Limits

Quick Revision

If x approches a i.e. $x \to a$, then $f(x)$ approaches l i.e. $f(x) \to l$, where l is a real number, then l is called **limit** of the function $f(x)$. In symbolic form, it can be written as $\lim_{x \to a} f(x) = l$.

Left Hand and Right Hand Limits

If values of the function at the point which are very near to a on the left tends to a definite unique number as x tends to a, then the unique number so obtained is called the Left Hand Limit (LHL) of $f(x)$ at $x = a$, we write it as

$$f(a-0) = \lim_{x \to a^-} f(x) = \lim_{h \to 0} f(a-h)$$

Similarly, Right Hand Limit (RHL) is

$$f(a+0) = \lim_{x \to a^+} f(x) = \lim_{h \to 0} f(a+h)$$

Existence of Limit

If the right hand limit and left hand limit coincide (i.e. same), then we say that limit exists and their common value is called the limit of $f(x)$ at $x = a$ and denoted it by $\lim_{x \to a} f(x)$.

Algebra of Limits

Let f and g be two functions such that both $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist, then

(i) $\lim_{x \to a} [f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$

(ii) $\lim_{x \to a} kf(x) = k \lim_{x \to a} f(x)$

(iii) $\lim_{x \to a} f(x) \cdot g(x) = \lim_{x \to a} f(x) \times \lim_{x \to a} g(x)$

(iv) $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$, where $\lim_{x \to a} g(x) \neq 0$

Limits of a Polynomial Function

A function f is said to be a polynomial function if $f(x)$ is zero function

or if $f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$,

where a_i's are real number such that $a_n \neq 0$.

Then, limit of polynomial functions is

$$f(x) = \lim_{x \to a} f(x) = \lim_{x \to a} [a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n]$$

$$= a_0 + a_1 a + a_2 a^2 + \ldots + a_n a^n = f(a)$$

Limits of Rational Functions

A function f is said to be a rational function, if

$f(x) = \dfrac{g(x)}{h(x)}$, where $g(x)$ and $h(x)$ are polynomial functions such that $h(x) \neq 0$.

Then, $\lim_{x \to a} f(x) = \lim_{x \to a} \dfrac{g(x)}{h(x)} = \dfrac{\lim_{x \to a} g(x)}{\lim_{x \to a} h(x)} = \dfrac{g(a)}{h(a)}$

However, if $h(a) = 0$, then there are two cases arise,

(i) $g(a) \neq 0$ (ii) $g(a) = 0$.

In the first case, we say that the limit does not exist.

In the second case, we can find limit.

Limit of a rational function can be find with the help of following methods

1. **Direct Substitution Method** In this method, we substitute the point, to which the variable tends to in the given limit. If it give us a real number, then the number so obtained is the limit of the function and if it does not give us a real number, then use other methods.

2. **Factorisation Method** Let $\lim_{x \to a} \frac{f(x)}{g(x)}$ reduces to the form $\frac{0}{0}$, when we substitute $x = a$. Then, we factorise $f(x)$ and $g(x)$ and then cancel out the common factor to evaluate the limit.

3. **Rationalisation Method** If we get $\frac{0}{0}$ form and numerator or denominator or both have radical sign, then we rationalise the numerator or denominator or both by multiplying their conjugate to remove $\frac{0}{0}$ form and then find limit by direct substitution method.

Some Standard Limits

(i) $\lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}$

(ii) $\lim_{x \to 0} \frac{\sin x}{x} = 1$

(iii) $\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$

(iv) $\lim_{x \to 0} \frac{\tan x}{x} = 1$

(v) $\lim_{x \to 0} \frac{a^x - 1}{x} = \log_e a$

(vi) $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$

(vii) $\lim_{x \to 0} \frac{\log(1 + x)}{x} = 1$

(viii) $\lim_{x \to 0} \frac{\log(1 - x)}{-x} = 1$

Objective Questions

Multiple Choice Questions

1. The value of $\lim_{x \to 3} (4x^3 - 2x^2 - x + 1)$ is equal to

(a) 40 (b) 20
(c) 38 (d) 88

2. If $f(x) = \begin{cases} 1 + x^2, \text{ if } 0 \le x \le 1 \\ 2 - x^2, \text{ if } x > 1 \end{cases}$, then right hand limit of $f(x)$ at $x = 1$ is equal to

(a) 1 (b) 2
(c) 3 (d) 4

3. If $f(x) = \begin{cases} 2x + 3, \text{ if } x \le 2 \\ x + 5, \text{ if } x > 2 \end{cases}$, then the left hand limit of $f(x)$ at $x = 2$ is equal to

(a) 6 (b) 7
(c) 9 (d) 8

4. If $f(x) = \begin{cases} \frac{|x - 3|}{x - 3}, & x \ne 3 \\ 0, & x = 3 \end{cases}$, then left hand limit of $f(x)$ at $x = 3$ is equal to

(a) 1 (b) –1
(c) 2 (d) 0

5. The value of $\lim_{x \to 0} \frac{\sqrt{2 + x} + \sqrt{2 - x}}{2 + x}$ is equal to

(a) 2 (b) $\sqrt{2}$
(c) $2\sqrt{2}$ (d) $\sqrt{3}$

6. The value of $\lim_{x \to 1} \frac{x - 4}{3 - \sqrt{13 - x}}$ is equal to

(a) $3+2\sqrt{3}$ (b) $3-2\sqrt{3}$
(c) $2+\sqrt{3}$ (d) $2-\sqrt{3}$

7. $\lim_{x \to 1} \frac{(\sqrt{x} - 1)(2x - 3)}{2x^2 + x - 3}$ is equal to **[NCERT Exemplar]**

(a) $\frac{1}{10}$ (b) $\frac{-1}{10}$
(c) 1 (d) None of these

8. The value of $\lim_{x \to 1/2} \left[\frac{4x^2 - 1}{2x - 1}\right]$ is equal to

(a) 1 (b) 2
(c) 3 (d) 4

9. The value of $\lim_{x \to 2} \frac{x^3 - 8}{(x - 2)}$ is equal to

(a) 10 (b) 11
(c) 12 (d) 13

10. The value of $\lim_{x\to 1}\left(\frac{2}{1-x^2}+\frac{1}{x-1}\right)$ is equal to

(a) $\frac{1}{2}$ (b) $\frac{1}{3}$

(c) $\frac{1}{4}$ (d) 1

11. The value of $\lim_{x\to \frac{1}{2}}\frac{4x^2-1}{2x-1}$ is equal to

(a) 1 (b) 2

(c) 3 (d) 4

12. The value of $\lim_{x\to 0}\frac{x}{\sqrt{1+x}+1}$ is equal to

(a) 0 (b) 1

(c) 2 (d) 3

13. The value of $\lim_{x\to 1}\left[\frac{x^{15}-1}{x^{10}-1}\right]$ is equal to

(a) $\frac{3}{2}$ (b) $\frac{2}{3}$

(c) $\frac{4}{3}$ (d) $\frac{3}{4}$

14. If $\lim_{x\to 2}\frac{x^n-2^n}{x-2}=80$, then n is equal to

(a) 1 (b) 3

(c) 5 (d) 7

15. The value of $\lim_{x\to 0}\frac{(x+2)^{1/3}-2^{1/3}}{x}$ is equal to

(a) $\frac{1}{3(2)^{3/2}}$ (b) $\frac{1}{3(2)^{2/3}}$

(c) $\frac{1}{2(3)^{2/3}}$ (d) $\frac{1}{3\sqrt{3}}$

16. The value of $\lim_{x\to 0}\left[\frac{\sqrt{1+x}-1}{x}\right]$ is equal to

(a) $\frac{1}{3}$ (b) $\frac{1}{2}$

(c) $\frac{1}{4}$ (d) $\frac{1}{5}$

17. The value of $\lim_{x\to 0}\frac{\sin 7x}{\tan 5x}$ is equal to

(a) $\frac{5}{7}$ (b) $\frac{7}{5}$

(c) $\frac{2}{7}$ (d) $\frac{7}{2}$

18. The value of $\lim_{x\to 0}\frac{\sin 4x}{\sin 2x}$ is equal to

(a) 1 (b) 2

(c) 3 (d) 4

19. $\lim_{x\to 0}\frac{\tan x^\circ}{x^\circ}$ is equal to

(a) 1 (b) 3

(c) 2 (d) –1

20. $\lim_{x\to 0}\frac{1-\cos 4x}{x}$ is equal to

(a) 0 (b) 1

(c) 2 (d) 3

21. $\lim_{x\to 0}\frac{\sin x}{x(1+\cos x)}$ is equal to

(a) 0 (b) $\frac{1}{2}$

(c) 1 (d) –1

22. $\lim_{x\to 0}\frac{\sin(2+x)-\sin(2-x)}{x}$ is equal to $p\cos q$, where p and q are respectively

(a) 1, 2 (b) 2, 1

(c) 1, 1 (d) 2, 2

23. $\lim_{x\to 0}\frac{\tan x-\sin x}{\sin^3 x}$ is equal to

(a) $\frac{1}{2}$ (b) 0

(c) 1 (d) Not defined

24. $\lim_{x\to 0}\frac{\tan 2x-x}{3x-\sin x}$ is equal to ***[NCERT Exemplar]***

