3.7 MOTION OF CHARGED PARTICLES IN ELECTRIC AND MAGNETIC FIELDS

3.372 Let the electron leave the negative plate of the capacitor at time $t = 0$

As, $$E_x = -\frac{d\varphi}{dx},\ E = \frac{\varphi}{l} = \frac{at}{l},$$

and, therefore, the acceleration of the electron,

$$w = \frac{eE}{m} = \frac{eat}{ml} \quad \text{or,} \quad \frac{dv}{dt} = \frac{eat}{ml}$$

or, $$\int_0^v dv = \frac{ea}{ml}\int_0^t t\,dt, \quad \text{or,} \quad v = \frac{1}{2}\frac{ea}{ml}t^2 \tag{1}$$

But, from $s = \int v\,dt$,

$$l = \frac{1}{2}\frac{ea}{ml}\int_0^t t^2\,dt = \frac{eat^3}{6\,ml} \quad \text{or,} \quad t = \left(\frac{6\,ml^2}{ea}\right)^{\frac{1}{3}}$$

Putting the value of t in (1),

$$v = \frac{1}{2}\frac{ea}{ml}\left(\frac{6\,ml^2}{ea}\right)^{\frac{2}{3}} = \left(\frac{9}{2}\frac{ale}{m}\right)^{\frac{1}{3}} = 16\ \text{km/s}.$$

3.373 The electric field inside the capacitor varies with time as,

$$E = at.$$

Hence, electric force on the proton,

$$F = eat$$

and subsequently, acceleration of the proton,

$$w = \frac{eat}{m}$$

Now, if t is the time elapsed during the motion of the proton between the plates, then $t = \frac{l}{v_{\parallel}}$, as no acceleration is effective in this direction. (Here $v_{\parallel}$ is velocity along the length of the plate.)

From kinematics, $\frac{dv_{\perp}}{dt} = w$

so, $$\int_0^{v_{\perp}} dv_{\perp} = \int_0^t w\,dt,$$

(as initially, the component of velocity in the direction, $\perp$ to plates, was zero.)

or $$v_\perp = \int_0^t \frac{ea}{m}\frac{t^2}{2m} = \frac{ea}{2m}\frac{l^2}{v_\parallel^2}$$

Now, $$\tan\alpha = \frac{v_\perp}{v_\parallel} = \frac{e\,al^2}{2\,m\,v_\parallel^3}$$

$$= \frac{eal^2}{2m\left(\frac{2\,eV}{m}\right)^{\frac{3}{2}}},\ \text{as}\ v_\parallel = \left(\frac{2\,eV}{m}\right)^{\frac{1}{2}},\ \text{from energy conservation.}$$

$$= \frac{al^2}{4}\sqrt{\frac{m}{2\,eV^3}}$$

3.374 The equation of motion is,

$$\frac{dv}{dt} = v\,\frac{dv}{dx} = \frac{q}{m}(E_0 - ax)$$

Integrating

$$\frac{1}{2}v^2 - \frac{q}{m}(E_0 x - \frac{1}{2}ax^2) = \text{constant.}$$

But initially $v = 0$ when $x = 0$, so "constant" = 0

Thus, $$v^2 = \frac{2q}{m}\left(E_0 x - \frac{1}{2}ax^2\right)$$

Thus, $v = 0$, again for $x = x_m = \frac{2\,E_0}{a}$

The corresponding acceleration is,

$$\left(\frac{dv}{dt}\right)_{x_m} = \frac{q}{m}(E_0 - 2\,E_0) = -\frac{qE_0}{m}$$

3.375 From the law of relativistic conservation of energy

$$\frac{m_0 c^2}{\sqrt{1 - (v^2/c^2)}} - e\,Ex = m_0 c^2.$$

as the electron is at rest ($v = 0$ for $x = 0$) initially.

Thus clearly $$T = e\,Ex.$$

On the other hand, $$\sqrt{1 - (v^2/c^2)} = \frac{m_0 c^2}{m_0 c^2 + e\,Ex}$$

or, $$\frac{v}{c} = \frac{\sqrt{(m_0 c^2 + e\,Ex)^2 - m_0^2 c^4}}{m_0 c^2 + eEx}$$

or, $$ct = \int c\,dt = \int \frac{(m_0 c^2 + eEx)\,dx}{\sqrt{(m_0 c^2 + eEx)^2 - m_0^2 c^4}}$$

$$= \frac{1}{2\,eE}\int \frac{dy}{\sqrt{y - m_0^2 c^4}} = \frac{1}{eE}\sqrt{(m_0 c^2 + eEx)^2 - m_0^2 c^4} + \text{constant}$$

The "constant" = 0, at $t = 0$, for $x = 0$,

So, $$ct = \frac{1}{eE}\sqrt{(m_0 c^2 + eEx)^2 - m_0^2 c^4}.$$

Finally, using $T = e\,E\,x$,

$$c\,e\,E\;t_0 = \sqrt{T\,(T + 2\,m_0\,c^2)} \quad \text{or,} \quad t_0 = \frac{\sqrt{T\,(T + 2\,m_0\,c^2)}}{e\,Ec}$$

3.376 As before, $T = e\,E\,x$

Now in linear motion,

$$\frac{d}{dt}\frac{m_0\,v}{\sqrt{1 - v^2/c^2}} = \frac{m_0\,w}{\sqrt{1 - v^2/c^2}} + \frac{m_0\,w}{(1 - v^2/c^2)^{3/2}}\frac{v}{c^2}\,w$$

$$= \frac{m_0}{(1 - v^2/c^2)^{3/2}}\,w = \frac{(T + m_0\,c^2)^3}{m_0^2\,c^6}\,w = e\,E,$$

