Chapter – 9

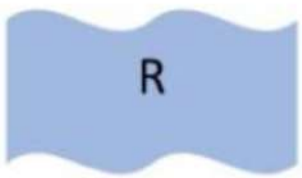
Areas of Parallelograms and Triangles

Introduction to Area between Parallelogram and Triangles

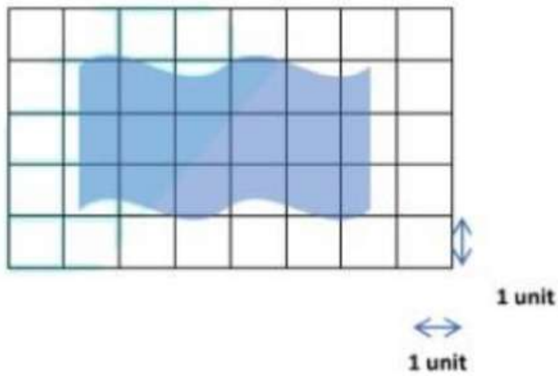
Introduction to Area

Area: Area can be defined as the space occupied by a closed two- dimensional shape or object.

The area of a closed figure is the number of unit-squares it covers.



Area of the figure is the region covered in blue colour.



Units of the area: Area is measured in square units such as square centimetres, square feet, square inches, etc.

The area of a closed figure/region R is denoted by ar (R).

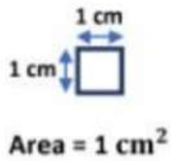
Suppose, this region R covers x number of unit squares, then its area will be represented as,

Ar(R) = x square units or x unit^2

How do we get the units of measurement of the area? Let us understand this.

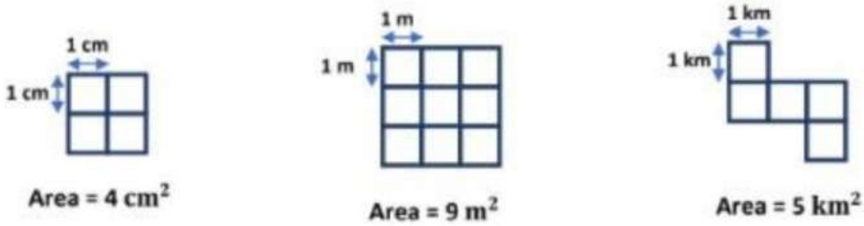
Let us find the area of a square with side 1 cm.

We know, the area of a square = side × side.



The area of the square with side 1 centimeter (cm) = 1 cm × 1 cm = 1 cm^2 .

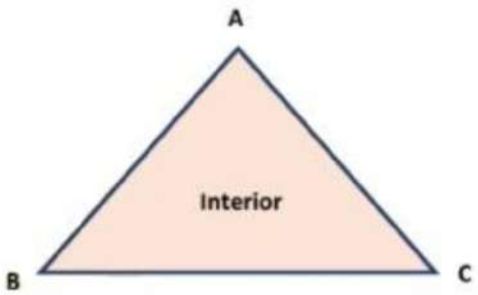
Similarly, the area of the figures below can be found by simply counting the number of unit squares.



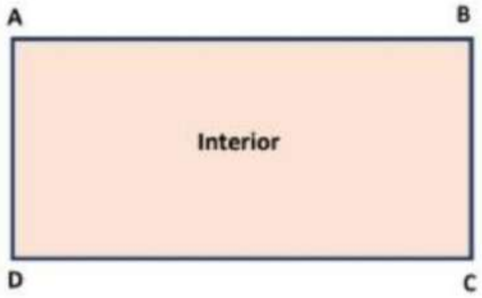
In this chapter, we will restrict our discussion to the area of the polygonal region. For example, the area of triangles, quadrilaterals, parallelograms.

We often find the area of the room floor to calculate the quantity of the carpet to be bought. Covering the floor with tiles, covering the wall with paint or wallpaper or building are other examples, wherein the computation of area is required.

A triangle is a closed, two- dimensional figure formed by three intersecting lines. The region enclosed by a triangle is its area.



The part of the plane enclosed by the triangle is called the interior region of the triangle. Hence, the area of a triangle is a triangular region formed by the union of a triangle and its interior.



Similarly, the area of a parallelogram ABCD is the union of the parallelogram and its interior.

We have already learned the formulae for finding the areas of different plane figures such as triangle, rectangle, parallelogram, square, etc.

In this chapter, we will study some relationships between the areas of these geometric figures under the condition like when they lie on the same base and between the same parallels.

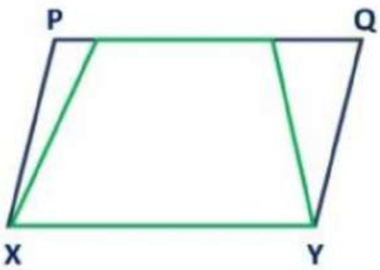
Figures on the same base and between the same parallels

Figures on the same base and between the same parallels

Two triangles with common base EF.

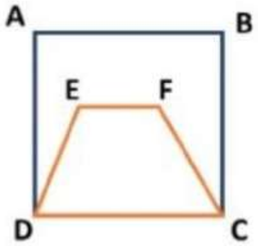
Two figures are between two parallels AB & CD but have different bases' length.

Two figures with common base XY and between two parallels XY & PQ.

