05

Inverse of a Matrix and Linear Equations

5.01 Non-singular matrix

If the determinant of any square matrix A is non-zero i.e. $|A| \neq 0$ then matrix A is termed as non-singular matrix.

For Example : $A = \begin{bmatrix} 2 & 4 \\ 3 & 5 \end{bmatrix}$ is a non-singular matrix

$\because$ $|A| = \begin{vmatrix} 2 & 4 \\ 3 & 5 \end{vmatrix} = 10 - 12 = -2 \neq 0$

5.02 Singular matrix

If the determinant of any square matrix A is zero i.e. $|A| = 0$ then matrix A is termed as singular matrix.

For Example : $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ is a singular matrix as $|A| = \begin{vmatrix} 1 & 2 \\ 3 & 6 \end{vmatrix} = 6 - 6 = 0$

5.03 Adjoint of a square matrix

The adjoint of a square matrix $A = [a_{ij}]_{m\times n}$ is defined as the transpose of the matrix $[F_{ij}]$ where F_{ij} is the cofactor of the element a_{ij} Adjoint of the matrix A is denoted by $adjA$.

i.e. $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then $|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

Cofactors of elements of $|A|$

$$[F_{ij}] = \begin{bmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{bmatrix}$$

$$\therefore \quad [F_{ij}]^T = \begin{bmatrix} F_{11} & F_{21} & F_{31} \\ F_{12} & F_{22} & F_{32} \\ F_{13} & F_{23} & F_{33} \end{bmatrix} = AdjA$$

For Example : (i) Matrix $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}_{2\times 2} \Rightarrow |A| = \begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix}_{2\times 2}$

$\therefore$ Elements of $|A|$ = cofactor of $a_{11}(=2), = |5| = 5$

cofactor of $a_{12}(=3), \; = -|4| = -4$

cofactor of $a_{21}(=4), \; = -|3| = -3$

cofactor of $a_{22}(=5), \; = |2| = 2$

$\therefore$ Matrix of cofactors of determinant $|A|$ is $B = \begin{bmatrix} 5 & -4 \\ -3 & 2 \end{bmatrix}_{2\times 2}$

$\therefore$ Adjoint matrix of matrix A is $adjA = B^T = \begin{bmatrix} 5 & -3 \\ -4 & 2 \end{bmatrix}$

Note: The adjoint can be found directly of a 2×2 matrix by interchanging the diagonal elements and changing the sign of the off-digonal elements.

(ii) Matrix $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & -1 & 1 \\ 4 & 6 & 4 \end{bmatrix} \Rightarrow |A| = \begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 1 \\ 4 & 6 & 4 \end{vmatrix}$

$\therefore$ Cofactors of $a_{11}(=1)$ is $= \begin{vmatrix} -1 & 1 \\ 6 & 4 \end{vmatrix} = -10$

Cofactors of $a_{12}(=2)$ is $= -\begin{vmatrix} 3 & 1 \\ 4 & 4 \end{vmatrix} = -8$

Cofactors of $a_{13}(=0)$ is $= \begin{vmatrix} 3 & -1 \\ 4 & 6 \end{vmatrix} = 22$

Cofactors of $a_{21}(=3)$ is $= -\begin{vmatrix} 2 & 0 \\ 6 & 4 \end{vmatrix} = -8$

Cofactors of $a_{22}(=-1)$ is $= \begin{vmatrix} 1 & 0 \\ 4 & 4 \end{vmatrix} = 4$

Cofactors of $a_{23}(=1)$ is $= -\begin{vmatrix} 1 & 2 \\ 4 & 6 \end{vmatrix} = 2$

Cofactors of $a_{31}(=4)$ is $= \begin{vmatrix} 2 & 0 \\ -1 & 1 \end{vmatrix} = 2$

Cofactors of $a_{32}(=6)$ is $= -\begin{vmatrix} 1 & 0 \\ 3 & 1 \end{vmatrix} = -1$

Cofactors of $a_{33}(=4)$ is $=\begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix} = -7$

$\therefore$ Matrix of cofactors $$B = \begin{bmatrix} -10 & -8 & 22 \\ -8 & 4 & 2 \\ 2 & -1 & -7 \end{bmatrix}$$

Adjoint of a matrix $$adjA = B^T = \begin{bmatrix} -10 & -8 & 2 \\ -8 & 4 & -1 \\ 22 & 2 & -7 \end{bmatrix}$$

5.04 Inverse of a matrix of invertible matrix

If A is a square matrix of order *m*, and if there exists another square matrix B of the same order *m*, such that $AB = I = BA$, then B is called the inverse matrix of A and it is denoted by A^{-1}. In that case A is said to be invertible.

Thus, $B = A^{-1} \Rightarrow AA^{-1} = I = A^{-1}A$, from the relation $AB = BA$ it is clear that A is the inverse of B i.e. if two matrices A and B are such that $AB = I = BA$ then matrix A and B are inverse matrices of each other.

