha^2 + b\alpha + c) = 0 \quad (\because 1 + \cos^8 \alpha \neq 0)$$

$$\text{or } a\alpha^2 + b\alpha + c = 0$$

i.e., α is a root of the equation $ax^2 + bx + c = 0$.

Similarly, β is a root of the equation $ax^2 + bx + c = 0$.

But equation $ax^2 + bx + c = 0$ being a quadratic equation cannot have more than two roots.

Hence, equation $ax^2 + bx + c = 0$ has one root α between 0 and 1, and other root β between 1 and 2.

3. a. $I = \int_0^{\pi/2} \frac{\sqrt{\cot x}}{\sqrt{\cot x} + \sqrt{\tan x}} dx \quad (1)$

$$\therefore I = \int_0^{\pi/2} \frac{\sqrt{\tan x}}{\sqrt{\tan x} + \sqrt{\cot x}} dx \quad (2)$$

$$[\text{Using } \int_0^a f(x) dx = \int_0^a f(a-x) dx]$$

$$\text{Adding equations (1) and (2), we get } 2I = \int_0^{\pi/2} 1 dx$$

$$\therefore I = \pi/4$$

4. c. $I = \int_0^{\pi} e^{\cos^2 x} \cos^3(2n+1)x dx, n \in \mathbb{Z} \quad (1)$

$$= \int_0^{\pi} e^{\cos^2(\pi-x)} \cos^3[(2n+1)(\pi-x)] dx$$

$$[\text{Using } \int_0^a f(x) dx = \int_0^a f(a-x) dx]$$

$$= \int_0^{\pi} e^{\cos^2 x} \cos^3[(2n+1)\pi - (2n+1)x] dx$$

$$= -\int_0^{\pi} e^{\cos^2 x} \cos^3(2n+1)x dx$$

$$= -I$$

$$\therefore I = 0$$

5. d. Since $h(x) = (f(x) + f(-x))(g(x) - g(-x))$

$$\therefore h(-x) = (f(-x) + f(x))(g(-x) - g(x))$$

$$\therefore h(-x) = -h(x)$$

$h(x)$ is odd function.

$$\therefore \int_{-\pi/2}^{\pi/2} (f(x) + f(-x))(g(x) - g(-x)) dx = 0$$

6. d. Let $I = \int_0^{\pi/2} \frac{dx}{1 + \tan^3 x}$

$$= \int_0^{\pi/2} \frac{\cos^3 x}{\sin^3 x + \cos^3 x} dx \quad (1)$$

$$= \int_0^{\pi/2} \frac{\cos^3\left(\frac{\pi}{2} - x\right)}{\sin^3\left(\frac{\pi}{2} - x\right) + \cos^3\left(\frac{\pi}{2} - x\right)} dx$$

$$= \int_0^{\pi/2} \frac{\sin^3 x}{\cos^3 x + \sin^3 x} dx \quad (2)$$

Adding equations (1) and (2), we get

$$2I = \int_0^{\pi/2} 1 dx$$

$$\therefore I = \frac{\pi}{4}$$

7. d. $f(x) = A \sin(\pi x/2) + B$

$$\therefore f'(x) = \frac{A\pi}{2} \cos\left(\frac{\pi x}{2}\right)$$

$$\therefore f'\left(\frac{1}{2}\right) = \frac{A\pi}{2} \cos \frac{\pi}{4} = \sqrt{2} \quad (\text{given})$$

$$\therefore A = 4/\pi$$

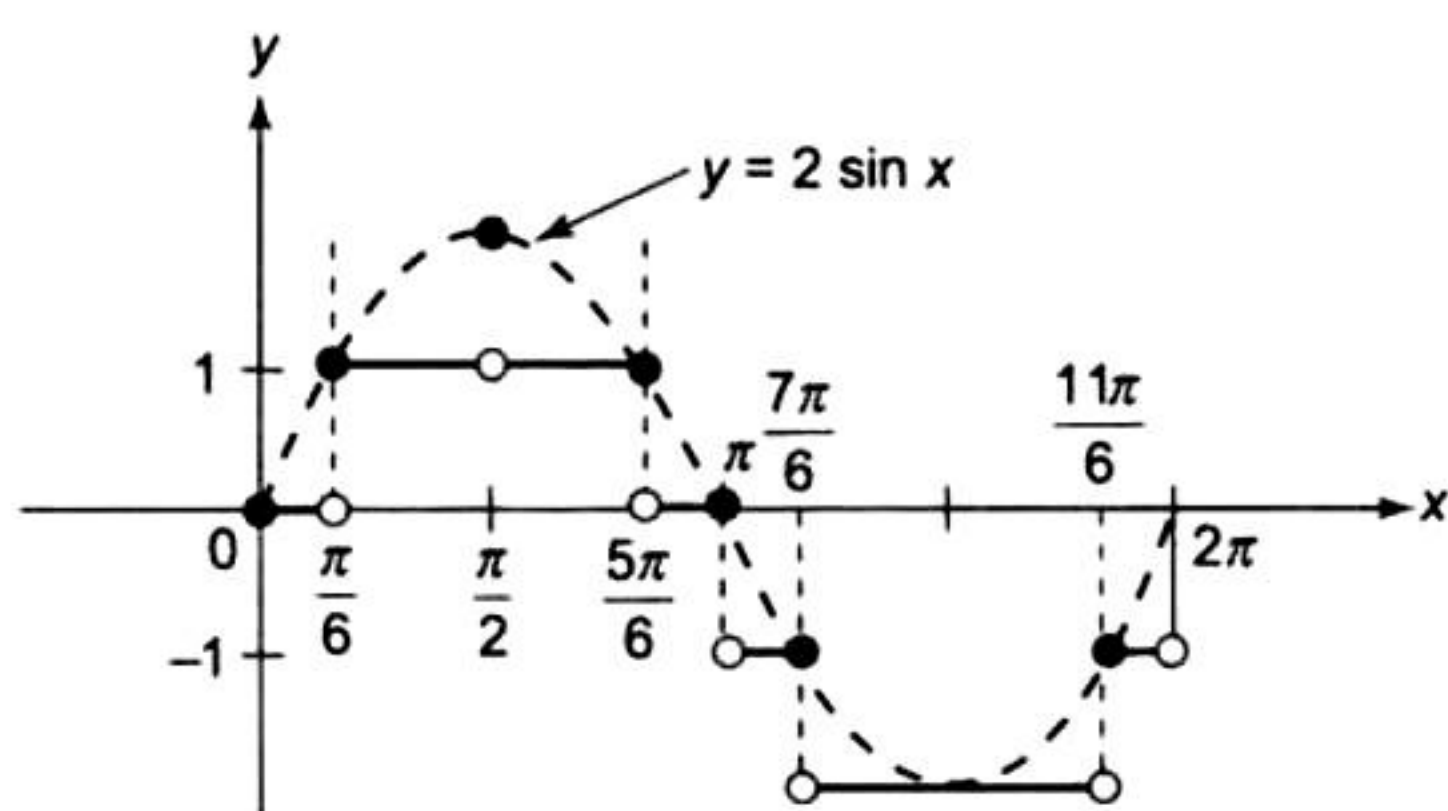
$$\text{Also, given } \int_0^1 f(x) dx = \frac{2A}{\pi}$$

$$\therefore \int_0^1 \left[A \sin\left(\frac{\pi x}{2}\right) + B \right] dx = \frac{2A}{\pi}$$

$$\therefore \left[-\frac{2A}{\pi} \cos\left(\frac{\pi x}{2}\right) + Bx \right]_0^1 = \frac{2A}{\pi}$$

$$\therefore B + \frac{2A}{\pi} = \frac{2A}{\pi} \quad \therefore B = 0$$

8. b. $I = \int_0^{2\pi} [2 \sin x] dx$



From the graph,

$$\begin{aligned}
 I &= \int_{\pi/6}^{5\pi/6} 1 dx + \int_{\pi}^{7\pi/6} -1 dx + \int_{7\pi/6}^{11\pi/6} -2 dx + \int_{11\pi/6}^{2\pi} -1 dx \\
 &= \left(\frac{5\pi}{6} - \frac{\pi}{6} \right) + \left(-\frac{7\pi}{6} + \pi \right) + 2 \left(-\frac{11\pi}{6} + \frac{7\pi}{6} \right) \\
 &\quad + \left(-2\pi + \frac{11\pi}{6} \right) \\
 &= \frac{2\pi}{3} - \frac{\pi}{6} - \frac{8\pi}{6} - \frac{\pi}{6} = -\pi
 \end{aligned}$$

9. c. Given f is a positive function, and

$$I_1 = \int_{1-k}^k x f[x(1-x)] dx$$

$$I_2 = \int_{1-k}^k f[x(1-x)] dx$$

$$\text{Now, } I_1 = \int_{1-k}^k x f[x(1-x)] dx \quad (1)$$

$$= \int_{1-k}^k (1-x) f[(1-x)x] dx \quad (2)$$

$$\left[\text{Using the property } \int_a^b f(x) dx = \int_a^b f(a+b-x) dx \right]$$

Adding equations (1) and (2), we get

$$2I_1 = \int_{1-k}^k f[x(1-x)] dx = I_2 \text{ or } \frac{I_1}{I_2} = \frac{1}{2}$$

$$10. a. g(x) = \int_0^x \cos^4 t dt$$

$$\begin{aligned}
 \therefore g(x+\pi) &= \int_0^{x+\pi} \cos^4 t dt \\
 &= \int_0^x \cos^4 t dt + \int_x^{x+\pi} \cos^4 t dt \\
 &= g(x) + \int_0^\pi \cos^4 t dt \quad [\because \text{period of } \cos^4 t \text{ is } \pi] \\
 &= g(x) + g(\pi)
 \end{aligned}$$

$$11. a. I = \int_{\pi/4}^{3\pi/4} \frac{dx}{1+\cos x} \quad (1)$$

$$\begin{aligned}
 &= \int_{\pi/4}^{3\pi/4} \frac{dx}{1+\cos(\pi-x)} \\
 &\quad \left[\text{Using the property } \int_a^b f(x) dx = \int_a^b f(a+b-x) dx \right]
 \end{aligned}$$

$$= \int_{\pi/4}^{3\pi/4} \frac{dx}{1-\cos x} \quad (2)$$

Adding (1) and (2), we get

$$\begin{aligned}
 2I &= \int_{\pi/4}^{3\pi/4} \left(\frac{1}{1+\cos x} + \frac{1}{1-\cos x} \right) dx \\
 &= \int_{\pi/4}^{3\pi/4} 2 \operatorname{cosec}^2 x dx \\
 &= 2 (-\cot x)_{\pi/4}^{3\pi/4} \\
 &= -2 [\cot 3\pi/4 - \cot \pi/4] \\
 &= -2 (-1-1) = 4 \\
 \therefore I &= 2
 \end{aligned}$$

12. c. Refer to graph of Question 8. Then

$$\begin{aligned}
 &\int_{\pi/2}^{3\pi/2} [2 \sin x] dx \\
 &= \int_{\pi/2}^{5\pi/6} 1 dx + \int_{\pi}^{7\pi/6} -1 dx + \int_{7\pi/6}^{3\pi/2} -2 dx \\
 &= \left[\frac{5\pi}{6} - \frac{\pi}{2} \right] - \left[\frac{7\pi}{6} - \pi \right] - 2 \left[\frac{3\pi}{2} - \frac{7\pi}{6} \right] \\
 &= \frac{-\pi}{2}
 \end{aligned}$$

$$13. b. g(x) = \int_0^x f(t) dt,$$

$$\therefore g(2) = \int_0^2 f(t) dt = \int_0^1 f(t) dt + \int_1^2 f(t) dt$$

$$\text{Now, } \frac{1}{2} \leq f(t) \leq 1 \text{ for } t \in [0, 1]$$

$$\therefore \int_0^1 \frac{1}{2} dt \leq \int_0^1 f(t) dt \leq \int_0^1 1 dt$$

$$\text{or } \frac{1}{2} \leq \int_0^1 f(t) dt \leq 1 \quad (1)$$

$$\text{Again, } 0 \leq f(t) \leq \frac{1}{2} \text{ for } t \in [1, 2]$$

$$\therefore \int_1^2 0 dt \leq \int_1^2 f(t) dt \leq \int_1^2 \frac{1}{2} dt$$

$$\therefore 0 \leq \int_1^2 f(t) dt \leq \frac{1}{2} \quad (2)$$

From equations (1) and (2), we get

$$\frac{1}{2} \leq \int_0^1 f(t) dt + \int_1^2 f(t) dt \leq \frac{3}{2}$$

$$\therefore \frac{1}{2} \leq g(2) \leq \frac{3}{2}$$

$$14. c. \text{ If } f(x) = \begin{cases} e^{\cos x} \sin x, & \text{for } |x| \leq 2 \\ 2, & \text{otherwise} \end{cases}$$

$$\begin{aligned}
 \text{or } \int_{-2}^3 f(x) dx &= \int_{-2}^2 f(x) dx + \int_2^3 f(x) dx \\
 &= \int_{-2}^2 e^{\cos x} \sin x dx + \int_2^3 2 dx = 0 + 2[x]_2^3 = 2
 \end{aligned}$$

$[\because e^{\cos x} \sin x \text{ is an odd function}]$

$$15. b. \text{ Let } I = \int_{e^{-1}}^{e^2} \left| \frac{\log_e x}{x} \right| dx$$

For $\frac{1}{e} < x < 1$, $\log_e x < 0$. Hence, $\frac{\log_e x}{x} < 0$.

For $1 < x < e^2$, $\log x > 0$. Hence, $\frac{\log_e x}{x} > 0$.

$$\begin{aligned}\therefore I &= \int_{1/e}^1 -\frac{\log_e x}{x} dx + \int_1^{e^2} \frac{\log_e x}{x} dx \\&= -\frac{1}{2} \left[(\log_e x)^2 \right]_{1/e}^1 + \frac{1}{2} \left[(\log_e x)^2 \right]_1^{e^2} \\&= -\frac{1}{2} [0 - (-1)^2] + \frac{1}{2} [(2)^2 - 0] \\&= \frac{1}{2} + 2 = \frac{5}{2}\end{aligned}$$

$$16. \text{ c. } I = \int_{-\pi}^{\pi} \frac{\cos^2 x}{1+a^x} dx \quad (1)$$

$$\begin{aligned}&= \int_{-\pi}^{\pi} \frac{\cos^2(0-x)}{1+a^{(0-x)}} dx \\&\quad \left[\text{Using the property } \int_a^b f(x) dx = \int_a^b f(a+b-x) dx \right] \\&\therefore I = \int_{-\pi}^{\pi} \frac{a^x \cos^2 x}{1+a^x} dx \quad (2)\end{aligned}$$