(a) 2 (b) $\frac{1}{2}$ (c) $\frac{-1}{2}$ (d) $\frac{1}{4}$

25. The value of $\lim_{x\to \frac{\pi}{2}}\frac{\tan 2x}{x-\frac{\pi}{2}}$ is equal to

(a) 1 (b) 3

(c) 4 (d) 2

26. $\lim_{x \to \pi/4} \dfrac{\sec^2 x - 2}{\tan x - 1}$ is *[NCERT Exemplar]*

(a) 3 (b) 1
(c) 0 (d) 2

27. $\lim_{x \to 0} \dfrac{\operatorname{cosec} x - \cot x}{x}$ is equal to *[NCERT Exemplar]*

(a) $\dfrac{-1}{2}$ (b) 1 (c) $\dfrac{1}{2}$ (d) 1

28. $\lim_{x \to 0} \dfrac{x^2 \cos x}{1 - \cos x}$ is equal to *[NCERT Exemplar]*

(a) 2 (b) $\dfrac{3}{2}$
(c) $\dfrac{-3}{2}$ (d) 1

29. $\lim_{x \to 0} \dfrac{e^{3x} - 1}{x}$ is equal to

(a) 1 (b) 2
(c) 3 (d) 4

30. $\lim_{x \to 3} \dfrac{e^x - e^3}{x - 3}$ is equal to

(a) e (b) e^2
(c) e^3 (d) e^4

31. $\lim_{x \to 0} \dfrac{e^{\sin x} - 1}{x}$ is equal to

(a) 1 (b) 2
(c) 3 (d) 4

32. $\lim_{x \to 0} \dfrac{e^x + e^{-x} - 2}{x^2}$ is equal to

(a) 1 (b) 2
(c) 3 (d) 4

33. $\lim_{x \to 0} \dfrac{3^x - 2^x}{x}$ is equal to

(a) $\log \dfrac{3}{2}$ (b) $\log \dfrac{2}{3}$
(c) $\log \dfrac{1}{2}$ (d) $\log \dfrac{1}{3}$

34. $\lim_{x \to 0} \dfrac{2^x - 1}{\sqrt{1 + x} - 1}$ is equal to

(a) log2 (b) 2log2
(c) 3log2 (d) 4log2

35. $\lim_{x \to 0} \dfrac{\log_e (1 + 2x)}{x}$ is equal to

(a) 1 (b) 2 (c) 3 (d) 4

Assertion-Reasoning MCQs

Directions (Q. Nos. 36-50) Each of these questions contains two statements Assertion (A) and Reason (R). Each of the questions has four alternative choices, any one of the which is the correct answer. You have to select one of the codes (a), (b), (c) and (d) given below.

(a) A is true, R is true; R is a correct explanation of A.
(b) A is true, R is true; R is not a correct explanation of A.
(c) A is true; R is false
(d) A is false; R is true.

36. **Assertion (A)** $\lim_{x \to 1} \dfrac{ax^2 + bx + c}{cx^2 + bx + a}$ is equal to 1, where $a + b + c \neq 0$.

Reason (R) $\lim_{x \to -2} \dfrac{\frac{1}{x} + \frac{1}{2}}{x + 2}$ is equal to $\dfrac{1}{4}$.

37. **Assertion (A)** $\lim_{x \to 0} \dfrac{\sin ax}{bx}$ is equal to $\dfrac{a}{b}$.

Reason (R) $\lim_{x \to 0} \dfrac{\sin x}{x} = 1$.

38. **Assertion (A)** $\lim_{x \to 0} \dfrac{\sin ax + bx}{ax + \sin bx}$ is equal to -2.

Reason (R) $\lim_{x \to 1} (5x^3 + 5x + 1)$ is equal to 11.

39. **Assertion (A)** $\lim_{x \to \pi} \dfrac{\sin(\pi - x)}{\pi(\pi - x)}$ is equal to π.

Reason (R) $\lim_{x \to 0} \dfrac{\cos x}{\pi - x}$ is equal to $\dfrac{1}{\pi}$.

40. **Assertion (A)** $\lim_{x \to 0} \dfrac{\sin ax}{\sin bx}$ is equal to $\dfrac{a}{b}$.

Reason (R) $\lim_{x \to 0} \dfrac{\sin x}{x} = 1$.

41. **Assertion (A)** $\lim_{x \to 0} \frac{\cos 2x - 1}{\cos x - 1}$ is equal to 4.

Reason (R) $\lim_{x \to 0} \frac{\tan x}{x} = 1.$

42. **Assertion (A)** $\lim_{x \to 0} \frac{ax + x\cos x}{b \sin x}$ is equal to $\frac{a+1}{b}$.

Reason (R) $\lim_{x \to 0} x \sec x$ is equal to 1.

43. **Assertion (A)** $\lim_{x \to 0} \frac{e^{3+x} - \sin x - e^3}{x}$ is equal to $e^3 + 1.$

Reason (R) $\lim_{x \to 0} \frac{\tan 4x}{\sin 2x}$ is equal to 2.

44. **Assertion (A)** $\lim_{x \to 0} \frac{e^x - e^{-x}}{x}$ is equal to 2.

Reason (R) $\lim_{x \to 0} \frac{e^x - 1}{x} = 1.$

45. **Assertion (A)** $\lim_{x \to 0} \frac{e^{\tan x} - 1}{x}$ is equal to 1.

Reason (R) $\lim_{x \to 0} \left(\frac{e^{4x} - 1}{x} \right)$ is equal to 2.

46. **Assertion (A)** $\lim_{x \to 0} \frac{e^x - e^{\sin x}}{x - \sin x}$ is equal to -1.

Reason (R) $\lim_{x \to 0} \left(\frac{3^{2x} - 2^{3x}}{x} \right)$ is equal to $\log \left(\frac{9}{8} \right)$.

47. **Assertion (A)** $\lim_{x \to 0} \frac{e^x - 1}{\sqrt{1 - \cos 2x}}$ is equal to $\frac{1}{\sqrt{2}}$.

Reason (R) $\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$ is equal to 1.

48. **Assertion (A)** $\lim_{x \to 0} \frac{3^x - 2^x}{\tan x}$ is equal to $\log \left(\frac{3}{2} \right)$.

Reason (R) $\lim_{x \to 0} \frac{\log(1+x)}{\tan x}$ is equal to 2.

49. **Assertion (A)** $\lim_{x \to 1} \frac{x - 1}{\log_e x}$ is equal to 1.

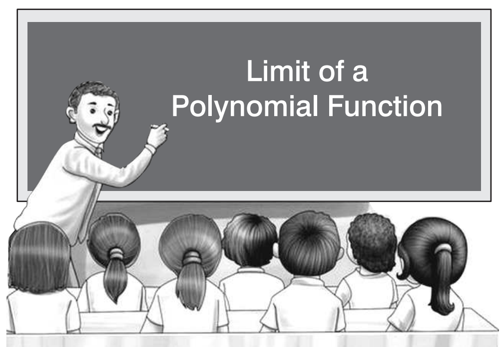
Reason (R) $\lim_{x \to 0} \frac{\log(\sin x + 1)}{x}$ is equal to 0.

50. **Assertion (A)** $\lim_{x \to 0} \frac{3^{2+x} - 9}{x}$ is equal to $9 \log 2$.

Reason (R) $\lim_{x \to 0} \frac{a^{\sin x} - 1}{\sin x}$ is equal to $\log a$.

Case Based MCQs

51. Raj was learning limit of a polynomial function from his tutor Rajesh.
His tutor told that a function f is said to be a polynomial function, if $f(x)$ is zero function.



Now, let
$f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$ be a polynomial function, where a_i's are real numbers and $a_n \neq 0$.

Then, limit of a polynomial function $f(x)$

$= \lim_{x \to a} f(x)$

$= \lim_{x \to a} [a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n]$

$= \lim_{x \to a} a_0 + \lim_{x \to a} a_1 x + \lim_{x \to a} a_2 x^2 + \ldots + \lim_{x \to a} a_n x^n$

$= a_0 + a_1 \lim_{x \to a} x + a_2 \lim_{x \to a} x^2 + \ldots + a_n \lim_{x \to a} x^n$

$= a_0 + a_1 a + a_2 a^2 + \ldots + a_n a^n = f(a)$

Based on above information, answer the following questions.

(i) $\lim_{x \to -1} (1 + x + x^2 + \ldots x^9)$ is equal to

(a) 0 (b) 1
(c) 2 (d) 3

(ii) $\lim_{x \to 5} [x^2 (x-1)]$ is equal to

(a) 10 (b) 100
(c) 25 (d) 125

(iii) $\lim_{x \to 2} (x^3 + x^2 + x - 1)$ is equal to

(a) 9 (b) 11
(c) 10 (d) 13

(iv) $\lim_{x \to -3} (x^3 + x + 2)$ is equal to

(a) 28 (b) –28
(c) 30 (d) –15

(v) $\lim_{x \to 4} (x^4 - x^3)$ is equal to

(a) 192 (b) 180
(c) 50 (d) 165

52. A function f is said to be a rational function, if $f(x) = \frac{g(x)}{h(x)}$, where $g(x)$ and $h(x)$ are polynomial functions such that $h(x) \neq 0$.

Then, $\lim_{x \to a} f(x) = \lim_{x \to a} \frac{g(x)}{h(x)}$

$= \frac{\lim_{x \to a} g(x)}{\lim_{x \to a} h(x)} = \frac{g(a)}{h(a)}$

However, if $h(a) = 0$, then there are two cases arise,

(i) $g(a) \neq 0$ (ii) $g(a) = 0$.

