PART THREE

ELECTRODYNAMICS

3.1 CONSTANT ELECTRIC FIELD IN VACUUM

3.1 F_{el} (for electorns) $= \dfrac{q^2}{4\pi\varepsilon_0 r^2}$ and $F_{gr} = \dfrac{\gamma m^2}{r^2}$

Thus $\dfrac{F_{el}}{F_{gr}}$ (for electrons) $= \dfrac{q^2}{4\pi\varepsilon_0 \gamma m^2}$

$$= \frac{(1{\cdot}602\times10^{-19}\,\text{C})^2}{\left(\dfrac{1}{9\times10^9}\right)\times 6{\cdot}67\times10^{-11}\,\text{m}^3/(\text{kg}\cdot\text{s}^2)\times(9{\cdot}11\times10^{-31}\,\text{kg})^2} = 4\times10^{42}$$

Similarly $\dfrac{F_{el}}{F_{gr}}$ (for proton) $= \dfrac{q^2}{4\pi\varepsilon_0\gamma m^2}$

$$= \frac{(1{\cdot}602\times10^{-19}\,\text{C})^2}{\left(\dfrac{1}{9\times10^9}\right)\times 6{\cdot}67\times10^{-11}\,\text{m}^3/(\text{kg}\cdot\text{s}^2)\times(1{\cdot}672\times10^{-27}\,\text{kg})^2} = 1\times10^{36}$$

For $F_{el} = F_{gr}$

$$\frac{q^2}{4\pi\varepsilon_0 r^2} = \frac{\gamma m^2}{r^2} \quad\text{or}\quad \frac{q}{m} = \sqrt{4\pi\varepsilon_0\gamma}$$

$$= \sqrt{\frac{6{\cdot}67\times10^{-11}\,\text{m}^3\,(\text{kg}-\text{s}^2)}{9\times10^9}} = 0{\cdot}86\times10^{-10}\,\text{C/kg}$$

3.2 Total number of atoms in the sphere of mass 1 gm $= \dfrac{1}{63{\cdot}54}\times 6{\cdot}023\times10^{23}$

So the total nuclear charge $\lambda = \dfrac{6{\cdot}023\times10^{23}}{63{\cdot}54}\times1{\cdot}6\times10^{-19}\times29$

Now the charge on the sphere = Total nuclear charge – Total electronic charge

$$= \frac{6{\cdot}023 \times 10^{23}}{63{\cdot}54} \times 1{\cdot}6 \times 10^{-19} \times \frac{29 \times 1}{100} = 4{\cdot}298 \times 10^{2}\,\mathrm{C}$$

Hence force of interaction between these two spheres,

$$F = \frac{1}{4\pi\varepsilon_0} \cdot \frac{[4{\cdot}398 \times 10^2]^2}{1^2}\,\mathrm{N} = 9 \times 10^9 \times 10^4 \times 19{\cdot}348\,\mathrm{N} = 1{\cdot}74 \times 10^{15}\,\mathrm{N}$$

3.3 Let the balls be deviated by an angle θ, from the vertical when separtion between them equals x.

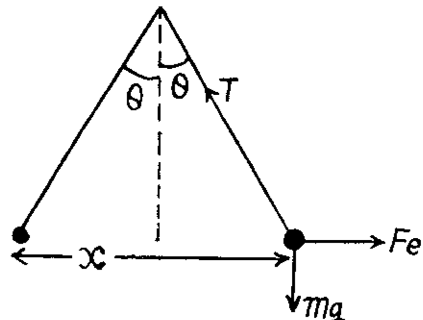
Applying Newton's second law of motion for any one of the sphere, we get,

$$T\cos\theta = mg \qquad (1)$$

and $$T\sin\theta = F_e \qquad (2)$$

From the Eqs. (1) and (2)

$$\tan\theta = \frac{F_e}{mg} \qquad (3)$$



But from the figure

$$\tan\theta = \frac{x}{2\sqrt{l^2 - \left(\frac{x}{2}\right)^2}} \approx \frac{x}{2l} \text{ as } x << l \qquad (4)$$

From Eqs. (3) and (4)

$$F_e = \frac{mg\,x}{2l} \quad \text{or} \quad \frac{q^2}{4\pi\varepsilon_0 x^2} = \frac{mg\,x}{2l}$$

Thus $$q^2 = \frac{2\pi\varepsilon_0\, mg\, x^3}{l} \qquad (5)$$

Differentiating Eqn. (5) with respect to time

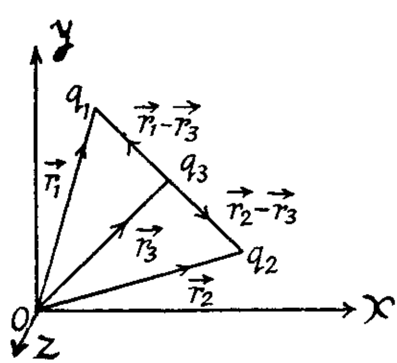
$$2q\frac{dq}{dt} = \frac{2\pi\varepsilon_0\, mg}{l} 3x^2 \frac{dx}{dt}$$

According to the problem $\frac{dx}{dt} = v = a/\sqrt{x}$ (approach velocity is $\frac{dx}{dt}$)

so, $$\left(\frac{2\pi\varepsilon_0\, mg}{l}x^3\right)^{1/2} \frac{dq}{dt} = \frac{3\pi\varepsilon_0\, mg}{l} x^2 \frac{a}{\sqrt{x}}$$

Hence, $$\frac{dq}{dt} = \frac{3}{2} a \sqrt{\frac{2\pi\varepsilon_0\, mg}{l}}$$

3.4 Let us choose coordinate axes as shown in the figure and fix three charges, q_1, q_2 and q_3 having position vectors $\vec{r}_1$, $\vec{r}_2$ and $\vec{r}_3$ respectively.



Now, for the equilibrium of q_3

$$\frac{+\,q_2 q_3 (\vec{r}_2 - \vec{r}_3)}{|\vec{r}_2 - \vec{r}_3|^3} + \frac{q_1 q_3 (\vec{r}_1 - \vec{r}_3)}{|\vec{r}_1 - \vec{r}_3|^3} = 0$$

or, $$\frac{q_2}{|\vec{r_2}-\vec{r_3}|^2} = \frac{q_1}{|\vec{r_1}-\vec{r_3}|^2}$$

because $$\frac{\vec{r_2}-\vec{r_3}}{|\vec{r_2}-\vec{r_3}|} = -\frac{\vec{r_1}-\vec{r_3}}{|\vec{r_1}-\vec{r_3}|}$$

or, $$\sqrt{q_2}\,(\vec{r_1}-\vec{r_3}) = \sqrt{q_1}\,(\vec{r_3}-\vec{r_2})$$

or, $$\vec{r_3} = \frac{\sqrt{q_2}\,\vec{r_1}+\sqrt{q_1}\,\vec{r_2}}{\sqrt{q_1}+\sqrt{q_2}}$$

Also for the equilibrium of q_1,

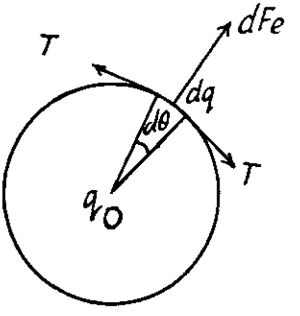
$$\frac{q_3(\vec{r_3}-\vec{r_1})}{|\vec{r_3}-\vec{r_1}|^3} + \frac{q_2(\vec{r_2}-\vec{r_1})}{|\vec{r_2}-\vec{r_1}|^3} = 0$$

or, $$q_3 = \frac{-q_2}{|\vec{r_2}-\vec{r_1}|^2}|\vec{r_1}-\vec{r_3}|^2$$

Substituting the value of $\vec{r_3}$, we get,

$$q_3 = \frac{-q_1 q_2}{(\sqrt{q_1}+\sqrt{q_2})^2}$$

3.5 When the charge q_0 is placed at the centre of the ring, the wire get stretched and the extra tension, produced in the wire, will balance the electric force due to the charge q_0. Let the tension produced in the wire, after placing the charge q_0, be T. From Newton's second law in projection form $F_n = mw_n$.



$$T d\theta - \frac{1}{4\pi\varepsilon_0}\frac{q_0}{r^2}\left(\frac{q}{2\pi r}r\,d\theta\right) = (dm)\,0,$$

or, $$T = \frac{q\,q_0}{8\pi^2\varepsilon_0 r^2}$$

3.6 Sought field strength

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{|\vec{r}-\vec{r_0}|^2}$$

$= 4{\cdot}5$ kV/m on putting the values.