Adding equations (1) and (2), we get

$$\begin{aligned}2I &= \int_{-\pi}^{\pi} \cos^2 x dx \\&= 2 \int_0^{\pi} \cos^2 x dx \\&\therefore 2I = 4 \int_0^{\pi/2} \cos^2 x dx \quad (3)\end{aligned}$$

$$\left[\because \int_0^{2a} f(x) dx = 2 \int_0^a f(x) dx \text{ if } f(2a-x) = f(x) \right]$$

$$= 4 \int_0^{\pi/2} \sin^2 x dx \quad (4)$$

Adding equations (3) and (4), we get

$$\begin{aligned}4I &= 4 \int_0^{\pi/2} 1 dx \\ \text{or } I &= \pi/2\end{aligned}$$

$$17. \text{ a. Here, } f(x) = \int_1^x \sqrt{2-t^2} dt$$

$$\therefore f'(x) = \sqrt{2-x^2}$$

Now, the given equation $x^2 - f'(x) = 0$ becomes

$$x^2 - \sqrt{2-x^2} = 0$$

$$\text{or } x^2 = \sqrt{2-x^2}$$

$$\therefore x = \pm 1$$

$$18. \text{ c. Let } I_1 = \int_3^{3+3T} f(2x) dx.$$

$$\text{Put } 2x = y, \text{ so that } I_1 = \frac{1}{2} \int_6^{6+6T} f(y) dy$$

$$\begin{aligned}&= \frac{1}{2} 6 \int_0^T f(y) dy \quad [\because f(x) \text{ has period } T] \\&= 3I\end{aligned}$$

$$\begin{aligned}19. \text{ a. } I &= \int_{-1/2}^{1/2} \left([x] + \ln \left(\frac{1+x}{1-x} \right) \right) dx \\&= \int_{-1/2}^{1/2} [x] dx + \int_{-1/2}^{1/2} \ln \left(\frac{1+x}{1-x} \right) dx \\&= \int_{-1/2}^0 -1 dx + \int_0^{1/2} 0 dx + 0 \\&\quad \left[\because \log \left(\frac{1+x}{1-x} \right) \text{ is an odd function} \right] \\&= [-x]_{-1/2}^0 = 0 - \left(-\frac{1}{2} \right) = -1/2\end{aligned}$$

$$20. \text{ a. Given } L(m, n) = \int_0^1 t^m (1+t)^n dt$$

Integrating by parts considering $(1+t)^n$ as first function, we get

$$\begin{aligned}L(m, n) &= \left[\frac{t^{m+1}}{m+1} (1+t)^n \right]_0^1 - \frac{n}{m+1} \int_0^1 t^{m+1} (1+t)^{n-1} dt \\&= \frac{2^n}{m+1} - \frac{n}{m+1} L(m+1, n-1)\end{aligned}$$

$$21. \text{ d. We have } f(x) = \int_{x^2}^{x^2+1} e^{-t^2} dt$$

$$\begin{aligned}\therefore f'(x) &= e^{-(x^2+1)^2} \cdot 2x - e^{-x^4} \cdot 2x \\&= 2x \left[e^{-(x^2+1)^2} - e^{-x^4} \right]\end{aligned}$$

$$\because (x^2+1)^2 > x^4$$

$$\therefore e^{-(x^2+1)^2} < e^{-x^4} \therefore e^{-(x^2+1)^2} - e^{-x^4} < 0$$

$$\therefore e^{-(x^2+1)^2} - e^{-x^4} < 0$$

$$\therefore f'(x) \leq 0 \quad \forall x \leq 0$$

Therefore, $f(x)$ increases when $x \leq 0$.

$$22. \text{ a. } \int_0^{t^2} xf(x) dx = \frac{2}{5} t^5 \quad (\text{Here, } t > 0)$$

Differentiating both sides w.r.t. t , we get

$$t^2 f(t^2) \times 2t = \frac{2}{5} \times 5t^4$$

$$\therefore f(t^2) = t$$

$$\text{Put } t = \frac{2}{5}. \text{ Then } f\left(\frac{4}{25}\right) = \frac{2}{5}.$$

$$23. \text{ b. } I = \int_0^1 \sqrt{\frac{1-x}{1+x}} dx$$

$$= \int_0^1 \frac{1-x}{\sqrt{1-x^2}} dx$$

$$= \sin^{-1} x \Big|_0^1 - \int_0^1 \frac{x}{\sqrt{1-x^2}} dx$$

$$= \frac{\pi}{2} + \left[\sqrt{1-x^2} \right]_0^1$$

$$= \frac{\pi}{2} + (0 - 1) = \frac{\pi}{2} - 1$$

$$24. \text{ c. } I = \int_{-2}^0 [x^3 + 3x^2 + 3x + 3 + (x+1)\cos(x+1)] dx \quad (1)$$

$$= \int_{-2}^0 [(x+1)^3 + 2 + (x+1)\cos(x+1)] dx$$

$$= \int_{-2}^0 [(-2-x+1)^3 + 2 + (-2-x+1)\cos(-2-x+1)] dx$$

$$\therefore I = \int_{-2}^0 [-(1+x)^3 + 2 - (1+x)\cos(1+x)] dx \quad (2)$$

Adding (1) and (2)

$$2I = 2 \int_{-2}^0 2 \quad \text{or} \quad I = 4$$

$$25. \text{ a. } \lim_{x \rightarrow \frac{\pi}{4}} \frac{\int_{\frac{\pi}{4}}^{\sec^2 x} f(t) dt}{x^2 - \frac{\pi^2}{16}} \quad \left(\frac{0}{0} \text{ form} \right)$$

$$= \lim_{x \rightarrow \frac{\pi}{4}} \frac{f(\sec^2 x) 2 \sec x \sec x \tan x}{2x}$$

(Applying L'Hospital's Rule)

$$= \frac{2f(2)}{\pi/4} = \frac{8f(2)}{\pi}$$

$$26. \text{ c. } \int_0^x \sqrt{1 - (f'(t))^2} dt = \int_0^x f(t) dt$$

Differentiating w.r.t. x , we get

$$\sqrt{1 - (f'(x))^2} = f(x)$$

$$\therefore 1 - (f'(x))^2 = (f(x))^2$$

$$\therefore (f'(x))^2 = 1 - (f(x))^2$$

$$\therefore f'(x) = \pm \sqrt{1 - (f(x))^2}$$

$$\therefore \int \frac{f'(x)}{\sqrt{1 - (f(x))^2}} dx = \pm \int 1 dx$$

$$\therefore \int \frac{dt}{\sqrt{1 - t^2}} = \pm \int 1 dx$$

$$\therefore \sin^{-1} t = \pm (x + c)$$

$$\therefore t = f(x) = \pm \sin(x + c)$$

Now, $f(0) = 0$

$$\therefore c = 0$$

Thus, $f(x) = \sin x$ (as $f(x)$ is non-negative function in $[0, 1]$).
Hence, correct option is (c) as $x > \sin x$ for $\forall x > 0$.

$$27. \text{ a } \int_0^1 \left(x^6 - 4x^5 + 5x^4 - 4x^2 + 4 - \frac{4}{1+x^2} \right) dx$$

$$= \left[\frac{x^7}{7} - \frac{2x^6}{3} + x^5 - \frac{4x^3}{3} + 4x \right]_0^1 - 4[\tan^{-1} x]_0^1$$

$$= \frac{1}{7} - \frac{2}{3} + 1 - \frac{4}{3} + 4 - \pi = \frac{22}{7} - \pi$$

$$28. \text{ b. } \lim_{x \rightarrow 0} \frac{1}{x^3} \int_0^x \frac{t \ln(1+t) dt}{t^4 + 4}$$

$$= \lim_{x \rightarrow 0} \frac{\int_0^x \frac{t \ln(1+t) dt}{t^4 + 4}}{\frac{x^3}{x^3}}$$

$$= \lim_{x \rightarrow 0} \frac{x \ln(1+x)}{(x^4 + 4) 3x^2}$$

$$= \lim_{x \rightarrow 0} \frac{1 \log_e(1+x)}{3x(x^4 + 4)}$$

$$= \frac{1}{12}$$

$\left(\frac{0}{0} \text{ form} \right)$

$$29. \text{ b } e^{-x} f(x) = 2 + \int_0^x \sqrt{t^4 + 1} dt \quad (1)$$

Now, $f(f^{-1}(x)) = x$

$$\therefore f'(f^{-1}(x))(f^{-1}(x))' = 1$$

$$\therefore (f^{-1})'(2) = \frac{1}{f'(f^{-1}(2))}$$

$$f(0) = 2 \therefore f^{-1}(2) = 0$$

$$\therefore (f^{-1})'(2) = \frac{1}{f'(0)}$$

$$e^{-x} f(x) = 2 + \int_0^x \sqrt{t^4 + 1} dt$$

Differentiating w.r.t. x , we get

$$e^{-x} (f'(x) - f(x)) = \sqrt{x^4 + 1}$$

Put $x = 0$

$$\therefore f'(0) - 2 = 1$$

$$\text{or } f'(0) = 3$$

$$\therefore (f^{-1})'(2) = 1/3$$

$$30. \text{ a } \text{Put } x^2 = t \text{ or } 2x dx = dt$$

$$\therefore I = \frac{1}{2} \int_{\ln 2}^{\ln 3} \frac{\sin t}{\sin t + \sin(\ln 6 - t)} dt \quad (1)$$

$$\therefore I = \frac{1}{2} \int_{\ln 2}^{\ln 3} \frac{\sin(\ln 6 - t)}{\sin(\ln 6 - t) + \sin t} dt \quad (2)$$

$$\text{Using } \left(\int_a^b f(x) dx = \int_a^b f(a+b-x) dx \text{ and } \ln 2 + \ln 3 = \ln 6 \right)$$

Adding (1) and (2)

$$2I = \frac{1}{2} \int_{\ln 2}^{\ln 3} 1 dt \quad \text{or} \quad I = \frac{1}{4} \ln \frac{3}{2}$$

$$31. \text{ c } R_1 = \int_{-1}^2 x f(x) dx = \int_{-1}^2 (2-1-x) f(2-1-x) dx$$

$$= \int_{-1}^2 (1-x) f(1-x) dx = \int_{-1}^2 (1-x) f(x) dx$$

Hence, $2R_1 = \int_{-1}^2 f(x) dx = R_2$.

32. b.

$$\begin{aligned} & \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left\{ x^2 + \log_e \left(\frac{\pi+x}{\pi-x} \right) \right\} \cos x \, dx \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^2 \cos x \, dx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log_e \left(\frac{\pi+x}{\pi-x} \right) \cos x \, dx \\ &= 2 \int_0^{\frac{\pi}{2}} x^2 \cos x \, dx + 0 \left(\because \log_e \left(\frac{\pi+x}{\pi-x} \right) \cos x \text{ is an odd function} \right) \\ &= 2 \left[x^2 \sin x + 2x \cos x - 2 \sin x \right]_0^{\frac{\pi}{2}} \\ &= 2 \left[\frac{\pi^2}{4} - 2 \right] = \frac{\pi^2}{2} - 4 \end{aligned}$$

33. d Given $f'(x) - 2f(x) < 0 \therefore f'(x)e^{-2x} - 2e^{-2x}f(x) < 0$

$$\therefore \frac{d}{dx}(f(x)e^{-2x}) < 0$$

Thus, $g(x) = f(x)e^{-2x}$ is decreasing function.

Also, $f(1/2) = 1$.

For $x > \frac{1}{2}$

$$g(x) < g(1/2) \text{ or } f(x)e^{-2x} < f(1/2)e^{-1} \text{ or } f(x) < e^{2x-1}$$

$$\therefore 0 < \int_{1/2}^1 f(x) dx < \int_{1/2}^1 e^{2x-1} dx \text{ or } 0 < \int_{1/2}^1 f(x) dx < \frac{e-1}{2}$$

34. b. $Fx = \int_0^x f(\sqrt{t}) dt$

$$F(0) = 0$$

$$F'(x) = 2x f(x) = f'(x)$$

$$\Rightarrow \int \frac{f'(x)}{f(x)} dx = \int 2x dx$$

$$\Rightarrow \log_e f(x) = x^2 + c$$

$$\Rightarrow f(x) = e^{x^2+c}$$

$$\Rightarrow f(x) = e^{x^2} \quad (\because f(0) = 1)$$

$$\Rightarrow F(x) = \int_0^x e^{t^2} dt$$

$$\Rightarrow F(x) = e^{x^2} - 1 \quad (\because F(0) = 0)$$

$$\Rightarrow F(2) = e^4 - 1$$

35. a. $I = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (2 \operatorname{cosec} x)^{17} dx$

$$\text{Let } e^u + e^{-u} = 2 \operatorname{cosec} x,$$

$$\text{For } x = \frac{\pi}{4}, u = \ln(1 + \sqrt{2})$$

$$\text{For } x = \frac{\pi}{2}, u = 0$$

$$\text{Also, } \operatorname{cosec} x + \cot x = e^u \text{ and } \operatorname{cosec} x - \cot x = e^{-u}$$

$$\Rightarrow \cot x = \frac{e^u - e^{-u}}{2}$$

$$\text{Also } (e^u - e^{-u}) du = -2 \operatorname{cosec} x \cot x \, dx$$

$$\begin{aligned} \Rightarrow I &= - \int_{\ln(1+\sqrt{2})}^0 (e^u + e^{-u})^{17} \frac{(e^u - e^{-u})}{2 \operatorname{cosec} x \cot x} du \\ &= -2 \int_{\ln(1+\sqrt{2})}^0 (e^u + e^{-u})^{16} du \\ &= \int_0^{\ln(1+\sqrt{2})} 2(e^u + e^{-u})^{16} du \end{aligned}$$

36. d. $f'(x) = \frac{192x^3}{2 + \sin^4(\pi x)} \forall x \in R; f\left(\frac{1}{2}\right) = 0$

$$\text{Now, } 64x^3 \leq f'(x) \leq 96x^3 \forall x \in \left[\frac{1}{2}, 1\right]$$

$$\int_{1/2}^1 64x^3 dx \leq \int_{1/2}^1 f'(x) dx \leq \int_{1/2}^1 96x^3 dx$$

$$\text{So, } 16x^4 - 1 \leq f(x) \leq 24x^4 - \frac{3}{2} \forall x \in \left[\frac{1}{2}, 1\right]$$

$$\int_{1/2}^1 (16x^4 - 1) dx \leq \int_{1/2}^1 f(x) dx \leq \int_{1/2}^1 \left(24x^4 - \frac{3}{2}\right) dx$$

$$\frac{16}{5} \cdot \frac{31}{32} - \frac{1}{2} \leq \int_{1/2}^1 f(x) dx \leq \frac{24}{5} \cdot \frac{31}{32} - \frac{3}{4}$$