5.05 Some Important Theorems

Thorem 1. A square matrix A is invertible if and only if A is non singular matrix i.e. $|A| \neq 0$

Proof : Let A be invertible matrix of order *n* and I be the identity matrix of order *n*. Then, there exists a square matrix B of order *n* such that $AB = BA = I$

$\Rightarrow \quad |AB| = |I|$

$\Rightarrow \quad |A|.|B| = 1 \qquad [\because |I| = 1]$

$\Rightarrow \quad |A| \neq 0$

let A be non singualr. Then $|A| \neq 0$,

$$A \cdot (adjA) = |A| I = (adjA).A$$

diving by $|A|$

$$A.\frac{adjA}{|A|} = I = \frac{(adjA)}{|A|}.A \qquad [\because |A| \neq O]$$

which is of the form $A.B = I = B.A$

Hence $$A^{-1} = B = \frac{adjA}{|A|}$$

$$\Rightarrow \quad A^{-1} = \frac{adjA}{|A|}$$

Thus A is an invertible matrix.

Theorem 2. If A is a square matrix of order 3 then

$$A\cdot(adjA)=|A|I_3=(adjA)\cdot A, \qquad \text{where } I_3 \text{ is an identity matrix of order 3}$$

Proof : Let $A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is a third order matrix

$$\therefore \quad adjA=\begin{bmatrix} F_{11} & F_{21} & F_{31} \\ F_{12} & F_{22} & F_{32} \\ F_{13} & F_{23} & F_{33} \end{bmatrix}$$

$$\therefore \quad A.(adjA)=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} F_{11} & F_{21} & F_{31} \\ F_{12} & F_{22} & F_{32} \\ F_{13} & F_{23} & F_{33} \end{bmatrix}=\begin{bmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{bmatrix}$$

$$=|A|\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}=|A|I_3 \qquad (1)$$

similarly, we can prove that

$$(adjA)\cdot A=|A|I_3 \qquad (2)$$

Hence from (1) and (2), we have

$$A\cdot(adjA)=|A|I_3=(adjA)\cdot A$$

Note: If A and B are square matrices of order n then

(i) $A.(adjA)=|A|I_n=(adjA).A$

(ii) $adj(adjA)=|A|^{n-2}A$

(iii) $adjA^T=(adjA)^T$

(iv) $adj(AB)=adjB.adjA$

Theorem 3. Inverse matrix of non-singular matrix is unique.

Proof : Let $A=[aij]$ be a non-singular matrix of order m. If possible, let B and C be two inverse matrices of A. We shall show that B = C. We know that Since B is the inverse of A.

$$AB=BA=I \qquad (1)$$

and $AC=CA=I$ (2)

then $AB=I \Rightarrow C(AB)=CI \Rightarrow (CA)B=CI$

$\Rightarrow \quad I\,B=CI$ [using (2)]

$\Rightarrow \quad B=C$

Thus Inverse of a non-singular matrix, is unique

Theorem 4. If A and B are non-singular matrices of the same order, then $(AB)^{-1} = B^{-1}A^{-1}$.

Proof : $\because$ A are B are non-signualr matrices

$\therefore$ multiplication AB is possible

$\because$ A are B are non-singular matrices

$\therefore$ $|A| \neq 0$ and $|B| \neq 0$

$\Rightarrow$ $|AB| = |A||B| \neq 0$

$\Rightarrow$ AB is non-singular square matrix.

let a matrix C be such that $C = B^{-1}A^{-1}$

$\therefore$ $(AB)C = (AB)(B^{-1}A^{-1})$

$= A.(BB^{-1})A^{-1}$

$= A.I.A^{-1}$ $[\because BB^{-1} = I]$

$= AA^{-1} = I$

similarly $C(AB) = (B^{-1}A^{-1})(AB)$

$= B^{-1}(A^{-1}A)B = B^{-1}IB$ $[\because A^{-1}A = I]$

$= B^{-1}B = I$

$\therefore$ $(AB)C = C(AB)$

$\therefore$ $(AB)^{-1} = B^{-1}A^{-1}$

Generalisation : $(ABC...XYZ)^{-1} = Z^{-1}Y^{-1}X^{-1}...B^{-1}A^{-1}$

Theorem 5. If A is a non-singualr matrix then matrix A^T will also be non singualr matrix and $(A^T)^{-1} = (A^{-1})^T$

Proof : $\because$ $|A| = |A^T|$ $|A| \neq 0$ $(\because A$ is non-singular)

$\therefore$ $|A^T| \neq 0$

Thus matrix A^T is also non-singular

$\because$ A is non-singular $\Rightarrow A^{-1}$ exists such that

$AA^{-1} = I = A^{-1}A$

$\Rightarrow$ $(AA^{-1})^T = I^T = (A^{-1}A)^T$

$\Rightarrow$ $(A^{-1})^T A^T = I = A^T (A^{-1})^T$ $[\because (AB)^T = B^T A^T]$