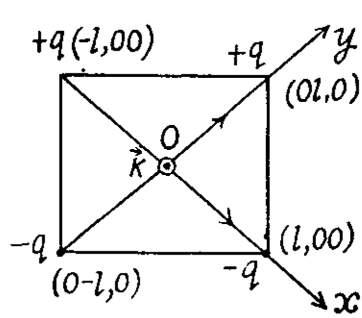
3.7 Let us fix the coordinate system by taking the point of intersection of the diagonals as the origin and let $\vec{k}$ be directed normally, emerging from the plane of figure. Hence the sought field strength :

$$\vec{E} = \frac{q}{4\pi\varepsilon_0}\frac{l\vec{i}+x\vec{k}}{(l^2+x^2)^{3/2}} + \frac{-q}{4\pi\varepsilon_0}\frac{l(-\vec{i})+x\vec{k}}{(l^2+x^2)^{3/2}}$$

$$+\frac{-q}{4\pi\varepsilon_0}\frac{l\vec{j}+x\vec{k}}{(l^2+x^2)^{3/2}} + \frac{q}{4\pi\varepsilon_0}\frac{l(-\vec{j})+x\vec{k}}{(l^2+x^2)^{3/2}}$$

$$= \frac{q}{4\pi\varepsilon_0(l^2+x^2)^{3/2}}[2l\vec{i} - 2l\vec{j}]$$

Thus $E = \dfrac{ql}{\sqrt{2}\,\pi\varepsilon_0(l^2+x^2)^{3/2}}$



3.8 From the symmetry of the problem the sought field.

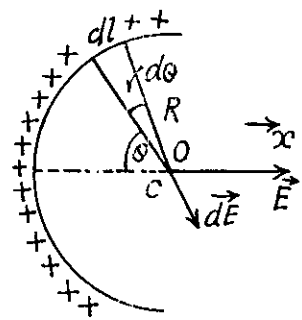
$$E = \int dE_x$$

where the projection of field strength along x – axis due to an elemental charge is

$$dE_x = \frac{dq\cos\theta}{4\pi\varepsilon_0 R^2} = \frac{qR\cos\theta\, d\theta}{4\pi^2\varepsilon_0 R^3}$$

Hence

$$E = \frac{q}{4\pi^2\varepsilon_0 R^2}\int_{\pi/2}^{\pi/2}\cos\theta\, d\theta \;\; \frac{q}{2\pi^2\varepsilon_0 R^2}$$



3.9 From the symmetry of the condition, it is clear that, the field along the normal will be zero

i.e. $E_n = 0$ and $E = E_l$

Now $dE_l = \dfrac{dq}{4\pi\varepsilon_0(R^2+l^2)}\cos\theta$

But $dq = \dfrac{q}{2\pi R}dx$ and $\cos\theta = \dfrac{l}{(R^2+l^2)^{1/2}}$

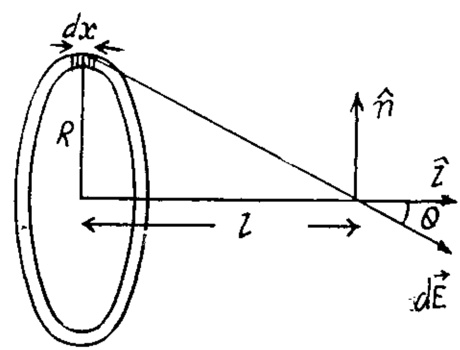
Hence

$$E = \int dE_l = \int_0^{2\pi R}\frac{ql}{2\pi R}\cdot\frac{dx}{4\pi\varepsilon_0(R^2+l^2)^{3/2}}$$

or $E = \dfrac{1}{4\pi\varepsilon_0}\dfrac{ql}{(l^2+R^2)^{3/2}}$

and for $l >> R$, the ring behaves like a point charge, reducing the field to the value,

$$E \approx \frac{1}{4\pi\varepsilon_0}\frac{q}{l^2}$$



For E_{max}, we should have $\frac{dE}{dl} = 0$

So, $(l^2+R^2)^{3/2} - \frac{3}{2} l (l^2+R^2)^{1/2} 2l = 0$ or $l^2 + R^2 - 3l^2 = 0$

Thus $l = \frac{R}{\sqrt{2}}$ and $E_{max} = \frac{q}{6\sqrt{3}\,\pi \varepsilon_0 R^2}$

3.10 The electric potential at a distance x from the given ring is given by,

$$\varphi(x) = \frac{q}{4\pi\varepsilon_0 x} - \frac{q}{4\pi\varepsilon_0 (R^2+x^2)^{1/2}}$$

Hence, the field strength along x-axis (which is the net field strength in our case),

$$E_x = -\frac{d\varphi}{dx} = \frac{q}{4\pi\varepsilon_0}\frac{1}{x^2} - \frac{qx}{4\pi\varepsilon_0 (R^2+x^2)^{3/2}}$$

$$= \frac{\frac{q}{4\pi\varepsilon_0} x^3 \left[\left(1+\frac{R^2}{x^2}\right)^{3/2} - 1\right]}{x^2 (R^2+x^2)^{3/2}}$$

$$= \frac{\frac{q}{4\pi\varepsilon_0} x^3 \left[1 + \frac{3}{2}\frac{R^2}{x^2} + \frac{3}{8}\frac{R^4}{x^4} + \dots\right]}{x^2 (R^2+x^2)^{3/2}}$$

Neglecting the higher power of R/x, as $x >> R$.

$$E = \frac{3qR^2}{8\pi\varepsilon_0 x^4}.$$

Note : Instead of $\varphi(x)$, we may write $E(x)$ directly using 3.9

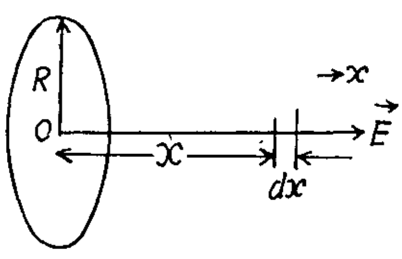
3.11 From the solution of 3.9, the electric field strength due to ring at a point on its axis (say x-axis) at distance x from the centre of the ring is given by :

$$E(x) = \frac{qx}{4\pi\varepsilon_0 (R^2+x^2)^{3/2}}$$

And from symmetry $\vec{E}$ at every point on the axis is directed along the x–axis (Fig.).

Let us consider an element (dx) on thread which carries the charge $(\lambda\, dx)$. The electric force experienced by the element in the field of ring.

$$dF = (\lambda\, dx) E(x) = \frac{\lambda q x\, dx}{4\pi\varepsilon_0 (R^2+x^2)^{3/2}}$$



Thus the sought interaction

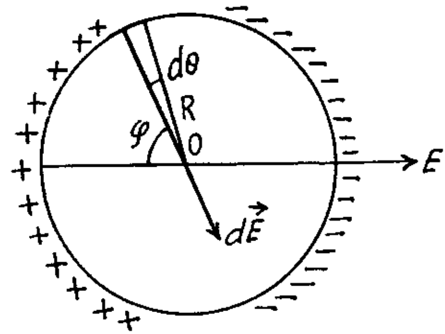
$$F = \int_0^\infty \frac{\lambda q x\, dx}{4\pi\varepsilon_0 (R^2+x^2)^{3/2}}$$

On integrating we get, $F = \frac{\lambda q}{4\pi\varepsilon_0 R}$

3.12 (a) The given charge distribution is shown in Fig. The symmetry of this distribution implies that vector $\vec{E}$ at the point O is directed to the right, and its magnitude is equal to the sum of the projection onto the direction of $\vec{E}$ of vectors $d\vec{E}$ from elementary charges dq. The projection of vector $d\vec{E}$ onto vector $\vec{E}$ is

$$dE \cos\varphi = \frac{1}{4\pi\varepsilon_0} \frac{dq}{R^2} \cos\varphi,$$

where $dq = \lambda R\, d\varphi = \lambda_0\, R \cos\varphi\, d\varphi$.



Integrating (1) over φ between 0 and 2π we find the magnitude of the vector E:

$$E = \frac{\lambda_0}{4\pi\varepsilon_0 R} \int_0^{2\pi} \cos^2\varphi\, d\varphi = \frac{\lambda_0}{4\varepsilon_0 R}.$$

It should be noted that this integral is evaluated in the most simple way if we take into account that $\langle\cos^2\varphi\rangle = 1/2$. Then

$$\int_0^{2\pi} \cos^2\varphi\, d\varphi = \langle\cos^2\varphi\rangle\, 2\pi = \pi.$$

(b) Take an element S at an azimuthal angle φ from the x-axis, the element subtending an angle $d\varphi$ at the centre.

The elementary field at P due to the element is

$$\frac{\lambda_0 \cos\varphi\, d\varphi\, R}{4\pi\varepsilon_0 (x^2 + R^2)} \text{ along } SP \text{ with components}$$

$$\frac{\lambda_0 \cos\varphi\, d\varphi\, R}{4\pi\varepsilon_0 (x^2 + R^2)} \times \{\cos\theta \text{ along } OP, \sin\theta \text{ along } OS\}$$

where
$$\cos\theta = \frac{x}{(x^2 + R^2)^{1/2}}$$

The component along OP vanishes on integration as $\int_0^{2\pi} \cos\varphi\, d\varphi = 0$

The component alon OS can be broken into the parts along OX and OY with

$$\frac{\lambda_0 R^2 \cos\varphi\, d\varphi}{4\pi\varepsilon_0 (x^2 + R^2)^{3/2}} \times \{\cos\varphi \text{ along } OX, \sin\varphi \text{ along } OY\}$$

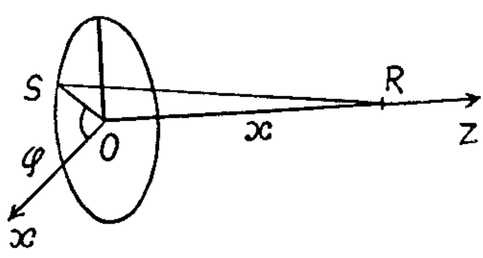
On integration, the part along OY vanishes.

