Principle of Mathematical Induction

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- The principle of mathematical induction can be stated as
  Suppose there is a given statement P(n) involving the natural number n such that

  (i) The statement is true for $n = 1$, i.e., P(1) is true, and
  (ii) If the statement is true for $n = k$ (where k is some positive integer), then the statement is true for $n = k + 1$, i.e., truth of P(k) implies the truth of P($k + 1$).

  Then, P(n) is true for all natural numbers n.

- Here, our assumption that the statement is true for $n = k$ is called **inductive hypothesis**.

Solved Examples

Example 1:

Prove that $9^{n+1} - 8n - 9$ is divisible by 8.

Solution:

Let the given statement be P(n), i.e.,

P(n) : $9^{n+1} - 8n - 9$ is divisible by 8

For $n = 1$, P(1): $9^{1+1} - 8(1) - 9 = 81 - 8 - 9 = 64$, which is divisible by 8

Thus, P(n) is true for $n = 1$.

Let P(n) be true for $n = k$, i.e., $9^{k+1} - 8k - 9$ is divisible by 8 for some natural number k.

Let $9^{k+1} - 8k - 9 = 8m$, where m is a natural number.

Now, we have to prove that P($k + 1$) is true whenever P(k) is true.

P($k + 1$) = $9^{(k+1)+1} - 8(k + 1) - 9$

$= 9^{(k+2)} - 8k - 8 - 9$

$= 9^{(k+1)}.9 - 8k - 8 - 9$

$= 9^{(k+1)}(8 + 1) - 8k - 8 - 9$

$= 9^{(k+1)}(8 + 1) - 8k - 8 - 9$

$= 8.\ 9^{(k+1)} + 9^{(k+1)} - 8k - 8 - 9$

$= \{9^{(k+1)} - 8k - 9\} + 8\ (9^{(k+1)} - 1)$

$= 8m + 8\ (9^{(k+1)} - 1)$

$= 8\{m + (9^{(k+1)} - 1)\}$

Thus, $P(k + 1)$ is divisible by 8.

Thus, by the principle of mathematical induction, the given statement is true for every positive integer n.

Example 2:

Prove the following by the principle of mathematical induction.

$2^n > n^2$, where n is a positive integer such that $n > 4$.

Solution:

Let the given statement be $P(n)$, i.e.,

$P(n) : 2^n > n^2$ where $n > 4$

For $n = 5$,

$2^5 = 32$ and $5^2 = 25$

$\therefore 2^5 > 5^2$

Thus, $P(n)$ is true for $n = 5$.

Let $P(n)$ be true for $n = k$, i.e.,

$2^k > k^2$ … (1)

Now, we have to prove that $P(k + 1)$ is true whenever $P(k)$ is true, i.e. we have to prove that $2^{k+1} > (k + 1)^2$.

From equation (1), we get

$2^k > k^2$

Multiplying both sides with 2, we obtain

$2 \times 2^k > 2 \times k^2$

$2^{k+1} > 2k^2$

$\therefore$To prove $2^{k+1} > (k+1)^2$, we only need to prove that $2k^2 > (k+1)^2$.

Let us assume $2k^2 > (k+1)^2$.

$\Rightarrow 2k^2 > k^2 + 2k + 1$

$\Rightarrow k^2 > 2k + 1$

$\Rightarrow k^2 - 2^k - 1 > 0$

$\Rightarrow (k-1)^2 - 2 > 0$

$\Rightarrow (k-1)^2 > 2$, which is true as $k > 4$

Hence, our assumption $2k^2 > (k+1)^2$ is correct and we have $2^{k+1} > (k+1)^2$.

Thus, P(n) is true for $n = k + 1$.

Thus, by the principle of mathematical induction, the given mathematical statement is true for every positive integer n.

Example 3:

Prove the following by the principle of mathematical induction for every positive integer n.

2.5 + 4.7 + 6.9 + … + 2*n*(2*n* + 3) = $\frac{1}{3}n(n+1)(4n+11)$

Solution:

Let the given statement be P(n), i.e., P(n): $2.5 + 4.7 + 6.9 + \ldots + 2n(2n + 3) = \frac{1}{3}n(n+1)(4n+11)$

For $n = 1$, $2.5 = \frac{1}{3}(1+1)(4+11) = \frac{1}{3}2.15 = 2.5$, which is true.

Let P(n) be true for $n = k$, i.e.,

$2.5 + 4.7 + 6.9 + \ldots + 2k(2k + 3) = \frac{1}{3}k(k+1)(4k+11)$ ... (1)

We now have to prove that P(k + 1) is true whenever P(k) is true.

$2.5 + 4.7 + 6.9 + \ldots + 2k(2k + 3) + 2(k + 1)\{2(k + 1) + 3\}$

$=\{2.5 + 4.7 + 6.9 + \ldots + 2k(2k + 3)\} + 2(k + 1)\{2(k + 1) + 3\}$

$= \frac{1}{3}k(k+1)(4k+11) + 2(k + 1)\{2(k + 1) + 3\}$ {from equation (1)}

$$=(k+1)\left\{\frac{1}{3}k(4k+11)+2(2k+5)\right\}$$
$$=\frac{1}{3}(k+1)\left\{4k^2+23k+30\right\}$$
$$=\frac{1}{3}(k+1)\left\{(k+2)(4k+15)\right\}$$
$$=\frac{1}{3}(k+1)\left[\left\{(k+1)+1\right\}\left\{4(k+1)+11\right\}\right]$$

Thus, P(n) is true for $n = k + 1$.

Thus, by the principle of mathematical induction, the given statement is true for every positive integer n.