CHAPTER 10

FACTORIZATION

10.1 Factors

In the previous classes we have learnt the prime factorization of natural numbers

Like -

$$24 = \underline{2} \times 2 \times 2 \times \underline{3}$$
$$30 = \underline{2} \times \underline{3} \times 5$$

We see that 2 and 3 are the common prime factors in the factorization of 24 and 30.these type of prime factors are known as common factors.

Example-1 Find the common factors of 32, 56 and 72

Solution: The factors of 32, 56 and 72 is

$$32 = \underline{2 \times 2 \times 2} \times 2 \times 2$$
$$56 = \underline{2 \times 2 \times 2} \times 7$$
$$72 = \underline{2 \times 2 \times 2} \times 3 \times 3$$

Common factors $= 2 \times 2 \times 2 = 8$

Example-2 Find the common factors of 25 and 27.

Solution: The factors of 25 and 27 is

$$25 = 5 \times 5$$
$$27 = 3 \times 3 \times 3$$

They do not have any common factors therefore the common factors among them is 1

In the same way we can find the common factors of the Algebraic expression by expressing them in their factors.

10.2 Factors of Algebraic Expressions

We have seen in Class VII that the terms of algebraic expressions can be shown in tree graph as products of factors.

For example, in the algebraic expression $3xy + 5x$

The term $3xy$ has been formed by the factors 3, x and y i.e.

$$3xy = 3 \times x \times y$$
$$5x = 5 \times x$$

Observe that the factors 3, x and y of $3xy$ cannot be further factorized.
We may say that 3, x and y are 'prime' factors of $3xy$. In algebraic expressions, we use the word 'indivisible' in place of 'prime'. We say that $3 \times x \times y$ is the indivisible form of $3xy$

In the same way

$$5x\,(x+3) = 5 \times x \times (x+3)$$
$$10xy\,(x+2)(x+9) = 10 \times x \times y \times (x+2) \times (x+9)$$

Prime factorized form of $3xy$ is $3 \times x \times y$

10.2.1 Method of Factorization

When we factorize an algebraic expression, we write it as a product of factors. These factors may be numbers, algebraic variables or algebraic expressions. Expressions like $5xy$, $2x^2$, $2x^2y$, $2x(y+2)$, $6(y+1)$ are already in factor form. Their factors can be just read off from them, as we already know. On the other hand consider expressions like $6x+3$, $2a+4b$, $y+5y$, $x^2+7x+12$, It is not obvious what their factors are. In order to find out the factors of this expression we have the following two methods.

1) Method of common factors
2) Factorization by regrouping terms

10.2.1.1 Method of Common Factors

We begin with a simple example: Factories $6x+9$
We shall write each term as a product of irreducible factors

$$6x = 2 \times 3 \times x$$
$$9 = 3 \times 3$$
$$6x + 9 = (2 \times 3 \times x) + (3 \times 3)$$

Notice that factor 3 is common to both the terms. Hence

$= 3[2 \times x + 3] = 3(2x+3)$ these are the irreducible factor of $6x+9$

For Binomial $ka + kb = k\,(a + b)$

For Trinomial $ka + kb + kc = k\,(a + b + c)$

10.2.1.2 Factorization by regrouping terms

There is no common factor in the term $3ab + 3b + 2a + 2$ .so we find the common factors from the given expression by making the group of different terms. For example the expression $3ab + 3b + 2a + 2$ have two group, the first one is $3ab + 3b$ and second is $2a + 2$.

$= \underline{3ab + 3b} + \underline{2a + 2}$

$= 3b(a+1) + 2(a+1)$ ← Here a+1 is the common factor. Let's repeat the process.

$= (a+1)(3b+2)$

Now if the given expression is written in the form

$3ab + 2 + 3b + 2a$ ← We don't get a common factor after making groups of 2 terms

Then it will not be easy to see the factorization. Rearranging the expression as

$3ab + 2 + 3b + 2a = 3ab + 3b + 2a + 2$ → When there is no common factor, We take 1 as the common factor.

$= 3b(a+1) + 2(a+1)$

$= (a+1)(3b+2)$ → (3b+2) is the common factor in both expression. So let's repeat the process.

Please note that $(a+1)(3b+2) = (3b+2)(a+1)$

i.e. multiplication follow commutative property.