So, $$w = \frac{e\,E\,m_0^2\,c^6}{(T + m_0\,c^2)^3} = \frac{eE}{m_0}\left(1 + \frac{T}{m_0\,c^2}\right)^{-3}$$

3.377 The equations are,

$$\frac{d}{dt}\left(\frac{m_0\,v_x}{\sqrt{1 - (v^2/c^2)}}\right) = 0 \quad \text{and} \quad \frac{d}{dt}\left(\frac{m_0\,v_y}{\sqrt{1 - v^2/c^2}}\right) = e\,E$$

Hence, $$\frac{v_x}{\sqrt{1 - v^2/c^2}} = \text{constant} = \frac{v_0}{\sqrt{1 - (v_0^2/c^2)}}$$

Also, by energy conservation,

$$\frac{m_0\,c^2}{\sqrt{1 - (v^2/c^2)}} = \frac{m_0\,c^2}{\sqrt{1 - (v_0^2/c^2)}} + e\,E\,y$$

Dividing $$v_x = \frac{v_0\,\varepsilon_0}{\varepsilon_0 + eEy},\ \varepsilon_0 = \frac{m_0\,c^2}{\sqrt{1 - (v_0^2/c^2)}}$$

Also, $$\frac{m_0}{\sqrt{1 - (v^2/c^2)}} = \frac{\varepsilon_0 + e\,E\,y}{c^2}$$

Thus, $$(\varepsilon_0 + e\,Ey)\,v_y = c^2\,e\,E\,t + \text{constant.}$$

"constant" = 0 as $v_y = 0$ at $t = 0$.

Integrating again,

$$\varepsilon_0\,y + \frac{1}{2}\,e\,E\,y^2 = \frac{1}{2}\,c^2\,E\,t^2 + \text{constant.}$$

"constant" = 0, as $y = 0$, at $t = 0$.

Thus, $(ceEt)^2 = (eyE)^2 + 2\varepsilon_0 eEy + \varepsilon_0^2 - \varepsilon_0^2$

or, $$ceEt = \sqrt{(\varepsilon_0 + eEy)^2 - \varepsilon_0^2}$$

or, $$\varepsilon_0 + eEy = \sqrt{\varepsilon_0^2 + c^2 e^2 E^2 t^2}$$

Hence, $$v_x = \frac{v_0 \varepsilon_0}{\sqrt{\varepsilon_0^2 + c^2 e^2 E^2 t^2}} \text{ also, } v_y = \frac{c^2 eEt}{\sqrt{\varepsilon_0^2 + c^2 e^2 E^2 t^2}}$$

and $$\tan\theta = \frac{v_y}{v_x} = \frac{eEt}{m_0 v_0}\sqrt{1 - (v_0^2/c^2)}.$$

3.378 From the figure,

$$\sin\alpha = \frac{d}{R} = \frac{dqB}{mv},$$

As radius of the arc $R = \frac{mv}{qB}$, where v is the velocity of the particle, when it enteres into the field. From initial condition of the problem,

$$qV = \frac{1}{2}mv^2 \quad \text{or,} \quad v = \sqrt{\frac{2qV}{m}}$$

Hence, $$\sin\alpha = \frac{dqB}{m\sqrt{\frac{2qV}{m}}} = dB\sqrt{\frac{q}{2mV}}$$

and $\alpha = \sin^{-1}\left(dB\sqrt{\frac{q}{2mV}}\right) = 30°$, on putting the values.

3.379 (a) For motion along a circle, the magnetic force acted on the particle, will provide the centripetal force, necessary for its circular motion.

i.e. $$\frac{mv^2}{R} = evB \quad \text{or,} \quad v = \frac{eBR}{m}$$

and the period of revolution, $T = \frac{2\pi}{\omega} = \frac{2\pi R}{v} = \frac{2\pi m}{eB}$

(b) Generally, $\frac{d\vec{p}}{dt} = \vec{F}$

But, $$\frac{d\vec{p}}{dt} = \frac{d}{dt}\frac{m_0\vec{v}}{\sqrt{1-(v^2/c^2)}} = \frac{m_0\dot{\vec{v}}}{\sqrt{1-(v^2/c^2)}} + \frac{m_0}{\left(1-(v^2/c^2)\right)^{3/2}}\frac{\vec{v}(\vec{v}\cdot\dot{\vec{v}})}{c^2}$$

For transverse motion, $\vec{v}\cdot\dot{\vec{v}} = 0$ so,

$$\frac{d\vec{p}}{dt} = \frac{m_0\dot{\vec{v}}}{\sqrt{1-(v^2/c^2)}} = \frac{m_0}{\sqrt{1-(v^2/c^2)}}\frac{v^2}{r}, \text{ here.}$$

Thus, $$\frac{m_0 v^2}{r\sqrt{1-(v^2/c^2)}} = B\,ev \quad \text{or,} \quad \frac{v/c}{\sqrt{1-(v^2/c^2)}} = \frac{Ber}{m_0 c}$$

or, $$\frac{v}{c} = \frac{Ber}{\sqrt{B^2 e^2 r^2 + m_0^2 c^2}}$$

Finally, $$T = \frac{2\pi r}{v} = \frac{2\pi m_0}{eB\sqrt{1-v^2/c^2}} = \frac{2\pi}{cBe}\sqrt{B^2 e^2 r^2 + m_0^2 c^2}$$

3.380 (a) As before, $p = B\,qr$.