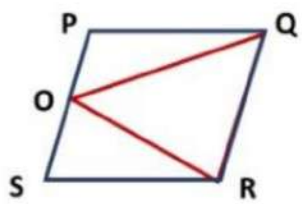


"So, two figures are said to be on the same base and between same parallels, if they have a common base (side) and the vertices (or the vertex) opposite to the common base of each figure lie on a line parallel to the base". [As shown in the figure above]

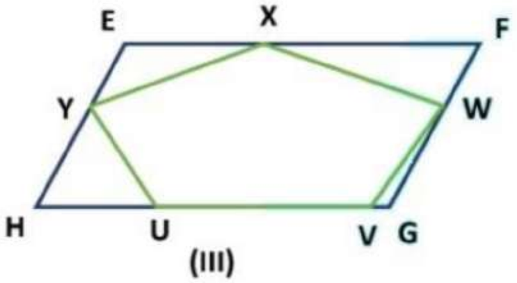
Example: Choose which of the following figures lie on the same base and between the same parallels. For such a figure, write the common base and the two parallels.

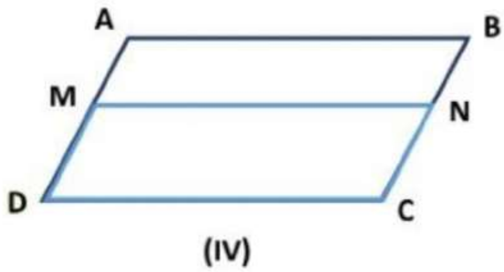


(I)



(II)





Solution:

I. Rectangle ABCD and trapezium EFCD are on the same base CD but they are not between the same parallels.

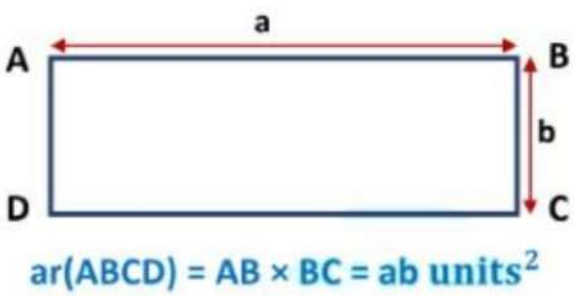
II. Parallelogram PQRS and Δ ROQ are on the same base RQ and between the same parallel lines PS and QR.

III. Parallelogram EFGH and pentagon UVWXY are between the same parallel EF and HG but they are not on the same base.

IV. Parallelogram ABCD and parallelogram MNCD lie between the same parallels AB and CD and they have common base CD.

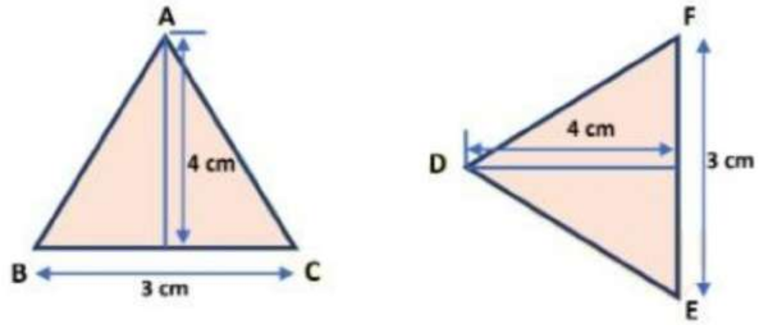
RECTANGLE AREA AXIOM

If ABCD is a rectangular region such that AB = a units and BC = b units in length, then



CONGRUENT AREA AXIOM

Consider two Δ ABC and Δ DEF which are congruent to each other.



Now, we calculate the area of Δ ABC, then we get

ar(ΔABC) = 1/2× base × height.

$= \frac{1}{2} \times 3 \text{ cm} \times 4 \text{ cm}.$

$= 3 \text{ cm} \times 2 \text{ cm}.$
$= 6 \text{ cm}^2.$

Also, we calculate the area of Δ DEF, then we get

ar(ΔDEF) = $\frac{1}{2}$ × base × height.

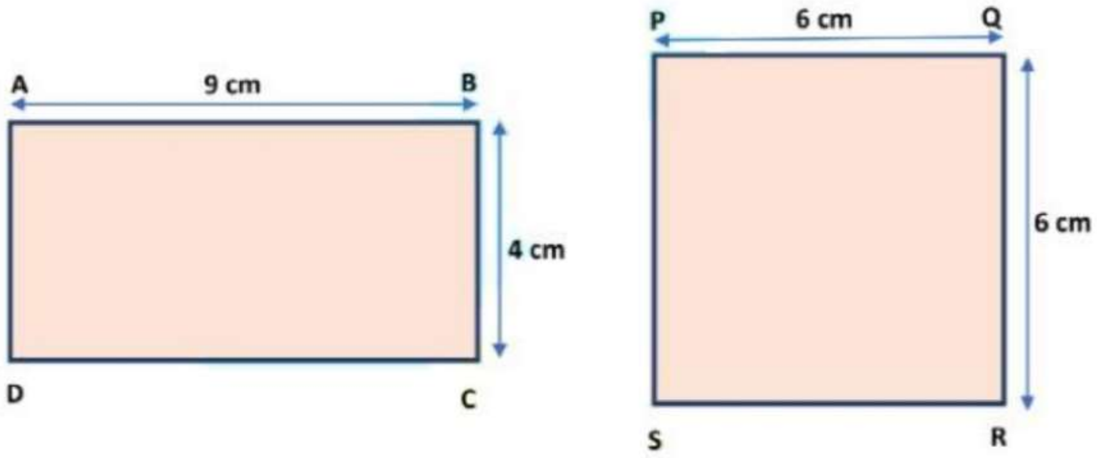
$= \frac{1}{2} \times 3 \text{ cm} \times 4 \text{ cm}.$

$= 3 \text{ cm} \times 2 \text{ cm}.$
$= 6 \text{ cm}^2.$

Here, we also see that the area of triangles ABC and DEF are equal.