> We can write it in the form as
>
> $Ka + Kb + Pa + Pb$
>
> $= K(a+b) + P(a+b)$
>
> $= (K+P)(a+b)$

Example 3: Factorise $6xy - 4y + 6 - 9x$.

Solution:

Step 1 Check if there is a common factor among all terms. There is none.

Step 2 Two group forms, the first is $6xy - 4y$ and second is $6 - 9x$.

$6xy - 4y$	$6 - 9x$
$= 2y(3x-2)$	$= 3(2-3x)$

Step 3 Both the group is not same so rearranging second group we get

$6 - 9x = -(9x-6)$

$= -3(2x-3)$

Putting them together, we get

$6xy - 4y + 6 - 9x = \underline{6xy - 4y} - \underline{9x + 6}$ (by rearranging)

$= 2y(3x-2) - 3(3x-2)$

$= (3x-2)(2y-3)$

The factors of $(6xy - 4y + 6 - 9x)$ are $(3x-2)$ and $(2y-3)$.

EXERCISE 10.1

1. Find the common factors of the given terms.

(i) $12x, 36$ (ii) $14\,pq, 28p^2q^2$ (iii) $6abc, 24ab^2, 12a^2b$
(iv) $16x^3, -4x^2, 32x$ (v) $10pq, 20qr, 30rp$ (vi) $3x^2y^3, 10x^3y^2, 6x^2y^2z$

2. Factorize the following expressions.(by common factors method)

(i) $6p - 12q$ (ii) $7a^2 + 14a$ (iii) $10a^2 - 15b^2 + 20c^2$
(iv) $ax^2y + bxy^2 + cxyz$ (v) $x^2yz + xy^2z + xyz^2$ (vi) $-16z + 20z^3$

3. Factorize(by regrouping method)

(i) $2xy + 3 + 2y + 3x$ (ii) $z - 7 - 7xy + xyz$
(iii) $6xy - 4y + 6 - 9x$ (iv) $15pq + 15 + 9q + 25p$

10.3 Factorization by using identities

We know that

(i) $(a+b)^2 = a^2 + 2ab + b^2$ (ii) $(a-b)^2 = a^2 - 2ab + b^2$ (iii) $(a+b)(a-b) = a^2 - b^2$

We use these identities to find the factor of the given expressions. With the help of these identities we can find the factors of the trinomial terms.

Example 4: Factorize the following

(1) $x^2 + 6x + 9$

Make the given expression in the form,

$(a+b)^2 = a^2 + 2ab + b^2$
$= x^2 + 2 \times 3 \times x + (3)^2$
$= (a)^2 + 2 \times (a)(b) + (b)^2$
$= (x+3)^2$

NOTE THAT

(i) The given expression has three terms

(ii) Expression is of the form $a^2 + 2ab + b^2 = (x+3)^2$ where a = x, and b = 3.

Grouping Method

$x^2 + 6x + 9$
$= x^2 + 3x + 3x + 9$
$= x(x+3) + 3(x+3)$
$= (x+3)(x+3)$
$= (x+3)^2$

(2) $9a^2 - 30ab + 25b^2$
$= (3a)^2 - 2 \times 3a \times 5b + (5b)^2$
$= (a)^2 - 2 \times a \times b + (b)^2$
$= (3a-5b)^2$

NOTE THAT

(i) the given expression has three terms

(ii) expression is of the form $a^2 - 2ab + b^2$ where a $= 3a$ and $b = 5b$

Grouping method :

$9a^2 - 30ab + 25b^2$
$= 9a^2 - 15ab - 15ab + 25b^2$
$= 3a(30-5b) - 5b(3a-5b)$
$= (3a-5b) - 5b(3a-5b)$
$= (3a-5b)(3a-5b)$
$= (3a-5b)^2$

(3) $4x^2 - 9a^2$
$= (2x)^2 - (3a)^2$
$= (2x + 3a)(2x - 3a)$

(i) This expression contains two terms therefore we use the identity
$a^2 - b^2 = (a + b)(a - b)$
Here $a^2 = 4x^2$ and $b^2 = 9a^2$
$a = 2x$ and $b = 3a$

(4) $2x^2 + 16x + 32$
$= 2[x^2 + 18x + 16]$
$= 2[(x^2) + 2x \times 4 + (4)^2]$
$= 2(x + 4)^2$

(i) In this expression first and last term is not prefect square
(ii) 2 is common in the expression
(iii) this expression is of $(a^2 + 2ab + b^2)$

10.3.1 Factors of the form $(x + a)(x + b)$

Let us now discuss how we can factorize expressions in one variable, like $x^2 + 8x + 12$, $y^2 - 5y + 6$, $z^2 - 4z - 12$, $x^2 + 2x - 15$ etc. Observe that these expressions are not of the type $(a + b)^2$ or $(a - b)^2$, i.e., they are not perfect squares. These expressions obviously also do not fit the type $(a^2 - b^2)$ either.