(b) $T = \sqrt{c^2 p^2 + m_0^2 c^4} = \sqrt{c^2 B^2 q^2 r^2 + m_0^2 c^4}$

(c) $w = \dfrac{v^2}{r} = \dfrac{c^2}{r\left[1 + (m_0 c / Bqr)^2\right]}$

using the result for v from the previous problem.

3.381 From (3.279),

$$T = \frac{2\pi\varepsilon}{c^2 eB}\text{(relativistic)}, \qquad T_0 = \frac{2\pi m_0 c^2}{c^2 eB}\text{ (nonrelativistic)},$$

Here, $$m_0 c^2 \big/ \sqrt{1-v^2/c^2} = E$$

Thus, $$\delta T = \frac{2\pi T}{c^2 eB},\ (T = K.E.)$$

Now, $$\frac{\delta T}{T_0} = \eta = \frac{T}{m_0 c^2}, \quad \text{so,} \quad T = \eta\, m_0 c^2$$

3.382 $T = eV = \dfrac{1}{2} mv^2$

(The given potential difference is not large enough to cause significant deviations from the nonrelativistic formula).

Thus, $$v = \sqrt{\frac{2eV}{m}}$$

So, $$v_\parallel = \sqrt{\frac{2eV}{m}}\cos\alpha, \quad v_\perp = \sqrt{\frac{2eV}{m}}\sin\alpha$$

Now, $$\frac{m v_\perp^2}{r} = Bev_\perp \quad \text{or,} \quad r = \frac{mv_\perp}{Be},$$

and $$T = \frac{2\pi r}{v_\perp} = \frac{2\pi m}{Be}$$

Pitch $$p = v_\parallel T = \frac{2\pi m}{Be}\sqrt{\frac{2eV}{m}}\cos\alpha = 2\pi\sqrt{\frac{2mV}{eB^2}}\cos\alpha$$

3.383 The charged particles will traverse a helical trajectory and will be focussed on the axis after traversing a number of turns. Thus

$$\frac{l}{v_0} = n \cdot \frac{2\pi m}{qB_1} = (n+1)\frac{2\pi m}{qB_2}$$

So,
$$\frac{n}{B_1} = \frac{n+1}{B_2} = \frac{1}{B_2 - B_1}$$

Hence,
$$\frac{l}{v_0} = \frac{2\pi m}{q(B_2 - B_1)}$$

or,
$$\frac{l^2}{2qV/m} = \frac{(2\pi)^2}{(B_2 - B_1)^2} \times \frac{1}{(q/m)^2}$$

or,
$$\frac{q}{m} = \frac{8\pi^2 V}{l^2 (B_2 - B_1)^2}$$

3.384 Let us take the point A as the origin O and the axis of the solenoid as z-axis. At an arbitrary moment of time let us resolve the velocity of electron into its two rectangular components, $\vec{v}_{\parallel}$ along the axis and $\vec{v}_{\perp}$ to the axis of solenoid. We know the magnetic force does no work, so the kinetic energy as well as the speed of the electron $|\vec{v}_{\perp}|$ will remain constant in the x-y plane. Thus $\vec{v}_{\perp}$ can change only its direction as shown in the Fig.. $\vec{v}_{\parallel}$ will remain constant as it is parallel to $\vec{B}$.

Thus at $t = t$

$$v_x = v_{\perp} \cos \omega t = v \sin\alpha \cos \omega t,$$

$$v_y = v_{\perp} \sin \omega t = v \sin\alpha \sin \omega t$$

and
$$v_z = v \cos\alpha, \quad \text{where } \omega = \frac{eB}{m}$$

As at $t = 0$, we have $x = y = z = 0$, so the motion law of the electron is.

$$\left.\begin{aligned} z &= v \cos\alpha \, t \\ x &= \frac{v \sin\alpha}{\omega} \sin \omega t \\ y &= \frac{v \sin\alpha}{\omega} (\cos \omega t - 1) \end{aligned}\right]$$

(The equation of the helix)

On the screen, $z = l$, so $t = \dfrac{l}{v \cos\alpha}$.

Then,
$$r^2 = x^2 + y^2 = \frac{2 v^2 \sin^2\alpha}{\omega^2}\left(1 - \cos \frac{\omega l}{v \cos\alpha}\right)$$

$$r = \frac{2 v \sin\alpha}{\omega} \left| \sin \frac{\omega l}{2 v \cos\alpha} \right| = 2\frac{mv}{eB} \sin\alpha \left| \sin \frac{leB}{2 mv \cos\alpha} \right|$$

3.385 Choose the wire along the z-axis, and the initial direction of the electron, along the x-axis. Then the magnetic field in the $x - z$ plane is along the y-axis and outside the wire it is,

$$B = B_y = \frac{\mu_0 I}{2\pi x}, \ (B_x = B_z = 0, \text{ if } y = 0)$$

The motion must be confined to the $x - z$ plane. Then the equations of motion are,

$$\frac{d}{dt} m v_x = -e v_z B_y$$

$$\frac{d(m v_z)}{dt} = +e v_x B_y$$

Multiplying the first equation by v_x and the second by v_z and then adding,

$$v_x \frac{dv_x}{dt} + v_z \frac{dv_z}{dt} = 0$$

or, $$v_x^2 + v_z^2 = v_0^2, \text{ say, or, } v_z = \sqrt{v_0^2 - v_x^2}$$

Then, $$v_x \frac{dv_x}{dx} = -\frac{e}{m}\sqrt{v_0^2 - v_x^2}\,\frac{\mu_0 I}{2\pi x}$$

or, $$-\frac{v_x\, dv_x}{\sqrt{v_0^2 - v_x^2}} = \frac{\mu_0 I e}{2\pi m}\frac{dx}{x}$$

Integrating, $$\sqrt{v_0^2 - v_x^2} = \frac{\mu_0 I e}{2\pi m}\ln\frac{x}{a}$$

on using, $v_x = v_0$, if $x = a$ (i.e. initially).