In the first case, we say that the limit does not exist.

In the second case, we can find limit.

Based on above information, answer the following questions.

(i) $\lim_{x \to -1} \left(\frac{x^{10} + x^5 + 1}{x - 1} \right)$ is equal to

(a) $\frac{1}{2}$ (b) $\frac{-1}{2}$
(c) 2 (d) $\frac{3}{2}$

(ii) $\lim_{x \to -1} \frac{(x-1)^2 + 3x^2}{(x^4 + 1)^2}$ is equal to

(a) $\frac{7}{4}$ (b) $\frac{6}{5}$
(c) $\frac{4}{7}$ (d) $\frac{3}{4}$

(iii) The value of $\lim_{x \to 2} \left[\frac{x^2 - 4}{x^3 - 4x^2 + 4x} \right]$ is

(a) 0 (b) 1
(c) 2 (d) Does not exist

(iv) $\lim_{x \to 1} \frac{x^7 - 2x^5 + 1}{x^3 - 3x^2 + 2}$ is equal to

(a) 0 (b) 1
(c) 2 (d) 3

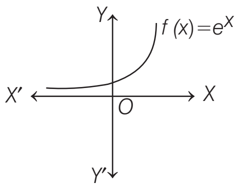
(v) $\lim_{x \to 0} \frac{\sqrt{1 + x^3} - \sqrt{1 - x^3}}{x^2}$ is equal to

(a) 1 (b) 0
(c) –1 (d) 2

53. The great Swiss Mathematician Leonhard Euler (1707-1783) introduced the number e, whose value lies between 2 and 3. This number is useful in defining exponential function.

A function of the form of $f(x) = e^x$ is called **exponential** function.

The graph of the function is given below



(i) Domain of $f(x) = (-\infty, \infty)$

(ii) Range of $f(x) = (0, \infty)$

To find the limit of a function involving exponential function, we use the following theorem

Theorem $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$

Based on above information, answer the following questions.

(i) $\lim_{x \to 4} \frac{e^x - e^4}{x - 4}$ is equal to

(a) e (b) e^2
(c) e^3 (d) e^4

(ii) $\lim_{x \to 0} \frac{e^{kx} - 1}{x}$ is equal to

(a) $\frac{k}{2}$ (b) k
(c) $-k$ (d) 1

(iii) $\lim_{x \to 0} \left(\frac{e^{-x} - 1}{x} \right)$ is equal to

(a) 1 (b) −1
(c) 0 (d) 2

(iv) $\lim_{x \to 0} \left(\frac{e^{5x} - e^{4x}}{x} \right)$ is equal to

(a) 1 (b) 2
(c) 3 (d) 4

(v) $\lim_{x \to 0} \left(\frac{2e^x - 3x - 2}{x} \right)$ is equal to

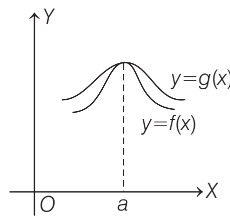
(a) −1 (b) 0
(c) 1 (d) 2

54. To find the limits of trigonometric functions, we use the following theorems

Theorem 1 Let f and g be two real valued functions with the same domain such that $f(x) \le g(x)$ for all x in the domain of definition. For some real number a, if both $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist, then

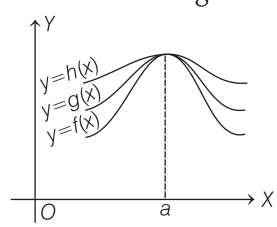
$$\lim_{x \to a} f(x) \le \lim_{x \to a} g(x).$$

This is shown in the figure



Theorem 2 (**Sandwich theorem**) Let f, g and h be real functions such that $f(x) \le g(x) \le h(x)$ for all x in the common domain of definition. For some real number a, if $\lim_{x \to a} f(x) = l = \lim_{x \to a} h(x)$, then $\lim_{x \to a} g(x) = l$.

This is shown in the figure



Theorem 3 Three important limits are

(i) $\lim_{x \to 0} \frac{\sin x}{x} = 1$

(ii) $\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$

(iii) $\lim_{x \to 0} \frac{\tan x}{x} = 1$

Based on above information, answer the following questions.

(i) $\lim_{x \to 0} \frac{\sin 3x}{5x}$ is equal to

(a) $\frac{1}{5}$ (b) $\frac{2}{5}$

(c) $\frac{3}{5}$ (d) $\frac{4}{5}$

(ii) $\lim_{\theta \to b} \frac{\tan(\theta - b)}{\theta - b}$ is equal to

(a) 0 (b) 1

(c) 2 (d) 3

(iii) $\lim_{x \to 0} \frac{\tan 2x - \sin 2x}{x^3}$ is equal to

(a) 4 (b) 3

(c) 2 (d) 1

(iv) $\lim_{x \to 0} \frac{2\sin x - \sin 2x}{x^3}$ is equal to

(a) 0 (b) 1

(c) 2 (d) 3

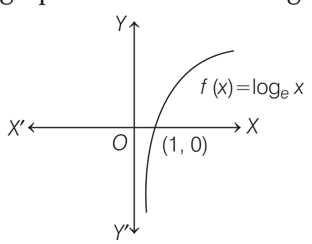
(v) $\lim_{x \to \frac{\pi}{4}} \frac{\sin x - \cos x}{x - \frac{\pi}{4}}$ is equal to

(a) $\sqrt{2}$ (b) 3

(c) 1 (d) $\sqrt{3}$

55. The logarithmic function expressed as $\log_e R^+ \to R$ and given by $\log_e x = y$ iff $e^y = x$.

The graph of the function is given below



(i) Domain of $f(x) = (0, \infty)$ or R^+

(ii) Range of $f(x) = (-\infty, \infty)$ or R

To find the limit of functions involving logarithmic function, we use the following theorem

Theorem $\lim_{x \to 0} \frac{\log_e(1 + x)}{x} = 1$

Based on above information, answer the following questions.

(i) $\lim_{x \to 0} \frac{\log_e(1 + 5x)}{x}$ is equal to

(a) 5 (b) 4

(c) 3 (d) 1

(ii) $\lim_{x \to 0} \frac{\log_e(1 + 6x) - 5x^2}{x}$ is equal to

(a) 1 (b) 2

(c) 3 (d) 6

(iii) $\lim_{x \to 0} \frac{\sqrt{1 + x} - 1}{\log(1 + x)}$ is equal to

(a) 1 (b) $\frac{1}{2}$

(c) $\frac{1}{3}$ (d) $\frac{3}{2}$

(iv) $\lim_{x \to 5} \frac{\log x - \log 5}{x - 5}$ is equal to

(a) $\frac{1}{5}$ (b) $\frac{3}{5}$

(c) $\frac{1}{4}$ (d) $\frac{2}{3}$

(v) $\lim_{x \to 0} \frac{\log(5 + x) - \log(5 - x)}{x}$ is equal to

(a) $\frac{1}{5}$ (b) $\frac{2}{5}$

(c) $\frac{3}{5}$ (d) $\frac{4}{5}$

ANSWERS

Multiple Choice Questions

1. (d)	*2. (a)*	*3. (b)*	*4. (b)*	*5. (b)*	*6. (a)*	*7. (b)*	*8. (b)*	*9. (c)*	*10. (a)*
11. (b)	*12. (a)*	*13. (a)*	*14. (c)*	*15. (b)*	*16. (b)*	*17. (b)*	*18. (b)*	*19. (a)*	*20. (a)*
21. (b)	*22. (d)*	*23. (a)*	*24. (b)*	*25. (d)*	*26. (d)*	*27. (c)*	*28. (a)*	*29. (c)*	*30. (c)*
31. (a)	*32. (a)*	*33. (a)*	*34. (b)*	*35. (b)*					

Assertion-Reasoning MCQs

36. (c)	*37. (a)*	*38. (d)*	*39. (d)*	*40. (a)*	*41. (b)*	*42. (c)*	*43. (d)*	*44. (a)*	*45. (c)*
46. (d)	*47. (c)*	*48. (c)*	*49. (c)*	*50. (d)*					

Case Based MCQs

51. (i) - (a); (ii) - (b); (iii) - (d); (iv) - (b); (v) - (a)

52. (i) - (b); (ii) - (a); (iii) - (d); (iv) - (b); (v) - (b)

53. (i) - (d); (ii) - (b); (iii) - (b); (iv) - (a); (v) - (a)

54. (i) - (c); (ii) - (b); (iii) - (a); (iv) - (b); (v) - (a)

55. (i) - (a); (ii) - (d); (iii) - (b); (iv) - (a); (v) - (b)

SOLUTIONS

1. $\lim_{x\to 3}(4x^3 - 2x^2 - x + 1)$

$= 4\lim_{x\to 3} x^3 - 2\lim_{x\to 3} x^2 - \lim_{x\to 3} x + \lim_{x\to 3} 1$

$= 4(3)^3 - 2(3)^2 - 3 + 1$

$= 108 - 18 - 2 = 88$

2. We have, $f(x) = \begin{cases} 1 + x^2, & \text{if } 0 \le x \le 1 \\ 2 - x^2, & \text{if } x > 1 \end{cases}$

$\text{RHL} = \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}(2 - x^2)$ $[\because f(x) = 2 - x^2, \text{ if } x > 1]$