Finally

$$E = E_x = \frac{\lambda_0 R^2}{4\varepsilon_0 (x^2 + R^2)^{3/2}}$$

For $x \gg R$

$$E_x = \frac{p}{4\pi\varepsilon_0 x^3} \text{ where } p = \lambda_0 \pi R^2$$



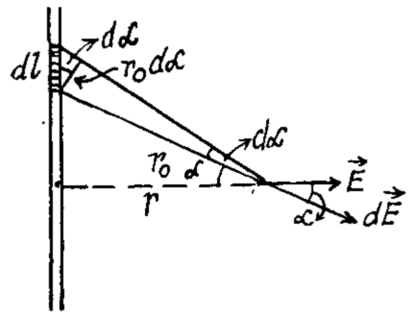
3.13 (a) It is clear from symmetry considerations that vector $\vec{E}$ must be directed as shown in the figure. This shows the way of solving this problem : we must find the component dE_r of the field created by the element dl of the rod, having the charge dq and then integrate the result over all the elements of the rod. In this case

$$dE_r = dE \cos\alpha = \frac{1}{4\pi\varepsilon_0}\frac{\lambda\, dl}{r_0^2}\cos\alpha,$$

where $\lambda = \frac{q}{2a}$ is the linear charge density. Let us reduce this equation of the form convenient for integration. Figure shows that $dl\cos\alpha = r_0\, d\alpha$ and $r_0 = \frac{r}{\cos\alpha}$;

Consequently,

$$dE_r = \frac{1}{4\pi\varepsilon_0}\frac{\lambda\, r_0\, d\alpha}{r_0^2} = \frac{\lambda}{4\pi\varepsilon_0 r}\cos\alpha\, d\alpha$$



This expression can be easily integrated :

$$E = \frac{\lambda}{4\pi\varepsilon_0 r}\, 2\int_0^{\alpha_0}\cos\alpha\, d\alpha = \frac{\lambda}{4\pi\varepsilon_0 r}\, 2\sin\alpha_0$$

where α_0 is the maximum value of the angle α,

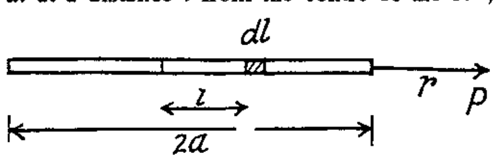
$$\sin\alpha_0 = a\Big/\sqrt{a^2+r^2}$$

Thus, $E = \frac{q/2a}{4\pi\varepsilon_0 r}\, 2\frac{a}{\sqrt{a^2+r^2}} = \frac{q}{4\pi\varepsilon_0 r\sqrt{a^2+r^2}}$

Note that in this case also $E \approx \frac{q}{4\pi\varepsilon_0 r^2}$ *for* $r >> a$ *as of the field of a point charge.*

(b) Let, us consider the element of length dl at a distance l from the centre of the rod, as shown in the figure.

Then field at P, due to this element.



$$dE = \frac{\lambda\, dl}{4\pi\varepsilon_0 (r-l)^2},$$

if the element lies on the side, shown in the diagram, and $dE = \frac{\lambda\, dl}{4\pi\varepsilon_0 (r+l)^2}$, if it lies on other side.

Hence $$E = \int dE = \int_0^a \frac{\lambda\, dl}{4\pi\varepsilon_0 (r-l)^2} + \int_0^a \frac{\lambda\, dl}{4\pi\varepsilon_0 (r+l)^2}$$

On integrating and putting $\lambda = \frac{q}{2a}$, we get, $E = \frac{q}{4\pi\varepsilon_0}\frac{1}{(r^2-a^2)}$

For $$r >> a, \quad E \approx \frac{q}{4\pi\varepsilon_0 r^2}$$

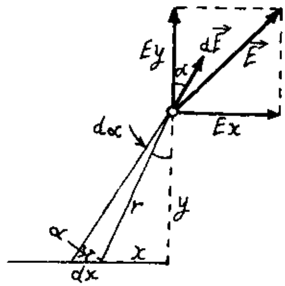
3.14 The problem is reduced to finding E_x and E_y viz. the projections of $\vec{E}$ in Fig, where it is assumed that $\lambda > 0$.

Let us start with E_x. The contribution to E_x from the charge element of the segment dx is

$$dE_x = \frac{1}{4\pi\varepsilon_0}\frac{\lambda\, dx}{r^2}\sin\alpha \qquad (1)$$

Let us reduce this expression to the form convenient for integration. In our case, $dx = r\, d\alpha/\cos\alpha$, $r = y/\cos\alpha$. Then

$$dE_x = \frac{\lambda}{4\pi\varepsilon_0 y}\sin\alpha\, d\alpha.$$



Integrating this expression over α between θ and $\pi/2$, we find

$$E_x = \lambda/4\pi\varepsilon_0 y.$$

In order to find the projection E_y it is sufficient to recall that dE_y differs from dE_x in that $\sin\alpha$ in (1) is simply replaced by $\cos\alpha$.

This gives

$dE_y = (\lambda\cos\alpha\, d\alpha)/4\pi\varepsilon_0 y$ and $E_y = \lambda/4\pi\varepsilon_0 y$.

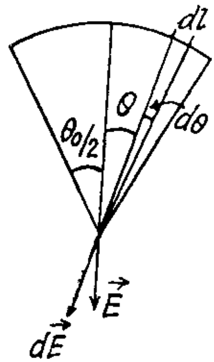
We have obtained an interesting result :

$E_x = E_y$ independently of y,

i.e. $\vec{E}$ is oriented at the angle of 45° to the rod. The modulus of $\vec{E}$ is

$$E = \sqrt{E_x^2 + E_y^2} = \lambda\sqrt{2}\,/\,4\pi\varepsilon_0 y.$$

3.15 (a) Using the solution of 3.14, the net electric field strength at the point O due to straight parts of the thread equals zero. For the curved part (arc) let us derive a general expression i.e. let us calculate the field strength at the centre of arc of radius R and linear charge density λ and which subtends angle θ_0 at the centre.



From the symmetry the sought field strength will be directed along the bisector of the angle θ_0 and is given by

$$E = \int_{-\theta_0/2}^{+\theta_0/2}\frac{\lambda\,(R\, d\theta)}{4\pi\varepsilon_0 R^2}\cos\theta = \frac{\lambda}{2\pi\varepsilon_0 R}\sin\frac{\theta_0}{2}$$

In our problem $\theta_0 = \pi/2$, thus the field strength due to the turned part at the point

$E_0 = \dfrac{\sqrt{2}\,\lambda}{4\pi\varepsilon_0 R}$ which is also the sought result.

(b) Using the solution of 3.14 (a), net field strength at O due to stright parts equals $\sqrt{2}\left(\dfrac{\sqrt{2}\,\lambda}{4\pi\varepsilon_0 R}\right) = \dfrac{\lambda}{2\pi\varepsilon_0 R}$ and is directed vertically down. Now using the solution of 3.15

(a), field strength due to the given curved part (semi-circle) at the point O becomes $\frac{\lambda}{2\pi\varepsilon_0 R}$ and is directed vertically upward. Hence the sought net field strengh becomes zero.

3.16 Given charge distribution on the surface $\sigma = \vec{a}\cdot\vec{r}$ is shown in the figure. Symmetry of this distribution implies that the sought $\vec{E}$ at the centre O of the sphere is opposite to $\vec{a}$

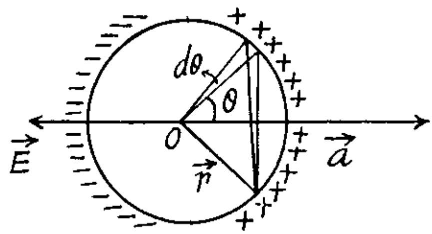
$dq = \sigma\,(2\pi r\sin\theta)\,r\,d\theta = (\vec{a}\cdot\vec{r})\,2\pi r^2\sin\theta\,d\theta = 2\pi a r^3\sin\theta\cos\theta\,d\theta$

Again from symmetry, field strength due to any ring element $d\vec{E}$ is also opposite to $\vec{a}$ i.e. $d\vec{E}\uparrow\downarrow\vec{a}$. Hence

$$d\vec{E} = \frac{dq\,r\cos\theta}{4\pi\varepsilon_0(r^2\sin^2\theta + r^2\cos^2\theta)^{3/2}}\frac{-\vec{a}}{a} \text{(Using the result of 3.9)}$$

$$= \frac{(2\pi a r^3\sin\theta\cos\theta\,d\theta)\,r\cos\theta}{4\pi\varepsilon_0 r^3}\frac{(-\vec{a})}{a}$$

$$= \frac{-\vec{a}\,r}{2\varepsilon_0}\sin\theta\cos^2 d\theta$$



Thus

$$\vec{E} = \int d\vec{E} = \frac{(-\vec{a})\,r}{2\varepsilon_0}\int_0^\pi \sin\theta\cos^2\theta\,d\theta$$

Integrating, we get $\vec{E} = -\frac{\vec{a}\,r}{2\varepsilon_0}\frac{2}{3} = -\frac{\vec{a}\,r}{3\varepsilon_0}$