Now, $$v_x = 0, \text{ when } x = x_m,$$

so, $$x_m = a\, e^{v_0/b}, \text{ where } b = \frac{\mu_0 I e}{2\pi m}.$$

3.386 Inside the capacitor, the electric field follows a $\frac{1}{r}$ law, and so the potential can be written as

$$\varphi = \frac{V \ln r/a}{\ln b/a}, \ E = \frac{-V}{\ln b/a}\frac{1}{r}.$$

Here r is the distance from the axis of the capacitor.

Also, $$\frac{mv^2}{r} = \frac{qV}{\ln b/a}\frac{1}{r} \text{ or } mv^2 = \frac{qV}{\ln b/a}$$

On the other hand,

$$mv = qBr \text{ in the magnetic field.}$$

Thus, $$v = \frac{V}{Br\ln b/a} \text{ and } \frac{q}{m} = \frac{v}{Br} = \frac{V}{B^2 r^2 \ln(b/a)}$$

3.387 The equations of motion are,

$$m\frac{dv_x}{dt} = -qBv_z,\ m\frac{dv_y}{dt} = qE \text{ and } m\frac{dv_z}{dt} = q\,v_x B$$

These equations can be solved easily.

First, $$v_y = \frac{qE}{m}t,\ y = \frac{qE}{2m}t'^2$$

Then, $$v_x^2 + v_z^2 = \text{constant} = v_0^2 \text{ as before.}$$

In fact, $v_x = v_0 \cos \omega t$ and $v_z = v_0 \sin \omega t$ as one can check.

Integrating again and using $x = z = 0$, at $t = 0$

$$x = \frac{v_0}{\omega}\sin \omega t,\ z = \frac{v_0}{\omega}(1 - \cos \omega t)$$

Thus, $$x = z = 0 \text{ for } t = t_n = n\frac{2\pi}{\omega}$$

At that instant, $y_n = \frac{qE}{2m} \times \frac{2\pi}{qB/m} \times n^2 \times \frac{2\pi}{qB/m} = \frac{2\pi^2 m E n^2}{qB^2}$

Also, $$\tan \alpha_n = \frac{v_x}{v_y},\ (v_z = 0 \text{ at this moment})$$

$$= \frac{mv_0}{qE\,t_n} = \frac{mv_0}{qE} \times \frac{qB}{m} \times \frac{1}{2\pi n} = \frac{B\,v_0}{2\pi E n}.$$

3.388 The equation of the trajectory is,

$$x = \frac{v_0}{\omega}\sin \omega t,\ z = \frac{v_0}{\omega}(1 - \cos \omega t),\ y = \frac{qE}{2m}t^2 \text{ as before see (3.384).}$$

Now on the screen $x = l$, so

$$\sin \omega t = \frac{\omega l}{v_0} \quad \text{or,} \quad \omega t = \sin^{-1}\frac{\omega l}{v_0}$$

At that moment,

$$y = \frac{qE}{2m\omega^2}\left(\sin^{-1}\frac{\omega l}{v_0}\right)^2$$

so, $$\frac{\omega l}{v_0} = \sin\sqrt{\frac{2m\omega^2 y}{qE}} = \sin\sqrt{\frac{2qB^2y}{Em}}$$

and $$z = \frac{v_0}{\omega}2\sin^2\frac{\omega t}{2} = l\tan\frac{\omega t}{2}$$

$$= l\tan\frac{1}{2}\left[\sin^{-1}\frac{\omega l}{v_0}\right] = l\tan\sqrt{\frac{qB^2y}{2mE}}$$

For small

$$z,\ \frac{qB^2y}{2mE} = \left(\tan^{-1}\frac{z}{l}\right)^2 \approx \frac{z^2}{l^2}$$

or, $$y = \frac{2mE}{qB^2l^2}\cdot z^2 \text{ is a parabola.}$$

3.389 In crossed field,

$$eE = evB, \text{ so } v = \frac{E}{B}$$

Then, F = force exerted on the plate $= \frac{I}{e} \times m \frac{E}{B} = \frac{m I E}{eB}$

3.390 When the electric field is switched off, the path followed by the particle will be helical. and pitch, $\Delta l = v_{\parallel} T$, (where $v_{\parallel}$ is the velocity of the particle, parallel to $\vec{B}$, and T, the time period of revolution.)

$$= v \cos(90 - \varphi)\, T = v \sin \varphi\, T$$

$$= v \sin \varphi \frac{2\pi m}{qB} \left(\text{as } T = \frac{2\pi}{qB}\right) \quad (1)$$

Now, when both the fields were present, $qE = qvB \sin(90 - \varphi)$, as no net force was effective on the system.

or, $$v = \frac{E}{B \cos \varphi} \quad (2)$$

From (1) and (2), $$\Delta l = \frac{E}{B} \frac{2\pi m}{qB} \tan \varphi = 6 \text{ cm}.$$