Therefore, we conclude that if two figures are congruent with each other, they must have equal areas.

Now, consider a rectangle ABCD and square PQRS which are equal in area.



We calculate the area of rectangle ABCD, then we get ar(ABCD) = Length × Breath.

= 9 cm × 4 cm.
= 36 cm^2
.

Also, we calculate the area of square PQRS, then we get
ar(PQRS) = Side × Side.
= 6 cm × 6 cm.
= 36 cm^2.

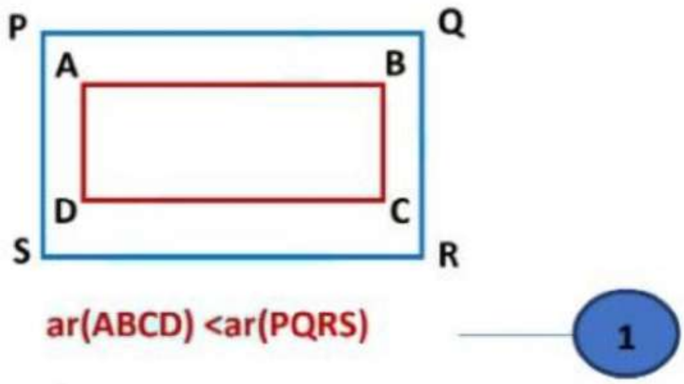
Here, we see that areas of triangles ABC and DEF are equal. Rectangles ABCD and square PQRS are not congruent, but still, they have equal areas. Therefore, we can conclude that if two figures are congruent, they will have equal areas whereas, if two figures have equal areas, they need not be congruent with each other.

AREA MONOTONE AXIOM:

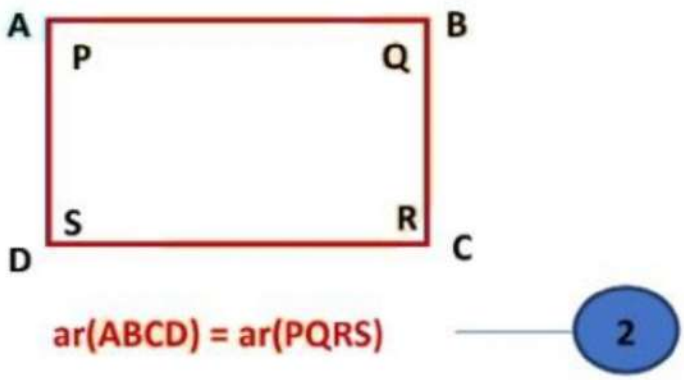
If R1, R2 are two polygonal regions such that R1 is a part of R2, then

ar(R1) ≤ ar(R2)

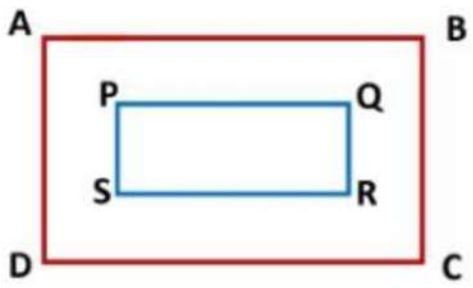
i.e. In the given figure region, ABCD is the part of region PQRS, that means



In the next figure (below) the region, the vertices A, B, C, D of parallelogram ABCD coincide with the vertices P, Q, R, S of the parallelogram PQRS.



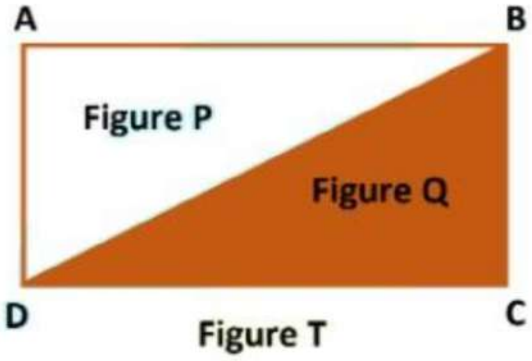
The third possibility for figure ABCD and PQRS is



But in this case, ar(ABCD) >ar(PQRS) and PQRS is the part of ABCD which is not held by the statement of the axiom. So, this case shall not be considered.

AREA ADDITION AXIOM:

If a planer region formed by a figure T is made up of two non-overlapping planar regions formed by figures P and Q, then ar (T)=ar (P)+ar (Q).



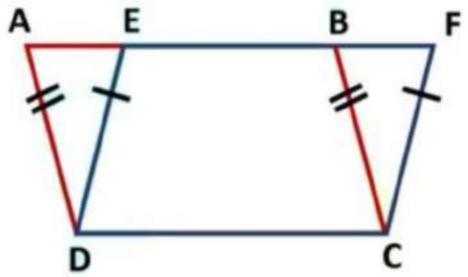
In the figure above, ar(ABCD) = ar(ΔADB) + ar(ΔBCD)

Parallelograms on the same base and between the same parallels

Parallelograms on the same base and between the same parallels.
Theorem 1: Parallelograms on the same base (or equal bases) and between the same parallels are equal in area.

Given: Two parallelograms ABCD and ABPQ, which have the same base AB and which are between the same parallel lines AB and QC.