They, however, seem to be of the type $x^2 + (a + b)x + ab$.

Therefore in order to factorize these type of expression we use the identity,

$$(x + a)(x + b) = x^2 + (a + b)x + ab$$

For that we have to look at the coefficients of x and the constant term. Let us see how it is done in the following example.

Do and learn

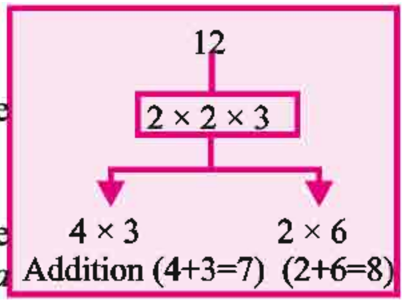
Find two integers a and b such that

$a+b$	ab	a	b
8	15	5	3
13	12		
-1	-20	-5	4
-5	4		
10	21		
-1	-12		
-11	10		
-7	10		

Table - 2

Example 5: Factories the following

Solution: (i) $x^2 + 8x + 12$



by comparing the given expression with the identity $x^2+(a+b)x+ab$ we get

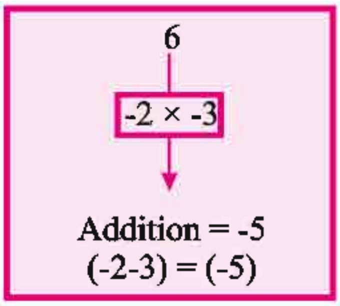
$a + b = 8$ and $ab = 12$

for finding the value of a and b we must select the value of a and b in such a way that the product of a and b is equal to 12 and their sum must be 8

Therefore we put $a = 2$ and $b = 6$ so that $ab = 12$ and $a+b=8$

Therefore $x^2 + 8x + 12= (x+2)(x+6)$

(ii) $y^2 - 5y + 6$



by comparing we get

$a+b = -5$ $ab = 6$

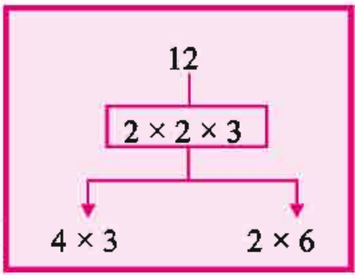
$y^2 - (2+3)y + 6$

$= \underline{y^2 - 2y} - \underline{3y + 6}$ (After grouping)

$= y(y-2) - 3(y-2)$

$= (y-2)(y-3)$

(iii) $z^2 - 4z - 12$



$a + b = -4$ $ab = -12$

$ab = -12$ that means that one among digits a or b is a negative integer. Further, $a + b = -4$, this means the one with larger numerical value is negative

Hence by taking $a = -6$ and $b = 2$

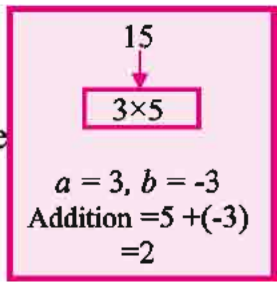
$z^2 - 4z - 12 = z^2 - 6z + 2z - 12$

$= z(z-6) + 2(z-6)$

$= (z-6)(z+2)$

Example 6: Find the factors of $x^2 + 2x - 15$

Solution: $x^2 + 2x - 15$, by comparing we get



$a + b = 2$ and $ab = -15$

$ab = -15$, means that one of the a and b is negative

$a + b = 2$, means that greater number should be of positive

$x^2 + 2x - 15 = \underline{x^2 + 5x} - \underline{3x - 15}$

$= x(x+5) - 3(x+5)$

$= (x+5)(x-3)$

EXERCISE 10.2

(1) Factorize the following expressions.