3.17 We start from two charged spherical balls each of radius R with equal and opposite charge densities $+\rho$ and $-\rho$. The centre of the balls are at $+\frac{\vec{a}}{2}$ and $-\frac{\vec{a}}{2}$ respectively so the equation of their surfaces are $\left|\vec{r} - \frac{\vec{a}}{2}\right| = R$ or $r - \frac{a}{2}\cos\theta \approx R$ and $r + \frac{a}{2}\cos\theta \approx R$, considering a to be small. The distance between the two surfaces in the radial direction at angle θ is $|\,a\cos\theta\,|$ and does not depend on the azimuthal angle. It is seen from the diagram that the surface of the sphere has in effect a surface density $\sigma = \sigma_0\cos\theta$ when

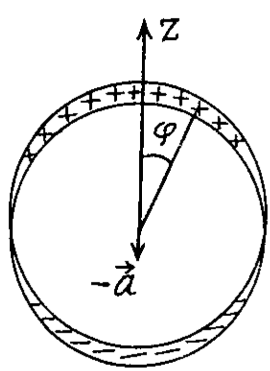
$$\sigma_0 = \rho a.$$

Inside any uniformly charged spherical ball, the field is radial and has the magnitude given by Gauss's theorm

$$4\pi r^2 E = \frac{4\pi}{3}r^3\rho/\varepsilon_0$$

or

$$E = \frac{\rho r}{3\varepsilon_0}$$



In vector notation, using the fact the V must be measured from the centre of the ball, we get, for the present case

$$\vec{E} = \frac{\rho}{3\varepsilon_0}\left(\vec{r} - \frac{\vec{a}}{2}\right) - \frac{\rho}{3\varepsilon_0}\left(\vec{r} + \frac{\vec{a}}{2}\right)$$

$$= -\rho\vec{a}/3\varepsilon_0 = \frac{\sigma_0}{3\varepsilon_0}\vec{k}$$

When $\vec{k}$ is the unit vector along the polar axis from which θ is measured.

3.18 Let us consider an elemental spherical shell of thickness dr. Thus surface charge density of the shell $\sigma = \rho\, dr = (\vec{a}\cdot\vec{r})\, dr$.

Thus using the solution of 3.16, field strength due to this sperical shell

$$d\vec{E} = -\frac{\vec{a}\,r}{3\,\varepsilon_0}\,dr$$

Hence the sought field strength

$$\vec{E} = -\frac{\vec{a}}{3\,\varepsilon_0}\int_0^R r\,dr = -\frac{\vec{a}R^2}{6\,\varepsilon_0}.$$

3.19 From the solution of 3.14 field strength at a perpendicular distance $r < R$ from its left end

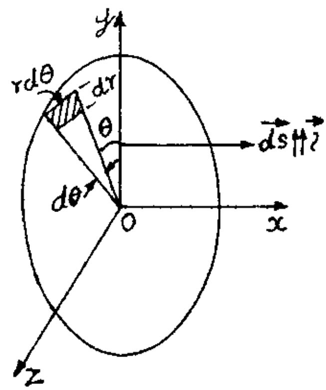
$$\vec{E}(r) = \frac{\lambda}{4\pi\varepsilon_0 r}(-\vec{i}) + \frac{\lambda}{4\pi\varepsilon_0 r}(\hat{e}_r)$$

Here $\hat{e}_r$ is a unit vector along radial direction.

Let us consider an elemental surface, $dS = dy\,dz = dz\,(r\,d\theta)$ a figure. Thus flux of $\vec{E}(r)$ over the element $d\vec{S}$ is given by

$$d\Phi = \vec{E}\cdot d\vec{S} = \left[\frac{\lambda}{4\pi\varepsilon_0 r}(-\vec{i}) + \frac{\lambda}{4\pi\varepsilon_0 r}(\hat{e}_r)\right]\cdot dr\,(r\,d\theta)\,\vec{i}$$

$$= -\frac{\lambda}{4\pi\varepsilon_0}dr\,d\theta\left(\text{as } \vec{e}_r \perp \vec{i}\right)$$



$$\text{The sought flux, } \Phi = -\frac{\lambda}{4\pi\varepsilon_0}\int_0^R dr\int_0^{2\pi} d\theta = -\frac{\lambda R}{2\,\varepsilon_0}.$$

If we have taken $d\vec{S}\uparrow\uparrow(-\vec{i})$, then Φ were $\frac{\lambda R}{2\,\varepsilon_0}$

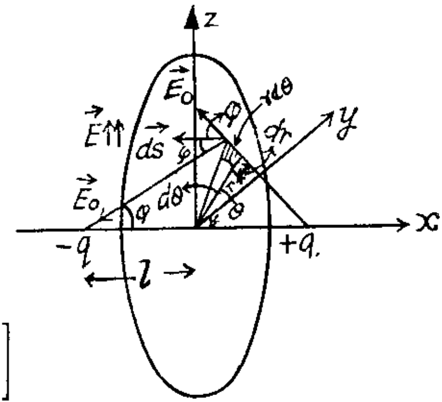
Hence $\qquad |\Phi| = \frac{\lambda R}{2\,\varepsilon_0}$

3.20 Let us consider an elemental surface area as shown in the figure. Then flux of the vector $\vec{E}$ through the elemental area,

$$d\Phi = \vec{E}\cdot d\vec{S} = E\,dS = 2\,E_0\cos\varphi\,dS \;(\text{as } \vec{E}\uparrow\uparrow d\vec{S})$$

$$= \frac{2\,q}{4\pi\varepsilon_0(l^2+r^2)}\,\frac{l}{(l^2+r^2)^{1/2}}(r\,d\theta)\,dr = \frac{2ql\,r\,dr\,d\theta}{4\pi\varepsilon_0(r^2+l^2)^{3/2}}$$

where $E_0 = \dfrac{q}{4\pi\varepsilon_0(l^2+r^2)}$ is magnitude of field strength due to any point charge at the point of location of considered elemental area.



$$\text{Thus } \Phi = \frac{2ql}{4\pi\varepsilon_0}\int_0^R \frac{r\,dr}{(r^2+l^2)^{3/2}}\int_0^{2\pi} d\theta$$

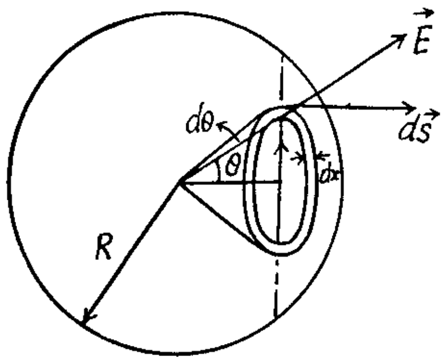
$$= \frac{2ql\times 2\pi}{4\pi\varepsilon_0}\int_0^R \frac{r\,dr}{(r^2+l^2)^{3/2}} = \frac{q}{\varepsilon_0}\left[1-\frac{l}{\sqrt{l^2+R^2}}\right]$$

It can also be solved by considering a ring element or by using solid angle.

3.21 Let us consider a ring element of radius x and thickness dx, as shown in the figure. Now, flux over the considered element,

$$d\Phi = \vec{E}\cdot\vec{dS} = E_r\,dS\cos\theta$$

But $E_r = \dfrac{\rho r}{3\varepsilon_0}$ from Gauss's theorem,



and $dS = 2\pi x\,dx$, $\cos\theta = \dfrac{r_0}{r}$

Thus $d\Phi = \dfrac{\rho r}{3\varepsilon_0}2\pi x\,dx\dfrac{r_0}{r} = \dfrac{\rho r_0}{3\varepsilon_0}2\pi x\,dx$

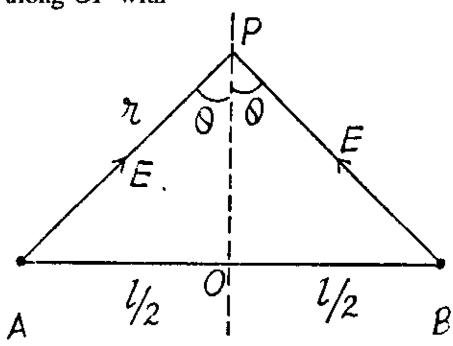
Hence sought flux

$$\Phi = \frac{2\pi\rho r_0}{3\varepsilon_0}\int_0^{\sqrt{R^2-r_0^2}} x\,dx = \frac{2\pi\rho r_0}{3\varepsilon_0}\frac{(R^2-r_0^2)}{2} = \frac{\pi\rho r_0}{3\varepsilon_0}(R^2-r_0^2)$$

3.22 The field at P due to the threads at A and B are both of magnitude $\dfrac{\lambda}{2\pi\varepsilon_0(x^2+l^2/4)^{1/2}}$

and directed along AP and BP. The resultant is along OP with

$$E = \frac{2\lambda\cos\theta}{2\pi\varepsilon_0(\pi^2+\pi^{1/2})^{1/2}} = \frac{\lambda x}{\pi\varepsilon_0(x^2+l^2/4)}$$