$$\Rightarrow \frac{26}{10} \leq \int_{1/2}^1 f(x) dx \leq \frac{78}{20}$$

Multiple Correct Answers Type

1. a. $\int_0^x f(t) dt = x + \int_x^1 t f(t) dt$

Differentiating both sides w.r.t. x , we get

$$f(x) = 1 + 0 - xf(x)$$

$$\therefore (x+1)f(x) = 1$$

$$\therefore f(x) = \frac{1}{x+1}$$

$$\text{or } f(1) = \frac{1}{2}$$

2. a. $\int_{-1}^1 f(x) dx = \int_{-1}^1 (x - [x]) dx$

$$= \int_{-1}^1 x dx - \int_{-1}^1 [x] dx$$

$$= 0 - \int_{-1}^1 [x] dx \quad (1)$$

$[\because x \text{ is an odd function}]$

$$= -\int_{-1}^0 (-1) dx - \int_0^1 0 dx$$

$$= 1$$

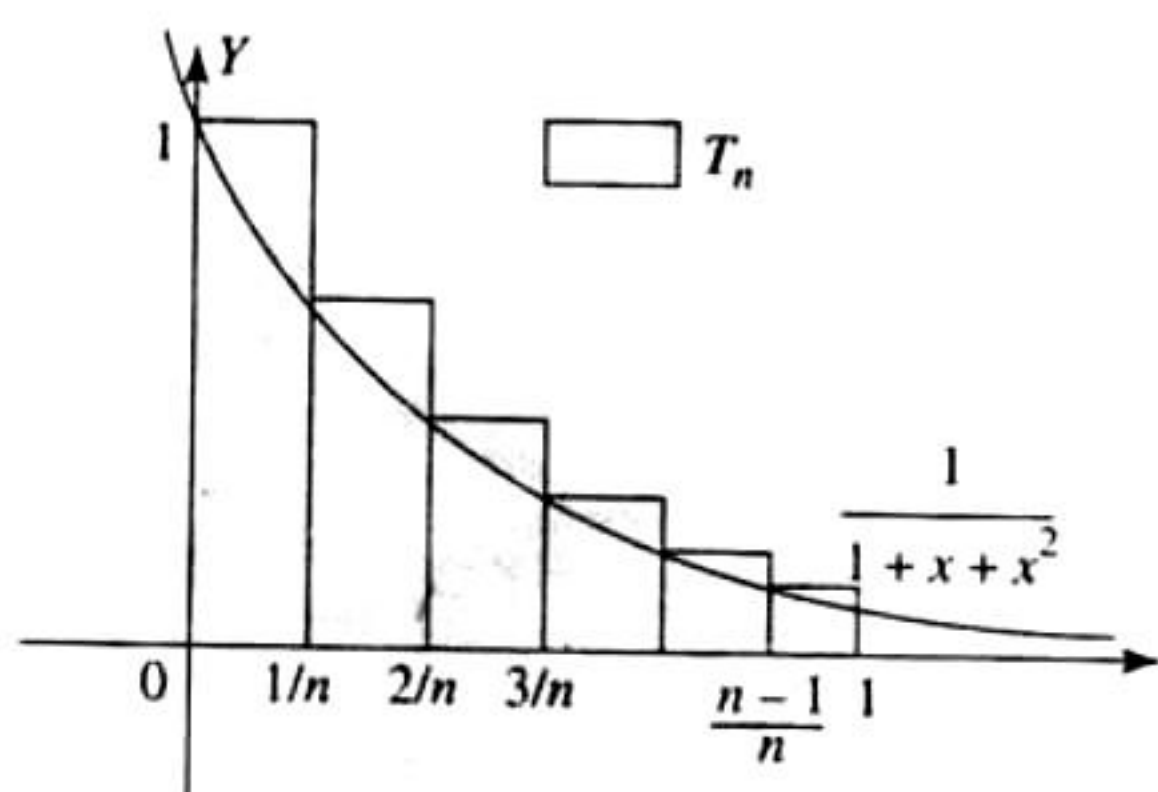
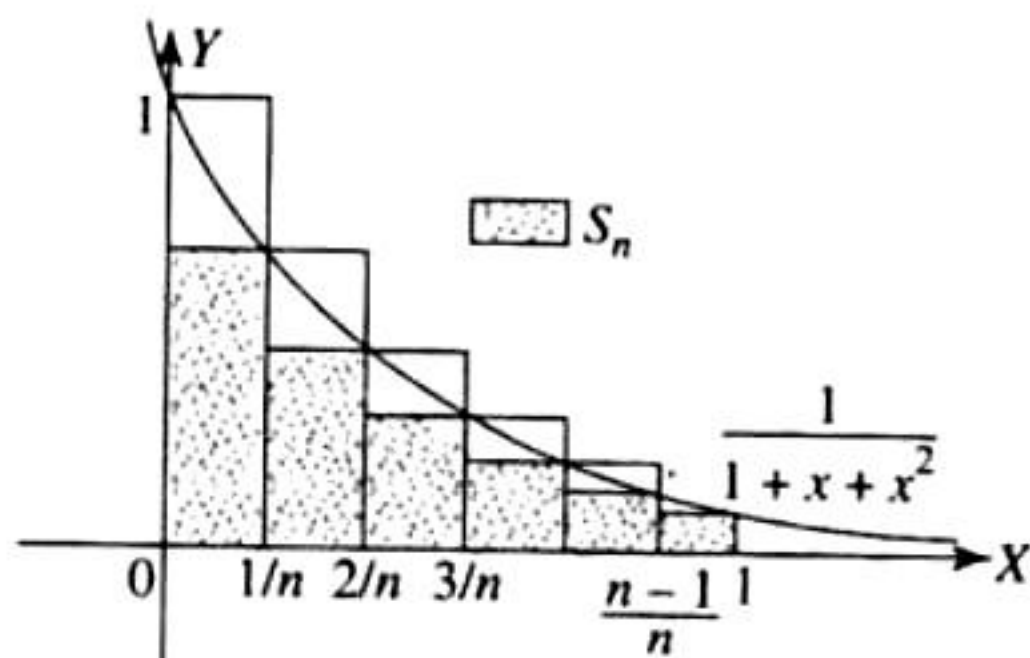
3. a., d.

$$S_n = \sum_{k=1}^n \frac{n}{n^2 + kn + k^2}$$

$$= \frac{1}{n} \sum_{k=1}^n \frac{1}{1 + \left(\frac{k}{n}\right) + \left(\frac{k}{n}\right)^2}$$

and $T_n = \frac{1}{n} \sum_{k=0}^{n-1} \frac{1}{1 + \left(\frac{k}{n}\right) + \left(\frac{k}{n}\right)^2}$

So, consider function $f(x) = \frac{1}{1+x+x^2}$



From the graph $S_n < \int_0^1 \frac{1}{1+x+x^2} dx < T_n$

Now, $\int_0^1 \frac{dx}{1+x+x^2}$

$$= \int_0^1 \frac{dx}{\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}}$$

$$= \left[\frac{2}{\sqrt{3}} \tan^{-1} \left(\frac{x + \frac{1}{2}}{\frac{\sqrt{3}}{2}} \right) \right]_0^1$$

$$= \frac{\pi}{3\sqrt{3}}$$

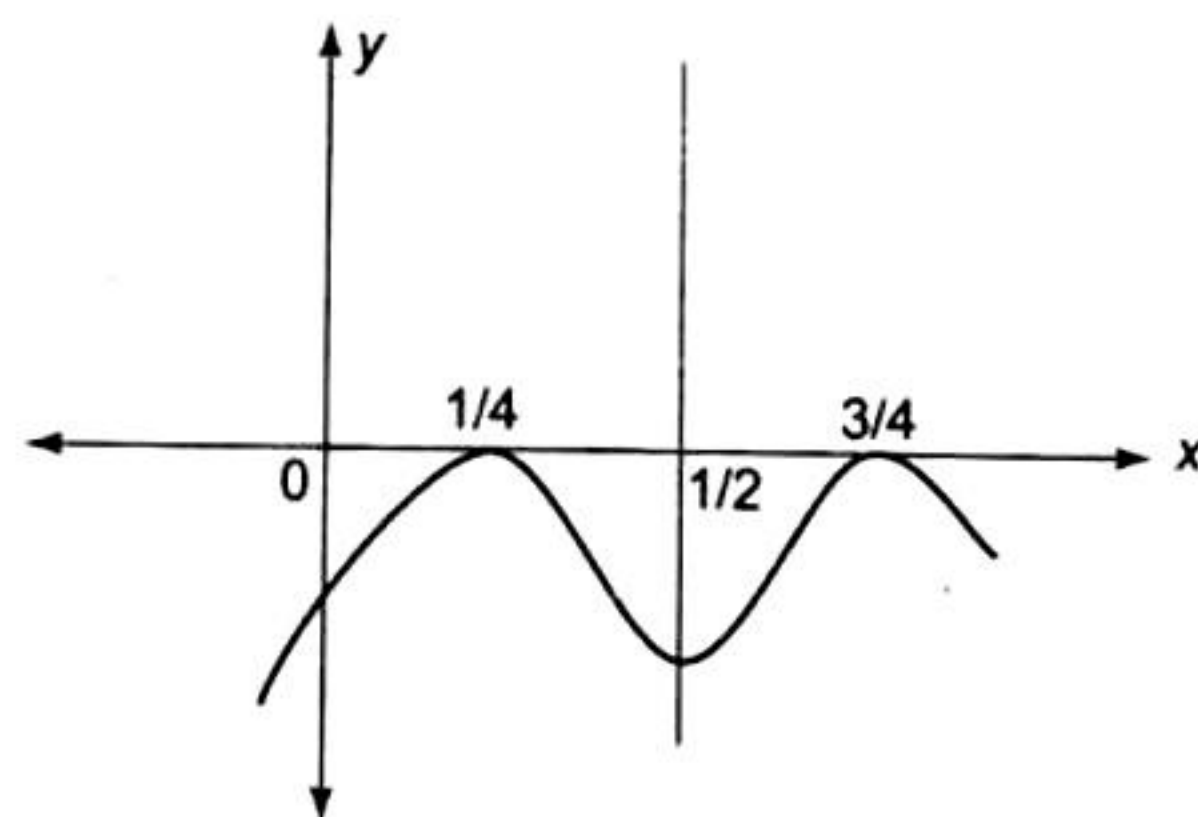
$$\therefore S_n < \frac{\pi}{3\sqrt{3}} < T_n$$

4. a., b., c., d.

$$f(x) = f(1-x)$$

Replacing x by $\frac{1}{2} + x$, we get

$$f\left(\frac{1}{2} + x\right) = f\left(\frac{1}{2} - x\right) \quad (1)$$



$$\int_{-1/2}^{1/2} f\left(x + \frac{1}{2}\right) \sin x \, dx = \int_{-1/2}^{1/2} f\left(-x + \frac{1}{2}\right) \sin(-x) \, dx$$

$$= - \int_{-1/2}^{1/2} f\left(x + \frac{1}{2}\right) \sin x \, dx$$

$$\therefore \int_{-1/2}^{1/2} f\left(x + \frac{1}{2}\right) \sin x \, dx = 0$$

Also, $f'(x) = -f'(1-x)$ and for $x = 1/2$, we have $f'(1/2) = 0$. (2)

$$\text{Also, } \int_{1/2}^1 f(1-t) e^{\sin \pi t} dt = - \int_{1/2}^0 f(y) e^{\sin \pi y} dy$$

(by putting, $1-t=y$)

Since $f'(1/4) = 0$, $f'(3/4) = 0$. [from equation (2)]

Also, $f'(1/2) = 0$. [from equation (2)]

Thus, $f'(x) = 0$ at least twice in $[0, 1]$. (by Rolle's theorem)

5. a., b., c.

$$I_n = \int_{-\pi}^{\pi} \frac{\sin nx}{(1 + \pi^x) \sin x} dx \quad (1)$$

$$= \int_{-\pi}^{\pi} \frac{\sin(-nx)}{(1 + \pi^{-x}) \sin(-x)} dx$$

$$\therefore I_n = \int_{-\pi}^{\pi} \frac{\pi^x \sin nx}{(\pi^x + 1) \sin x} dx \quad (2)$$

Adding (1) and (2)

$$2I_n = \int_{-\pi}^{\pi} \frac{\sin nx}{\sin x} dx$$

$$= 2 \int_0^{\pi} \frac{\sin nx}{\sin x} dx \quad \left(\because \frac{\sin nx}{\sin x} \text{ is an even function} \right)$$

$$\text{Now, } I_{n+2} - I_n = \int_0^{\pi} \frac{\sin(n+2)x - \sin nx}{\sin x} dx$$

$$= \int_0^{\pi} \frac{2 \cos(n+1)x \sin x}{\sin x} dx = 0$$

$$\Rightarrow I_{n+2} = I_n$$

$$\therefore I_1 = \pi, I_2 = \int_0^{\pi} 2 \cos x \, dx = 0$$

$$\therefore I_1 = I_3 = I_5 = \dots = \pi$$

$$\text{and } I_2 = I_4 = I_6 = \dots = 0$$

6. b., c. $f(x) = \ln x + \int_0^x \sqrt{1 + \sin t} dt$

$$\Rightarrow f'(x) = \frac{1}{x} + \sqrt{1 + \sin x}$$

$$\Rightarrow f''(x) = -\frac{1}{x^2} + \frac{\cos x}{2\sqrt{1 + \sin x}}$$

$f''(x)$ is not defined at for $x = 2n\pi - \frac{\pi}{2}, n \in \mathbb{N}$.

So, option (a) is wrong.

$f'(x)$ exists for all $x > 0$. ($\because 1 + \sin x \geq 0$)

Also $f'(x)$ is continuous and differentiable as $\frac{1}{x}$ and $\sqrt{1 + \sin x}$ are continuous and differentiable.

For $x > 1, f(x) = \ln x + \int_0^x \sqrt{1 + \sin t} dt > 0$

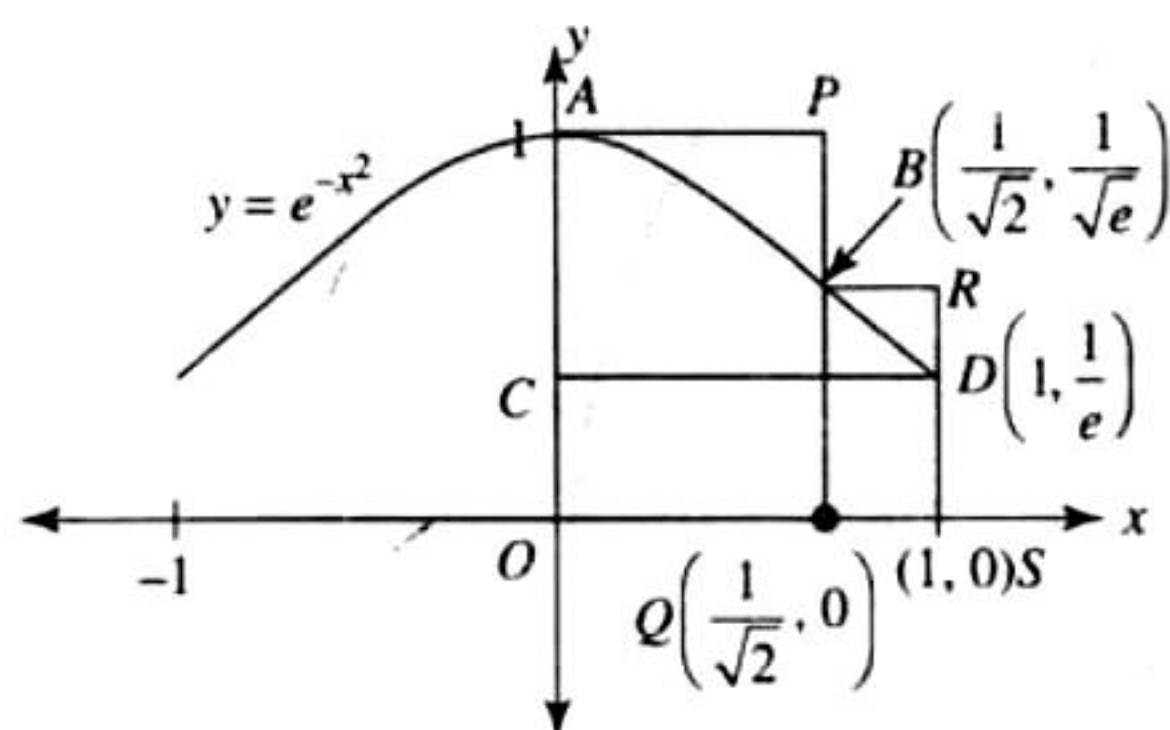
Also, $f'(x) = \frac{1}{x} + \sqrt{1 + \sin x} > 0$

Further, $\frac{1}{x} + \sqrt{1 + \sin x}$ is bounded and $\ln x + \int_0^x \sqrt{1 + \sin t} dt$ is unbounded for $x \rightarrow \infty$.

Thus, there exists $\alpha > 1$ such that $|f''(x)| < |f(x)|$ for $x \in (\alpha, \infty)$.

$|f(x)| + |f''(x)| \leq \beta$ is wrong as $f(x)$ is monotonically increasing and it is unbounded, while β is finite.

7. a., b., d.



$S > \frac{1}{e}$ (As area of rectangle $OAPQ = 1/e$)

Since $e^{-x^2} \geq e^{-x} \forall x \in [0, 1]$, we have

$$S > \int_0^1 e^{-x} dx = \left(1 - \frac{1}{e}\right)$$

Area of rectangle $OAPQ$ + Area of rectangle $QBRD > S$

$$\text{or } S < \frac{1}{\sqrt{2}}(1) + \left(1 - \frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{e}}\right)$$

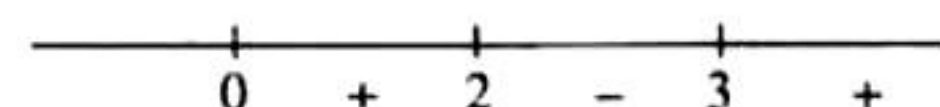
$$\text{Since } \frac{1}{4}\left(1 + \frac{1}{\sqrt{e}}\right) < 1 - \frac{1}{e}$$

Option (c) is incorrect.

8. a., b., c., d.

$$f(x) = \int_0^x e^{t^2} \cdot (t-2)(t-3) dt$$

$$\therefore f'(x) = e^{x^2} (x-2)(x-3)$$



Clearly, maxima at $x = 2$, minima at $x = 3$ and decreasing in $x \in (2, 3)$.

$f'(x) = 0$ for $x = 2$ and $x = 3$ (Rolle's theorem)

So, there exist $c \in (2, 3)$ for which $f''(c) = 0$.