3.391 When there is no deviation,

$$-q\vec{E} = q(\vec{v} \times \vec{B})$$

or, in scalar from, $E = vB$ (as $\vec{v} \perp \vec{B}$) or, $v = \frac{E}{B}$ (1)

Now, when the magnetic field is switched on, let the deviation in the field be x. Then,

$$x = \frac{l}{2}\left(\frac{q\,vB}{m}\right) t^2,$$

where t is the time required to pass through this region.

also, $$t = \frac{a}{v}$$

Thus $$x = \frac{1}{2}\left(\frac{qvB}{m}\right)\left(\frac{a}{v}\right)^2 = \frac{1}{2} \frac{q}{m} \frac{a^2 B^2}{E} \quad (2)$$

For the region where the field is absent, velocity in upward direction

$$= \left(\frac{qvB}{m}\right) t = \frac{q}{m} a B \quad (3)$$

Now, $$\Delta x - x = \frac{qaB}{m} t'$$

$$= \frac{q}{m} \frac{aB^2 b}{E} \text{ when } t' = \frac{b}{v} = \frac{bB}{E} \quad (4)$$

From (2) and (4),

$$\Delta x - \frac{1}{2} \frac{q}{m} \frac{a^2 B^2}{E} = \frac{q}{m} \frac{a B^2 b}{E}$$

or, $$\frac{q}{m} = \frac{2 E \Delta x}{a B^2 (a + 2b)}$$

3.392 (a) The equation of motion is,

$$m\frac{d^2\vec{r}}{dt^2} = q(\vec{E} + \vec{v}\times\vec{B})$$

Now, $$\vec{v}\times\vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dot{x} & \dot{y} & \dot{z} \\ 0 & 0 & B \end{vmatrix} = \vec{i}B\dot{y} - \vec{j}B\dot{x}$$

So, the equation becomes,

$$\frac{dv_x}{dt} = \frac{qBv_y}{m},\ \frac{dv_y}{dt} = \frac{qE}{m} - \frac{qB}{m}v_x,\ \text{and}\ \frac{dv_z}{dt} = 0$$

Here, $v_x = \dot{x}$, $v_y = \dot{y}$, $v_z = \dot{z}$. The last equation is easy to integrate;

$$v_z = \text{constant} = 0,$$

since v_z is zero initially. Thus integrating again,

$$z = \text{constant} = 0,$$

and motion is confined to the $x - y$ plane. We now multiply the second equation by i and add to the first equation.

$$\xi = v_x + iv_y$$

we get the equation,

$$\frac{d\xi}{dt} = i\omega\frac{E}{B} - i\omega\xi,\ \omega = \frac{qB}{m}.$$

This equation after being multiplied by $e^{i\omega t}$ can be rewritten as,

$$\frac{d}{dt}(\xi e^{i\omega t}) = i\omega e^{i\omega t}\frac{E}{B}$$

and integrated at once to give,

$$\xi = \frac{E}{B} + Ce^{-i\omega t - i\alpha},$$

where C and α are two real constants. Taking real and imaginary parts.

$$v_x = \frac{E}{B} + C\cos(\omega t + \alpha)\ \text{ and }\ v_y = -C\sin(\omega t + \alpha)$$

Since $v_y = 0$, when $t = 0$, we can take $\alpha = 0$, then $v_x = 0$ at $t = 0$ gives, $C = -\frac{E}{B}$

and we get,

$$v_x = \frac{E}{B}(1 - \cos\omega t)\ \text{ and }\ v_y = \frac{E}{B}\sin\omega t.$$

Integrating again and using $x = y = 0$, at $t = 0$, we get

$$x(t) = \frac{E}{B}\left(t - \frac{\sin\omega t}{\omega}\right),\ y(t) = \frac{E}{\omega B}(1 - \cos\omega t).$$

This is the equation of a cycloid.

(b) The velocity is zero, when $\omega t = 2n\pi$. We see that

$$v^2 = v_x^2 + v_y^2 = \left(\frac{E}{B}\right)^2(2 - 2\cos\omega t)$$

or, $$v = \frac{ds}{dt} = \frac{2E}{B}\left|\sin\frac{\omega t}{2}\right|$$

The quantity inside the modulus is positive for $0 < \omega t < 2\pi$. Thus we can drop the modulus and write for the distance traversed between two successive zeroes of velocity.

$$S = \frac{4E}{\omega B}\left(1 - \cos\frac{\omega t}{2}\right)$$

Putting $\omega t = 2\pi$, we get

$$S = \frac{8E}{\omega B} = \frac{8\,mE}{q\,B^2}$$

(c) The drift velocity is in the x–direction and has the magnitude,

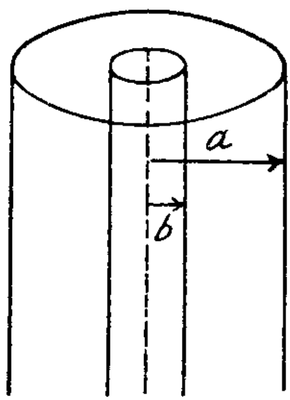
$$< v_x > = \left< \frac{E}{B}(1 - \cos\omega t) \right> = \frac{E}{B}.$$

3.393 When a current I flows along the axis, a magnetic field $B_\varphi = \frac{\mu_0 I}{2\pi\rho}$ is set up where $\rho^2 = x^2 + y^2$. In terms of components,

$$B_x = -\frac{\mu_0 I y}{2\pi\rho^2},\ B_y = \frac{\mu_0 I x}{2\pi\rho^2} \text{ and } B_z = 0$$