(i) $a^2 - 4$ (ii) $a^2 - 49b^2$ (iii) $p^3 - 121p$ (iv) $(a-b)^2 - c^2$
(v) $a^4 - b^4$ (vi) $5x^3 - 125x$ (vii) $63a^2 - 112b^2$ (viii) $9x^2y^2 - 16$
(ix) $(l+m)^2 - (l-m)^2$

(2) Factorize the following expressions.

(i) $lx^2 + mx$ (ii) $2x^3 + 2xy^2 + 2xz^2$
(iii) $a(a+b) + 4(a+b)$ (iv) $(xy+y) + x + 1$
(v) $5a^2 - 15a - 6c + 2ac$ (vi) $am^2 + bm^2 + bn^2 + an^2$

(3) Factorize the following expressions.

(i) $x^2 + 5x + 6$ (ii) $q^2 + 11q + 24$ (iii) $m^2 - 10m + 21$
(iv) $x^2 + 6x - 16$ (v) $x^2 - 7x - 18$ (vi) $k^2 - 11k - 102$
(vii) $y^2 + 2y - 48$ (viii) $d^2 - 4d - 45$ (ix) $m^2 + 16m + 63$
(x) $n^2 - 19n - 92$ (xi) $p^2 - 10p + 16$ (xii) $x^2 + 4x - 45$

10.4 Division of Algebraic Expressions

We have learnt how to add and subtract algebraic expressions. We also know how to multiply two expressions. We have not however, looked at division of one algebraic expression by another..

We recall that division is the inverse operation of multiplication. Thus,

$5 \times 8 = 40$ gives

$40 \div 8 = 5$ or $40 \div 5 = 8$.

We may similarly follow the division of algebraic expressions. For example

(i) $3x \times 5x^2 = 15x^3$

Therefore $15x^3 \div 3x = 5x^2$

And also $15x^3 \div 5x^2 = 3x$

(ii) Similarly $5x(x+3) = 5x^2 + 15x$

$(5x^2 + 15x) \div 5x = x + 3$

$(5x^2 + 15x) \div (x+3) = 5x$

10.4.1 Division of a Monomial by Another Monomial

$$8x^3 \div 2x = \frac{2\times2\times2\times x\times x\times x}{2\times x} = 2\times2\times x\times x = 4x^2$$

Example:7 Do the following divisions.

(i) $-20x^5 \div 5x^2$ (ii) $7a^2b^2c^2 \div 21abc$ (iii) $63p^2q^3r \div -3p^4q$

Solution (i) $-20x^5 = -2 \times 2 \times 5 \times x \times x \times x \times x \times x$

$5x^2 = 5 \times x \times x$

Therefore $-20x^5 \div 5x^2 = \dfrac{-2 \times 2 \times 5 \times x \times x \times x \times x \times x}{5 \times x \times x} = -2 \times 2 \times x \times x \times x$

$= -4x^3$

(ii) $7a^2b^2c^2 \div 21abc$

$7a^2b^2c^2 = 7\times a\times a\times b\times b\times c\times c$

$21abc = 21\times a\times b\times c$

Therefore $7a^2b^2c^2 \div 21abc = \dfrac{7\times a\times a\times b\times b\times c\times c}{21\times a\times b\times c} = \dfrac{a\times b\times c}{3} = \dfrac{abc}{3}$

(iii) $63p^2q^3r \div -3p^4q$

$63p^2q^3r = 3\times 3\times 7\times p\times p\times q\times q\times q\times r$

$-3p^4q = -3\times p\times p\times p\times p\times q$

Therefore

$63p^2q^3r \div -3p^4q = \dfrac{3\times 3\times 7\times p\times p\times q\times q\times q\times r}{-3\times p\times p\times p\times p\times q} = \dfrac{-21q^2r}{p^2}$

10.4.2 Division of a Polynomial by a Monomial

Let us consider the division of the trinomial $8y^3 + 6y^2 + 12y$ by the monomial 2y.

$$\frac{8y^3 + 6y^2 + 12y}{2y} = \frac{2y(4y^2 + 3y + 6)}{2y} = 4y^2 + 3y + 6$$

This can also be written as

$$\frac{8y^3 + 6y^2 + 12y}{2y} = \frac{8y^3}{2y} + \frac{6y^2}{2y} + \frac{12y}{2y} = 4y^2 + 3y + 6$$

Each term of the numerator is divided by the monomial of the denominator

Example:8 Divide $(18x + 12x^3 - 6x^2)$ by $(-3x)$

Solution: $\dfrac{18x + 12x^3 - 6x^2}{-3x} = \dfrac{6x(3 + 2x^2 - x)}{-3x} = -2(3 + 2x^2 - x)$