9. b., d. $\lim_{n \rightarrow \infty} \frac{(1^a + 2^a + \dots + n^a)}{(n+1)^{a-1} [(na+1) + (na+2) + \dots + (na+n)]}$

$$\begin{aligned} &= \frac{\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^n \left(\frac{r}{n}\right)^a}{\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^{a-1} \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^n \left(a + \frac{r}{n}\right)} \\ &= \frac{\int_0^1 x^a dx}{\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^{a-1} \int_0^1 (a+x) dx} \quad \left(\because \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^{a-1} = 1^{a-1} = 1\right) \\ &= \frac{\frac{1}{a+1}}{\int_0^1 (a+x) dx} = \frac{2}{(2a+1)(a+1)} = \frac{2}{120} \end{aligned}$$

$\therefore a = 7$ or $-\frac{17}{2}$ (Given)

10. a., c.

For continuity at $x = a$

$$\lim_{x \rightarrow a} g(x) = 0$$

$$g(a) = \int_a^a f(t) dt = 0$$

$$\lim_{x \rightarrow a^+} g(x) = \lim_{x \rightarrow a^+} \int_a^x f(t) dt = 0$$

Hence, $g(x)$ is continuous at $x = a$.

For continuity at $x = b$

$$\begin{aligned} \lim_{x \rightarrow b^-} g(x) &= \lim_{x \rightarrow b^-} \int_a^x f(t) dt = \int_a^b f(t) dt \\ &= \lim_{x \rightarrow b^-} g(x) = g(b) \end{aligned}$$

Thus, $f(x)$ is continuous at $x = b$.

$$g'(x) = \begin{cases} 0, & x < a \\ f(x), & a < x < b \\ 0, & x > b \end{cases}$$

Since $f(x) \geq 1$ for $x \in [a, b]$, $g(x)$ is non-differentiable at $x = a$ and $x = b$.

11. a., c., d.

$$f(x) = \int_{1/x}^x e^{-\left(t+\frac{1}{t}\right)} \frac{dt}{t}$$

$$\begin{aligned} \Rightarrow f'(x) &= \frac{e^{-\left(x+\frac{1}{x}\right)}}{x} - \frac{e^{-\left(\frac{1}{x}+x\right)}}{\frac{1}{x}} \left(-\frac{1}{x^2}\right) \\ &= \frac{2e^{-\left(x+\frac{1}{x}\right)}}{x} > 0 \text{ for } x \in [1, \infty) \end{aligned}$$

Therefore, $f(x)$ is increasing in $[1, \infty)$.

$$f'(x) > 0 \text{ for } x \in (0, 1).$$

Hence, $f(x)$ is increasing.

Also, $f(x) + f\left(\frac{1}{x}\right)$

$$= \int_{1/x}^x e^{-\left(t+\frac{1}{t}\right)} \frac{dt}{t} + \int_x^{1/x} e^{-\left(t+\frac{1}{t}\right)} \frac{dt}{t}$$

$$= 0$$

$$g(x) = f(2^x) = \int_{2^{-x}}^{2^x} \frac{e^{-\left(t+\frac{1}{t}\right)}}{t} dt$$

$$\therefore g(-x) = \int_{2^x}^{2^{-x}} \frac{e^{-\left(t+\frac{1}{t}\right)}}{t} dt = -g(x)$$

Hence, $f(2^x)$ is an odd function.

12. a., c.

Let $\int_0^\pi e^t (\sin^6 at + \cos^4 at) dt = I_1$

$$I_2 = \int_\pi^{2\pi} e^t (\sin^6 at + \cos^4 at) dt$$

Put $t = x + \pi$

$\therefore dt = dx$

For $a = 2$ and $a = 4$

$$\therefore I_2 = \int_0^\pi e^{x+\pi} (\sin^6 ax + \cos^4 ax) dx$$

$$= e^\pi I_1$$

Similarly,

$$\int_{2\pi}^{3\pi} e^t (\sin^6 at + \cos^4 at) dt = e^{2\pi} I_1$$

and

$$\int_{4\pi}^{5\pi} e^t (\sin^6 at + \cos^4 at) dt = e^{3\pi} I_1$$

$$\therefore \frac{\int_{4\pi}^{5\pi} e^t (\sin^6 at + \cos^4 at) dt}{\int_0^\pi e^t (\sin^6 at + \cos^4 at) dt}$$

$$= \frac{\int_0^\pi e^t (\sin^6 at + \cos^4 at) dt + \int_\pi^{2\pi} e^t (\sin^6 at + \cos^4 at) dt + \int_{2\pi}^{3\pi} e^t (\sin^6 at + \cos^4 at) dt + \int_{3\pi}^{4\pi} e^t (\sin^6 at + \cos^4 at) dt}{\int_0^\pi e^t (\sin^6 at + \cos^4 at) dt}$$

$$= 1 + e^\pi + e^{2\pi} + e^{3\pi}$$

$$= \frac{e^{4\pi} - 1}{e^\pi - 1}$$

13. a., b.

$$f(x) = (7 \tan^6 x - 3 \tan^2 x) \cdot \sec^2 x$$

$$\therefore \int_0^{\frac{\pi}{4}} f(x) dx = \int_0^1 (7t^6 - 3t^2) dt = (t^7 - t^3)_0^1 = 0$$

Now, $\int_0^{\frac{\pi}{4}} xf(x) dx = \int_0^1 (7t^6 - 3t^2) \tan^{-1} t dt$

$$= (\tan^{-1} t \cdot (t^7 - t^3))_0^1 - \int_0^1 (t^7 - t^3) \frac{1}{1+t^2} dt$$

$$= \int_0^1 \frac{t^3(1-t^4)}{1+t^2} dt$$

$$= \int_0^1 t^3(1-t^2) dt$$

$$= \frac{1}{4} - \frac{1}{6} = \frac{1}{12}$$

Linked Comprehension Type

1. a. $\int_0^{\pi/2} \sin x dx = \frac{\left(\frac{\pi}{2} - 0\right)}{4} \left(\sin 0 + \sin \frac{\pi}{2} + 2 \sin \frac{\pi}{4}\right)$

$$= \frac{\pi}{8} (1 + \sqrt{2})$$

2. d. $\lim_{x \rightarrow a} \frac{\int_a^x f(x) dx - \left(\frac{x-a}{2}\right)(f(x) + f(a))}{(x-a)^3} = 0$

$$\lim_{h \rightarrow 0} \frac{\int_a^{a+h} f(x) dx - \frac{h}{2}(f(a+h) + f(a))}{h^3} = 0$$

$$\therefore \lim_{h \rightarrow 0} \frac{f(a+h) - \frac{1}{2}[f(a) + f(a+h)] - \frac{h}{2}(f'(a+h))}{3h^2} = 0$$

[Using L' Hospital's rule]

$$\therefore \lim_{h \rightarrow 0} \frac{\frac{1}{2}f(a+h) - \frac{1}{2}f(a) - \frac{h}{2}f'(a+h)}{3h^2} = 0$$

$$\therefore \lim_{h \rightarrow 0} \frac{\frac{1}{2}f'(a+h) - \frac{1}{2}f'(a) - \frac{h}{2}f''(a+h)}{6h} = 0$$

[Using L' Hospital's rule]

$$\therefore \lim_{h \rightarrow 0} \frac{-f''(a+h)}{12} = 0$$

$$\therefore f''(a) = 0 \quad \forall a \in R$$

Thus, $f(x)$ must be of maximum degree 1.

3. b. $f''(x) < 0 \quad \forall x \in (a, b)$, for $c \in (a, b)$

$$F(c) = \frac{c-a}{2} (f(a) + f(c)) + \frac{b-c}{2} (f(b) + f(c))$$

$$= \frac{b-a}{2} f(c) + \frac{c-a}{2} f(a) + \frac{b-c}{2} f(b)$$

$$\begin{aligned}\text{or } F'(c) &= \frac{b-a}{2} f'(c) + \frac{1}{2} f(a) - \frac{1}{2} f(b) \\ &= \frac{1}{2} [(b-a)f'(c) + f(a) - f(b)]\end{aligned}$$

$$\begin{aligned}\therefore F''(c) &= \frac{1}{2}(b-a)f''(c) < 0 \\ &[\because f''(x) < 0 \forall x \in (a, b) \text{ and } b > a]\end{aligned}$$

Therefore, $F(c)$ is maximum at the point $(c, f(c))$ where

$$F'(c) = 0 \text{ or } f'(c) = 2 \left(\frac{f(b) - f(a)}{b-a} \right).$$

$$4. \text{ b. } f'(x) = \frac{2a(x^2-1)}{(x^2+ax+1)^2}; 0 < a < 2$$

$$\begin{aligned}g'(x) &= \frac{f'(e^x)e^x}{1+e^{2x}} \\ &= \frac{2a(e^{2x}-1)e^x}{(e^{2x}+ae^x+1)^2(1+e^{2x})}\end{aligned}$$

Now, $g'(x) > 0$ if $e^{2x} - 1 > 0$ or $e^{2x} > 1$ or $x > 0$

And, $g'(x) < 0$ if $e^{2x} - 1 < 0$ or $e^{2x} < 1$ or $x < 0$

$$5. \text{ b. } g'(x) = \left(\frac{2(x-1)}{x+1} - \log_e x \right) f(x).$$

$$\begin{aligned}f(x) &= \frac{x^2 - ax + 1}{x^2 + ax + 1} \\ &= 1 - \frac{2ax}{x^2 + ax + 1} \\ &= 1 - \frac{2a}{\left(x + \frac{1}{x}\right) + a}\end{aligned}$$

For $x > 1$, $x + \frac{1}{x} > 2$

$$\begin{aligned}\therefore \frac{2a}{\left(x + \frac{1}{x}\right) + a} &< 1 \quad (\because 0 < a < 2) \\ \therefore f(x) &> 0\end{aligned}$$

$$\text{Let } h(x) = \left(\frac{2(x-1)}{x+1} - \log_e x \right)$$

$$\Rightarrow h'(x) = \left(\frac{4}{(x+1)^2} - \frac{1}{x} \right) = \frac{-(x-1)^2}{(x+1)^2} < 0$$

Also, $h(1) = 0$ so, $h(x) < 0$. $\therefore g'(x) < 0$

Therefore, $g(x)$ is decreasing on $(1, \infty)$.

6. 2.

$$\begin{aligned}g\left(\frac{1}{2}\right) &= \lim_{h \rightarrow 0^+} \int_h^{1-h} t^{-1/2} (1-t)^{-1/2} dt \\ &= \int_0^1 \frac{dt}{\sqrt{t-t^2}}\end{aligned}$$

$$\begin{aligned}&= \int_0^1 \frac{dt}{\sqrt{\frac{1}{4} - \left(t - \frac{1}{2}\right)^2}} \\ &= \sin^{-1} \left(\frac{t - \frac{1}{2}}{\frac{1}{2}} \right) \Big|_0^1 \\ &= \sin^{-1} 1 - \sin^{-1}(-1) = \pi\end{aligned}$$

$$7. \text{ d. } g(a) = \lim_{h \rightarrow 0^+} \int_h^{1-h} t^{-a} (1-t)^{a-1} dt$$

$$\begin{aligned}\Rightarrow g(1-a) &= \lim_{h \rightarrow 0^+} \int_h^{1-h} t^{-(1-a)} (1-t)^{(1-a)-1} dt \\ &= \lim_{h \rightarrow 0^+} \int_h^{1-h} t^{a-1} (1-t)^{-a} dt \\ &= \lim_{h \rightarrow 0^+} \int_h^{1-h} (1-t)^{a-1} (1-(1-t))^{-a} dt \\ &= \lim_{h \rightarrow 0^+} \int_h^{1-h} (1-t)^{a-1} t^{-a} dt\end{aligned}$$

Thus, $g(a) = g(1-a)$

$$\begin{aligned}\Rightarrow g'(a) &= -g'(1-a) \\ \Rightarrow g'(1/2) &= -g'(1-1/2) \\ \Rightarrow g'(1/2) &= 0\end{aligned}$$

8. a., b., c.

$$f(x) = x F(x) \quad (1)$$

$$\therefore f'(x) = x F'(x) + F(x) \quad (2)$$

$$\Rightarrow f'(1) = F'(1) + F(1) = F'(1) < 0$$

$$F(1) = 0 \text{ and } F(3) = -4$$

Also, $F'(x) < 0$ for all $x \in (1/2, 3)$.

So, $F(x)$ is decreasing and hence $F(2) < 0$.

$$\therefore f(2) = 2F(2) < 0$$

Also, for $x \in (1, 3)$,

$$f'(x) = x F'(x) + F(x) < 0$$

9. c., d.

$$\int_1^3 x^3 F''(x) dx = 40$$

$$\Rightarrow [x^3 F'(x)]_1^3 - \int_1^3 3x^2 F'(x) dx = 40$$

$$\Rightarrow [x^2 f'(x) - x f(x)]_1^3 - 3(-12) = 40 \quad (\text{Using (1) and (2)})$$

$$\Rightarrow 9f'(3) - 3f(3) - f'(1) + f(1) = 4$$

$$\Rightarrow 9f'(3) + 36 - f'(1) + 0 = 4 \quad (\because F(1), \therefore f(1) = 0)$$

$$\Rightarrow 9f'(3) - f'(1) + 32 = 0$$

$$\int_1^3 f(x) dx$$

$$= \int_1^3 x F(x) dx$$

$$= \left[\frac{x^2}{2} F(x) \right]_1^3 - \frac{1}{2} \int_1^3 x^2 F'(x) dx$$

$$= \frac{9}{2} F(3) - \frac{1}{2} F(1) + 6$$

$$= -18 + 6 = -12$$

Matching Column Type

1. (iii) – (b), (c)

$$\left| \int_0^1 (1-y^2) dy \right| + \left| \int_1^0 (y^2-1) dy \right|$$

$$= 2 \int_0^1 (1-y^2) dy = \frac{4}{3}$$

Also, $\left| \int_0^1 \sqrt{1-x} dx \right| + \left| \int_{-1}^0 \sqrt{1+x} dx \right|$

$$= 2 \int_0^1 \sqrt{1-x} dx$$

$$= 2 \int_0^1 \sqrt{x} dx$$

$$= 2 \cdot \frac{2}{3} \cdot x^{3/2} \Big|_0^1 = \frac{4}{3}$$

Note: Solutions of the remaining parts are given in their respective chapters.

2. (i) – (a)

$$I = \int_0^{\pi/2} (\sin x)^{\cos x} (\cos x \cdot \cot x - \log(\sin x)^{\sin x}) dx$$

$$\Rightarrow I = \int_0^{\pi/2} \frac{d}{dx} (\sin x)^{\cos x} dx = [(\sin x)^{\cos x}]_0^{\pi/2} = 1$$

Note: Solutions of the remaining parts are given in their respective chapters.

3. (a) – (s); (b) – (s); (c) – (p); (d) – (r).

a. $\int_{-1}^1 \frac{dx}{1+x^2} = [\tan^{-1} x]_{-1}^1 = \tan^{-1}(1) - \tan^{-1}(-1)$

$$= \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{2\pi}{4} = \frac{\pi}{2}$$

b. $\int_0^1 \frac{dx}{\sqrt{1-x^2}} = [\sin^{-1} x]_0^1 = \sin^{-1}(1) - \sin^{-1}(0)$

$$= \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

c. $\int_2^3 \frac{dx}{1-x^2} = \left[\frac{1}{2} \log \left| \frac{1+x}{1-x} \right| \right]_2^3 = \frac{1}{2} [\log 2 - \log 3]$

$$= \frac{1}{2} \log 2/3$$

d. $\int_1^2 \frac{dx}{x\sqrt{x^2-1}} = [\sec^{-1} x]_1^2 = \sec^{-1} 2 - \sec^{-1} 1 = \frac{\pi}{3}$

4. (b) – (p), (t)

$$I = \int_1^5 (x-1)(x-2)(x-3)(x-4)(x-5) dx$$

Putting $x = t + 3$

$$I = \int_{-2}^2 (t+2)(t+1)t(t-1)(t-2) dt$$

$$= \int_{-2}^2 t(t^2-1)(t^2+1) dt$$

$$= 0 \text{ (as } t(t^2-1)(t^2+1) \text{ is an odd function)}$$

Note: Solutions of the remaining parts are given in their respective chapters.