To prove: ar(∥ gm ABCD) = ar(∥ gm EFCD)



Proof: In ΔADE and ΔBCF

∠DAE = ∠CBF (Corresponding angles from AD ∥ BC and transversal AF)

∠AED = ∠BFC (Corresponding angles from ED ∥ FC and transversal AF)

Also, AD = BC (Opposite sides of the parallelogram ABCD)

So, $\Delta ADE \cong \Delta BCF$ [By AAS criteria]

Therefore, $ar(\Delta ADE) = ar(\Delta BCF)$ (Congruent figures have equal areas)

Now, $ar(ABCD) = ar(ADE) + ar(EDCB)$

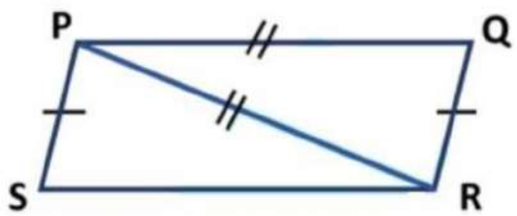
$= ar(BCF) + ar(EDCB)$ [From 1]

$= ar(EFCD)$

So, parallelograms ABCD and EFCD are equal in area.

Theorem 2: A diagonal of a parallelogram divides it into two triangles of equal area.

Given: A parallelogram PQRS in which PR is one of the diagonals.



To prove: $ar(\Delta PSR) = ar(\Delta RQP)$

Proof: In ΔPSR and ΔRQP, we have

$PS = QR$ (Opposite side of the parallelogram PQRS)

$SR = QP$ (Opposite side of the parallelogram PQRS)

$PR = RP$ (Common side)

So, by SSS criterion of congruence, we have

$\Delta PSR \cong \Delta RQP$

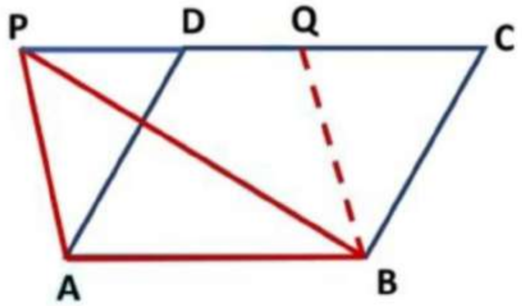
And we know that congruent figures have equal areas.

Hence, $ar(\Delta PSR) = ar(\Delta RQP)$.

Theorem 3: If a triangle and a parallelogram are on the same base and between the same parallels, then the area of the triangle is equal to half of the area of the

parallelogram.

Given: A ΔABP and $\|^{gm}ABCD$ on the same base AB and between the same parallels AB and PC.



To prove: $ar(\Delta PAB) = \frac{1}{2} ar(\|^{gm}ABCD)$

Construction: Draw $BQ \| AP$ to obtain another parallelogram ABQP.

Proof: Parallelogram ABCD and ABQP are on the same base AB and between the same parallels AB and PC.

Therefore, $ar(ABQP) = ar(ABCD)$ [By Theorem 1]

Now, in $\|^{gm}ABQP$, PB is a diagonal which divides it into two equal

parts.

So, $ar(\Delta PAB) = ar(\Delta BQP)$ [By Theorem 2]

$ar(ABPQ) = ar(\Delta PAB) + ar(\Delta BQP)$

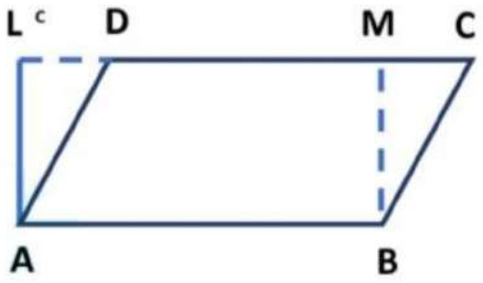
$ar(ABPQ) = ar(\Delta PAB) + ar(\Delta PAB)$ [From 2]

$ar(ABCD) = 2ar(\Delta PAB)$ [From 1]

$ar(\Delta PAB) = \frac{1}{2} ar(\|^{gm}ABCD)$

Theorem 4: The area of a parallelogram is the product of its base and the corresponding altitude.

Given: A parallelogram ABCD in which AB is the base and AL is the corresponding altitude.



To prove: ar($\|^{gm}$ ABCD) = AB × AL

Construction: Complete the rectangle

ALMB by drawing BM ⊥ CD.

Proof: In ΔADL and ΔBCM

∠ALD = ∠BMC (Each is equal to 90°)

∠ADL = ∠BCM

(Corresponding angles from AD ∥ BC and transversal LC)

AD = BC (Opposite side of the parallelogram ABCD)

So, ΔADL ≅ ΔBCM [By AAS criterion of congruence]

And we know that congruence figures have equal areas

Therefore, ar (ΔADL) = ar (ΔBCM)ar(∥gmABCD) = ar(ADMB) + ar(ΔBMC) [By area addition axiom]

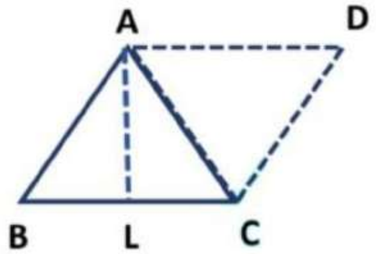
ar($\|^{gm}$ ABCD) = ar(ADMB) + ar(ΔADL) [From 1]

ar($\|^{gm}$ ABCD) = ar(ABML) [ar(ABML) = AB × AL]

ar($\|^{gm}$ ABCD) = AB × AL

Hence, area of the ∥gm is the product of its side and the corresponding altitude [ar (∥gm) = Base× Height].

Theorem 5: The area of a triangle is half the product of any of its sides and the corresponding altitude.



Given: ΔABC in which AL is the altitude to the side BC.

To prove: $ar(\Delta ABC) = \frac{1}{2}(BC \times AL)$

Construction: Through C and A draw CD∥BA and AD∥BC respectively, meeting each other at D.