Suppose a p.d. V is set up between the inner cathode and the outer anode. This means a potential function of the form

$$\varphi = V\frac{\ln\rho/b}{\ln a/b},\ a > \rho > b,$$



as one can check by solving Laplace equation. The electric field corresponding to this is,

$$E_x = -\frac{Vx}{\rho^2\ln a/b},\ E_y = -\frac{Vy}{\rho^2\ln a/b},\ E_z = 0.$$

The equations of motion are,

$$\frac{d}{dt}mv_x = +\frac{|e|\,Vz}{\rho^2\ln a/b} + \frac{|e|\,\mu_0 I}{2\pi\rho^2}x\dot{z}$$

$$\frac{d}{dt}mv_y = +\frac{|e|\,Vy}{\rho^2\ln a/b} + \frac{|e|\,\mu_0 I}{2\pi\rho^2}y\dot{z}$$

and $$\frac{d}{dt}mv_z = -|e|\frac{\mu_0 I}{2\pi\rho^2}(x\dot{x} + y\dot{y}) = -|e|\frac{\mu_0 I}{2\pi}\frac{d}{dt}\ln\rho$$

$(-|e|)$ is the charge on the electron.

Integrating the last equation,

$$mv_z = -|e|\frac{\mu_0 I}{2\pi}\ln\rho/a = m\dot{z}.$$

since $v_z = 0$ where $\rho = a$. We now substitute this $\dot{z}$ in the other two equations to get

$$\frac{d}{dt}\left(\frac{1}{2}mv_x^2 + \frac{1}{2}mv_y^2\right)$$

$$= \left[\frac{|e|V}{\ln a/b} - \frac{|e|^2}{m}\left(\frac{\mu_0 I}{2\pi}\right)^2 \ln \rho/b\right] \cdot \frac{x\dot{x} + y\dot{y}}{\rho^2}$$

$$= \left[\frac{|e|V}{\ln \frac{a}{b}} - \frac{|e|^2}{m}\left(\frac{\mu_0 I}{2\pi}\right)^2 \ln \frac{\rho}{b}\right] \cdot \frac{1}{2\rho^2}\frac{d}{dt}\rho^2$$

$$= \left[\frac{|e|V}{\ln \frac{a}{b}} - \frac{|e|^2}{m}\left(\frac{\mu_0 I}{2\pi}\right)^2 \ln \frac{\rho}{b}\right] \frac{d}{dt}\ln \frac{\rho}{b}$$

Integrating and using $v^2 = 0$, at $\rho = b$, we get,

$$\frac{1}{2}mv^2 = = \left[\frac{|e|V}{\ln \frac{a}{b}}\ln \frac{\rho}{b} - \frac{1}{2m}|e|^2\left(\frac{\mu_0 I}{2\pi}\right)^2 \left(\ln \frac{\rho}{b}\right)\right]$$

The RHS must be positive, for all $a > \rho > b$. The condition for this is,

$$V \geq \frac{1}{2}\frac{|e|}{m}\left(\frac{\mu_0 I}{2\pi}\right)^2 \ln \frac{a}{b}$$

3.394 This differs from the previous problem in $(a \leftrightarrow b)$ and the magnetic field is along the z-direction. Thus $B_x = B_y = 0$, $B_z = B$

Assuming as usual the charge of the electron to be $-|e|$, we write the equation of motion

$$\frac{d}{dt}mv_x = \frac{|e|V_x}{\rho^2 \ln \frac{b}{a}} - |e|B\dot{y}, \quad \frac{d}{dt}mv_y = \frac{|e|V_y}{\rho^2 \ln \frac{b}{a}} + |e|B\dot{x}$$

and
$$\frac{d}{dt}mv_z = 0 \Rightarrow z = 0$$

The motion is confined to the plane $z = 0$. Eliminating B from the first two equations,

$$\frac{d}{dt}\left(\frac{1}{2}mv^2\right) = \frac{|e|V}{\ln b/a}\frac{x\dot{x} + y\dot{y}}{\rho^2}$$

or,
$$\frac{1}{2}mv^2 = |e|V\frac{\ln \rho/a}{\ln b/a}$$

so, as expected, since magnetic forces do not work,

$$v = \sqrt{\frac{2|e|V}{m}}, \text{ at } \rho = b.$$

On the other hand, eliminating V, we also get,

$$\frac{d}{dt} m\,(x v_y - y v_x) = |e|\,B\,(x\dot{x} + y\dot{y})$$

i.e.
$$(x v_y - y v_x) = \frac{|e|B}{2m}\rho^2 + \text{constant}$$

The constant is easily evaluated, since v is zero at $\rho = a$. Thus,

$$(x v_y - y v_x) = \frac{|e|B}{2m}(\rho^2 - a^2) > 0$$

At
$$\rho = b,\ (x v_y - y v_x) \le vb$$

Thus,
$$vb \ge \frac{|e|B}{2m}(b^2 - a^2)$$

or,
$$B \le \frac{2\,mb}{b^2 - a^2}\sqrt{\frac{2|e|V}{m}} \times \frac{1}{|e|}$$

or,
$$B \le \frac{2b}{b^2 - a^2}\sqrt{\frac{2m\,B}{|e|}}$$

3.395 The equations are as in 3.392.

$$\frac{dv_x}{dt} = \frac{qB}{m} v_y,\ \frac{dv_y}{dt} = \frac{qE_m}{m}\cos\omega t - \frac{qB}{m} v_x \text{ and } \frac{dv_z}{dt}$$