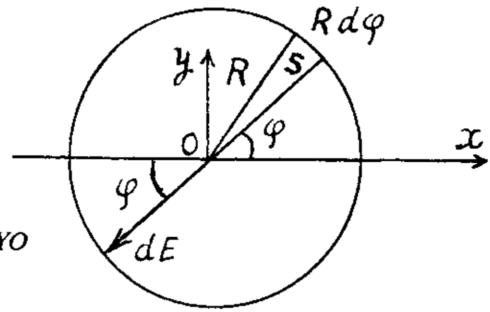


$$= \frac{\lambda}{\pi\varepsilon_0\left[x+\dfrac{l^2}{4x}-2\cdot\dfrac{l}{2\sqrt{x}}\cdot\sqrt{x}+l\right]}$$

$$= \frac{\lambda}{\pi\varepsilon_0\left[\left(\sqrt{x}-\dfrac{l}{2\sqrt{x}}\right)^2+l\right]}$$

This is maximum when $x = l/2$ and then $E = E_{max} = \dfrac{\lambda}{\pi\varepsilon_0 l}$

3.23 Take a section of the cylinder perpendicular to its axis through the point where the electric field is to be calculated. (All points on the axis are equivalent.) Consider an element S with azimuthal angle φ. The length of the element is $R\,d\varphi$, R being the radius of cross section of the cylinder. The element itself is a section of an infinite strip. The electric field at O due to this strip is



$$\frac{\sigma_0 \cos\varphi\,(R\,d\varphi)}{2\pi\varepsilon_0 R} \text{ along } SO$$

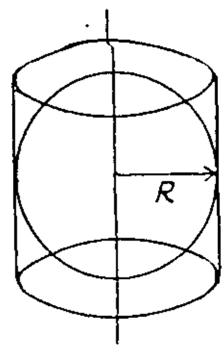
This can be resolved into

$$\frac{\sigma_0 \cos\varphi\,d\varphi}{2\pi\varepsilon_0}\begin{cases}\cos\varphi \text{ along } OX \text{ towards } O\\ \sin\varphi \text{ along } YO\end{cases}$$

On integration the component along YO vanishes. What remains is

$$\int_0^{2\pi} \frac{\sigma_0 \cos^2\varphi\,d\varphi}{2\pi\varepsilon_0} = \frac{\sigma_0}{2\varepsilon_0} \text{ along } XO \text{ i.e. along the direction } \varphi = \pi.$$

3.24 Since the field is axisymmetric (as the field of a uniformly charged filament), we conclude that the flux through the sphere of radius R is equal to the flux through the lateral surface of a cylinder having the same radius and the height $2R$, as arranged in the figure.



Now, $\Phi = \oint \vec{E}\cdot d\vec{S} = E_r S$

But $E_r = \dfrac{a}{R}$

Thus $\Phi = \dfrac{a}{R} S = \dfrac{a}{R} 2\pi R \cdot 2R = 4\pi a R$

3.25 (a) Let us consider a sphere of radius $r < R$ then charge, inclosed by the considered sphere,

$$q_{inclosed} = \int_0^r 4\pi r^2\,dr\,\rho = \int_0^r 4\pi r^2 \rho_0\left(1 - \frac{r}{R}\right)dr \qquad (1)$$

Now, applying Gauss' theorem,

$E_r\,4\pi r^2 = \dfrac{q_{inclosed}}{\varepsilon_0}$, (where E_r is the projection of electric field along the radial line.)

$$= \frac{\rho_0}{\varepsilon_0}\int_0^r 4\pi r^2\left(1 - \frac{r}{R}\right)dr$$

or,
$$E_r = \frac{\rho_0}{3\varepsilon_0}\left[r^2 - \frac{3r^2}{4R}\right]$$

And for a point outside the sphere $r > R$.

$$q_{inclosed} = \int_0^R 4\pi r^2 \, dr \, \rho_0 \left(1 - \frac{r}{R}\right) \text{(as there is no charge outside the ball)}$$

Again from Gauss' theorem,

$$E_r \, 4\pi r^2 = \int_0^R \frac{4\pi r^2 \, dr \, \rho_0 \left(1 - \frac{r}{R}\right)}{\varepsilon_0}$$

or,

$$E_r = \frac{\rho_0}{r^2 \varepsilon_0}\left[\frac{R^3}{3} - \frac{R^4}{4R}\right] = \frac{\rho_0 R^3}{12 r^2 \varepsilon_0}$$

(b) As magnitude of electric field decreases with increasing r for $r > R$, field will be maximum for $r < R$. Now, for E_r to be maximum,

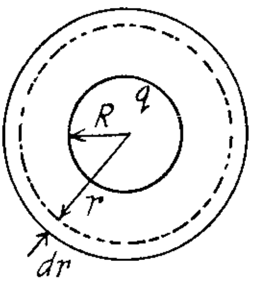
$$\frac{d}{dr}\left(r - \frac{3r^2}{4R}\right) = 0 \quad \text{or} \quad 1 - \frac{3r}{2R} = 0 \quad \text{or} \quad r = r_m = \frac{2R}{3}$$

Hence

$$E_{max} = \frac{\rho_0 R}{9\varepsilon_0}$$

3.26 Let the charge carried by the sphere be q, then using Gauss' theorem for a spherical surface having radius $r > R$, we can write.

$$E \, 4\pi r^2 = \frac{q_{inclosed}}{\varepsilon_0} = \frac{q}{\varepsilon_0} + \frac{1}{\varepsilon_0}\int_R^r \frac{\alpha}{r} 4\pi r^2 \, dr$$



On integrating we get,

$$E \, 4\pi r^2 = \frac{(q - 2\pi\alpha R^2)}{\varepsilon_0} + \frac{4\pi\alpha r^2}{2\varepsilon_0}$$

The intensity E does not depend on r when the experession in the parentheses is equal to zero. Hence

$$q = 2\pi\alpha R^2 \text{ and } E = \frac{\alpha}{2\varepsilon_0}$$

3.27 Let us consider a spherical layer of radius r and thickness dr, having its centre coinciding with the centre of the system. Then using Gauss' theorem for this surface,

$$E_r \, 4\pi r^2 = \frac{q_{inclosed}}{\varepsilon_0} = \int_0^r \frac{\rho \, dV}{\varepsilon_0}$$

$$= \frac{1}{\varepsilon_0}\int_0^r \rho_0 e^{-\alpha r^3} 4\pi r^2 \, dr$$

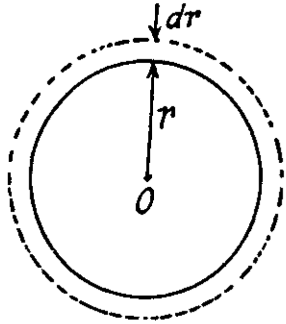
After integration

$$E_r 4\pi r^2 = \frac{\rho 4\pi}{3\varepsilon_0 \alpha}[1 - e^{-\alpha r^3}]$$

or, $$E_r = \frac{\rho_0}{3\varepsilon\alpha r^2}[1 - e^{-\alpha r^3}]$$

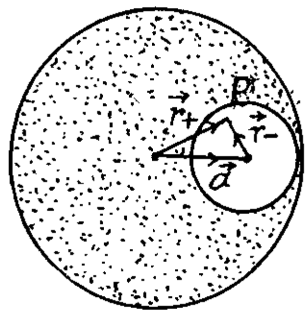
Now when $\alpha r^3 << 1$, $E_r \approx \frac{\rho_0 r}{3\varepsilon_0}$

And when $\alpha r^3 >> 1$, $E_r \approx \frac{\rho_0}{3\varepsilon_0 \alpha r^2}$



3.28 Using Gauss theorem we can easily show that the electric field strength within a uniformly charged sphere is $\vec{E} = \left(\frac{\rho}{3\varepsilon_0}\right)\vec{r}$

The cavity, in our problem, may be considered as the superposition of two balls, one with the charge density ρ and the other with $-\rho$.



Let P be a point inside the cavity such that its position vector with respect to the centre of cavity be $\vec{r}_-$ and with respect to the centre of the ball $\vec{r}_+$. Then from the principle of superposition, field inside the cavity, at an arbitrary point P,

$$\vec{E} = \vec{E}_+ + \vec{E}_-$$

$$= \frac{\rho}{3\varepsilon_0}(\vec{r}_+ - \vec{r}_-) = \frac{\rho}{3\varepsilon_0}\vec{a}$$

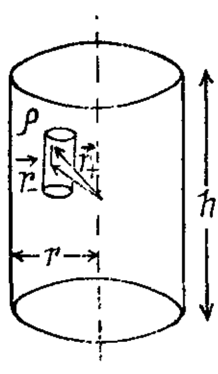
Note : Obtained expression for $\vec{E}$ shows that it is valid regardless of the ratio between the radii of the sphere and the distance between their centres.