$= \lim_{h\to 0}[2 - (1 + h)^2]$

[putting $x = 1 + h$ and when $x \to 1^+$, then $h \to 0$]

$= 2 - 1 = 1$

3. Given, $f(x) = \begin{cases} 2x + 3, & \text{if } x \le 2 \\ x + 5, & \text{if } x > 2 \end{cases}$

$\text{LHL} = \lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} 2x + 3$ $[\because f(x) = 2x + 3, \text{ if } x \le 2]$

$= \lim_{h\to 0}[2(2 - h) + 3] = 2(2 - 0) + 3$

[putting $x = 2 - h$ and when $x \to 2^-$, then $h \to 0$]

$= 4 + 3 = 7$

4. Given, $f(x) = \begin{cases} \dfrac{|x - 3|}{x - 3}, & x \ne 3 \\ 0, & x = 3 \end{cases}$

$\therefore$ Left hand limit at $x = 3$ is

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} \frac{|x - 3|}{x - 3} \quad \text{...(i)}$$

On putting $x = 3 - h$ and changing the limit $x \to 3^-$ by $h \to 0$ in Eq. (i), we get

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} \frac{|x - 3|}{x - 3} = \lim_{h\to 0} \frac{|-h|}{-h}$$

$$\Rightarrow \lim_{x\to 3^-} f(x) = \lim_{h\to 0} \frac{h}{(-h)} \quad [\because |x| = x]$$

$= -1$

5. $\lim_{x\to 0} \dfrac{\sqrt{2 + x} + \sqrt{2 - x}}{2 + x} = \dfrac{\sqrt{2 + 0} + \sqrt{2 - 0}}{2 + 0}$

$= \dfrac{\sqrt{2} + \sqrt{2}}{2} = \dfrac{2\sqrt{2}}{2} = \sqrt{2}$

6. $\lim_{x\to 1} \dfrac{x - 4}{3 - \sqrt{13 - x}} = \dfrac{-3}{3 - \sqrt{12}}$

$= \dfrac{-3}{3 - 2\sqrt{3}} = \dfrac{-3(3 + 2\sqrt{3})}{(3 - 2\sqrt{3})(3 + 2\sqrt{3})}$

$= \dfrac{-3(3 + 2\sqrt{3})}{9 - 12}$

$= \dfrac{-3(3 + 2\sqrt{3})}{-3} = 3 + 2\sqrt{3}$

7. Given, $\lim_{x\to 1}\frac{(\sqrt{x}-1)(2x-3)}{2x^2+x-3}$

$$= \lim_{x\to 1}\frac{(\sqrt{x}-1)(2x-3)}{(2x+3)(x-1)}$$

$$= \lim_{x\to 1}\frac{(\sqrt{x}-1)(2x-3)}{(2x+3)(\sqrt{x}-1)(\sqrt{x}+1)}$$

$$= \lim_{x\to 1}\frac{2x-3}{(2x+3)(\sqrt{x}+1)} = \frac{-1}{5\times 2} = \frac{-1}{10}$$

8. Given, $\lim_{x\to 1/2}\frac{4x^2-1}{2x-1}$

$$= \lim_{x\to 1/2}\frac{(2x)^2-(1)^2}{2x-1}$$

$$= \lim_{x\to 1/2}\frac{(2x+1)(2x-1)}{(2x-1)}$$

$$= \lim_{x\to 1/2}(2x+1)$$

$$= 2\times\frac{1}{2}+1 = 1+1 = 2$$

9. We have, $L = \lim_{x\to 2}\frac{x^3-8}{x-2}$.

Let $f(x) = x^3-8$ and $g(x) = x-2$

Here, $\lim_{x\to 2} f(x) = \lim_{x\to 2} x^3-8 = 2^3-8 = 0$

and $\lim_{x\to 2} g(x) = \lim_{x\to 2} x-2 = 2-2 = 0$

Thus, we get $\frac{0}{0}$ form.

Now, factorise $f(x)$ and $g(x)$ such that $(x-2)$ is a common factor.

Here, $f(x) = x^3-8 = (x^3-2^3)$

$= (x-2)(x^2+4+2x)$

and $g(x) = x-2$

$\therefore \quad L = \lim_{x\to 2}\frac{(x-2)(x^2+4+2x)}{(x-2)}$

On cancelling the common factor $(x-2)$, we get

$$L = \lim_{x\to 2}(x^2+4+2x) = (2)^2+4+2(2)$$

$$= 4+4+4 = 12$$

Hence, $\lim_{x\to 2}\frac{x^3-8}{x-2} = 12$

10. We have,

$$\lim_{x\to 1}\left(\frac{2}{1-x^2}+\frac{1}{x-1}\right) = \lim_{x\to 1}\left(\frac{2}{1-x^2}-\frac{1}{1-x}\right)$$

$[\infty-\infty \text{ form}]$

$$= \lim_{x\to 1}\frac{2-(1+x)}{1-x^2} \qquad \left[\frac{0}{0}\text{ form}\right]$$

$$= \lim_{x\to 1}\frac{1-x}{1-x^2} = \lim_{x\to 1}\frac{1}{1+x} = \frac{1}{2}.$$

11. On putting $x = \frac{1}{2}$, we get the form $\frac{0}{0}$.

So, let us first factorise it.

Consider,

$$\lim_{x\to\frac{1}{2}}\frac{4x^2-1}{2x-1} = \lim_{x\to\frac{1}{2}}\frac{(2x+1)(2x-1)}{(2x-1)}$$

[using factorisation method]

$$= \lim_{x\to\frac{1}{2}}(2x+1)$$

$$= 2\left(\frac{1}{2}\right)+1 = 2$$

12. $\lim_{x\to 0}\frac{x}{\sqrt{1+x}+1} = \lim_{x\to 0}\frac{x}{\sqrt{1+x}+1}\times\frac{\sqrt{1+x}-1}{\sqrt{1+x}-1}$

[multiplying numerator and denominator by $\sqrt{1+x}-x$]

$$= \lim_{x\to 0}\frac{x(\sqrt{1+x}-1)}{(1+x)-(1)^2} \quad [\because (a+b)(a-b) = a^2-b^2]$$

$$= \lim_{x\to 0}\frac{x(\sqrt{1+x}-1)}{x} = \lim_{x\to 0}(\sqrt{1+x}-1)$$

[cancel out x from numerator and denominator]

$$= \sqrt{1+0}-1 = 1-1 = 0 \qquad [\text{put } x = 0]$$

13. Given, $\lim_{x\to 1}\frac{x^{15}-1}{x^{10}-1} = \lim_{x\to 1}\left[\frac{x^{15}-1}{x-1}\div\frac{x^{10}-1}{x-1}\right]$

$$= \lim_{x\to 1}\left[\frac{x^{15}-1}{x-1}\right]\div\lim_{x\to 1}\left[\frac{x^{10}-1}{x-1}\right]$$

$$= 15(1)^{14}\div 10(1)^9 = 15\div 10 = \frac{3}{2}$$

14. Given, $\lim_{x\to 2}\frac{x^n-2^n}{x-2} = 80$

$\Rightarrow \quad n(2)^{n-1} = 80$

$\left[\because \lim_{x\to a}\frac{x^n-a^n}{x-a} = na^{n-1}\right]$

$\Rightarrow \quad n(2)^{n-1} = 5\times 16$

$\Rightarrow \quad n\times 2^{n-1} = 5\times(2)^4$

$\Rightarrow \quad n\times 2^{n-1} = 5\times(2)^{5-1}$

$\therefore \quad n = 5$

15. Given, $\lim_{x\to 0}\frac{(x+2)^{1/3}-2^{1/3}}{x}$

$$= \lim_{x\to 0}\frac{(x+2)^{1/3}-2^{1/3}}{(x+2)-2}$$

$$= \frac{1}{3}\times 2^{\frac{1}{3}-1}$$

$$= \frac{1}{3}\times(2)^{-2/3} = \frac{1}{3(2)^{2/3}}$$

16. Put $y = 1 + x$, so that $y \to 1$ as $x \to 0$.

Then, $\lim_{x\to 0}\frac{\sqrt{1+x}-1}{x} = \lim_{y\to 1}\frac{\sqrt{y}-1}{y-1}$

$$= \lim_{y\to 1}\frac{y^{\frac{1}{2}}-1^{\frac{1}{2}}}{y-1}$$

$$= \frac{1}{2}(1)^{\frac{1}{2}-1} = \frac{1}{2}$$

17. $\lim_{x\to 0}\frac{\sin 7x}{\tan 5x} = \lim_{x\to 0}\frac{7x\left(\frac{\sin 7x}{7x}\right)}{5x\left(\frac{\tan 5x}{5x}\right)} = \frac{7}{5}\,\frac{\lim_{x\to 0}\frac{\sin 7x}{7x}}{\lim_{x\to 0}\frac{\tan 5x}{5x}}$

$$= \frac{7}{5}\times\frac{1}{1} = \frac{7}{5}$$

$$\left[\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1 \text{ and } \lim_{\theta\to 0}\frac{\tan\theta}{\theta} = 1\right]$$

18. $\lim_{x\to 0}\frac{\sin 4x}{\sin 2x} = \lim_{x\to 0}\left[\frac{\sin 4x}{4x}\cdot\frac{2x}{\sin 2x}\cdot 2\right]$

$$= 2\cdot\left(\lim_{x\to 0}\left[\frac{\sin 4x}{4x}\right]\div\left[\frac{\sin 2x}{2x}\right]\right)$$

$$= 2\cdot\left(\lim_{4x\to 0}\left[\frac{\sin 4x}{4x}\right]\div\lim_{2x\to 0}\left[\frac{\sin 2x}{2x}\right]\right)$$