5. (b) – (p)

$$\int_a^b (f(x) - 3x) dx = a^2 - b^2$$

$$\int_a^b f(x) dx = \frac{3}{2}(b^2 - a^2) + a^2 - b^2 = \left(\frac{b^2 - a^2}{2} \right)$$

$$\Rightarrow f(x) = x \Rightarrow f\left(\frac{\pi}{6}\right) = \frac{\pi}{6}$$

(c) – (s)

$$\frac{\pi^2}{\log_e 3} \int_{7/6}^{5/6} \sec(\pi x) dx$$

$$= \frac{\pi^2}{\log_e 3} \left(\frac{\log_e |(\sec \pi x + \tan \pi x)|^{5/6}}{\pi} \right)$$

$$= \frac{\pi}{\log_e 3} \left(\log_e \left| \sec \frac{5\pi}{6} + \tan \frac{5\pi}{6} \right| - \log_e \left| \sec \frac{7\pi}{6} + \tan \frac{7\pi}{6} \right| \right)$$

$$= \pi$$

Note: Solutions of the remaining parts are given in their respective chapters.

6. d.

(p) Let $f(x) = ax^2 + bx$, ($\because f(0) = 0$)

Given $\int_0^1 f(x) dx = 1$

$$\Rightarrow 2a + 3b = 6$$

$$\Rightarrow (a, b) \equiv (0, 2) \text{ and } (3, 0)$$

(q) $f(x) = \sin(x^2) + \cos(x^2)$

$$= \sqrt{2} \cos\left(x^2 - \frac{\pi}{4}\right)$$

For maximum value, $x^2 - \frac{\pi}{4} = 2n\pi, n \in \mathbb{Z}$

$$\Rightarrow x^2 = 2n\pi + \frac{\pi}{4}, n \in \mathbb{Z}$$

$$\Rightarrow x = \pm \sqrt{\frac{x}{4}}, \pm \sqrt{\frac{9\pi}{4}} \text{ as } x \in [-\sqrt{13}, \sqrt{13}]$$

(r) $I = \int_{-2}^2 \frac{3x^2}{(1+e^x)} dx$ (1)

$$= \int_{-2}^2 \frac{3(-x)^2}{1+e^{-x}} dx$$

$$\therefore I = \int_{-2}^2 \frac{e^x(3x^2)}{e^x+1} dx$$
 (2)

Adding (1) and (2)

$$\Rightarrow I + I = \int_{-2}^2 \frac{3x^2}{(1+e^x)} dx + \int_{-2}^2 \frac{e^x(3x^2)}{e^x+1} dx$$

$$= \int_{-2}^2 3x^2 dx$$

$$= 2 \int_0^2 3x^2 dx$$

$$= 16$$

$$\Rightarrow I = 8$$

$$(s) \text{ We have } I = \frac{\int_{-1/2}^{1/2} \cos 2x \cdot \log\left(\frac{1+x}{1-x}\right) dx}{\int_{-1/2}^{1/2} \cos 2x \cdot \log\left(\frac{1+x}{1-x}\right) dx}$$

$$\text{Let, } f(x) = \cos 2x \ln\left(\frac{1+x}{1-x}\right)$$

$$\begin{aligned} \therefore f(-x) &= \cos(-2x) \ln\left(\frac{1-x}{1+x}\right) \\ &= -\cos(2x) \ln\left(\frac{1+x}{1-x}\right) \\ &= -f(x) \end{aligned}$$

Thus, $f(x)$ is an odd function.

$$\Rightarrow I = 0$$

Integer Answer Type

$$1. (0) f(x) = \int_0^x f(t) dt, \text{ i.e., } f(0) = 0$$

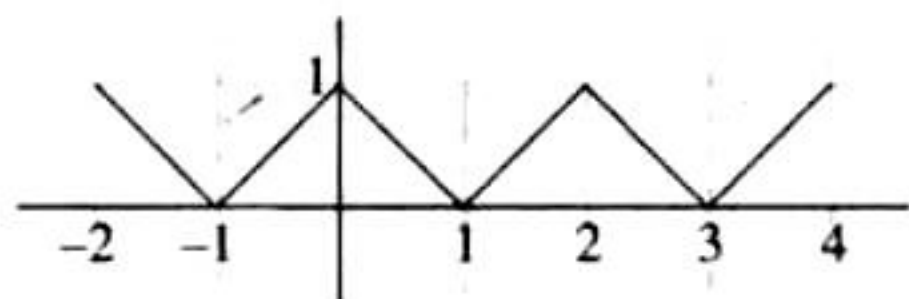
$$\text{Also, } f'(x) = f(x), x > 0$$

$$\text{or } f(x) = ke^x, x > 0$$

Since $f(0) = 0$ and $f(x)$ is continuous, $f(x) = 0 \forall x > 0$

$$\therefore f(\ln 5) = 0$$

$$2. (4) f(x) = \begin{cases} x-1, & 1 \leq x < 2 \\ 1-x, & 0 \leq x < 1 \end{cases}$$



From the graph $f(x)$ is periodic with period 2 and also even function.

Also $\cos \pi x$ has period 2

$$\begin{aligned} \therefore I &= \int_{-10}^{10} f(x) \cos \pi x dx \\ &= 2 \int_0^{10} f(x) \cos \pi x dx \quad (\because f(x) \cos \pi x \text{ is an even function}) \end{aligned}$$

$$= 2 \times 5 \int_0^2 f(x) \cos \pi x dx$$

$$= 10 \left[\int_0^1 (1-x) \cos \pi x dx + \int_1^2 (x-1) \cos \pi x dx \right] = 10(I_1 + I_2)$$

$$I_2 = \int_1^2 (x-1) \cos \pi x dx \quad (\text{put } x-1 = t)$$

$$I_2 = - \int_0^1 t \cos \pi t dt$$

$$I_1 = \int_0^1 (1-x) \cos \pi x dx = - \int_0^1 x \cos(\pi x) dx$$

$$\begin{aligned} \therefore I &= 10 \left[-2 \int_0^1 x \cos \pi x dx \right] \\ &= -20 \left[x \frac{\sin \pi x}{\pi} + \frac{\cos \pi x}{\pi^2} \right]_0^1 \\ &= -20 \left[-\frac{1}{\pi^2} - \frac{1}{\pi^2} \right] = \frac{40}{\pi^2} \end{aligned}$$

$$\therefore \frac{\pi^2}{10} I = 4$$

$$\begin{aligned} 3. (2) \int_0^1 4x^3 \left\{ \frac{d^2}{dx^2} (1-x^2)^5 \right\} dx \\ = \left[4x^3 \frac{d}{dx} (1-x^2)^5 \right]_0^1 - \int_0^1 12x^2 \frac{d}{dx} (1-x^2)^5 dx \end{aligned}$$

(Integrating using by parts)

$$\begin{aligned} &= [4x^3 \times 5(1-x^2)^4 (-2x)]_0^1 \\ &\quad - 12 \left[x^2 (1-x^2)^5 \right]_0^1 - \int_0^1 2x(1-x^2)^5 dx \\ &= 0 - 0 - 12[0 - 0] + 12 \int_0^1 2x(1-x^2)^5 dx \\ &= 12 \left[-\frac{(1-x^2)^6}{6} \right]_0^1 \\ &= 12 \left[0 + \frac{1}{6} \right] = 2 \end{aligned}$$

4. (7) $f(x)$ is continuous odd function and vanishes exactly at one point.

$$\therefore f(0) = 0$$

$$\begin{aligned} F(x) &= \int_{-1}^x f(t) dt \\ &= \int_{-1}^1 f(t) dt + \int_1^x f(t) dt \\ &= 0 + \int_1^x f(t) dt \quad (\text{as } f(t) \text{ is an odd function}) \end{aligned}$$

$f(t)$ is odd function

$\therefore f(f(t))$ is also odd function

$\therefore |f(f(t))|$ is an even function

$\therefore t|f(f(t))|$ is an odd function

$$\therefore G(x) = \int_{-1}^x t|f(f(t))| dt = \int_1^x t|f(f(t))| dt$$

$$\text{Now, } \lim_{x \rightarrow 1} \frac{F(x)}{G(x)} \quad \left(\frac{0}{0} \text{ form} \right)$$

$$= \lim_{x \rightarrow 1} \frac{f(x)}{x|f(f(x))|} \quad (\text{Using L'Hospital's Rule})$$

$$= \frac{1}{2} \left| f\left(\frac{1}{2}\right) \right| = \frac{1}{14} \quad (\text{Given})$$

$$\Rightarrow f\left(\frac{1}{2}\right) = 7$$

$$5. (9) \alpha = \int_0^1 e^{9x+3\tan^{-1}x} \left(\frac{12+9x^2}{1+x^2} \right) dx$$

$$\text{Let } 9x + 3 \tan^{-1} x = t$$

$$\Rightarrow \left(9 + \frac{3}{1+x^2} \right) dx = dt$$

$$\Rightarrow \left(\frac{12+9x^2}{1+x^2} \right) dx = dt$$

$$\Rightarrow \alpha = \int_0^{9+\frac{3\pi}{4}} e^t dt = e^{9+\frac{3\pi}{4}} - 1$$

$$\Rightarrow \log_e(1 + \alpha) = 9 + \frac{3\pi}{4}$$

$$6. (3) F(x) = \int_x^{x^2+\frac{\pi}{6}} 2 \cos^2 t dt$$

$$\therefore F'(x) = 2 \left(\cos \left(x^2 + \frac{\pi}{6} \right) \right)^2 2x - 2 \cos^2 x$$

According to the question,

$$\therefore F'(a) + 2 = \int_0^a f(x) dx$$

$$\Rightarrow 2 \left(\cos \left(a^2 + \frac{\pi}{6} \right) \right)^2 2a - 2 \cos^2 a + 2 = \int_0^a f(x) dx$$

Differentiating w.r.t. a , we get

$$4 \cos^2 \left(a^2 + \frac{\pi}{6} \right) + 4a \times 2 \cos \left(a^2 + \frac{\pi}{6} \right) \left(-\sin \left(a^2 + \frac{\pi}{6} \right) \right) \times 2a + 4 \cos a \sin a = f(a)$$

$$\therefore f(0) = 4 \left(\frac{\sqrt{3}}{2} \right)^2 = 3$$

$$\begin{aligned} 7. (0) \quad I &= \int_{-1}^2 \frac{x[x^2]}{2+[x+1]} dx \\ &= \int_{-1}^2 \frac{x[x^2]}{3+[x]} dx \\ &= \int_{-1}^0 \frac{0}{3-1} dx + \int_0^1 \frac{0}{3+0} dx + \int_1^{\sqrt{2}} \frac{x \cdot 1}{3+1} dx + 0 \\ &= \frac{1}{4} \left[\frac{x^2}{2} \right]_1^{\sqrt{2}} \\ &= \frac{2-1}{8} = \frac{1}{8} \end{aligned}$$

$$\therefore 4I - 1 = 0$$

Fill in the Blanks Type

1. Given that

$$f(x) = \begin{vmatrix} \sec x & \cos x & \sec^2 x + \cot x \operatorname{cosec} x \\ \cos^2 x & \cos^2 x & \operatorname{cosec}^2 x \\ 1 & \cos^2 x & \cos^2 x \end{vmatrix}$$

Operating $R_1 \rightarrow R_1 - \sec x \cdot R_3$, we get

$$f(x) = \begin{vmatrix} 0 & 0 & \sec^2 x + \cot x \operatorname{cosec} x - \cos x \\ \cos^2 x & \cos^2 x & \operatorname{cosec}^2 x \\ 1 & \cos^2 x & \cos^2 x \end{vmatrix}$$

Expanding along R_1 , we get

$$f(x) = (\sec^2 x + \cot x \operatorname{cosec} x - \cos x) (\cos^4 x - \cos^2 x)$$

$$= \left(\frac{1}{\cos^2 x} + \frac{\cos x}{\sin^2 x} - \cos x \right) \cos^2 x (\cos^2 x - 1)$$

$$= - \left[\frac{\sin^2 x + \cos^3 x - \cos^3 x \sin^2 x}{\cos^2 x \sin^2 x} \right] \cos^2 x \sin^2 x$$

$$= -\sin^2 x - \cos^3 x (1 - \sin^2 x)$$

$$= -\sin^2 x - \cos^5 x$$

$$\therefore \int_0^{\pi/2} f(x) dx = \int_0^{\pi/2} (\sin^2 x + \cos^5 x) dx$$

$$= - \int_0^{\pi/2} \left[\frac{1 - \cos 2x}{2} + \cos x (1 - \sin^2 x)^2 \right] dx$$

$$= - \left[\frac{x + \frac{\sin 2x}{2}}{2} \right]_0^{\pi/2} - \left(t - \frac{2t^3}{3} + \frac{t^5}{5} \right)_1^1,$$

where $t = \sin x$

$$= -\frac{\pi}{4} - \left(1 - \frac{2}{3} + \frac{1}{5} \right)$$

$$= -\left(\frac{15\pi + 32}{60} \right)$$

2. When $x = 0$, $x^2 = 0$, and when $x = 1.5$, $x^2 = 2.25$.

Thus, $[x^2]$ is discontinuous when $x^2 = 1$ and $x^2 = 2$ or $x = 1$ and $x = \sqrt{2}$.

$$\therefore \int_0^{1.5} [x^2] dx = \int_0^1 [x^2] dx + \int_1^{\sqrt{2}} [x^2] dx + \int_{\sqrt{2}}^{1.5} [x^2] dx$$

$$= 0 + \int_1^{\sqrt{2}} 1 dx + \int_{\sqrt{2}}^{1.5} 2 dx$$

$$= 1(\sqrt{2} - 1) + 2(1.5 - \sqrt{2}) = (2 - \sqrt{2})$$

$$3. \text{ Let } I = \int_{-2}^2 |1 - x^2| dx = 2 \int_0^2 |1 - x^2| dx$$

$$= 2 \int_0^1 (1 - x^2) dx + 2 \int_1^2 (x^2 - 1) dx$$

$$= 2 \left[x - \frac{x^3}{3} \right]_0^1 + 2 \left[\frac{x^3}{3} - x \right]_1^2$$

$$= 2 \left[1 - \frac{1}{3} \right] + 2 \left[\frac{8}{3} - 2 - \frac{1}{3} + 1 \right]$$

$$= \frac{4}{3} + \frac{8}{3} = 4$$

$$4. I = \int_{\pi/4}^{3\pi/4} \frac{\phi}{1 + \sin \phi} d\phi$$

(1)

$$= \int_{\pi/4}^{3\pi/4} \frac{\pi - \phi}{1 + \sin(\pi - \phi)} d\phi$$

$$\left[\text{Using } \int_a^b f(x) dx = \int_a^b f(a+b-x) dx \right]$$

$$\therefore I = \int_{\pi/4}^{3\pi/4} \frac{\pi - \phi}{1 + \sin \phi} d\phi \quad (2)$$

Adding equations (1) and (2), we get

$$\begin{aligned} 2I &= \int_{\pi/4}^{3\pi/4} \frac{\pi}{1 + \sin \phi} d\phi \\ &= \pi \int_{\pi/4}^{3\pi/4} \frac{1 - \sin \phi}{1 - \sin^2 \phi} d\phi \\ &= \pi \int_{\pi/4}^{3\pi/4} \frac{1 - \sin \phi}{\cos^2 \phi} d\phi \\ &= \pi \int_{\pi/4}^{3\pi/4} (\sec^2 \phi - \sec \phi \tan \phi) d\phi \\ &= \pi [\tan \phi - \sec \phi]_{\pi/4}^{3\pi/4} \\ &= \pi [\tan 3\pi/4 - \sec 3\pi/4 - \tan \pi/4 + \sec \pi/4] \\ &= \pi [-1 + \sqrt{2} - 1 + \sqrt{2}] \\ &= 2\pi(\sqrt{2} - 1) \\ \therefore I &= \pi(\sqrt{2} - 1) \end{aligned}$$

$$5. \text{ Let } I = \int_2^3 \frac{\sqrt{x}}{\sqrt{5-x} + \sqrt{x}} dx \quad (1)$$

$$\begin{aligned} &= \int_2^3 \frac{\sqrt{5-x}}{\sqrt{5-(5-x)} + \sqrt{5-x}} dx \\ &\quad \left[\text{Using } \int_a^b f(x) dx = \int_a^b f(a+b-x) dx \right] \end{aligned}$$

$$\therefore I = \int_2^3 \frac{\sqrt{5-x}}{\sqrt{x} + \sqrt{5-x}} dx \quad (2)$$