$= 6 - 4x^2 + 2x$

$= -4x^2 + 2x - 6$

10.4.3 Division of Algebraic Expressions Continued (Polynomial ÷ Polynomial)

Consider $(7x^2 + 14x) \div (x + 2)$

$$\frac{(7x^2 + 14x)}{(x + 2)} = \frac{7x(x + 2)}{(x + 2)} = 7x$$

Example:9 Divide $(x^2 + 7x + 10)$ by $(x+2)$

Solution: $\dfrac{x^2 + 7x + 10}{x + 2} = \dfrac{x^2 + 5x + 2x + 10}{x + 2}$

$= \dfrac{x(x+5) + 2(x+5)}{x + 2} = \dfrac{(x+5)(x+2)}{x + 2} = x + 5$

10
5 2
Adding 5+2 = 7

Example: 10 Divide $p(5p^2-80)$ by $5p(p-4)$

Solution: $\dfrac{p(5p^2 - 80)}{5p(p - 4)} = \dfrac{5p(p^2 - 16)}{5p(p - 4)}$

$\dfrac{5p[(p)^2 - (4)^2]}{5p(p - 4)} = \dfrac{5p(p - 4)(p + 4)}{5p(p - 4)} = p + 4$

Do and Learn

Find the error

(I) $3x + x + 4x = 56$

$7x = 56$

$x = \dfrac{56}{7}$

Find the error

(2) Find the value of $5x$ at $x = -2$

$= 5 - 2 = 3$

Find the error and also find the correct value

(3) Solution of the expression is given in the column A and B. Find which of the solution is correct

We generally do not show 1 as a coefficient of any term but in addition of like term we add it with the respective terms

Expression	A	B
$3(x-4)$	$3x-4$	$3x-12$
$(2x)^2$	$2x^2$	$4x^2$
$(x+4)^2$	x^2+16	$x^2+8x+16$
$(x-3)^2$	x^2-9	x^2-6x+9
$\dfrac{y+1}{5}$	$y+1$	$\dfrac{y}{5}+1$

EXERCISE 10.3

1. Carry out the following divisions.

(i) $28x^4$ by $56x$ (ii) $-36y^3$ by $9y^2$

(iii) $34x^3y^3z^3$ by $51xy^2z^3$ (iv) $12a^8b^8$ by $(-6a^6b^4)$

2. Divide the given polynomial by the given monomial.

(i) $(5x^2 - 6x)$ by $3x$ (ii) $(x^3 + 2x^2 + 3x)$ by $2x$

(iii) $(p^3q^6 - p^6q^3)$ by p^3q^3 (iv) $(3x^8 - 4x^6 + 5x^4)$ by x^4

3. Work out the following divisions.

(i) $10y(6y+21) \div 5(2y+7)$ (ii) $9x^2y^2(3z-24) \div 27xy(z-8)$

(iii) $(10y+14) \div 2$ (iv) $(6x-5) \div (2x-5)$

4. Factorize the expressions and divide them as directed

(i) $(y^2+7y+10) \div (y+5)$ (ii) $(5x^2-25x+20) \div (x-1)$

(iii) $12xy(9x^2 -16y^2) \div 4xy(3x+4y)$ (iv) $4yz(z^2+6z-16) \div 2y(z+8)$

WE LEARNT

1. When we factorize an expression, we write it as a product of factors.

2. A prime factor is a factor which cannot be expressed further as a product of factors.

3. Number of expressions to be factorized are of the form or can be put into the form :

(i) $(a+b)^2 = a^2 + 2ab + b^2$

(ii) $(a-b)^2 = a^2 - 2ab + b$

(iii) $(a+b)(a-b) = a^2 - b^2$

(iv) $x^2 + (a+b)x + ab = (x+a)(x+b)$

(v) If we have the expression of type $x^2 + (a+b)x + ab$. Then this will have the factor of $(x+a)(x+b)$ form.

4. We know that in the case of numbers, division is the inverse of multiplication. This idea is applicable also to the division of algebraic expressions.

5. In the case of divisions of algebraic expressions that we studied in this chapter, we have

Dividend = Divisor × Quotient.

In general, however, the relation is

Dividend = Divisor × Quotient + Remainder