[as $x\to 0$, $4x\to 0$ and $2x\to 0$]

$$= 2(1\div 1) = 2$$

19. $\lim_{x\to 0}\frac{\tan x^\circ}{x^\circ} = \lim_{x\to 0}\frac{\tan\frac{\pi x}{180}}{\frac{\pi x}{180}} = 1 \quad \left[\because 1^\circ = \frac{\pi}{180}\text{ rad}\right]$

20. $\lim_{x\to 0}\frac{1-\cos 4x}{x} = \lim_{x\to 0}\frac{2\sin^2 2x}{x}\times\frac{x}{x}$

$$[\because 1-\cos 2\theta = 2\sin^2\theta]$$

$$= \lim_{x\to 0} 2\left(\frac{\sin 2x}{2x}\right)^2\times 4x$$

$$= 2\times 1\times 0 = 0 \quad \left[\because \lim_{x\to 0}\frac{\sin x}{x} = 1\right]$$

21. We have, $\lim_{x\to 0}\frac{\sin x}{x(1+\cos x)} = \lim_{x\to 0}\frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{x\left(2\cos^2\frac{x}{2}\right)}$

$$= \frac{1}{2}\lim_{x\to 0}\frac{\tan\frac{x}{2}}{\frac{x}{2}} = \frac{1}{2}$$

22. We have, $\lim_{x\to 0}\frac{\sin(2+x)-\sin(2-x)}{x}$

$$= \lim_{x\to 0}\frac{2\cos\frac{(2+x+2-x)}{2}\sin\frac{(2+x-2+x)}{2}}{x}$$

$$= \lim_{x\to 0}\frac{2\cos 2\sin x}{x}$$

$$\left[\because \sin C - \sin D = 2\cos\frac{C+D}{2}\sin\frac{C-D}{2}\right]$$

$$= 2\cos 2\lim_{x\to 0}\frac{\sin x}{x} = 2\cos 2 \quad \left[\because \lim_{x\to 0}\frac{\sin x}{x} = 1\right]$$

Hence, $p = 2$ and $q = 2$

23. We have, $\lim_{x\to 0}\frac{\tan x - \sin x}{\sin^3 x}$

$$= \lim_{x\to 0}\frac{\sin x\left(\frac{1}{\cos x}-1\right)}{\sin^3 x}$$

$$= \lim_{x\to 0}\frac{1-\cos x}{\cos x\sin^2 x}$$

$$= \lim_{x\to 0}\frac{2\sin^2\frac{x}{2}}{\cos x\left(4\sin^2\frac{x}{2}\cdot\cos^2\frac{x}{2}\right)} = \frac{1}{2}$$

24. Given, $\lim_{x\to 0}\frac{\tan 2x - x}{3x - \sin x}$

$$= \lim_{x\to 0}\frac{x\left[\frac{\tan 2x}{x}-1\right]}{x\left[3-\frac{\sin x}{x}\right]}$$

$$= \frac{\lim_{x\to 0} 2\times\frac{\tan 2x}{2x}-1}{3-\lim_{x\to 0}\frac{\sin x}{x}} = \frac{2-1}{3-1} = \frac{1}{2}$$

25. Given, $\lim_{x\to\frac{\pi}{2}}\frac{\tan 2x}{x-\frac{\pi}{2}}$

Let $x - \frac{\pi}{2} = h$,

when $x\to\frac{\pi}{2}$, then $h\to 0$

Therefore, given limit $= \lim_{h \to 0} \dfrac{\tan 2\left(\dfrac{\pi}{2} + h\right)}{h}$

$$= \lim_{h \to 0} \frac{\tan(\pi + 2h)}{h}$$

$$= \lim_{h \to 0} \frac{\tan 2h}{h} \quad [\because \tan(\pi + \theta) = \tan \theta]$$

$$= \lim_{h \to 0} \frac{2 \tan 2h}{2h}$$

$$= 2 \times 1 = 2 \quad \left[\because \lim_{x \to 0} \frac{\tan x}{x} = 1\right]$$

26. Given, $\lim_{x \to \pi/4} \dfrac{\sec^2 x - 2}{\tan x - 1}$

$$= \lim_{x \to \pi/4} \frac{1 + \tan^2 x - 2}{\tan x - 1}$$

$$= \lim_{x \to \pi/4} \frac{\tan^2 x - 1}{\tan x - 1}$$

$$= \lim_{x \to \pi/4} \frac{(\tan x + 1)(\tan x - 1)}{(\tan x - 1)}$$

$$= \lim_{x \to \pi/4} (\tan x + 1)$$

$$= 2$$

27. Given, $\lim_{x \to 0} \dfrac{\operatorname{cosec} x - \cot x}{x}$

$$= \lim_{x \to 0} \frac{\dfrac{1}{\sin x} - \dfrac{\cos x}{\sin x}}{x} = \lim_{x \to 0} \frac{1 - \cos x}{x \cdot \sin x}$$

$$= \lim_{x \to 0} \frac{2 \sin^2 \dfrac{x}{2}}{x \cdot 2 \sin \dfrac{x}{2} \cos \dfrac{x}{2}} = \lim_{x \to 0} \frac{\tan \dfrac{x}{2}}{x}$$

$$= \lim_{x \to 0} \frac{\tan \dfrac{x}{2}}{\dfrac{x}{2}} \cdot \frac{1}{2} = \frac{1}{2} \quad \left[\because \lim_{\theta \to 0} \frac{\tan \theta}{\theta} = 1\right]$$

28. Given, $\lim_{x \to 0} \dfrac{x^2 \cos x}{1 - \cos x} = \lim_{x \to 0} \dfrac{x^2 \cos x}{2 \sin^2 \dfrac{x}{2}}$

$$\left[\because 1 - \cos x = 2 \sin^2 \frac{x}{2}\right]$$

$$= 2 \lim_{x \to 0} \frac{\left(\dfrac{x}{2}\right)^2}{\sin^2 \dfrac{x}{2}} \cdot \lim_{x \to 0} \cos x$$

$$= 2 \cdot 1 = 2$$

29. $\lim_{x \to 0} \dfrac{e^{3x} - 1}{x} = \lim_{x \to 0} \dfrac{e^{3x} - 1}{x} \times \dfrac{3}{3}$

[multiplying numerator and denominator by 3]

$$= 3 \lim_{x \to 0} \frac{e^{3x} - 1}{3x} \quad \text{...(i)}$$

Let $h = 3x$.

Then, $x \to 0 \Rightarrow h \to 0$

Now, from Eq. (i), we get

$$\lim_{x \to 0} \frac{e^{3x} - 1}{x} = 3 \lim_{h \to 0} \frac{e^h - 1}{h} = 3(1)$$

$$\left[\because \lim_{\theta \to 0} \frac{e^\theta - 1}{\theta} = 1\right]$$

$$= 3$$

30. We have, $\lim_{x \to 3} \dfrac{e^x - e^3}{x - 3}$

On putting $h = x - 3$ we get

$$\lim_{x \to 3} \frac{e^x - e^3}{x - 3} = \lim_{h \to 0} \frac{e^{h+3} - e^3}{h}$$

$$[\because x \to 3 \Rightarrow h \to 0]$$

$$= \lim_{h \to 0} \frac{e^h e^3 - e^3}{h} = e^3 \lim_{h \to 0} \frac{e^h - 1}{h}$$

$$= e^3 \times 1 = e^3 \quad \left[\because \lim_{\theta \to 0} \frac{e^\theta - 1}{\theta} = 1\right]$$

31. $\lim_{x \to 0} \dfrac{e^{\sin x} - 1}{x} = \lim_{x \to 0} \dfrac{e^{\sin x} - 1}{x} \times \dfrac{\sin x}{\sin x}$

[multiplying numerator and denominator by $\sin x$]

$$= \lim_{x \to 0} \left[\frac{e^{\sin x} - 1}{\sin x} \times \frac{\sin x}{x}\right]$$

$$= \lim_{x \to 0} \frac{e^{\sin x} - 1}{\sin x} \times \lim_{x \to 0} \frac{\sin x}{x}$$

$$= 1 \times 1 = 1$$

$$\left[\because \lim_{\theta \to 0} \frac{e^\theta - 1}{\theta} = 1 \text{ and } \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1\right]$$

32. $\lim_{x \to 0} \dfrac{e^x + e^{-x} - 2}{x^2} = \lim_{x \to 0} \dfrac{e^{2x} + 1 - 2e^x}{x^2 e^x}$

$$= \lim_{x \to 0} \left(\frac{e^x - 1}{x}\right)^2 \times e^{-x}$$

$$= \lim_{x \to 0} \left(\frac{e^x - 1}{x}\right)^2 \times \lim_{x \to 0} e^{-x}$$

$$= (1)^2 \times e^0 = 1$$

33. $\lim_{x \to 0} \left(\frac{3^x - 2^x}{x} \right)$

$= \lim_{x \to 0} \left(\frac{3^x - 1}{x} \right) - \lim_{x \to 0} \left(\frac{2^x - 1}{x} \right)$