Adding equations (1) and (2), we get

$$2I = \int_2^3 \frac{\sqrt{x} + \sqrt{5-x}}{\sqrt{5-x} + \sqrt{x}} dx$$

$$\text{or } I = \frac{1}{2} \int_2^3 1 dx = \frac{1}{2} (3-2) = \frac{1}{2}$$

6. Let us first find the functions satisfying

$$af(x) + bf\left(\frac{1}{x}\right) = \frac{1}{x} - 5. \quad (1)$$

$$\text{Replacing } x \text{ by } \frac{1}{x}, \text{ we have } af\left(\frac{1}{x}\right) + bf(x) = x - 5. \quad (2)$$

Eliminating $f\left(\frac{1}{x}\right)$ from equations (1) and (2), we get

$$\begin{aligned} \int_1^2 f(x) dx &= \int_1^2 \frac{\frac{a}{x} - 5a - bx + 5b}{a^2 - b^2} dx \\ &= \frac{1}{a^2 - b^2} \left[a \log x - b \frac{x^2}{2} + 5(b-a)x \right]_1^2 \end{aligned}$$

$$\begin{aligned} &= \frac{1}{a^2 - b^2} \left[a \log 2 - 2b + 10(b-a) + \frac{b}{2} - 5(b-a) \right] \\ &= \frac{1}{a^2 - b^2} \left[a \log 2 - 5a + \frac{7b}{2} \right] \end{aligned}$$

$$7. I = \int_0^{2\pi} \frac{x \cos^{2n} x}{\cos^{2n} x + \sin^{2n} x} dx \quad (1)$$

$$\begin{aligned} &= 2 \int_0^{2\pi} \frac{(2\pi - x) \cos^{2n} x}{\cos^{2n} x + \sin^{2n} x} dx \\ &\quad \left[\text{Using } \int_0^a f(x) dx = \int_0^a f(a-x) dx \right] \end{aligned} \quad (2)$$

Adding equations (1) and (2), we get

$$\begin{aligned} 2I &= \int_0^{2\pi} \frac{2\pi \cos^{2n} x}{\cos^{2n} x + \sin^{2n} x} dx = 4\pi \int_0^{\pi} \frac{\cos^{2n} x}{\cos^{2n} x + \sin^{2n} x} dx \\ &\quad \left[\int_0^{2a} f(x) dx = 2 \int_0^a f(x) dx \text{ if } f(2a-x) = f(x) \right] \\ &= 8\pi \int_0^{\pi/2} \frac{\cos^{2n} x}{\cos^{2n} x + \sin^{2n} x} dx \quad (3) \end{aligned}$$

[Using the above property again]

$$\begin{aligned} &= 8\pi \int_0^{\pi/2} \frac{\cos^{2n} \left(\frac{\pi}{2} - x \right)}{\cos^{2n} \left(\frac{\pi}{2} - x \right) + \sin^{2n} \left(\frac{\pi}{2} - x \right)} dx \\ &\quad \left[\text{Using } \int_0^a f(x) dx = \int_0^a f(a-x) dx \right] \\ &= 8\pi \int_0^{\pi/2} \frac{\sin^{2n} x}{\sin^{2n} x + \cos^{2n} x} dx \quad (4) \end{aligned}$$

Adding equations (3) and (4), we have

$$4I = 8\pi \int_0^{\pi/2} 1 dx$$

$$\text{or } I = \pi^2$$

$$8. \text{ Let } I = \int_1^{e^{37}} \frac{\pi \sin(\pi \ln x)}{x} dx$$

$$\text{Let } \pi \ln x = t$$

$$\therefore \frac{\pi}{x} dx = dt$$

$$\begin{aligned} \therefore I &= \int_0^{37\pi} \sin t dt = [-\cos t]_0^{37\pi} = -\cos 37\pi + 1 \\ &= -(-1) + 1 = 2 \end{aligned}$$

$$9. I = \int_1^4 \frac{2e^{\sin x^2}}{x} dx = F(k) - F(1)$$

$$\text{Put } x^2 = t \therefore 2x dx = dt$$

$$\therefore I = \int_1^{16} \frac{e^{\sin t}}{t} dt = F[(t)]_1^{16}$$

$$\therefore I = F(16) - F(1)$$

$$\therefore k = 16$$

True/False Type

$$1. \text{ Let } I = \int_0^{2a} \frac{f(x)}{f(x) + f(2a-x)} dx \quad (1)$$

$$= \int_0^{2a} \frac{f(2a-x)}{f(2a-x) + f(x)} dx \quad (2)$$

$$[\text{Using } \int_0^a f(x) dx = \int_0^a f(a-x) dx]$$

Adding equations (1) and (2), we get

$$2I = \int_0^{2a} \frac{f(x) + f(2a-x)}{f(x) + f(2a-x)} dx$$

$$= \int_0^{2a} 1 dx$$

$$= [x]_0^{2a} = 2a \text{ or } I = a$$

Therefore, the given statement is true.

Subjective Type

$$1. L = \lim_{n \rightarrow \infty} \left(\frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{6n} \right)$$

$$= \lim_{n \rightarrow \infty} \sum_{r=1}^{5n} \frac{1}{n+r} = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^{5n} \left(\frac{1}{1+r/n} \right)$$

$$\text{Now, Lower limit} = \lim_{n \rightarrow \infty} (r/n)_{r=1} = \lim_{n \rightarrow \infty} (1/n) = 0$$

$$\text{Upper limit} = \lim_{n \rightarrow \infty} (r/n)_{r=5n} = \lim_{n \rightarrow \infty} (5n/5n) = 5$$

$$\text{Then } L = \int_0^5 \frac{dx}{1+x} = [\log(1+x)]_0^5 = \log 6$$

$$2. \int_0^1 (tx + 1 - x)^n dx$$

$$= \int_0^1 [(t-1)x + 1]^n dx$$

$$= \left[\frac{[(t-1)x + 1]^{n+1}}{(t-1)(n+1)} \right]_0^1$$

$$= \frac{1}{n+1} \left[\frac{t^{n+1}}{t-1} - \frac{1}{t-1} \right]$$

$$\therefore \int_0^1 (tx + 1 - x)^n dx = \frac{t^{n+1} - 1}{(t-1)(n+1)} \quad (1)$$

$$\text{For } \int_0^1 x^k (1-x)^{n-k} dx = \left[{}^n C_k (n+1) \right]^{-1}$$

$$k = 0, 1, 2, \dots, n$$

$$\text{Now, } [tx + (1-x)]^n$$

$$= \sum_{k=0}^n {}^n C_k (tx)^k (1-x)^{n-k} \quad [\text{Using binomial theorem}]$$

$$= \sum_{k=0}^n \left[{}^n C_k x^k (1-x)^{n-k} \right] t^k$$

Integrating both sides from 0 to 1 w.r.t. x , we get

$$\int_0^1 [tx + (1-x)]^n dx = \sum_{k=0}^n t^k {}^n C_k \int_0^1 x^k (1-x)^{n-k} dx$$

$$\therefore \frac{t^{n+1} - 1}{(t-1)(n+1)} = \sum_{k=0}^n {}^n C_k t^k \left\{ \int_0^1 x^k (1-x)^{n-k} dx \right\}$$

[Using equation (1)]

$$\therefore \sum_{k=0}^n {}^n C_k t^k \left\{ \int_0^1 x^k (1-x)^{n-k} dx \right\}$$

$$= \frac{1}{n+1} [1 + t + t^2 + t^3 + \dots + t^n] \quad [\text{Using sum of G.P.}]$$

Equating the coefficients of t^k on both the sides, we get

$${}^n C_k \int_0^1 x^k (1-x)^{n-k} dx = \frac{1}{n+1}$$

$$\therefore \int_0^1 x^k (1-x)^{n-k} dx = \frac{1}{{}^n C_k (n+1)}$$

$$3. \text{ Let } I = \int_0^\pi x f(\sin x) dx \quad (1)$$

Now, using property $\int_a^b f(x) dx = \int_a^b f(a+b-x) dx$, we get

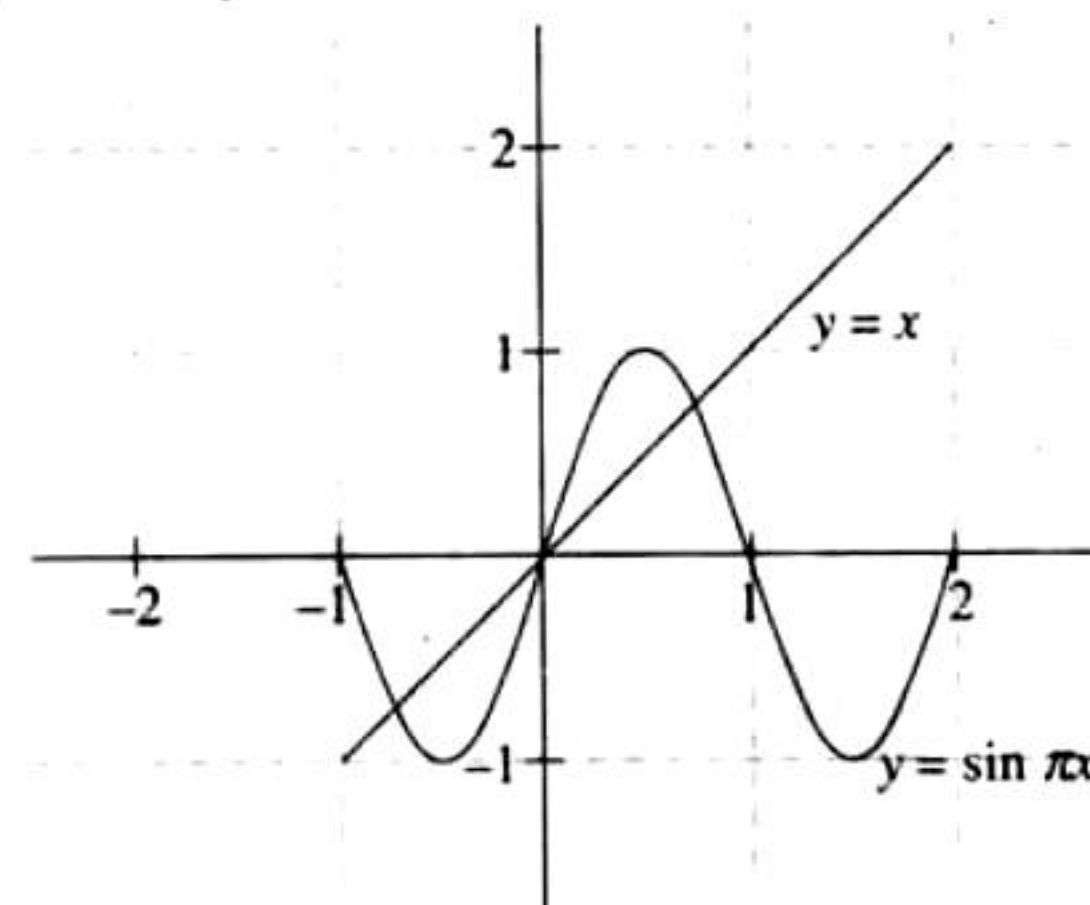
$$I = \int_0^\pi (\pi - x) f(\sin(\pi - x)) dx$$

$$\text{or } I = \int_0^\pi (\pi - x) f(\sin x) dx \quad (2)$$

$$\text{Thus, adding equations (1) and (2), we get } 2I = \pi \int_0^\pi f(\sin x) dx$$

$$\text{or } I = \frac{\pi}{2} \int_0^\pi f(\sin x) dx.$$

4. To understand the sign scheme of $y = x \sin \pi x$, let's first draw the graphs of $y = x$ and $y = \sin \pi x$.
The period of $y = \sin \pi x$ is '2'.



From the graphs,

$$|x \sin \pi x| = \begin{cases} (-x)(-\sin \pi x) & \text{if } -1 \leq x < 0 \\ x \sin \pi x & \text{if } 0 < x \leq 1 \\ x(-\sin \pi x) & \text{if } 1 < x \leq 3/2 \end{cases}$$

$$\therefore \int_{-1}^{3/2} |x \sin \pi x| dx$$

$$= \int_{-1}^0 x \sin \pi x dx + \int_0^1 x \sin \pi x dx + \int_1^{3/2} (-x \sin \pi x) dx$$

$$= \int_{-1}^1 x \sin \pi x dx - \int_1^{3/2} x \sin \pi x dx$$

$$= 2 \int_0^1 x \sin \pi x dx - \int_1^{3/2} x \sin \pi x dx$$

$$= 2 \left[\left\{ x \left(\frac{-1}{\pi} \right) \cos \pi x \right\}_0^1 - \int_0^1 \left(\frac{-1}{\pi} \right) \cos \pi x dx \right]$$

$$\begin{aligned}
& - \left\{ x \left(\frac{-1}{\pi} \right) \cos \pi x \right\}_1^{3/2} + \int_1^{3/2} 1 \left(\frac{-1}{\pi} \right) \cos \pi x \, dx \\
& = \left(\frac{2}{\pi} \right) + \left(\frac{2}{\pi^2} \right) [\sin \pi x]_0^1 + \left\{ \frac{3}{(2\pi)} \right\} \cos \frac{3}{2} \pi + \left(\frac{1}{\pi} \right) \\
& \quad - \left(\frac{1}{\pi^2} \right) [\sin \pi x]_1^{3/2} \\
& = (2/\pi) + 0 + 0 + (1/\pi) + (1/\pi^2) \\
& = (3\pi + 1)/\pi^2
\end{aligned}$$

5. $I = \int_0^{\pi/4} \frac{\sin x + \cos x}{9 + 16 \sin 2x} \, dx$

We know that $(\sin x - \cos x)^2 = 1 - \sin 2x$

or $\sin 2x = 1 - (\sin x - \cos x)^2$

$$\begin{aligned}
\therefore I &= \int_0^{\pi/4} \frac{\sin x + \cos x}{9 + 16(1 - (\sin x - \cos x)^2)} \, dx \\
&= \int_0^{\pi/4} \frac{\sin x + \cos x}{25 - 16(\sin x - \cos x)^2} \, dx
\end{aligned}$$

Let $\sin x - \cos x = t$

$$\begin{aligned}
\therefore I &= \int_{-1}^0 \frac{dt}{25 - 16t^2} \\
&= \frac{1}{16} \int_{-1}^0 \frac{dt}{\left(\frac{5}{4}\right)^2 - t^2} \\
&= \frac{1}{16} \cdot \frac{1}{2 \cdot \frac{5}{4}} \log \left[\frac{\frac{5}{4} + t}{\frac{5}{4} - t} \right]_{-1}^0 \\
&= \frac{1}{40} \left[\log 1 - \log \frac{1}{9} \right] \\
&= \frac{\log 9}{40} = \frac{1}{20} \log 3
\end{aligned}$$

6. Let $I = \int_0^{1/2} \frac{x \sin^{-1} x}{\sqrt{1-x^2}} \, dx$

Put $x = \sin \theta \therefore dx = \cos \theta \, d\theta$

Also, when $x = 0$, $\theta = 0$, and when $x = 1/2$, $\theta = \pi/6$.

$$\begin{aligned}
\text{Thus, } I &= \int_0^{\pi/6} \frac{\sin \theta \sin^{-1}(\sin \theta)}{\sqrt{1 - \sin^2 \theta}} \cos \theta \, d\theta \\
&= \int_0^{\pi/6} \theta \sin \theta \, d\theta
\end{aligned}$$

Integrating by parts, we get

$$\begin{aligned}
I &= [\theta (-\cos \theta)]_0^{\pi/6} + \int_0^{\pi/6} 1 \cos \theta \, d\theta \\
&= [-\theta \cos \theta + \sin \theta]_0^{\pi/6} \\
&= \frac{-\pi}{6} \cdot \frac{\sqrt{3}}{2} + \frac{1}{2} = \frac{6 - \pi\sqrt{3}}{12}
\end{aligned}$$

7. Given that $f(x)$ is integrable over any interval on real line and $f(t+x) = f(x)$ for all real x and a real t . (1)

$$\text{Now, } \int_a^{a+t} f(x) \, dx = \int_a^0 f(x) \, dx + \int_0^t f(x) \, dx + \int_t^{a+t} f(x) \, dx$$

In the last integral, put $x = t + y$ so that $dx = dy$.

$$\begin{aligned}
\text{Then } \int_t^{a+t} f(x) \, dx &= \int_0^a f(t+y) \, dy = \int_0^a f(y) \, dy \quad [\text{Using equation (1)}] \\
&= \int_0^a f(x) \, dx
\end{aligned}$$

$$\begin{aligned}
\text{Hence, } \int_a^{a+t} f(x) \, dx &= - \int_0^a f(x) \, dx + \int_0^t f(x) \, dx + \int_0^a f(x) \, dx \\
&= \int_0^t f(x) \, dx, \text{ which is independent of } a
\end{aligned}$$

Alternative Method:

$$g(x) = \int_a^{a+t} f(x) \, dx$$

Given that $f(x+t) = f(x)$ (1)

We have to prove that $g(x)$ is independent of 'a'.

i.e., we have to prove that $\frac{dg(x)}{da} = 0$

$$\text{Now, } \frac{dg(x)}{da} = f(a+t) - f(a)$$

From (1), $f(a+t) = f(a)$

$$\therefore \frac{dg(x)}{da} = 0$$

Thus, $g(x)$ is independent of 'a'.