3.29 Let us consider a cylinderical Gaussian surface of radius r and height h inside an infinitely long charged cylinder with charge density ρ. Now from Gauss theorem :

$$E_r 2\pi r h = \frac{q_{inclosed}}{\varepsilon_0}$$

(where E_r is the field inside the cylinder at a distance r from its axis.)

or, $E_r 2\pi r h = \frac{\rho\pi r^2 h}{\varepsilon_0}$ or $E_r = \frac{\rho r}{2\varepsilon_0}$



Now, using the method of 3.28 field at a point P, inside the cavity, is

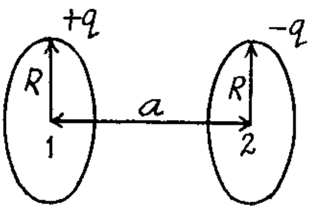
$$\vec{E} = \vec{E}_+ + \vec{E}_- = \frac{\rho}{2\varepsilon_0}(\vec{r}_+ - \vec{r}_-) = \frac{\rho}{2\varepsilon_0}\vec{a}$$

3.30 The arrangement of the rings are as shown in the figure. Now, potential at the point 1, φ_1 = potential at 1 due to the ring 1 + potential at 1 due to the ring 2.

$$= \frac{q}{4\pi\varepsilon_0 R} + \frac{-q}{4\pi\varepsilon_0 (R^2+a^2)^{1/2}}$$

Similarly, the potential at point 2,

$$\varphi_2 = \frac{-q}{4\pi\varepsilon_0 R} + \frac{q}{4\pi\varepsilon_0 (R^2+a^2)^{1/2}}$$



Hence, the sought potential difference,

$$\varphi_1 - \varphi_2 = \Delta\varphi = 2\left(\frac{q}{4\pi\varepsilon_0 R} + \frac{-q}{4\pi\varepsilon_0 (R^2+a^2)^{1/2}}\right)$$

$$= \frac{q}{2\pi\varepsilon_0 R}\left(1 - \frac{1}{\sqrt{1+(a/R)^2}}\right)$$

3.31 We know from Gauss theorem that the electric field due to an infinietly long straight wire, at a perpendicular distance r from it equals, $E_r = \frac{\lambda}{2\pi\varepsilon_0 r}$. So, the work done is

$$\int_1^2 E_r\,dr = \int_x^{\eta x} \frac{\lambda}{2\pi\varepsilon_0 r}\,dr$$

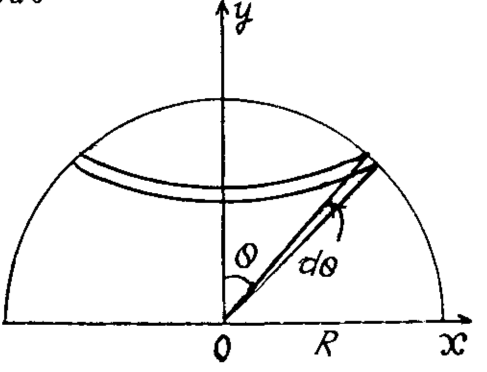
(where x is perpendicular distance from the thread by which point 1 is removed from it.)

Hence $$\Delta\varphi_{12} = \frac{\lambda}{2\pi\varepsilon_0}\ln\eta$$

3.32 Let us consider a ring element as shown in the figure. Then the charge, carried by the element, $dq = (2\pi R\sin\theta)\,R\,d\theta\,\sigma$,

Hence, the potential due to the considered element at the centre of the hemisphere,

$$d\varphi = \frac{1}{4\pi\varepsilon_0}\frac{dq}{R} = \frac{2\pi\sigma R\sin\theta\,d\theta}{4\pi\varepsilon_0} = \frac{\sigma R}{2\varepsilon_0}\sin\theta\,d\theta$$



So potential due to the whole hemisphere

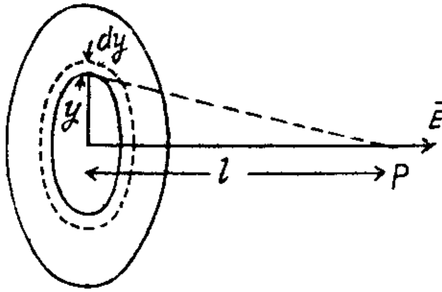
$$\varphi = \frac{R\sigma}{2\varepsilon_0}\int_0^{\pi/2}\sin\theta\,d\theta = \frac{\sigma R}{2\varepsilon_0}$$

Now from the symmetry of the problem, net electric field of the hemisphere is directed towards the negative y–axis. We have

$$dE_y = \frac{1}{4\pi\varepsilon_0}\frac{dq\cos\theta}{R^2} = \frac{\sigma}{2\varepsilon_0}\sin\theta\cos\theta\,d\theta$$

Thus $$E = E_y' = \frac{\sigma}{2\varepsilon_0}\int_0^{\pi/2}\sin\theta\cos\theta\,d\theta = \frac{\sigma}{4\varepsilon_0}\int_0^{\pi/2}\sin 2\theta\,d\theta = \frac{\sigma}{4\varepsilon_0},\ \text{along } YO$$

3.33 Let us consider an elementary ring of thickness dy and radius y as shown in the figure. Then potential at a point P, at distance l from the centre of the disc, is



$$d\varphi = \frac{\sigma\, 2\pi y\, dy}{4\pi\varepsilon_0 (y^2 + l^2)^{1/2}}$$

Hence potential due to the whole disc,

$$\varphi = \int_0^R \frac{\sigma\, 2\pi y\, dy}{4\pi\varepsilon_0 (y^2 + l^2)^{1/2}} = \frac{\sigma l}{2\varepsilon_0}\left(\sqrt{1 + (R/l)^2} - 1\right)$$

From symmetry

$$E = E_l = -\frac{d\varphi}{dl}$$

$$= -\frac{\sigma}{2\varepsilon_0}\left[\frac{2l}{2\sqrt{R^2 + l^2}} - 1\right] = \frac{\sigma}{2\varepsilon_0}\left[1 - \frac{1}{\sqrt{1 + (R/l)^2}}\right]$$

when $l \to 0$, $\varphi = \dfrac{\sigma R}{2\varepsilon_0}$, $E = \dfrac{\sigma}{2\varepsilon_0}$ and when $l >> R$,

$$\varphi = \frac{\sigma R^2}{4\varepsilon_0 l},\ E = \frac{\sigma R^2}{4\varepsilon_0 l^2}$$

3.34 By definition, the potential in the case of a surface charge distribution is defined by integral $\varphi = \dfrac{1}{4\pi\varepsilon_0}\displaystyle\int \frac{\sigma\, dS}{r}$. In order to simplify integration, we shall choose the area element dS in the form of a part of the ring of radius r and width dr in (Fig.). Then $dS = 2\theta\, r\, dr$, $r = 2R\cos\theta$ and $dr = -2R\sin\theta\, d\theta$. After substituting these expressions into integral $\varphi = \dfrac{1}{4\pi\varepsilon_0}\displaystyle\int \frac{\sigma\, dS}{r}$, we obtain the expression for φ at the point O:

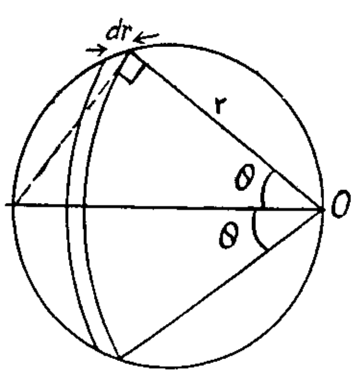
$$\varphi = -\frac{\sigma R}{\pi\varepsilon_0}\int_{\pi/2}^{0} \theta\sin\theta\, d\theta.$$

We integrate by parts,
denoting $\theta = u$ and $\sin\theta\, d\theta = dv$:

$\int \theta\sin\theta\, d\theta = -\theta\cos\theta$

$+\int \cos\theta\, d\theta = -\theta\cos\theta + \sin\theta$

which gives -1 after substituting the limits of integration. As a result, we obtain



$$\varphi = \sigma R/\pi\varepsilon_0.$$

3.35 In accordance with the problem $\varphi = \vec{a} \cdot \vec{r}$

Thus from the equation : $\vec{E} = -\vec{\nabla}\varphi$

$$\vec{E} = -\left[\frac{\partial}{\partial x}(a_x x)\vec{i} + \frac{\partial}{\partial_y}(a_y y)\vec{j} + \frac{\partial}{\partial_z}(a_z z)\vec{k}\right] = -[a_x \vec{i} + a_y \vec{j} + {}_z \vec{k}] = -\vec{a}$$

3.36 (a) Given, $\varphi = a(x^2 - y^2)$

So, $$\vec{E} = -\vec{\nabla}\varphi = -2a(x\vec{i} - y\vec{j})$$

The sought shape of field lines is as shown in the figure (a) of answersheet assuming $a > 0$:

(b) Since $\varphi = axy$

So, $$\vec{E} = -\vec{\nabla}\varphi = -ay\vec{i} - ax\vec{j}$$

Plot as shown in the figure (b) of answersheet.

3.37 Given, $\varphi = a(x^2 + y^2) + bz^2$

So, $$\vec{E} = -\vec{\nabla}\varphi = -[2ax\vec{i} + 2ay\vec{j} + 2bz\vec{k}]$$

Hence $$|\vec{E}| = 2\sqrt{a^2(x^2 + y^2) + b^2 z^2}$$

Shape of the equipotential surface :

Put $$\vec{\rho} = x\vec{i} + y\vec{j} \text{ or } \rho^2 = x^2 + y^2$$

Then the equipotential surface has the equation

$$a\rho^2 + bz^2 = \text{constant} = \varphi$$

If $a > 0$, $b > 0$ then $\varphi > 0$ and the equation of the equipotential surface is

$$\frac{\rho}{\varphi/a} + \frac{z^2}{\varphi/b} = 1$$

which is an ellipse in ρ, z coordinates. In three dimensions the surface is an ellipsoid of revolution with semi- axis $\sqrt{\varphi/a}$, $\sqrt{\varphi/a}$, $\sqrt{\varphi/b}$.