Proof: We have,

BA ∥ CD [By construction]

AD ∥BC [By construction]

Since a quadrilateral is a parallelogram if both pairs of opposite sides are parallel

∴ BCDA is a parallelogram.

As AC is a diagonal of ∥gm BCDA

$\therefore ar(\Delta ABC) = \frac{1}{2} ar(\parallel^{gm} BCDA)$

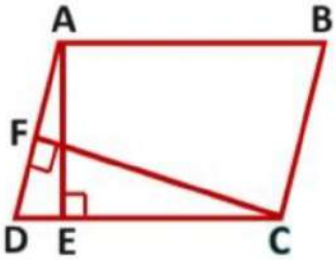
By theorem 4, the area of a ∥gm is the product of its base and the corresponding altitude.

$ar(\parallel^{gm} BCDA) = BC \times AL$

$\Rightarrow ar(\Delta ABC = \frac{1}{2}(BC \times AL)$

Example 1: In the figure, ABCD is a parallelogram, AE⊥DC and CF⊥AD. If AB=16 cm, AE=8 cm and CF=10 cm, find AD.

(REFERENCE: NCERT)



In parallelogram ABCD, CD=AB=16 cm(Opposite sides of a parallelogram are equal)

We know that

Area of a parallelogram =Base× altitude.

For altitude AE with base CD

$ar(\|^{gm}) = CD \times AE$

For altitude CF with base AD

$ar\ (\|^{gm}) = AD \times CF$

Now, by using equation (1) and (2), we get

$CD \times AE = AD \times CF$

$16 \times 8 = AD \times 10$

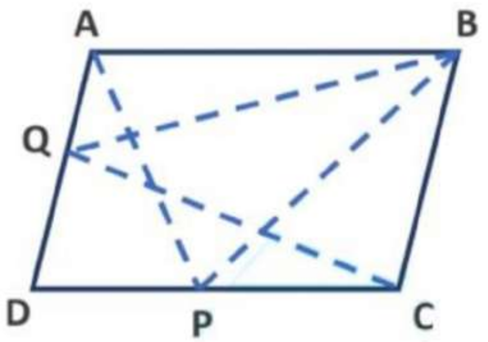
$128 = AD \times 10$

$\frac{128}{10} = AD$

$AD = 12.8\ cm$

Example 2: P and Q are any two points lying on the sides DC and AD respectively of a parallelogram ABCD. Show that ar(ΔAPB) = ar(ΔBQC).

(REFERENCE: NCERT)



Here, ΔAPB and parallelogram ABCD are on the same base AB and between the same parallels AB and DC.

$\therefore ar(\Delta APB) = \frac{1}{2} ar(\|^{gm} ABCD)$ [Theorem 3]

Similarly, ΔBQC and parallelogram ABCD are on the same base BC and between the same parallels BC and AD.

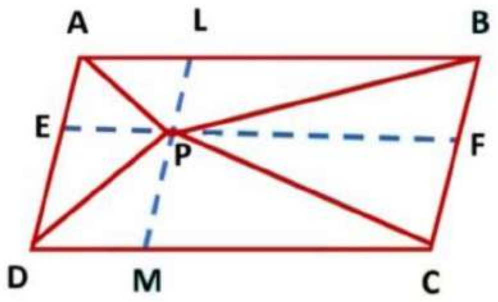
$\therefore ar(\Delta BQC) = \frac{1}{2} ar(\|^{gm} ABCD)$ [Theorem 3]

Now, by using equation (1) and (2), we get

$ar(\Delta APB) = ar(\Delta BQC)$

Example 3: In figure, P is a point in the interior of a parallelogram ABCD. Show that

1. $ar(APB) + ar(PCD) = \frac{1}{2} ar(ABCD)$
2. $ar(APD) + ar(PBC) = ar(APB) + ar(PCD)$



(REFERENCE: NCERT)

Construction: Draw a line segment EF parallel to the line segment AB passing through P. Draw a line segment LM parallel to the line segment AD passing through P.

(1). In parallelogram ABCD,

AB∥EF (By construction)

AD∥BC (Opposite side of a parallelogram ABCD)

⇒AE∥BF

From equation (1) and (2), we obtain

AB∥EF and AE∥BF

Therefore, quadrilateral ABFE is a parallelogram.

Since ΔAPB and parallelogram ABFE are on the same base AB and between the same parallels AB and EF.

$\therefore ar(\Delta APB) = \frac{1}{2} ar(\|gm\ ABFE)$ [By Theorem 3]

Similarly, ΔPCD and parallelogram EFCD are on the same base DC and between the same parallels DC and EF.

$\therefore ar(\Delta PCD) = \frac{1}{2} ar\ (\|gm\ EFCD)$ [By Theorem 3]

By adding (3) and (4), we get

$ar(\Delta APB) + ar(\Delta PCD) = \frac{1}{2}\ ar(\|gm\ ABFE) + 12ar(\|gm\ EFCD).$

$ar(\Delta APB) + ar\ (\Delta PCD) = \frac{1}{2}\ ar(\|gm\ ABCD).$

(2). Quadrilateral ALMD is formed the parallelogram.

Since ΔAPD and parallelogram ALMD are on the same base AB and between the same parallels AD and LM.

$\therefore ar(\Delta APD) = \frac{1}{2}\ ar(\|gm\ ALMD)$ [By Theorem 3]

Similarly, ΔPBC and parallelogram BLMC are on the same base BC and between the same parallels BC and LM.