with
$$\omega = \frac{qB}{m},\ \xi = v_x + i v_y, \text{ we get,}$$

$$\frac{d\xi}{dt} = i\frac{E_m}{B}\omega\cos\omega t - i\,\omega\,\xi$$

or multiplying by $e^{i\omega t}$,

$$\frac{d}{dt}(\xi\, e^{i\omega t}) = i\frac{E_m}{2B}\omega\,(e^{2i\omega t} + 1)$$

or integrating,
$$\xi\, e^{i\omega t} = \frac{E_m}{4B} e^{2i\omega t} + \frac{E_m}{2B} i\,\omega t$$

or,
$$\xi = \frac{E_m}{4B}(e^{i\omega t} + 2\,i\,\omega t\, e^{i\omega t}) + C\, e^{i\omega t}$$

since
$$\xi = 0 \text{ at } t = 0,\ C = -\frac{E_m}{4B}.$$

Thus,
$$\xi = i\frac{E_m}{2B}\sin\omega t + i\frac{E_m}{2B}\omega t\, e^{i\omega t}$$

or,
$$v_x = \frac{E_m}{2B}\omega t \sin\omega t \text{ and } v_y = \frac{E_m}{2B}\sin\omega t + \frac{E_m}{2B}\omega t\cos\omega t$$

Integrating again,

$$x = \frac{a}{2\,\omega^2}(\sin\omega t - \omega t\cos\omega t),\ y = \frac{a}{2\omega} t\sin\omega t.$$

where $a = \frac{qE_m}{m}$, and we have used $x = y = 0$, at $t = 0$.

The trajectory is an unwinding spiral.

3.396 We know that for a charged particle (proton) in a magnetic field,

$$\frac{mv^2}{r} = B\,ev \text{ or } mv = Ber$$

But, $$\omega = \frac{eB}{m},$$

Thus $$E = \frac{1}{2}mv^2 = \frac{1}{2}m\,\omega^2 r^2.$$

So, $$\Delta E = m\omega^2 r\,\Delta r = 4\,\pi^2 \nu^2 mr\,\Delta r$$

On the other hand $\Delta E = 2\,eV$, where V is the effective acceleration voltage, across the Dees, there being two crossings per revolution. So,

$$V \geq 2\,\pi^2 \nu^2 mr\,\Delta r/e$$

3.397 (a) From $\frac{mv^2}{r} = Bev$, or, $mv = Ber$

and $$T = \frac{(Ber)^2}{2m} = \frac{1}{2}mv^2 = 12\,\text{MeV}$$

(b) From $\frac{2\pi}{\omega} = \frac{2\pi\,r}{v}$

we get, $$f_{\min} = \frac{v}{2\,\pi r} = \frac{1}{\pi r}\sqrt{\frac{T}{2m}} = 15\,\text{MHz}$$

3.398 (a) The total time of acceleration is,

$$t = \frac{1}{2\,\nu}\cdot n,$$

where n is the number of passages of the Dees.

But, $$T = neV = \frac{B^2 e^2 r^2}{2m}$$

or, $$n = \frac{B^2 e\,r^2}{2mV}$$

So, $$t = \frac{\pi}{eB/m}\times\frac{B^2 er^2}{2mV} = \frac{\pi\,B\,r^2}{2V} = \frac{\pi^2\,m\nu\,r^2}{eV} = 30\,\mu\,\text{s}$$

(b) The distance covered is, $s = \sum v_n \cdot \frac{1}{2\,\nu}$

But, $$v_n = \sqrt{\frac{2eV}{m}}\,\sqrt{n},$$

So, $$s = \sqrt{\frac{eV}{2m\nu^2}}\sum\sqrt{n} = \sqrt{\frac{eV}{2m\nu^2}}\int\sqrt{n}\,dn = \sqrt{\frac{eV}{2m\nu^2}}\,\frac{2}{3}n^{3/2}$$

But, $$n = \frac{B^2 e^2 r^2}{2\, eV m} = \frac{2\,\pi^2 m\, \nu^2 r^2}{eV}$$

Thus, $$s = \frac{4\,\pi^3 \nu^2 m r^2}{3eV} = 1{\cdot}24\ \text{km}$$

3.399 In the nth orbit, $\frac{2\,\pi\, r_n}{v_n} = n\ T_0 = \frac{n}{\nu}\cdot$ We ignore the rest mass of the electron and write $v_n \approx c$. Also $W \approx cp = cBer_n$.

Thus, $$\frac{2\,\pi\, W}{Bec^2} = \frac{n}{\nu}$$

or, $$n = \frac{2\,\pi\, W\nu}{Bec^2} = 9$$

3.400 The basic condition is the relativistic equation,

$$\frac{mv^2}{r} = Bqv, \quad \text{or,} \quad mv = \frac{m_0 v}{\sqrt{1 - v^2/c^2}} = Bqr.$$

Or calling, $$\omega = \frac{Bq}{m},$$

we get, $$\omega = \frac{\omega_0}{\sqrt{1 + \frac{\omega_0^2 r^2}{c^2}}}, \quad \omega_0 = \frac{Bq}{m_0} r$$

is the radius of the instantaneous orbit.

The time of acceleration is,

$$t = \sum_{n=1}^{N} \frac{1}{2\nu_n} = \sum_{n=1}^{N} \frac{\pi}{\omega_n} = \sum_{n}^{N} \frac{\pi\, W_n}{q\, Bc^2}.$$

N is the number of crossing of either Dee.