8. Let $I = \int_0^{\pi/2} \frac{x \sin x \cos x}{\cos^4 x + \sin^4 x} \, dx$ (1)

$$= \int_0^{\pi/2} \frac{(\pi/2 - x) \sin(\pi/2 - x) \cos(\pi/2 - x)}{\cos^4(\pi/2 - x) + \sin^4(\pi/2 - x)} \, dx$$

[Using $\int_0^a f(x) \, dx = \int_0^a f(a-x) \, dx$]

$$\therefore I = \int_0^{\pi/2} \frac{(\pi/2 - x) \sin x \cos x}{\sin^4 x + \cos^4 x} \, dx \quad (2)$$

Adding equations (1) and (2), we get

$$2I = \frac{\pi}{2} \int_0^{\pi/2} \frac{\sin x \cos x}{\sin^4 x + \cos^4 x} \, dx$$

$$\text{or } I = \frac{\pi}{4} \int_0^{\pi/2} \frac{\sin x \cos x}{\sin^4 x + \cos^4 x} \, dx$$

$$= \frac{\pi}{4} \int_0^{\pi/2} \frac{\sec^2 x \tan x}{\tan^4 x + 1} \, dx \quad (\text{Dividing numerator and denominator by } \cos^4 x)$$

$$= \frac{\pi}{2 \times 4} \int_0^{\pi/2} \frac{2 \tan x \sec^2 x \, dx}{1 + (\tan^2 x)^2}$$

Put $\tan^2 x = t \therefore 2 \tan x \sec^2 x dx = dt$
 Also, as $x \rightarrow 0, t \rightarrow 0$; as $x \rightarrow \pi/2, t \rightarrow \infty$

$$\therefore I = \frac{\pi}{8} \int_0^\infty \frac{dt}{1+t^2}$$

$$= \frac{\pi}{8} \left[\tan^{-1} t \right]_0^\infty = \frac{\pi}{8} [\pi/2 - 0] = \pi^2/16.$$

9. Let $I = \int_0^\pi \frac{x dx}{1 + \cos \alpha \sin x}$ (1)

$$= \int_0^\pi \frac{(\pi - x) dx}{1 + \cos \alpha (\sin(\pi - x))}$$

[using $\int_0^a f(x) dx = \int_0^a f(a-x) dx$]

$$\therefore I = \int_0^\pi \frac{(\pi - x) dx}{1 + \cos \alpha \sin x} \quad (2)$$

Adding equations (1) and (2), we get

$$2I = \int_0^\pi \frac{x + \pi - x}{1 + \cos \alpha \sin x} dx$$

$$= \int_0^\pi \frac{\pi}{1 + \cos \alpha \sin x} dx$$

$$\therefore I = \frac{\pi}{2} \int_0^\pi \frac{1}{1 + \cos \alpha \sin x} dx$$

$$= \frac{\pi}{2} \times 2 \int_0^{\pi/2} \frac{1}{1 + \cos \alpha \sin x} dx$$

$$= \pi \int_0^{\pi/2} \frac{1}{1 + \cos \alpha \times \frac{2 \tan x/2}{1 + \tan^2 x/2}} dx$$

$$= \pi \int_0^{\pi/2} \frac{\sec^2 x/2}{1 + \tan^2 x/2 + 2 \cos \alpha \tan x/2} dx$$

Put $\tan x/2 = t$ or $\frac{1}{2} \sec^2 \frac{x}{2} dx = dt$

Also, when $x \rightarrow 0, t \rightarrow 0$
 and when $x \rightarrow \pi/2, t \rightarrow 1$

$$\therefore I = \pi \int_0^1 \frac{2dt}{t^2 + (2 \cos \alpha)t + 1}$$

$$= 2\pi \int_0^1 \frac{dt}{(t + \cos \alpha)^2 + 1 - \cos^2 \alpha}$$

$$= 2\pi \int_0^1 \frac{dt}{(t + \cos \alpha)^2 + \sin^2 \alpha}$$

$$= 2\pi \cdot \frac{1}{\sin \alpha} \left[\tan^{-1} \left(\frac{t + \cos \alpha}{\sin \alpha} \right) \right]_0^1$$

$$= \frac{2\pi}{\sin \alpha} \left[\tan^{-1} \left(\frac{1 + \cos \alpha}{\sin \alpha} \right) - \tan^{-1} \left(\frac{\cos \alpha}{\sin \alpha} \right) \right]$$

$$= \frac{2\pi}{\sin \alpha} \left[\tan^{-1} \left(\frac{\frac{1 + \cos \alpha}{\sin \alpha} - \frac{\cos \alpha}{\sin \alpha}}{1 + \left(\frac{1 + \cos \alpha}{\sin \alpha} \cdot \frac{\cos \alpha}{\sin \alpha} \right)} \right) \right]$$

$$= \frac{2\pi}{\sin \alpha} \left[\tan^{-1} \left(\frac{\sin \alpha}{1 + \cos \alpha} \right) \right]$$

$$= \frac{2\pi}{\sin \alpha} \left[\tan^{-1} \left(\tan \frac{\alpha}{2} \right) \right]$$

$$= \frac{\pi \alpha}{\sin \alpha}$$

10. $\int_0^a f(x)g(x)dx$

$$= \int_0^a f(a-x)g(a-x)dx$$

$$= \int_0^a f(x) \cdot \{2 - g(x)\} dx$$

$$= 2 \int_0^a f(x) dx - \int_0^a f(x)g(x)dx$$

$$\therefore 2 \int_0^a f(x)g(x)dx = 2 \int_0^a f(x) dx$$

$$\therefore \int_0^a f(x)g(x)dx = \int_0^a f(x) dx$$

11. Let $I = \int_0^{\pi/2} f(\sin 2x) \sin x dx$ (1)

$$= \int_0^{\pi/2} f \left\{ \sin 2 \left(\frac{1}{2} \pi - x \right) \right\} \sin \left(\frac{1}{2} \pi - x \right) dx$$

$$= \int_0^{\pi/2} f(\sin 2x) \cos x dx \quad (2)$$

Then adding equations (1) and (2), we have

$$2I = \int_0^{\pi/2} f(\sin 2x)(\sin x + \cos x) dx$$

$$= \sqrt{2} \int_0^{\pi/2} f(\sin 2x) \sin \left(x + \frac{\pi}{4} \right) dx$$

Now, from the result which we have to prove, it is clear that we have to substitute $\frac{\pi}{2} - 2\theta = 2x$.

$\therefore dx = -d\theta$. Also, when $x = 0, \theta = \pi/4$, and when $x = \pi/2, \theta = -\pi/4$.

$$\therefore 2I = \sqrt{2} \int_{-\pi/4}^{\pi/4} f(\cos 2\theta) \cos \theta d\theta$$

$$= 2\sqrt{2} \int_0^{\pi/4} f(\cos 2\theta) \cos \theta d\theta$$

[as $g(\theta) = f(\cos 2\theta) \cos \theta$ is an even function]

$$\therefore I = \sqrt{2} \int_0^{\pi/4} f(\cos 2x) \cos x dx \quad (3)$$

Equations (1), (2), (3) give the required result.

12. $S = \cos x + \cos 3x + \dots + \cos (2k-1)x$

$$\therefore (2 \sin x)S = 2 \sin x \cos x + 2 \sin x \cos 3x + \dots$$

$$+ 2 \sin x \cos (2k-1)x$$

$$= \sin 2x + (\sin 4x - \sin 2x) + (\sin 6x - \sin 4x) + \dots$$

$$+ (\sin 2kx - \sin(2k-2)x)$$

$$= \sin 2kx$$

$$\therefore 2S = \frac{\sin 2kx}{\sin x}$$

$$\Rightarrow \sin 2kx \cot x$$

$$= \frac{\sin 2kx}{\sin x} \cos x$$

$$= 2 \cos x (\cos x + \cos 3x + \dots + \cos (2k-1)x)$$

$$\begin{aligned}
&= 1 + \cos 2x + \cos 4x + \cos 2x + \cos 6x + \cos 4x + \dots + \cos 2kx + \cos (2k-2)x \\
\therefore \int_0^{\pi/2} \sin 2kx \cot x dx &= \int_0^{\pi/2} dx + \int_0^{\pi/2} (2 \cos 2x + 2 \cos 4x + \dots + 2 \cos (2k-2)x) \\
&\quad + \int_0^{\pi/2} \cos 2kx = \frac{\pi}{2} + 0 = \frac{\pi}{2}
\end{aligned}$$

(As other integrals than 1st one are zero)

13. We are given that f is a continuous function and

$$\int_0^x f(t) dt \rightarrow \infty \text{ as } |x| \rightarrow \infty$$

To show that every line $y = mx$ intersects the curve

$$y^2 + \int_0^x f(t) dt = 2$$

if possible, let $y = mx$ intersects the given curve. Then substituting $y = mx$ in the curves, we get

$$m^2 x^2 + \int_0^x f(t) dt = 2 \quad (1)$$

$$\text{Consider } F(x) = m^2 x^2 + \int_0^x f(t) dt - 2$$

Then $F(x)$ is a continuous function as $f(x)$ is given to be continuous.

Also, $F(x) \rightarrow \infty$ as $|x| \rightarrow \infty$

But $F(0) = -2$

Thus, $F(0) = -ve$ and $F(b) = +ve$ where b is some value of x and $F(x)$ is continuous.

Therefore, $F(x) = 0$ for some value of $x \in (0, b)$ or equation (1) is solvable for x .

Hence, $y = mx$ intersects the given curves.

$$14. \text{ Let } I = \int_0^{\pi} \frac{x \sin 2x \sin \left(\frac{\pi}{2} \cos x \right)}{2x - \pi} dx \quad (1)$$

Then

$$I = \int_0^{\pi} \frac{(\pi - x) \sin (2\pi - 2x) \sin \left(\frac{\pi}{2} \cos (\pi - x) \right)}{2(\pi - x) - \pi} dx$$

$$= \int_0^{\pi} \frac{(\pi - x) (-\sin 2x) \sin \left(-\frac{\pi}{2} \cos x \right)}{\pi - 2x}, \text{ or}$$

$$\text{or } I = \int_0^{\pi} \frac{(x - \pi) \sin 2x \sin \left(\frac{\pi}{2} \cos x \right)}{2x - \pi} dx \quad (2)$$

Adding equations (1) and (2), we get

$$2I = \int_0^{\pi} \frac{(2x - \pi) \sin 2x \sin \left(\frac{\pi}{2} \cos x \right)}{2x - \pi} dx$$

$$= \int_0^{\pi} \sin 2x \sin \left(\frac{\pi}{2} \cos x \right) dx$$

$$= \int_0^{\pi} 2 \sin x \cos x \sin \left(\frac{\pi}{2} \cos x \right) dx$$

$$\text{or } I = \int_0^{\pi} \sin x \cos x \sin \left(\frac{\pi}{2} \cos x \right) dx$$

Put $z = \frac{\pi}{2} \cos x$. Then $dz = -\frac{\pi}{2} \sin x dx$.

When $x = 0$, $z = \frac{\pi}{2}$, and when $x = \pi$, $z = -\frac{\pi}{2}$.

$$\therefore I = -\frac{2}{\pi} \int_{\pi/2}^{-\pi/2} \frac{2z}{\pi} \sin z dz$$

$$= \frac{8}{\pi^2} \int_0^{\pi/2} z \sin z dz = \frac{8}{\pi^2} [-z \cos z + \sin z]_0^{\pi/2} = \frac{8}{\pi^2}$$

$$15. \text{ Given } \int_0^1 e^x (x-1)^n dx = 16 - 6e$$

where $n \in N$ and $n \leq 5$.

To find the value of n , let

$$\begin{aligned}
I_n &= \int_0^1 e^x (x-1)^n dx \\
&= [(x-1)^n e^x]_0^1 - \int_0^1 n(x-1)^{n-1} e^x dx \\
&= -(-1)^n - \int_0^1 n(x-1)^{n-1} e^x dx
\end{aligned}$$

$$\text{or } I_n = (-1)^{n+1} - nI_{n-1} \quad (1)$$

$$\text{Also, } I_1 = \int_0^1 e^x (x-1) dx$$

$$= [e^x (x-1)]_0^1 - \int_0^1 e^x dx$$

$$= -(-1) - (e^x)_0^1$$

$$= 1 - (e - 1) = 2 - e$$

Using equation (1), we get

$$I_2 = (-1)^3 - 2I_1 = -1 - 2(2 - e) = 2e - 5$$

$$\text{Similarly, } I_3 = (-1)^4 - 3I_2 = 1 - 3(2e - 5)$$

$$= 16 - 6e$$

$$\therefore n = 3$$

$$16. I = \int_2^3 \frac{2x^5 + x^4 - 2x^3 + 2x^2 + 1}{(x^2 + 1)(x^4 - 1)} dx$$

$$= \int_2^3 \frac{2x^3(x^2 - 1) + (x^2 + 1)^2}{(x^2 + 1)^2(x^2 - 1)} dx$$

$$= \int_2^3 \frac{2x^3 dx}{(x^2 + 1)^2} + \int_2^3 \frac{1}{x^2 - 1} dx$$

$$= \int_2^3 \frac{x^2 2x dx}{(x^2 + 1)^2} + \left[\frac{1}{2} \log \frac{x-1}{x+1} \right]_2^3$$

$$= \int_5^{10} \frac{t-1}{t^2} dt + \left[\frac{1}{2} \left(\log \frac{2}{4} - \log \frac{1}{3} \right) \right]$$

Put $x^2 + 1 = t$ or $2x dx = dt$. When $x \rightarrow 2$, $t \rightarrow 5$, and when $x \rightarrow 3$, $t \rightarrow 10$.

$$\therefore I = \int_5^{10} \left(\frac{1}{t} - \frac{1}{t^2} \right) dt + \frac{1}{2} \log \frac{3}{2}$$

$$= \left(\log |t| + \frac{1}{t} \right)_5^{10} + \frac{1}{2} \log \frac{3}{2}$$

$$= \log 10 - \log 5 + \frac{1}{10} - \frac{1}{5} + \frac{1}{2} \log \frac{3}{2}$$

$$= \log 2 + \left(-\frac{1}{10}\right) + \frac{1}{2} \log \frac{3}{2}$$

$$= \frac{1}{2} \log 6 - \frac{1}{10}$$

17. Let $I = \int_0^{n\pi+v} |\sin x| dx$

$$= \int_0^v |\sin x| dx + \int_v^{n\pi+v} |\sin x| dx$$

$$= \int_0^v \sin x dx + n \int_0^\pi |\sin x| dx \quad [\because |\sin x| \text{ has period } \pi]$$

$$= (-\cos x)_0^v + n(-\cos x)_0^\pi$$

$$= 2n + 1 - \cos v = \text{R.H.S.}$$

18. $U_{n+2} - U_{n+1} = \int_0^\pi \frac{(1 - \cos(n+2)x) - (1 - \cos(n+1)x)}{1 - \cos x} dx$

$$= \int_0^\pi \frac{\cos(n+1)x - \cos(n+2)x}{1 - \cos x} dx$$

$$= \int_0^\pi \frac{2 \sin\left(n + \frac{3}{2}\right)x \sin \frac{x}{2}}{2 \sin^2 \frac{x}{2}} dx$$

or $U_{n+2} - U_{n+1} = \int_0^\pi \frac{\sin\left(n + \frac{3}{2}\right)x}{\sin \frac{x}{2}} dx \quad (1)$

or $U_{n+1} - U_n = \int_0^\pi \frac{\sin\left(n + \frac{1}{2}\right)x}{\sin \frac{x}{2}} dx \quad (2)$