$\therefore ar(\Delta PBC) = \frac{1}{2}\ ar\ (\|gm\ BLMC)$ [By Theorem 3]

By adding (6) and (7), we get

$ar\ (\Delta APD) + ar\ (\Delta PBC) = \frac{1}{2} ar\ (\|gm\ ALMD) + \frac{1}{2}\ ar(\|gm\ BLMC)$

$ar\ (\Delta APD) + ar(\Delta PBC) = \frac{1}{2}\ ar(\|gm\ ABCD)$

By comparing equation (5) and (8), we get

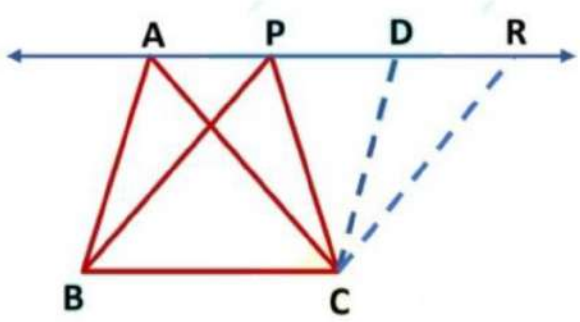
$ar(APD) + ar(PBC) = ar(APB) + ar(PCD)$

Triangles on the same base and between the same parallels

Triangles on the same base and between the same parallels.

Theorem 6: Two triangles on the same base (or equal bases) and between the same parallels are equal in area.

Given: Two triangles ABC and PBC on the same base BC and between the same parallels BC and AP.



To prove: ar(ΔABC) = ar(ΔPBC)

Construction: Draw CD ∥ BA and CR ∥ BP such that D and R lie on the same line AP.

Proof: By the construction, we get

Two parallelograms PBCR and ABCD on the same base BC and between the same parallels BC and AR.

∴ ar(∥^{gm} ABCD) = ar(∥^{gm} PBCR)   [Theorem 1]

Now, we know that the diagonal of a parallelogram divides it into two triangles of equal area.

∴ ar(ΔPBC) = $\frac{1}{2}$ ar(∥^{gm} PBCR)   (∥^{gm} PBCR with diagonal PC)

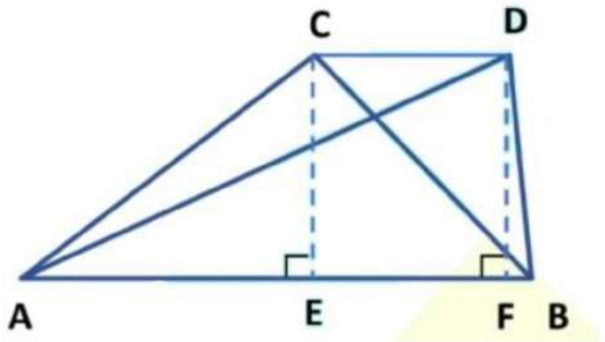
By multiplying both sides with $\frac{1}{2}$, we get

⇒ $\frac{1}{2}$ ar(∥gm ABCD) = $\frac{1}{2}$ ar(∥gm PBCR)

From equation (2) and (3), we obtain

ar(ΔABC) = ar(ΔPBC).

Theorem 7: Two triangles having the same base (or equal base) and equal areas lie between the same parallels.



Given: ΔABC and ΔADC both lie on the same base in a way that ar(ΔABC) = ar(ΔADC).

To prove: ΔABC and ΔADB lie between the same parallel. i.e. CD ∥ AB Construction: Draws altitudes CE and DF of ΔACB & ΔADB on AB respectively.

Proof: Now according to question, it's said that ΔABC and ΔABD both lie on the same base and both have equal area.

Now, by construction, we have CE perpendicular to AB and DF perpendicular to AB.

Now, we know that lines perpendicular to the same line are parallel to each other, therefore, CE ∥ DF.

And it is given that, ar (ΔABC) = ar (ΔABD)

We know that,

Area of Triangle = $\frac{1}{2}$ Base x Height).

Area Of ΔABC = Area of ΔABD.

$\Rightarrow \frac{1}{2}\ AB \times CE = \frac{1}{2} \times AB \times DF$

$\Rightarrow \qquad CE = DF.$

From equation (1) & (2), we have

CE = DF and CE ∥ DF.

So, we know that if in a quadrilateral if one pair of opposite sides are equal and parallel then the quadrilateral is a parallelogram.

Therefore, CDEF is a parallelogram.

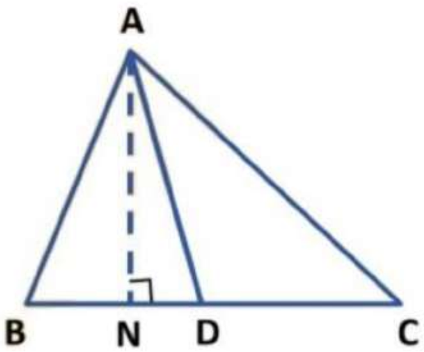
CD∥EF (Opposite sides of the parallelogram)

Hence, CD ∥ AB.

Theorem 8: The median of a triangle divides it into two triangles of equal areas.

Given: ΔABC with a median AD.

To prove: ar(ΔABD) = ar(ΔACD) Construction: Draw AN⊥BC.



Proof: By theorem 5, we know that, area of a triangle is half the product of its base and corresponding altitude.

ar(ΔABD) = $\frac{1}{2}$ × base × altitude of ΔABD.

ar(ΔABD) = $\frac{1}{2}$ × BD × AN

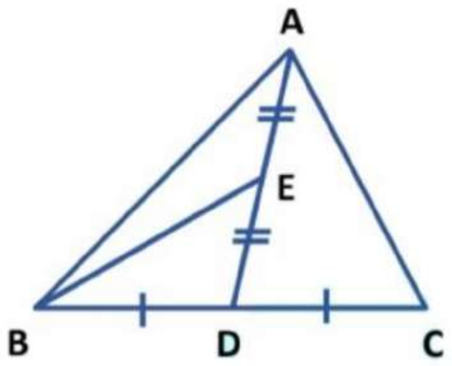
Now, ar(ΔACD) = $\frac{1}{2}$× CD × AN

ar(ΔACD) = $\frac{1}{2}$ × BD × AN (∵ BD = CD)

From 1 & 2, ar(ΔABD) = ar (ΔACD).