But, $W_n = m_0\, c^2 + \frac{n\, \Delta W}{2}$, there being two crossings of the Dees per revolution.

So, $$t = \sum \frac{\pi\, m_0 c^2}{qB\, c^2} + \sum \frac{\pi\, \Delta W_n}{2q\, B\, c^2}$$

$$= N \frac{\pi}{\omega_0} + \frac{N\,(N+1)}{4} \frac{\pi\, \Delta\, W}{qB\, c^2} \approx N^2 \frac{\pi\, \Delta W}{4\, q\, Bc^2}\ (N >> 1)$$

Also, $$r = r_N \frac{v_N}{\omega_N} = \frac{c}{\pi} \frac{\partial t}{\partial N} = \frac{\Delta W}{2q\, Bc} N$$

Hence finally, $$\omega = \frac{\omega_0}{\sqrt{1 + \frac{q^2 B^2}{m_0^2 c^2} \times \frac{\Delta W^2}{4 q^2 B^2 c^2} N^2}}$$

$$= \frac{\omega_0}{\sqrt{1 + \frac{(\Delta W)^2}{4 m_0^2 c^4} \times \frac{4 qBc^2}{\pi \Delta W} t}} = \frac{\omega_0}{\sqrt{1 + at}};$$

$$a = \frac{q B \Delta W}{\pi m_0^2 c^2}$$

3.401 When the magnetic field is being set up in the solenoid, and electric field will be induced in it, this will accelerate the charged particle. If $\dot{B}$ is the rate, at which the magnetic field is increasing, then.

$$\pi r^2 \dot{B} = 2 \pi r E \quad \text{or} \quad E = \frac{1}{2} r \dot{B}$$

Thus, $$m \frac{dv}{dt} = \frac{1}{2} r \dot{B} q, \quad \text{or} \quad v = \frac{qBr}{2m},$$

After the field is set up, the particle will execute a circular motion of radius ρ, where

$$mv = B q \rho, \quad \text{or} \quad \rho = \frac{1}{2} r$$

3.402 The increment in energy per revolution is $e \Phi$, so the number of revolutions is,

$$N = \frac{W}{e \Phi}$$

The distance traversed is, $s = 2 \pi r N$

3.403 On the one hand,

$$\frac{dp}{dt} = eE = \frac{e}{2\pi r} \frac{d\Phi}{dt} = \frac{e}{2\pi r} \frac{d}{dt} \int_0^r 2\pi r' B(r')\, dr'$$

On the other ,

$$p = B(r)\, er, \quad r = \text{constant}.$$

so, $$\frac{dp}{dt} = er \frac{d}{dt} B(r) = er \dot{B}(r)$$

Hence, $$er \dot{B}(r) = \frac{e}{2 \pi r} \pi r^2 \frac{d}{dt} \langle B \rangle$$

So, $$\dot{B}(r) = \frac{1}{2} \frac{d}{dt} \langle B \rangle$$

This equations is most easily satisfied by taking $B(r_0) = \frac{1}{2} \langle B \rangle$.

3.404 The condition, $B(r_0) = \frac{1}{2} \langle B \rangle = \frac{1}{2} \int_0^{r_0} B \cdot 2\pi r\, dr / \pi r_0^2$

or, $$B(r_0) = \frac{1}{r_0^2}\int_0^{r_0} Brdr$$

This gives r_0.

In the present case,

$$B_0 - ar_0^2 = \frac{1}{r_0^2}\int_0^{r_0}(B - ar^2)\, rdr = \frac{1}{2}\left(B_0 - \frac{1}{2}ar_0^2\right)$$

or, $$\frac{3}{4}ar_0^2 = \frac{1}{2}B_0 \quad \text{or} \quad r_0 = \sqrt{\frac{2B_0}{3a}}\,.$$

3.405 The induced electric field (or eddy current field) is given by,

$$E(r) = \frac{1}{2\pi r}\frac{d}{dt}\int_0^r 2\,\pi r'\,(r')\,B(r')\,dr'$$

Hence,

$$\frac{dE}{dr} = -\frac{1}{2\pi r^2}\frac{d}{dt}\int_0^r 2\pi r' B(r')\,dr' + \frac{dB(r)}{dt}$$

$$= -\frac{1}{2}\frac{d}{dt}< B > + \frac{dB(r)}{dt}$$

This vanishes for $r = r_0$ by the betatron condition, where r_0 is the radius of the equilibrium orbit.

3.406 From the betatron condition,

$$\frac{1}{2}\frac{d}{dt}< B > = \frac{dB}{dt}(r_0) = \frac{B}{\Delta t}$$

Thus, $$\frac{d}{dt}< B > = \frac{2B}{\Delta t}$$

and $$\frac{d\Phi}{dt} = \pi r^2\frac{d < B >}{dt} = \frac{2\,\pi r^2 B}{\Delta t}$$

So, energy increment per revolution is,

$$e\frac{d\,\Phi}{dt} = \frac{2\,\pi\, r^2 eB}{\Delta t}$$

3.407 (a) Even in the relativistic case, we know that : $p = Ber$

Thus, $W = \sqrt{c^2p^2 + m_0^2c^4} - m_0c^2 = m_0\,c^2\left(\sqrt{1 + (Ber/m_0c)^2} - 1\right)$

(b) The distance traversed is,

$$2\pi r\frac{W}{e\,\Phi} = 2\pi r\frac{W}{2\pi r^2 eB/\Delta t} = \frac{W\,\Delta t}{Ber},$$

on using the result of the previous problem.