$= \log 3 - \log 2 = \log (3/2)$

34. $\lim_{x \to 0} \frac{2^x - 1}{\sqrt{1+x} - 1} = \lim_{x \to 0} \frac{2^x - 1}{\sqrt{1+x} - 1} \times \frac{(\sqrt{1+x} + 1)}{(\sqrt{1+x} + 1)}$

$= \lim_{x \to 0} \frac{2^x - 1}{x} \times \{\sqrt{1+x} + 1\}$

$= \lim_{x \to 0} \frac{2^x - 1}{x} \times \lim_{x \to 0} (\sqrt{1+x} + 1)$

$= (\log 2)\, 2 = 2 \log 2$

35. We have, $\lim_{x \to 0} \frac{\log_e (1 + 2x)}{x} \times \frac{2}{2}$

[multiplying numerator and denominator by 2]

$= 2 \lim_{x \to 0} \frac{\log_e (1 + 2x)}{2x}$

On putting $h = 2x$, we get

$\lim_{x \to 0} \frac{\log_e (1 + 2x)}{x} = 2 \lim_{h \to 0} \frac{\log_e (1 + h)}{h}$

$[\because x \to 0 \Rightarrow h \to 0]$

$= 2\,(1) \quad \left[\because \lim_{x \to 0} \frac{\log_e (1 + x)}{x} = 1 \right]$

$= 2$

36. **Assertion** Given, $\lim_{x \to 1} \frac{ax^2 + bx + c}{cx^2 + bx + a}$

$= \frac{a \times (1)^2 + b \times 1 + c}{c \times (1)^2 + b \times 1 + a}$

$= \frac{a + b + c}{c + b + a} = 1$

Reason $\lim_{x \to -2} \frac{\frac{1}{x} + \frac{1}{2}}{x + 2}$

$= \lim_{x \to -2} \frac{(2 + x)}{2x\,(x + 2)} = \lim_{x \to -2} \frac{1}{2x}$

$= \frac{1}{2(-2)} = -\frac{1}{4}$

Hence, Assertion is true and Reason is false.

37. **Assertion** Given, $\lim_{x \to 0} \frac{\sin ax}{bx} = \lim_{x \to 0} \frac{(a) \sin ax}{b(ax)}$

[dividing and multiplying by a]

$= \frac{a}{b} \times 1 = \frac{a}{b} \quad \left[\because \lim_{x \to 0} \frac{\sin ax}{ax} = 1 \right]$

Hence, Assertion and Reason both are true and Reason is the correct explanation of Assertion.

38. **Assertion** $\lim_{x \to 0} \frac{\sin ax + bx}{ax + \sin bx}$

Dividing each term by x, we get

$= \lim_{x \to 0} \frac{\frac{\sin ax}{x} + \frac{bx}{x}}{\frac{ax}{x} + \frac{\sin bx}{x}} = \lim_{x \to 0} \frac{\frac{a \sin ax}{ax} + b}{a + \frac{b \sin bx}{bx}}$

$= \frac{a \times 1 + b}{a + b \times 1} = \frac{a + b}{a + b} = 1 \quad \left[\because \lim_{x \to 0} \frac{\sin x}{x} = 1 \right]$

Reason $\lim_{x \to 1} (5x^3 + 5x + 1)$

$= 5(1)^3 + 5(1) + 1 = 5 + 5 + 1 = 11$

Hence, Assertion is false and Reason is true.

39. **Assertion** Given, $\lim_{x \to \pi} \frac{\sin(\pi - x)}{\pi\,(\pi - x)}$

Let $\pi - x = h$, As $x \to \pi$, then $h \to 0$

$\therefore \lim_{x \to \pi} \frac{\sin(\pi - x)}{\pi(\pi - x)} = \lim_{h \to 0} \frac{\sin h}{\pi h}$

$= \lim_{h \to 0} \frac{1}{\pi} \times \frac{\sin h}{h}$

$= \frac{1}{\pi} \times 1 = \frac{1}{\pi} \quad \left[\because \lim_{h \to 0} \frac{\sin h}{h} = 1 \right]$

Reason Given, $\lim_{x \to 0} \frac{\cos x}{\pi - x}$

Put the limit directly, we get

$\frac{\cos 0}{\pi - 0} = \frac{1}{\pi}$

Hence, Assertion is false and Reason is true.

40. **Assertion** Given, $\lim_{x \to 0} \frac{\sin ax}{\sin bx}$

Multiplying and dividing by (ax) and (bx), we get

$= \lim_{x \to 0} \frac{\sin ax}{ax} \times \frac{bx}{\sin bx} \times \frac{ax}{bx}$

$= 1 \times 1 \times \frac{a}{b} = \frac{a}{b}$

$\left[\because \lim_{x \to 0} \frac{\sin ax}{ax} = \lim_{x \to 0} \frac{bx}{\sin bx} = 1 \right]$

Hence, Assertion and Reason both are true and Reason is the correct explanation of Assertion.

41. **Assertion** Given, $\lim_{x \to 0} \frac{\cos 2x - 1}{\cos x - 1}$

$$= \lim_{x \to 0} \frac{1 - \cos 2x}{1 - \cos x} = \lim_{x \to 0} \frac{2 \sin^2 x}{2 \sin^2 \frac{x}{2}}$$

$$\left[\because 1 - \cos 2x = 2 \sin^2 x \text{ and } 1 - \cos x = 2 \sin^2 \frac{x}{2}\right]$$

Multiplying and dividing by x^2 and then multiplying by $\frac{4}{4}$ in the numerater. we get

$$= \lim_{x \to 0} \frac{\sin^2 x}{x^2} \times \frac{4 \times \frac{x^2}{4}}{\sin^2 \frac{x}{2}}$$

$$= \lim_{x \to 0} \left(\frac{\sin x}{x}\right)^2 \times \left(\frac{\frac{x}{2}}{\sin \frac{x}{2}}\right)^2 \times 4$$

$$= 1 \times 1 \times 4 = 4$$

Hence, Assertion and Reason both are true and Reason is not the correct explanation of Assertion.

42. **Assertion** Given, $\lim_{x \to 0} \frac{ax + x \cos x}{b \sin x}$

Dividing each term by x, we get

$$= \lim_{x \to 0} \frac{\frac{ax}{x} + \frac{x \cos x}{x}}{\frac{b \sin x}{x}} = \lim_{x \to 0} \frac{a + \cos x}{b\left(\frac{\sin x}{x}\right)}$$

$$= \frac{a + \cos 0}{b \times 1} = \frac{a + 1}{b} \qquad \left[\because \lim_{x \to 0} \frac{\sin x}{x} = 1\right]$$

Reason $\lim_{x \to 0} x \sec x = 0 \times \sec 0 = 0 \times 1 = 0$

Hence Assertion is true and Reason is false.

43. **Assertion** Given, limit $= \lim_{x \to 0} \frac{e^{3+x} - \sin x - e^3}{x}$

$$= \lim_{x \to 0} \left[\frac{e^3 (e^x - 1)}{x} - \frac{\sin x}{x}\right]$$

$$= e^3 - 1$$

Reason Given limit $= \lim_{x \to 0} \frac{\tan 4x}{\sin 2x}$

$$= \lim_{x \to 0} \left(\frac{\tan 4x}{4x}\right)\left(\frac{2x}{\sin 2x}\right)\left(\frac{4x}{2x}\right)$$

$$= \lim_{4x \to 0} \frac{\tan 4x}{4x} \times \frac{1}{\lim_{2x \to 0} \frac{\sin 2x}{2x}} \times \frac{4x}{2x}$$

$$= 1 \times 1 \times 2 = 2$$

Hence, Assertion is false and Reason is true.

44. **Assertion** Given, limit $= \lim_{x \to 0} \frac{e^x - e^{-x}}{x}$

$$= \lim_{x \to 0} \frac{e^{2x} - 1}{xe^x}$$

$$= \lim_{2x \to 0} \frac{e^{2x} - 1}{2x} \times \lim_{x \to 0} \frac{2}{e^x}$$

$$= 1 \times \frac{2}{1} = 2$$

Hence, Assertion and Reason both are true and Reason is the correct explanation of Assertion.

45. **Assertion** Given limit $= \lim_{x \to 0} \frac{e^{\tan x} - 1}{\tan x} \cdot \frac{\tan x}{x}$

$$= 1 \cdot 1 = 1$$

Reason Given limit $= \lim_{x \to 0} \left(\frac{e^{4x} - 1}{x}\right)$

$$= \lim_{4x \to 0} 4\left(\frac{e^{4x} - 1}{4x}\right) = 4$$

Hence, Assertion is true and Reason is false.

46. **Assertion** Given limit $= \lim_{x \to 0} \left(\frac{e^x - e^{\sin x}}{x - \sin x}\right)$

$$= \lim_{x \to 0} e^{\sin x} \left(\frac{e^{x - \sin x} - 1}{x - \sin x}\right) = e^{\sin 0} \times 1 = 1$$

Reason Given limit $= \lim_{x \to 0} \left(\frac{3^{2x} - 2^{3x}}{x}\right)$

$$= \lim_{x \to 0} \left(\frac{(3^{2x} - 1) - (2^{3x} - 1)}{x}\right)$$

$$= 2 \cdot \lim_{x \to 0} \left(\frac{3^{2x} - 1}{2x}\right) - 3 \cdot \lim_{x \to 0} \left(\frac{2^{3x} - 1}{3x}\right)$$

$$= 2 \log 3 - 3 \log 2$$

$$= \log 9 - \log 8$$

$$= \log\left(\frac{9}{8}\right)$$

Hence Assertion is false and Reason is true.