From equations (1) and (2), we get

$$\therefore (U_{n+2} - U_{n+1}) - (U_{n+1} - U_n)$$

$$= \int \frac{\sin\left(n + \frac{3}{2}\right)x - \sin\left(n + \frac{1}{2}\right)x}{\sin \frac{x}{2}} dx$$

or $U_{n+2} + U_n - 2U_{n+1} = \int \frac{2 \cos(n+1)x \sin \frac{x}{2}}{\sin \frac{x}{2}} dx$

$$= 2 \int_0^\pi \cos(n+1)x dx = 2 \left(\frac{\sin(n+1)x}{n+1} \right)_0^\pi = 0$$

or $U_{n+2} + U_n = 2U_{n+1}$

$\therefore U_n, U_{n+1}, U_{n+2}$ are in A.P.

$$U_0 = \int_0^\pi \frac{1 - 1}{1 - \cos x} dx = 0, U_1 = \int_0^\pi \frac{1 - \cos x}{1 - \cos x} dx = \pi$$

$$U_1 - U_0 = \pi \quad (\text{Common difference})$$

$$\therefore U_n = U_0 + n\pi = n\pi$$

$$\therefore U_n = n\pi$$

Now, $I_n = \int_0^{\pi/2} \frac{\sin^2 n\theta}{\sin^2 \theta} d\theta = \int_0^{\pi/2} \frac{1 - \cos 2n\theta}{1 - \cos 2\theta} d\theta$

$$= \frac{1}{2} \int_0^\pi \frac{1 - \cos nx}{1 - \cos x} dx = \frac{1}{2} n\pi$$

19. Let $I = \int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{x^4}{1-x^4} \cos^{-1} \left(\frac{2x}{1+x^2} \right) dx$

Put $x = -y$, so that $dx = -dy$

and $I = \int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{y^4}{1-y^4} \cos^{-1} \left(\frac{-2y}{1+y^2} \right) dy$

But $\cos^{-1}(-x) = \pi - \cos^{-1} x$ for $-1 \leq x \leq 1$,

$$\therefore I = \int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{y^4}{1-y^4} \left[\pi - \cos^{-1} \left(\frac{2y}{1+y^2} \right) \right] dy$$

$$= \pi \int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{x^4}{1-x^4} dx - \int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{x^4}{1-x^4} \cos^{-1} \left(\frac{2x}{1+x^2} \right) dx$$

$$= \pi \int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{x^4}{1-x^4} dx - I$$

$$\therefore I = \pi \int_0^{1/\sqrt{3}} \frac{x^4}{1-x^4} dx$$

$$= \pi \int_0^{1/\sqrt{3}} \left[-1 + \frac{1}{1-x^4} \right] dx$$

$$= -\pi \int_0^{1/\sqrt{3}} dx + \pi \int_0^{1/\sqrt{3}} \frac{dx}{1-x^4}$$

$$= -\frac{\pi}{\sqrt{3}} + \frac{\pi}{2} \int_0^{1/\sqrt{3}} \left[\frac{1}{1-x^2} + \frac{1}{1+x^2} \right] dx$$

$$= -\frac{\pi}{\sqrt{3}} + \frac{\pi}{2} \left[-\frac{1}{2} \log_e \left| \frac{1-x}{1+x} \right| + \tan^{-1} x \right]_0^{1/\sqrt{3}}$$

$$= -\frac{\pi}{\sqrt{3}} - \frac{\pi}{4} \log_e \left| \frac{\sqrt{3}+1}{\sqrt{3}-1} \right| + \frac{\pi^2}{12}$$

20. Let $I = \int_0^{\pi/4} \ln(1 + \tan x) dx \quad (1)$

$$= \int_0^{\pi/4} \ln(1 + \tan(\pi/4 - x)) dx$$

$$\left[\because \int_0^a f(x) dx = \int_0^a f(a-x) dx \right]$$

$$= \int_0^{\pi/4} \ln \left[1 + \frac{1 - \tan x}{1 + \tan x} \right] dx$$

$$= \int_0^{\pi/4} \ln \left(\frac{2}{1 + \tan x} \right) dx$$

$$\therefore I = \int_0^{\pi/4} [\ln 2 - \ln(1 + \tan x)] dx \quad (2)$$

Adding equations (1) and (2), we get

$$2I = \int_0^{\pi/4} \ln 2 dx$$

$$= \ln 2 [x]_0^{\pi/4} = \ln 2 \left[\frac{\pi}{4} \right]$$

$$\therefore I = \frac{\pi}{8} \ln 2$$

21. Given $a + b = 4$ and $\frac{dg(x)}{dx} > 0$, i.e., $g(x)$ is an increasing function.

We have to prove that $\int_0^a g(x)dx + \int_0^b g(x)dx$ increases as $(b-a)$ increases.

Or we have to prove that $f(a) = \int_0^a g(x)dx + \int_0^{4-a} g(x)dx$ increases as $(4-2a)$ increases.

$$\text{i.e., } \frac{df(a)}{d(4-2a)} > 0$$

$$\text{or } \frac{df(a)}{-2da} > 0$$

$$\text{or } \frac{df(a)}{da} < 0$$

$$\text{Now, } \frac{df(a)}{da} = g(a) - g(4-a)$$

Given that $a < 2$

$$\therefore 2a < 4$$

$$\therefore a < 4-a$$

$$\therefore g(a) < g(4-a) \text{ (as given that } g(x) \text{ is increasing function)}$$

Thus, $\int_0^a g(x)dx + \int_0^b g(x)dx$ increases as $(b-a)$ increases.

$$\begin{aligned} 22. I &= \int_{-\pi}^{\pi} \frac{2x(1+\sin x)}{1+\cos^2 x} dx \\ &= \int_{-\pi}^{\pi} \frac{2x}{1+\cos^2 x} dx + 2 \int_{-\pi}^{\pi} \frac{x \sin x}{1+\cos^2 x} dx \end{aligned} \quad (1)$$

$$= 0 + 4 \int_0^{\pi} \frac{x \sin x}{1+\cos^2 x} dx$$

$$= 4 \int_0^{\pi} \frac{(\pi-x) \sin(\pi-x)}{1+\cos^2(\pi-x)} dx$$

$$= 4 \int_0^{\pi} \frac{(\pi-x) \sin x}{1+\cos^2 x} dx$$

$$= 4\pi \int_0^{\pi} \frac{\sin x}{1+\cos^2 x} dx - 4 \int_0^{\pi} \frac{x \sin x}{1+\cos^2 x} dx$$

$$\therefore 2I = 4\pi \int_0^{\pi} \frac{\sin x}{1+\cos^2 x} dx$$

$$\text{or } I = 2\pi \int_0^{\pi} \frac{\sin x}{1+\cos^2 x} dx$$

Put $\cos x = t$ so that $-\sin x dx = dt$.

When $x = 0$, $t = 1$; when $x = \pi$, $t = -1$.

$$\therefore I = 2\pi \int_1^{-1} \frac{-dt}{1+t^2}$$

$$= 4\pi \left[\tan^{-1} t \right]_0^1$$

$$= 4\pi \frac{\pi}{4} = \pi^2$$

$$\begin{aligned} 23. \int_0^1 \tan^{-1} \frac{1}{1-x+x^2} dx &= \int_0^1 \tan^{-1} \frac{x+(1-x)}{1-x(1-x)} dx \\ &= \int_0^1 [\tan^{-1} x + \tan^{-1} (1-x)] dx \\ &= \int_0^1 \tan^{-1} x dx + \int_0^1 \tan^{-1} (1-x) dx \\ &= \int_0^1 \tan^{-1} x dx + \int_0^1 \tan^{-1} [1-(1-x)] dx \\ &= 2 \int_0^1 \tan^{-1} x dx \end{aligned} \quad (1)$$

Now,

$$I = \int_0^1 \tan^{-1} (1-x+x^2) dx$$

$$= \int_0^1 \cot^{-1} \left(\frac{1}{1-x+x^2} \right) dx$$

$$= \int_0^1 \left[\frac{\pi}{2} - \tan^{-1} \left(\frac{1}{1-x+x^2} \right) \right] dx$$

$$= \frac{\pi}{2} - 2 \int_0^1 \tan^{-1} x dx \quad [\text{From equation (1)}]$$

$$= \frac{\pi}{2} - 2 \left\{ x \tan^{-1} x - \frac{1}{2} \log(1+x^2) \right\}_0^1 \quad (\text{integrating by parts})$$

$$= \log_e 2$$

$$24. \text{ Let } F(x) = f(x) + f\left(\frac{1}{x}\right)$$

$$= \int_1^x \frac{\log t}{1+t} dt + \int_1^{1/x} \frac{\log t}{1+t} dt$$

In second integral, let $t = 1/y \therefore dt = -\frac{1}{y^2} dy$

$$\therefore F(x) = \int_1^x \frac{\log t}{1+t} dt + \int_1^x \frac{-\log y}{1+\frac{1}{y}} \left(-\frac{dy}{y^2} \right)$$

$$= \int_1^x \frac{\log t}{1+t} dt + \int_1^x \frac{\log y}{y(1+y)} dy$$

$$= \int_1^x \frac{\log t}{1+t} dt + \int_1^x \frac{\log t}{t(1+t)} dt$$

$$= \int_1^x \frac{\log t}{t} dt = \frac{1}{2} (\log x)^2$$

$$\therefore F(e) = \frac{1}{2}$$

$$25. \text{ We have } y(x) = \int_{\pi^2/16}^{x^2} \frac{\cos x \cos \sqrt{\theta}}{1+\sin^2 \sqrt{\theta}} d\theta$$

$$= \cos x \int_{\pi^2/16}^{x^2} \frac{\cos \sqrt{\theta}}{1+\sin^2 \sqrt{\theta}} d\theta$$

$$\therefore \frac{dy}{dx} = -\sin x \int_{\pi^2/16}^{x^2} \frac{\cos \sqrt{\theta}}{1+\sin^2 \sqrt{\theta}} d\theta$$

$$+ \cos x \frac{d}{dx} \left[\int_{\pi^2/16}^{x^2} \frac{\cos \sqrt{\theta}}{1+\sin^2 \sqrt{\theta}} d\theta \right]$$

$$= -\sin x \int_{\pi^2/16}^{x^2} \frac{\cos \sqrt{\theta}}{1+\sin^2 \sqrt{\theta}} d\theta$$

$$+ \cos x \left[\frac{\cos x}{1+\sin^2 x} \cdot 2x - 0 \right]$$

$$\text{or } \frac{dy}{dx} \Big|_{x=\pi} = 0 + \frac{\cos^2 \pi}{1+\sin^2 \pi} \cdot 2\pi = 2\pi$$

$$26. \text{ Let } I = \int_{-\pi/3}^{\pi/3} \frac{\pi+4x^3}{2-\cos\left(x+\frac{\pi}{3}\right)} dx$$

$$= \int_{-\pi/3}^{\pi/3} \frac{\pi}{2 - \cos\left(x + \frac{\pi}{3}\right)} dx + \int_{-\pi/3}^{\pi/3} \frac{4x^3}{2 - \cos\left(x + \frac{\pi}{3}\right)} dx$$

The second integral becomes zero as integrand being an odd function of x .

$$\therefore I = 2\pi \int_0^{\pi/3} \frac{dx}{2 - \cos\left(x + \frac{\pi}{3}\right)}$$

Let $x + \pi/3 = y \therefore dx = dy$.

Also, as $x \rightarrow 0$, $y \rightarrow \pi/3$, and as $x \rightarrow \pi/3$, $y \rightarrow 2\pi/3$.

$$\begin{aligned} \therefore I &= 2\pi \int_{\pi/3}^{2\pi/3} \frac{dy}{2 - \cos y} \\ &= 2\pi \int_{\pi/3}^{2\pi/3} \frac{dy}{2 - \frac{1 - \tan^2 y/2}{1 + \tan^2 y/2}} \\ &= 2\pi \int_{\pi/3}^{2\pi/3} \frac{\sec^2 y/2}{3 \tan^2 y/2 + 1} dy \\ &= \frac{4\pi}{3} \int_{\pi/3}^{2\pi/3} \frac{\frac{1}{2} \sec^2 y/2}{\tan^2 y/2 + (1/\sqrt{3})^2} dy \\ &= \frac{4\pi\sqrt{3}}{3} \left[\tan^{-1} (\sqrt{3} \tan y/2) \right]_{\pi/3}^{2\pi/3} \\ &= \frac{4\pi}{\sqrt{3}} [\tan^{-1} 3 - \tan^{-1} 1] \\ &= \frac{4\pi}{\sqrt{3}} [\tan^{-1} 3 - \pi/4] \end{aligned}$$

$$\begin{aligned} 27. I &= \int_0^{\pi} e^{|\cos x|} \left(2 \sin\left(\frac{1}{2} \cos x\right) + 3 \cos\left(\frac{1}{2} \cos x\right) \right) \sin x dx \\ &= \int_0^{\pi} e^{|\cos x|} 2 \sin\left(\frac{1}{2} \cos x\right) \sin x dx \\ &\quad + \int_0^{\pi} e^{|\cos x|} 3 \cos\left(\frac{1}{2} \cos x\right) \sin x dx \\ &= I_1 + I_2 \end{aligned}$$

Now, using the property that

$$\int_0^{2a} f(x) dx = \begin{cases} 0 & \text{if } f(2a-x) = -f(x) \\ 2 \int_0^a f(x) dx & \text{if } f(2a-x) = f(x) \end{cases}$$

we get $I_1 = 0$ and

$$\begin{aligned} I_2 &= 2 \int_0^{\pi/2} e^{|\cos x|} 3 \cos\left(\frac{1}{2} \cos x\right) \sin x dx \\ &= 6 \int_0^{\pi/2} e^{\cos x} \cos\left(\frac{1}{2} \cos x\right) \sin x dx \end{aligned}$$

Put $\cos x = t \therefore -\sin x dx = dt$

$$\begin{aligned} \therefore I_2 &= 6 \int_0^1 e^t \cos \frac{t}{2} dt \\ &= 6 \left[\frac{e^t}{1 + \frac{1}{4}} \left(\frac{1}{2} \sin \frac{t}{2} + \cos \frac{t}{2} \right) \right]_0^1 \\ &\quad \left[\text{Using } \int e^{ax} \cos bx dx = \frac{e^{ax}}{a^2 + b^2} (b \sin bx + a \cos bx) \right] \\ &= \frac{24}{5} \left[e \cos\left(\frac{1}{2}\right) + \frac{1}{2} e \sin\left(\frac{1}{2}\right) - 1 \right] \end{aligned}$$

$$\begin{aligned} 28. \frac{5050 \int_0^1 (1-x^{50})^{100} dx}{\int_0^1 (1-x^{50})^{101} dx} &= 5050 \frac{I_{100}}{I_{101}} \\ I_{101} &= \int_0^1 (1-x^{50})(1-x^{50})^{100} dx \\ &= I_{100} - \int_0^1 x \cdot x^{49} (1-x^{50})^{100} dx \\ &= I_{100} - \left[\frac{-x(1-x^{50})^{101}}{101} \right]_0^1 - \int_0^1 \frac{(1-x^{50})^{101}}{5050} \\ \therefore I_{101} &= I_{100} - \frac{I_{101}}{5050} \therefore 5050 \frac{I_{100}}{I_{101}} = 5051 \end{aligned}$$