If $a > 0$, $b < 0$ then φ can be ≥ 0. If $\varphi > 0$ then the equation is

$$\frac{\rho^2}{\varphi/a} - \frac{z^2}{\varphi/|b|} = 1$$

This is a single cavity hyperboloid of revolution about z axis. If $\varphi = 0$ then

$$a\rho^2 - |b|z^2 = 0$$

or $$z = \pm\sqrt{\frac{a}{|b|}}\,\rho$$

is the equation of a right circular cone.

If $\varphi < 0$ then the equation can be written as

$$|b|z^2 - a\rho^2 = |\varphi|$$

or $$\frac{z^2}{|\varphi|/|b|} - \frac{\rho^2}{|\varphi|/a} = 1$$

This is a two cavity hyperboloid of revolution about z-axis.

3.38 From Gauss' theorem intensity at a point, inside the sphere at a distance r from the centre is given by, $E_r = \frac{\rho r}{3 \varepsilon_0}$ and outside it, is given by $E_r = \frac{1}{4 \pi \varepsilon_0} \frac{q}{r^2}$.

(a) Potential at the centre of the sphere,

$$\varphi_0 = \int_0^\infty \vec{E} \cdot d\vec{r} = \int_0^R \frac{\rho r}{3 \varepsilon_0} dr + \int_R^\infty \frac{q}{4 \pi \varepsilon_0 r^2} dr = \frac{\rho}{3 \varepsilon} \frac{R^2}{2} + \frac{q}{4 \pi \varepsilon_0 R}$$

as $$= \frac{q}{8 \pi \varepsilon_0 R} + \frac{q}{4 \pi \varepsilon_0 R} = \frac{3q}{8 \pi \varepsilon_0 R} \quad \left(\text{as } \rho = \frac{3 q}{4 \pi R^3}\right)$$

(b) Now, potential at any point, inside the sphere, at a distance r from it s centre.

$$\varphi(r) = \int_r^R \frac{\rho}{3 \varepsilon_0} r\, dr \int_r^\infty \frac{q}{4 \pi \varepsilon_0} \frac{dr}{r^2}$$

On integration : $\varphi(r) = \frac{3 q}{8 \pi \varepsilon_0 R} \left[1 - \frac{r^2}{3 R^2}\right] = \varphi_0 \left[1 - \frac{r^2}{3 R^2}\right]$

3.39 Let two charges $+q$ and $-q$ be separated by a distance l. Then electric potential at a point at distance $r >> l$ from this dipole,

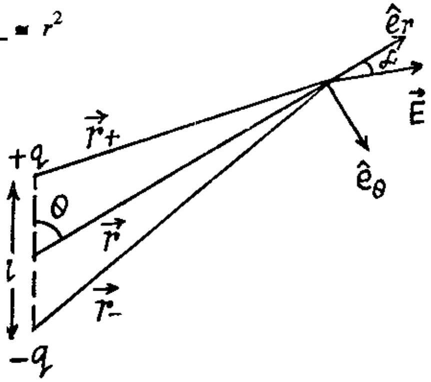
$$\varphi(r) = \frac{+q}{4 \pi \varepsilon_0 r_+} + \frac{-q}{4 \pi \varepsilon_0 r_-} = \frac{q}{4 \pi \varepsilon_0} \left(\frac{r_- - r_+}{r_+ r_-}\right) \qquad (1)$$

But $$r_- - r_+ = l \cos\theta \text{ and } r_+ r_- = r^2$$

From Eqs. (1) and (2),

$$\varphi(r) = \frac{q\, l \cos\theta}{4 \pi \varepsilon_0 r^2} = \frac{p \cos\theta}{4 \pi \varepsilon_0 r^2}\ \varphi = \frac{\vec{p} \cdot \vec{r}}{4 \pi \varepsilon_0 r^3},$$

where p is magnitude of electric moment vector.



Now, $E_r = -\frac{\partial \varphi}{\partial r} = \frac{2 p \cos\theta}{4 \pi \varepsilon_0 r^3}$

and $E_\theta = -\frac{\partial \varphi}{r\, \partial \theta} = \frac{p \sin\theta}{4 \pi \varepsilon_0 r^3}$

So $E = \sqrt{E_r^2 + E_0^2} = \frac{p}{4 \pi \varepsilon_0 r^3} \sqrt{4 \cos^2\theta + \sin^2\theta}$

3.40 From the results, obtained in the previous problem,

$$E_r = \frac{2 p \cos\theta}{4 \pi \varepsilon_0 r^3} \text{ and } E_\theta = \frac{p \sin\theta}{4 \pi \varepsilon_0 r^3}$$

From the given figure, it is clear that,

$$E_z = E_r \cos\theta - E_\theta \sin\theta = \frac{p}{4 \pi \varepsilon_0 r^3} (3 \cos^2\theta - 1)$$

and $$E_\perp = E_r \sin\theta + E_0 \cos\theta = \frac{3p\sin\theta\cos\theta}{4\pi\varepsilon_0 r^3}$$

When $\vec{E}\perp\vec{p}$, $|\vec{E}| = E_\perp$ and $E_z = 0$

So $$3\cos^2\theta = 1 \text{ and } \cos\theta = \frac{1}{\sqrt{3}}$$

Thus $\vec{E}_\perp \vec{p}$ at the points located on the lateral surface of the cone, having its axis, coinciding with the direction of z-axis and semi vertex angle $\theta = \cos^{-1} 1/\sqrt{3}$.

3.41 Let us assume that the dipole is at the centre of the one equipotential surface which is spherical (Fig.). On an equipotential surface the net electric field strength along the tangent of it becomes zero. Thus

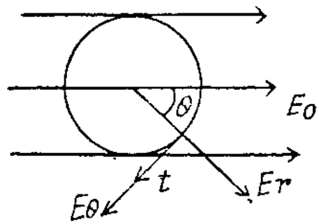
$$-E_0\sin\theta + E_\theta = 0 \quad \text{or} \quad -E_0\sin\theta + \frac{p\sin\theta}{4\pi\varepsilon_0 r^3} = 0$$

Hence $$r = \left(\frac{p}{4\pi\varepsilon_0 E_0}\right)^{1/3}$$

Alternate : Potential at the point, near the dipole is given by,

$$\varphi = \frac{\vec{p}\cdot\vec{r}}{4\pi\varepsilon_0 r^3} - \vec{E}_0\cdot\vec{r} + \text{constant},$$

$$= \left(\frac{p}{4\pi\varepsilon_0 r^3} - E_0\right)\cos\theta + \text{Const}$$



For φ to be constant,

$$\frac{p}{4\pi\varepsilon_0 r^3} - E_0 = 0 \quad \text{or,} \quad \frac{p}{4\pi\varepsilon_0 r^3} = E_0$$

Thus $$r = \left(\frac{p}{4\pi\varepsilon_0 E_0}\right)^{1/3}$$

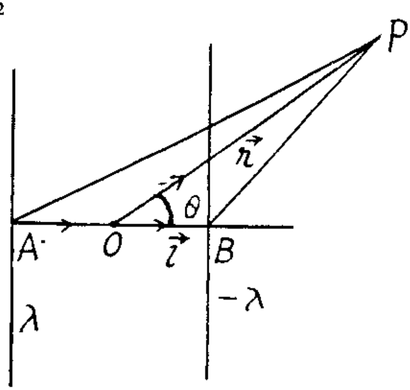
3.42 Let P be a point, at distace $r >> l$ and at an angle to θ the vector $\vec{l}$ (Fig.).

Thus $$\vec{E} \text{ at } P = \frac{\lambda}{2\pi\varepsilon_0}\frac{\vec{r}+\frac{\vec{l}}{2}}{\left|\vec{r}+\frac{\vec{l}}{2}\right|^2} - \frac{\lambda}{2\pi\varepsilon_0}\frac{\vec{r}-\frac{\vec{l}}{2}}{\left|\vec{r}-\frac{\vec{l}}{2}\right|^2}$$

$$= \frac{\lambda}{2\pi\varepsilon_0}\left[\frac{\vec{r}+\vec{l}/2}{r^2+\frac{l^2}{4}+r\,l\cos\theta} - \frac{\vec{r}-\vec{l}/2}{r^2+\frac{l^2}{4}-r\,l\cos\theta}\right]$$

$$= \frac{\lambda}{2\pi\varepsilon_0}\left(\frac{\vec{l}}{r^2} - \frac{2\,l\,\vec{r}}{r^3}\cos\theta\right)$$

Hence $$E = |\vec{E}| = \frac{\lambda\, l}{2\pi\varepsilon_0 r^2},\ r >> l$$



Also, $$\varphi = \frac{\lambda}{2\pi\varepsilon_0}\ln|\vec{r}+\vec{l}/2| - \frac{\lambda}{2\pi\varepsilon_0}\ln|\vec{r}-\vec{l}/2|$$

$$= \frac{\lambda}{4\pi\varepsilon_0}\ln\frac{r^2 + r\,l\cos\theta + l^2/4}{r^2 - r\,l\cos\theta + l^2/4} = \frac{\lambda\, l\cos\theta}{2\pi\varepsilon_0 r},\ r >> l$$

3.43 The potential can be calculated by superposition. Choose the plane of the upper ring as $x = l/2$ and that of the lower ring as $x = -l/2$.