47. **Assertion** Given, limit $= \lim_{x\to 0} \frac{e^x - 1}{\sqrt{1-\cos 2x}}$

$$= \lim_{x\to 0} \frac{e^x - 1}{\sqrt{2\sin^2 x}}$$

$$= \frac{1}{\sqrt{2}} \lim_{x\to 0} \frac{e^x - 1}{x} \times \frac{x}{\sin x}$$

$$= \frac{1}{\sqrt{2}} \times 1 \times 1 = \frac{1}{\sqrt{2}}$$

Reason Given, limit $= \lim_{x\to 1} \frac{(x^2-1)}{(x-1)}$

$$= \lim_{x\to 1} \frac{(x-1)(x+1)}{(x-1)}$$

$$= \lim_{x\to 1} (x+1) = 2$$

Hence Assertion is true and Reason is false.

48. **Assertion** Given limit $= \lim_{x\to 0} \frac{3^x - 2^x}{\tan x}$

$$= \lim_{x\to 0} \frac{(3^x-1)-(2^x-1)}{x} \times \frac{x}{\tan x}$$

$$= \left(\lim_{x\to 0} \frac{3^x-1}{x} - \lim_{x\to 0} \frac{2^x-1}{x} \right) \times \lim_{x\to 0} \frac{x}{\tan x}$$

$$= (\log 3 - \log 2) \times 1$$

$$= \log\left(\frac{3}{2}\right)$$

Reason $\lim_{x\to 0} \frac{\log(1+x)}{\tan x} = \lim_{x\to 0} \frac{\log(1+x)}{x} \times \frac{x}{\tan x}$

$$= 1 \times 1 = 1$$

Hence Assertion is true and Reason is false.

49. **Assertion** Given limit $= \lim_{x\to 1} \frac{x-1}{\log_e x}$

Put $x = 1+h$ as $x \to 1$, $h \to 0$

$$\therefore \lim_{h\to 0} \frac{1+h-1}{\log_e(1+h)} = \frac{1}{\lim_{h\to 0} \frac{\log(1+h)}{h}} = 1$$

Reason Given, limit $= \lim_{x\to 0} \frac{\log(\sin x + 1)}{x}$

$$= \lim_{x\to 0} \frac{\log(\sin x+1)}{\sin x} \times \frac{\sin x}{x} = 1$$

Hence Assertion is true and Reason is false.

50. **Assertion** Given, limit $= \lim_{x\to 0} \frac{3^{2+x} - 9}{x}$

$$= \lim_{x\to 0} \frac{3^2(3^x-1)}{x} = 9 \log 3$$

Reason Given, limit $= \lim_{x\to 0} \frac{a^{\sin x} - 1}{\sin x}$

Let $y = \sin x$

Then, $y \to 0$ as $x \to 0$

$$\therefore \lim_{x\to 0} \frac{a^{\sin x}-1}{\sin x} = \lim_{y\to 0} \frac{a^y - 1}{y} = \log a$$

Hence, Assertion is false and Reason is true.

51. (i) Given, limit

$$= \lim_{x\to -1} (1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9)$$

$$= 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 = 0$$

(ii) Given, limit $= \lim_{x\to 5} [x^2(x-1)]$

$$= \lim_{x\to 5} [x^3 - x^2]$$

$$= \lim_{x\to 5} x^3 - \lim_{x\to 5} x^2$$

$$= (5)^3 - (5)^2$$

$$= 125 - 25 = 100$$

(iii) Given, limit $= \lim_{x\to 2} (x^3 + x^2 + x - 1)$

$$= \lim_{x\to 2} x^3 + \lim_{x\to 2} x^2 + \lim_{x\to 2} x + \lim_{x\to 2} (-1)$$

$$= (2)^3 + (2)^2 + (2) - 1 = 8 + 4 + 2 - 1 = 13$$

(iv) Given, limit $= \lim_{x\to -3} (x^3 + x + 2)$

$$= \lim_{x\to -3} x^3 + \lim_{x\to -3} x + \lim_{x\to -3} 2$$

$$= (-3)^3 + (-3) + 2 = -27 - 3 + 2$$

$$= -30 + 2 = -28$$

(v) Given, limit $= \lim_{x\to 4} (x^4 - x^3)$

$$= \lim_{x\to 4} x^4 - \lim_{x\to 4} x^3 = (4)^4 - (4)^3$$

$$= 256 - 64 = 192$$

52. (i) $\lim_{x\to -1} \frac{x^{10} + x^5 + 1}{x - 1} = \frac{(-1)^{10} + (-1)^5 + 1}{-1 - 1}$

$$= \frac{1 - 1 + 1}{-2} = \frac{-1}{2}$$

(ii) $\lim_{x\to -1} \frac{(x-1)^2 + 3x^2}{(x^4+1)^2} = \frac{(-1-1)^2 + 3(-1)^2}{((-1)^4+1)^2}$

$$= \frac{(-2)^2 + 3(1)}{(1+1)^2}$$

$$= \frac{4+3}{2^2} = \frac{7}{4}$$

(iii) Consider $f(x) = \dfrac{x^2 - 4}{x^3 - 4x^2 + 4x}$

On putting $x = 2$, we get

$$f(2) = \frac{4-4}{8-16+8} = \frac{0}{0}$$

i.e. it is the form $\dfrac{0}{0}$.

So, let us first factorise it.

Consider, $\lim_{x \to 2} \dfrac{x^2 - 4}{x^3 - 4x^2 + 4x}$

$$= \lim_{x \to 2} \frac{(x+2)(x-2)}{x(x-2)^2}$$

$$= \lim_{x \to 2} \frac{(x+2)}{x(x-2)}$$

$$= \frac{2+2}{2\,(2-2)} = \frac{4}{0}$$

which is not defined.

$\therefore \quad \lim_{x \to 2} \left[\dfrac{x^2 - 4}{x^3 - 4x^2 + 4x}\right]$ does not exist.

(iv) Given, $\lim_{x \to 1} \dfrac{x^7 - 2x^5 + 1}{x^3 - 3x^2 + 2} \qquad \left[\dfrac{0}{0} \text{ form}\right]$

$$= \lim_{x \to 1} \frac{x^7 - x^5 - x^5 + 1}{x^3 - x^2 - 2x^2 + 2}$$

$$= \lim_{x \to 1} \frac{x^5(x^2 - 1) - 1(x^5 - 1)}{x^2(x-1) - 2(x^2 - 1)}$$

On dividing numerator and denominator by $(x-1)$, then

$$= \lim_{x \to 1} \frac{\dfrac{x^5(x^2-1)}{(x-1)} - \dfrac{1(x^5-1)}{(x-1)}}{\dfrac{x^2(x-1)}{(x-1)} - \dfrac{2(x^2-1)}{(x-1)}}$$

$$= \frac{\lim_{x \to 1} x^5(x+1) - \lim_{x \to 1}\left(\dfrac{x^5 - 1}{x - 1}\right)}{\lim_{x \to 1} x^2 - \lim_{x \to 1} 2(x+1)}$$

$$= \frac{1 \times 2 - 5 \times (1)^4}{1 - 2 \times 2} = \frac{2-5}{1-4}$$

$$\left[\because \lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}\right]$$

$$= \frac{-3}{-3} = 1$$

(v) Given, $\lim_{x \to 0} \dfrac{\sqrt{1 + x^3} - \sqrt{1 - x^3}}{x^2}$

$$= \lim_{x \to 0} \frac{\sqrt{1 + x^3} - \sqrt{1 - x^3}}{x^2} \cdot \frac{\sqrt{1 + x^3} + \sqrt{1 - x^3}}{\sqrt{1 + x^3} + \sqrt{1 - x^3}}$$

$$= \lim_{x \to 0} \frac{(1 + x^3) - (1 - x^3)}{x^2(\sqrt{1 + x^3} + \sqrt{1 - x^3})}$$

$$= \lim_{x \to 0} \frac{1 + x^3 - 1 + x^3}{x^2(\sqrt{1 + x^3} + \sqrt{1 - x^3})}$$

$$= \lim_{x \to 0} \frac{2x^3}{x^2(\sqrt{1 + x^3} + \sqrt{1 - x^3})}$$

$$= \lim_{x \to 0} \frac{2x}{(\sqrt{1 + x^3} + \sqrt{1 - x^3})}$$

$$= 0$$

53. (i) We have, $\lim_{x \to 4} \dfrac{e^x - e^4}{x - 4}$

Put, $h = x - 4$

$$\therefore \lim_{x \to 4} \frac{e^x - e^4}{x - 4} = \lim_{h \to 0} \frac{e^{h+4} - e^4}{h}$$

$$= \lim_{h \to 0} \frac{e^h e^4 - e^4}{h}$$

$$= e^4 \lim_{h \to 0} \frac{e^h - 1}{h}$$

$$= e^4 \times 1 = e^4$$

(ii) We have, $\lim_{x \to 0} \dfrac{e^{kx} - 1}{x}$

$$= \lim_{kx \to 0} \left(\frac{e^{kx} - 1}{kx}\right) \times (k)$$

$$= 1 \times k = k \quad [\because x \to 0 \Rightarrow kx \to 0]$$

(iii) We have, $\lim_{x \to 0} \left(\dfrac{e^{-x} - 1}{x}\right)$

Put, $-x = y$, as $x \to 0 \Rightarrow y \to 0$

$$\therefore \lim_{x \to 0} \left(\frac{e^{-x} - 1}{x}\right) = \lim_{y \to 0} \left(\frac{e^y - 1}{-y}\right)$$

$$= -\lim_{y \to 0} \left(\frac{e^y - 1}{y}\right) = -1$$

(iv) We have, $\lim_{x \to 0}\left(\frac{e^{5x} - e^{4x}}{x}\right)$

$$= \lim_{x \to 0}\left\{\frac{(e^{5x} - 1) - (e^{4x} - 1)}{x}\right\}$$

$$= 5 \lim_{5x \to 0}\left(\frac{e^{5x} - 1}{5x}\right) - 4 \lim_{4x \to 0}\left(\frac{e^{4x} - 1}{4x}\right)$$