Examples 1: In a triangle ABC, E is the mid-point of median AD.

Show that $ar(\Delta BED) = \frac{1}{4} ar(\Delta ABC)$.   [Reference: NCERT]



Given: ΔABC with median AD and E is the mid-point of median to AD.

To prove: $ar(\Delta BED) = \frac{1}{4} ar(\Delta ABC)$.

Proof: By theorem 7, AD is a median of ΔABC and median divides the triangle into two triangles of the equal area.

$\therefore ar(\Delta ABD) = ar(\Delta ADC)$

$\Rightarrow ar(\Delta ABD) = \frac{1}{2} ar(\Delta ABC)$

Similarly, in (ΔABD), BE is the median

$\therefore ar(\Delta BED) = ar(\Delta BAE)$

$\Rightarrow ar(\Delta BED) = \frac{1}{2} ar(\Delta ABD)$

$\Rightarrow ar(\Delta BED) = \frac{1}{2} \times \frac{1}{2} ar(\Delta ABC)$ [From (1)]

$\Rightarrow ar(\Delta BED) = \frac{1}{4} ar(\Delta ABC)$.

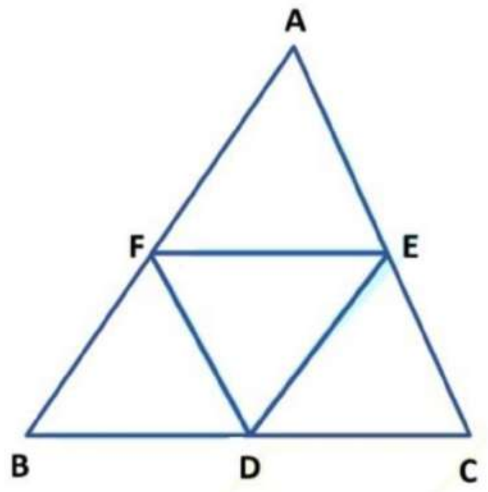
Hence, $ar(\Delta BED) = \frac{1}{4} ar(\Delta ABC)$.

Example 2: D, E, and F are respectively the mid-points of the sides BC, CA and AB of a ΔABC. Show that

(1) BDEF is a parallelogram

(2) $ar(DEF) = \frac{1}{4} ar(\Delta ABC)$

(3) $ar(BDEF) = \frac{1}{2} ar(\Delta ABC)$ [Reference: NCERT]



(1) In ΔABC, D and E are the mid-point of sides BC and AC respectively.

$\therefore DE \parallel BA$

$\Rightarrow DE \parallel BF$

[$\because$ Line joining the mid-point of two sides of a triangle is parallel to the third side and half of it.]

Similarly, $FE \parallel BC$

$\Rightarrow FE \parallel BD$

From (1) and (2), we obtain

BDEF is a parallelogram

(2). Now, DF is a diagonal of parallelogram BDEF.

By theorem 2, a diagonal of a parallelogram divides it into two triangles of equal area.

$\therefore$ ar(ΔBDF) = ar(ΔDEF)

Similarly, DE is a diagonal of ‖gm DCEF

$\therefore$ ar(Δ DCE) = ar(ΔDEF)

And, FE is a diagonal of ‖gm AFDE

$\therefore$ ar(Δ AFE) = ar(ΔDEF)

From equation (3), (4) and (5), we get

ar(Δ BDF) = ar(Δ DCE) = ar(Δ AFE) = ar(ΔDEF)

Now, ar(Δ ABC) = ar(Δ BDF) + ar(Δ DCE) + ar(Δ AFE) + ar(ΔDEF)

ar(Δ ABC) = 4 ar(ΔDEF) [Using equation (6)]

$\frac{1}{4}$ ar(Δ ABC) = ar(ΔDEF)

ar(ΔDEF) = $\frac{1}{4}$ ar(Δ ABC)

(3) ar(‖gm BDEF) = ar(ΔBDF) + ar(ΔDEF) [Using equation (3)]

ar(‖gm BDEF) = ar(ΔDEF) + ar(ΔDEF)

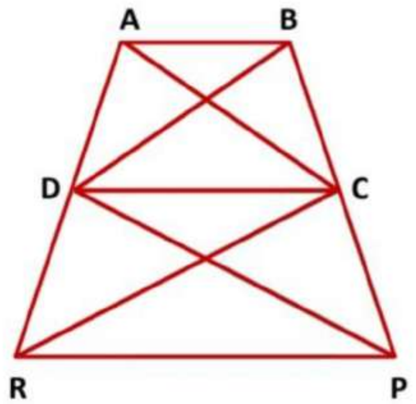
ar(‖gm BDEF) = 2 ar(ΔDEF)

ar(‖gm BDEF) = 2 × $\frac{1}{4}$

ar(Δ ABC) [From equation (7)]

ar(‖gm BDEF) =1/2 ar(Δ ABC)

Example 3: In figure, ar(ΔDRC) = ar(ΔDPC) and ar(ΔBDP) = ar(ΔARC). Show that both the quadrilateral ABCD and DCPR are trapeziums. [Reference: NCERT]



It is given that, ar(ΔDRC) = ar(ΔDPC).

Here, ΔDRC and ΔDPC are on the same base CD and are equal in area. Therefore, they must be between the same parallels.