Then $$\varphi = \frac{q}{4\pi\varepsilon_0[R^2+(x-l/2)^2]^{1/2}} - \frac{q}{4\pi\varepsilon_0[R^2+(x+l/2)^2]^{1/2}}$$

$$= \frac{q}{4\pi\varepsilon_0[R^2+x^2-lx]^{1/2}} - \frac{q}{4\pi\varepsilon_0[R^2+x^2+lx]^{1/2}}$$

$$= \frac{q}{4\pi\varepsilon_0(R^2+x^2)^{1/2}}\left(1+\frac{lx}{2(R^2+x^2)}\right) - \frac{q}{4\pi\varepsilon_0(R^2+x^2)^{1/2}}\left(1-\frac{lx}{2(R^2+x^2)}\right)$$

$$= \frac{q\,lx}{4\pi\varepsilon_0(R^2+x^2)^{3/2}}$$

For $$|x| >> R,\ \varphi = \frac{q\,l}{4\pi x^2}$$

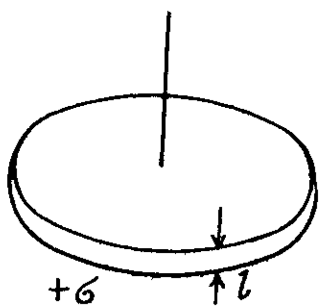
The electric field is $E = -\dfrac{\partial\varphi}{\partial x}$

$$= -\frac{ql}{4\pi\varepsilon_0(R^2+x^2)^{3/2}} + \frac{3}{2}\frac{ql}{(R^2+x^2)^{5/2}4\pi\varepsilon_0}\times 2x = \frac{ql(2x^2-R^2)}{4\pi\varepsilon_0(R^2+x^2)^{5/2}}$$

For $$|x| >> R,\ E = \frac{ql}{2\pi\varepsilon_0 x^3}.$$ The plot is as given in the book.

3.44 The field of a pair of oppositely charged sheets with holes can by superposition be reduced to that of a pair of uniform opposite charged sheets and discs with opposite charges. Now the charged sheets do not contribute any field outside them. Thus using the result of the previous problem

$$\varphi = \int_0^R \frac{(-\sigma)\,l\,2\pi r\,dr\,x}{4\pi\varepsilon_0(r^2+x^2)^{3/2}}$$



$$= -\frac{\sigma x l}{4\varepsilon_0}\int_{x^2}^{R^2+x^2}\frac{dy}{y^{3/2}} = \frac{\sigma x l}{2\varepsilon_0\sqrt{R^2+x^2}}$$

$$E_x = -\frac{\partial\varphi}{\partial x} = -\frac{\sigma l}{2\varepsilon_0}\left[\frac{1}{\sqrt{R^2+x^2}} - \frac{x^2}{(R^2+x^2)^{3/2}}\right] = -\frac{\sigma l R^2}{2\varepsilon_0(R^2+x^2)^{3/2}}.$$

The plot is as shown in the answersheet.

3.45 For $x > 0$ we can use the result as given above and write

$$\varphi = \pm \frac{\sigma l}{2 \varepsilon_0}\left(1 - \frac{|x|}{(R^2 + x^2)^{1/2}}\right)$$

for the solution that vanishes at α. There is a discontinuity in potential for $|x| = 0$. The solution for negative x is obtained by $\sigma \to -\sigma$. Thus

$$\varphi = -\frac{\sigma l x}{2 \varepsilon_0 (R + x^2)^{1/2}} + \text{constant}$$

Hence ignoring the jump

$$E = -\frac{\partial \varphi}{\partial x} = \frac{\sigma l R^2}{2 \varepsilon_0 (R^2 + x^2)^{3/2}}$$

for large $|x|$ $\quad \varphi = \pm \dfrac{p}{4 \pi \varepsilon_0 x^2}$ and $E = \dfrac{p}{2 \pi \varepsilon_0 |x|^3}$ (where $p = \pi R^2 \sigma l$)

3.46 Here $E_r = \dfrac{\lambda}{2 \pi \varepsilon_0 r}$, $E_\theta = E_\varphi = 0$ and $\vec{F} = p \dfrac{\partial \vec{E}}{\partial l}$

(a) $\vec{p}$ along the thread.

$\vec{E}$ does not change as the point of observation is moved along the thread.

$$\vec{F} = 0$$

(b) $\vec{p}$ along $\vec{r}$,

$$\vec{F} = F_r \vec{e}_r = -\frac{\lambda p}{2 \pi \varepsilon_0 r^2} \vec{e}_r = -\frac{\lambda \vec{p}}{2 \pi \varepsilon_0 r^2} \left(\text{On using } \frac{\partial}{\partial r} \vec{e}_r = 0\right)$$

(c) $\vec{p}$ along $\vec{e}_\theta$

$$\vec{F} = p \frac{\partial}{r \partial \theta} \frac{\lambda}{2 \pi \varepsilon_0 r} e_r$$

$$= \frac{p \lambda}{2 \pi \varepsilon_0 r^2} \frac{\partial \vec{e}_r}{\partial \theta} = \frac{p \lambda}{2 \pi \varepsilon_0 r^2} \vec{e}_\theta = \frac{\vec{p} \lambda}{2 \pi \varepsilon_0 r^2}.$$

3.47 Force on a dipole of moment p is given by,

$$F = \left| \varphi \frac{\partial \vec{E}}{\partial l} \right|$$

In our problem, field, due to a dipole at a distance l, where a dipole is placed,

$$|\vec{E}| = \frac{p}{2 \pi \varepsilon_0 l^3}$$

Hence, the force of interaction,

$$F = \frac{3 p^2}{2 \pi \varepsilon_0 l^4} = 2{\cdot}1 \times 10^{-16}\,\text{N}$$

3.48 $-d\varphi = \vec{E} \cdot d\vec{r} = a\,(y\,dx + x\,dy) = a\,d\,(xy)$

On integrating, $\qquad \varphi = -a\,xy + C$

3.49 $-d\varphi = \vec{E} \cdot d\vec{r} = [2\,a\,xy\,\vec{i} + 2\,(x^2 - y^2)\,\vec{j}\,] \cdot [dx\,\vec{i} + dy\,\vec{j}\,]$

or, $\qquad d\varphi = 2\,a\,x\,y\,dx + a\,(x^2 - y^2)\,dy = ad\,(x^2 y) - ay^2\,dy$

On integrating, we get,

$$\varphi = ay\left(\frac{y^2}{3} - x^2\right) + C$$

3.50 Given, again

$$-d\varphi = \vec{E}\cdot d\vec{r} = (ay\vec{i} + (ax+bz)\vec{j} + by\,\vec{k})\cdot(dx\,\vec{i} + dy\,\vec{j} + dx\,\vec{k})$$

$$= a\,(y\,dx + ax\,dy) + b\,(z\,dy + y\,dz) = ad\,(x\,y) + bd\,(yz)$$

On integrating,

$$\varphi = -(a\,xy + b\,yz) + C$$

3.51 Field intensity along x-axis.

$$E_x = -\frac{\partial\varphi}{\partial x} = 3\,a\,x^2 \qquad (1)$$

Then using Gauss's theorem in differential from

$$\frac{\partial E_x}{\partial x} = \frac{\rho\,(x)}{\varepsilon_0} \quad \text{so,}\ \ \rho\,(x) = 6a\,\varepsilon_0\,x.$$

3.52 In the space between the plates we have the Poisson equation

$$\frac{\partial^2\varphi}{\partial x^2} = -\frac{\rho_0}{\varepsilon_0}$$

or,

$$\varphi = -\frac{\rho_0}{2\,\varepsilon_0}x^2 + A\,x + B$$

where ρ_0 is the constant space charge density between the plates.

We can choose $\varphi\,(0) = 0$ so $B = 0$

Then $\varphi\,(d) = \Delta\,\varphi = Ad - \dfrac{\rho_0\,d^2}{2\,\varepsilon_0}$ or, $A = \dfrac{\Delta\,\varphi}{d} + \dfrac{\rho_0\,d}{2\,\varepsilon_0}$

Now $E = -\dfrac{\partial\,\varphi}{\partial\,x} = \dfrac{\rho_0}{\varepsilon_0}x - A = 0$ for $x = 0$

if $A = \dfrac{\Delta\,\varphi}{d} + \dfrac{\rho_0\,d}{2\,\varepsilon_0} = 0$

then $\rho_0 = -\dfrac{2\,\varepsilon_0\,\Delta\,\varphi}{d^2}$

Also $E\,(d) = \dfrac{\rho_0\,d}{\varepsilon_0}.$

3.53 Field intensity is along radial line and is

$$E_r = -\frac{\partial\,\varphi}{\partial r} = -2\,a\,r \qquad (1)$$

From the Gauss' theorem,

$$4\,\pi\,r^2\,E_r = \int\frac{dq}{\varepsilon_0,}$$

where dq is the charge contained between the sphere of radii r and $r + dr$.

Hence $4\,\pi\,r^2\,E_r = 4\,\pi\,r^2 \times (-2ar) = \dfrac{4\pi}{\varepsilon_0}\displaystyle\int_0^r r'^2\,\rho\,(r')\,dr'$ (2)

Differentiating (2) $\rho = -6\,\varepsilon_0\,a$