$[\because x \to 0 \Rightarrow 5x \to 0 \text{ and } 4x \to 0]$

$$= (5 \times 1) - (4 \times 1)$$

$$= 5 - 4 = 1$$

(v) We have, $\lim_{x \to 0}\left(\frac{2e^{x} - 3x - 2}{x}\right)$

$$= \lim_{x \to 0}\left(\frac{2(e^{x} - 1) - 3x}{x}\right)$$

$$= 2 \lim_{x \to 0}\left(\frac{e^{x} - 1}{x}\right) - 3 \lim_{x \to 0} \frac{x}{x}$$

$$= 2 \times 1 - 3 \lim_{x \to 0}(1)$$

$$= 2 \times 1 - 3 \times 1$$

$$= 2 - 3 = -1$$

54. (i) $\lim_{x\to 0} \frac{\sin 3x}{5x} = \lim_{x\to 0} \frac{\sin 3x}{5x \times \frac{3}{3}}$

$$= \lim_{x\to 0} \frac{3}{5} \cdot \frac{\sin 3x}{3x}$$

$$= \frac{3}{5} \cdot \lim_{3x\to 0} \frac{\sin 3x}{3x}$$

[as $x \to 0$, therefore $3x \to 0$]

$$= \frac{3}{5} \times 1 = \frac{3}{5} \qquad \left[\because \lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1\right]$$

(ii) We have, $\lim_{\theta \to b} \frac{\tan(\theta - b)}{\theta - b}$

Put $\theta - b = h \Rightarrow \theta = h + b$

Also, when $\theta \to b$, then $h \to b$

$$\therefore \quad \lim_{h\to b} \frac{\tan(\theta - b)}{\theta - b} = \lim_{h\to 0} \frac{\tan h}{h} = 1$$

(iii) $\lim_{x\to 0} \frac{\tan 2x - \sin 2x}{x^3}$

$$= \lim_{x\to 0} \frac{\frac{\sin 2x}{\cos 2x} - \sin 2x}{x^3}$$

$$= \lim_{x\to 0} \frac{\sin 2x - \sin 2x \cdot \cos 2x}{x^3 \cdot \cos 2x}$$

$$= \lim_{x\to 0} \frac{\sin 2x\,(1 - \cos 2x)}{x^3 \cdot \cos 2x}$$

$$= \lim_{x\to 0} \frac{\tan 2x}{x} \cdot \lim_{x\to 0} \frac{2\sin^2 x}{x^2}$$

[by using product of limits and $\cos 2\theta = 1 - 2\sin^2\theta$]

$$= 2 \cdot \lim_{x\to 0} \frac{\tan 2x}{2x} \times 2 \lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2$$

$$= 2\,(1) \times 2\,(1)^2$$

$$\left[\because \lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\tan x}{x} = 1\right]$$

$$= 4$$

(iv) Given, $\lim_{x \to 0} \frac{2\sin x - \sin 2x}{x^3}$

$$= \lim_{x \to 0} \frac{2\sin x - 2\sin x \cos x}{x^3}$$

$[\because \sin 2x = 2\sin x \cos x]$

$$= \lim_{x \to 0} \frac{2\sin x(1 - \cos x)}{x^3}$$

$$= 2 \lim_{x \to 0} \frac{\sin x}{x} \cdot \lim_{x \to 0}\left(\frac{1 - \cos x}{x^2}\right)$$

$$= 2 \cdot 1 \lim_{x \to 0} \frac{1 - \cos x}{x^2} \qquad \left[\because \lim_{x \to 0} \frac{\sin x}{x} = 1\right]$$

$$= 2 \lim_{x \to 0} \frac{2\sin^2 \frac{x}{2}}{4 \times \frac{x^2}{4}}$$

$$= \frac{2 \cdot 2}{4} \lim_{x \to 0}\left(\frac{\sin \frac{x}{2}}{\frac{x}{2}}\right)^2 = \lim_{x \to 0}\left(\frac{\sin \frac{x}{2}}{\frac{x}{2}}\right)^2 = 1$$

(v) Given,

$$\lim_{x \to \pi/4} \frac{\sqrt{2}\left(\sin x \cdot \frac{1}{\sqrt{2}} - \cos x \cdot \frac{1}{\sqrt{2}}\right)}{\left(x - \frac{\pi}{4}\right)}$$

$$= \lim_{x \to \pi/4} \frac{\sqrt{2}\left(\sin x \cos \frac{\pi}{4} - \cos x \cdot \sin \frac{\pi}{4}\right)}{\left(x - \frac{\pi}{4}\right)}$$

$$= \lim_{x \to \pi/4} \frac{\sqrt{2}\left\{\sin\left(x - \frac{\pi}{4}\right)\right\}}{\left(x - \frac{\pi}{4}\right)}$$

$[\because \sin A \cos B - \cos A \sin B = \sin(A - B)]$

$$= \sqrt{2} \lim_{x - \frac{\pi}{4} \to 0} \frac{\sin\left(x - \frac{\pi}{4}\right)}{\left(x - \frac{\pi}{4}\right)} = \sqrt{2}$$

$$\left[\because x \to \frac{\pi}{4} \Rightarrow \left(x - \frac{\pi}{4}\right) \to 0 \text{ and } \lim_{x \to 0} \frac{\sin x}{x} = 1\right]$$

55. (i) We have, $\lim_{x \to 0} \frac{\log_e(1 + 5x)}{x}$

$$= 5 \lim_{5x \to 0} \frac{\log_e(1 + 5x)}{5x} = 5 \times 1 = 5$$

$[\because x \to 0 \Rightarrow 5x \to 0]$

(ii) We have, $\lim_{x \to 0} \frac{\log_e(1 + 6x) - 5x^2}{x}$

$$= 6 \lim_{6x \to 0} \frac{\log_e(1 + 6x)}{6x} - 5 \lim_{x \to 0} x$$

$[\because x \to 0 \Rightarrow 6x \to 0]$

$$= 6 \times (1) - 5 \times (0) = 6$$

(iii) $\lim_{x \to 0} \frac{\sqrt{1 + x} - 1}{\log(1 + x)}$

On multiplying numerator and denominator by $\sqrt{1 + x} + 1$, we get

$$\lim_{x \to 0} \frac{\sqrt{1 + x} - 1}{\log(1 + x)} \times \frac{\sqrt{1 + x} + 1}{(\sqrt{1 + x} + 1)}$$

$$= \lim_{x \to 0} \frac{1 + x - 1}{(\sqrt{1 + x} + 1)\log(1 + x)}$$

$$= \lim_{x \to 0} \frac{x}{(\sqrt{1 + x} + 1)\log(1 + x)}$$

$$= \frac{1}{(\sqrt{1 + 0} + 1)} \lim_{x \to 0} \frac{1}{\frac{\log(1 + x)}{x}}$$

$$= \frac{1}{1 + 1} \times \frac{1}{\lim_{x \to 0} \frac{\log(1 + x)}{x}}$$

$$= \frac{1}{1 + 1} \times 1 = \frac{1}{2}$$

(iv) Put $x - 5 = h$ and as $x \to 5$, then $h \to 0$

$$\therefore \lim_{h \to 0} \frac{\log(h + 5) - \log 5}{h}$$

$$= \lim_{\frac{h}{5} \to 0} \frac{\log\left(1 + \frac{h}{5}\right)}{\frac{h}{5} \times 5} = \frac{1}{5}$$

$$\left[\begin{array}{r} \because \log m - \log n = \log \frac{m}{n}, \\ h \to 0 \Rightarrow \frac{h}{5} \to 0 \end{array}\right]$$

(v) $\lim_{x \to 0} \frac{\log\left\{5\left(1 + \frac{x}{5}\right)\right\} - \log\left\{5\left(1 - \frac{x}{5}\right)\right\}}{x}$

$$= \lim_{x \to 0} \frac{\left\{\log 5 + \log\left(1 + \frac{x}{5}\right)\right\} - \left\{\log 5 + \log\left(1 - \frac{x}{5}\right)\right\}}{x}$$

$$= \lim_{\frac{x}{5} \to 0} \frac{1}{5} \frac{\log\left(1 + \frac{x}{5}\right)}{\frac{x}{5}} - \lim_{\frac{x}{5} \to 0} \frac{\log\left(1 - \frac{x}{5}\right)}{-\frac{x}{5}} \cdot \frac{1}{(-5)}$$

$$\left[\because x \to 0 \Rightarrow \frac{x}{5} \to 0\right]$$

$$= \frac{1}{5} \times (1) + \frac{1}{5} \times (1) = \frac{2}{5}$$