∴ RP∥DC. Therefore, DCPR is a trapezium.

We also have, ar(ΔBDP) = ar(ΔARC)

Now, subtracting ar(ΔDRC) from both sides, we get

⇒ ar(ΔBDP) - ar(ΔDRC) = ar(ΔARC) - ar(ΔDRC) [From equation (1)]

⇒ ar(ΔBDP) - ar(ΔDPC) = ar(ΔARC) - ar(ΔDRC)

⇒ ar(ΔADC) = ar(ΔBDC)

∴ ΔADC and ΔBDC are on the same base DC and are equal in area.

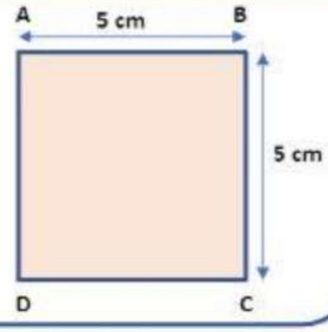
Therefore, they must be between the same parallel lines, i.e. AB∥DC.

Therefore, ABCD is a trapezium.

Summary of Area of Parallelograms and Triangles

Area

Area of figure ABCD is the number of unit square associated with the part of the plane enclosed by that figure. ar(ABCD) = 25 cm^2

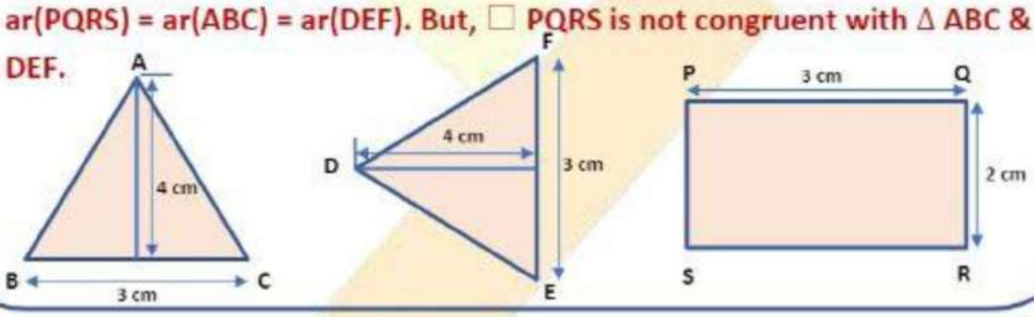


CONGRUENT AREA AXIOM

Two congruent figures have equal areas but the converse need not be always true.

Δ ABC ≅ Δ DEF & ar(ABC) = ar(DEF).

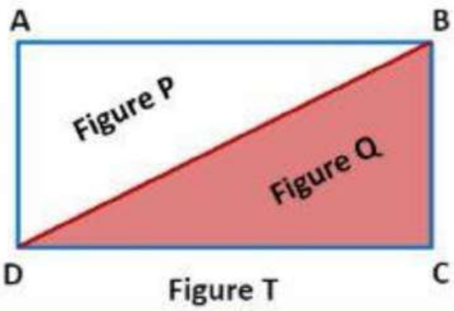
ar(PQRS) = ar(ABC) = ar(DEF). But, □ PQRS is not congruent with Δ ABC & Δ DEF.



AREA ADDITION AXIOM

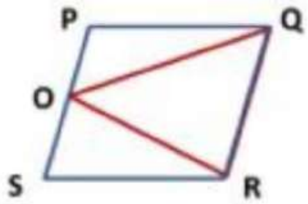
If a planer region formed by a figure T is made up of two non-overlapping planar regions formed by figures P and Q, then ar (T) = ar (P) + ar (Q).

ar(ABCD) = ar(ΔADB) + ar(ΔBCD).



Figures on the same base and between the same parallels

Two figures are said to be on the same base and between same parallels, if they have a common base (side) and the vertices (or the vertex) opposite to the common base of each figure lie on a line parallel to the base.

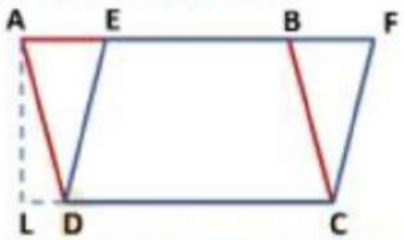


$\|^{gm}$ PQRS & Δ ROQ are on the same base RQ and between the same parallel's PS and QR.

Parallelograms on the same base and between the same parallels.

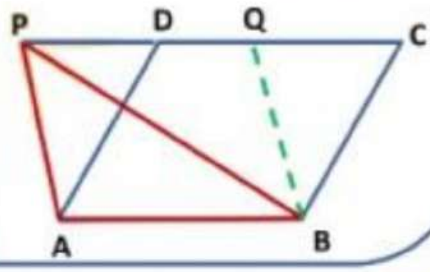
Parallelograms on the same base (or equal bases) and between the same parallels are equal in area.

ar($\|^{gm}$ ABCD) = ar($\|^{gm}$ EFCD)



The area of a parallelogram is the product of its base and the corresponding altitude. ar($\|^{gm}$ ABCD) = AB × AL.

If a triangle and a parallelogram are on the same base and between the same parallels, then the area of the triangle is equal to half of the area of the parallelogram. ar(ΔPAB) = $\frac{1}{2}$ ar($\|^{gm}$ ABCD)



Triangles on the same base and between the same parallels.

Two triangles on the same base (or equal bases) and between the same parallels are equal in area. ar(ΔABC) = ar(ΔPBC).

Two triangles having the same base (or equal base) and equal areas lie between the same parallels. ΔABC and ΔPBC lie between the same parallel. i.e. DE $\|$ BC.