$\Rightarrow$ The inverse of A^T is $(A^{-1})^T$

$\Rightarrow$ $(A^T)^{-1} = (A^{-1})^T$

Illustrative Examples

Example 1. If matrix $A=\begin{bmatrix}1 & 3\\ 2 & 4\end{bmatrix}$ then

(i) Find the adjoint of A $(adjA)$

(ii) Prove that $A.(adjA)=|A|I_2=(adjA).A$

(iii) Find A^{-1}

(iv) Prove that $(A^{-1})^T=(A^T)^{-1}$

Solution : (i) $\because$ Given matrix $A=\begin{bmatrix}1 & 3\\ 2 & 4\end{bmatrix}$

$\therefore$ Cofactor of $a_{11}(=1)$ is $=4$

Cofactor of $a_{12}(=3)$ is $=-2$

Cofactor of $a_{21}(=2)$ is $=-3$

Cofactor of $a_{22}(=4)$ is $=1$

$$\therefore \quad adjA=\begin{bmatrix}4 & -2\\ -3 & 1\end{bmatrix}^T=\begin{bmatrix}4 & -3\\ -2 & 1\end{bmatrix} \tag{1}$$

(ii) $|A|=\begin{vmatrix}1 & 3\\ 2 & 4\end{vmatrix}=4-6=-2.$

$$\therefore \quad A\cdot(adjA)=\begin{bmatrix}1 & 3\\ 2 & 4\end{bmatrix}\cdot\begin{bmatrix}4 & -3\\ -2 & 1\end{bmatrix}=\begin{bmatrix}4-6 & -3+3\\ 8-8 & -6+4\end{bmatrix}$$

$$=\begin{bmatrix}-2 & 0\\ 0 & -2\end{bmatrix}=-2\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=|A|I_2 \tag{2}$$

$$\therefore \quad (adjA).A=\begin{bmatrix}4 & -3\\ -2 & 1\end{bmatrix}\begin{bmatrix}1 & 3\\ 2 & 4\end{bmatrix}=\begin{bmatrix}4-6 & 12-12\\ -2+2 & -6+4\end{bmatrix}$$

$$=\begin{bmatrix}-2 & 0\\ 0 & -2\end{bmatrix}=-2\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=|A|I_2. \tag{3}$$

from (2) and (3) $\quad A\cdot(adjA)=|A|I_2=(adjA)\cdot A$ $\quad$ Hence Proved.

(iii) $$A^{-1}=\frac{adjA}{|A|}=\frac{-1}{2}\begin{bmatrix}4 & -3\\ -2 & 1\end{bmatrix}=\begin{bmatrix}-2 & 3/2\\ 1 & -1/2\end{bmatrix} \tag{4}$$

(iv) $\because \quad A^{-1} = \begin{bmatrix} -2 & 3/2 \\ 1 & -1/2 \end{bmatrix}$

$\therefore \quad (A^{-1})^T = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$ (5)

and $A^T = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \Rightarrow |A^T| = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = -2 \neq 0$

$\therefore \quad (A^T)^{-1}$ Exists.

$$adj\left(A^T\right) = \begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}^T = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$

$$\because \quad \left(A^T\right)^{-1} = \frac{adj\left(A^T\right)}{|A^T|} = -\frac{1}{2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix} \quad (6)$$

from (5) and (6) $\quad (A^{-1})^T = (A^T)^{-1}$. Hence Proved.

Example 2. If matrix $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ then find A^{-1}.

Solution : $\because \quad A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

$\therefore \quad |A| = \begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix} = \cos^2\theta + \sin^2\theta = 1$.

$\therefore \quad |A| \neq 0$ i.e. A^{-1} exists

$$adjA = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}^T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{adjA}{|A|} = \frac{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}{1} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Example 3. If matrix $A=\begin{bmatrix}1&2&3\\2&3&1\\3&1&2\end{bmatrix}$ then find A^{-1} and prove that $A^{-1}A=I_3$.

Solution : Given matrix $A=\begin{bmatrix}1&2&3\\2&3&1\\3&1&2\end{bmatrix}$

$$\therefore \quad |A|=\begin{vmatrix}1&2&3\\2&3&1\\3&1&2\end{vmatrix}=1(6-1)-2(4-3)+3(2-9)=5-2-21=-18\neq 0.$$

$\therefore \quad A^{-1}$ exists

$$\text{Now} \quad adjA=\begin{bmatrix}5&-1&-7\\-1&-7&5\\-7&5&-1\end{bmatrix}^T=\begin{bmatrix}5&-1&-7\\-1&-7&5\\-7&5&-1\end{bmatrix}$$

$$\therefore \quad A^{-1}=\frac{adjA}{|A|}=-\frac{1}{18}\begin{bmatrix}5&-1&-7\\-1&-7&5\\-7&5&-1\end{bmatrix}$$

$$\therefore \quad A^{-1}A=-\frac{1}{18}\begin{bmatrix}5&-1&-7\\-1&-7&5\\-7&5&-1\end{bmatrix}\begin{bmatrix}1&2&3\\2&3&1\\3&1&2\end{bmatrix}$$

$$=-\frac{1}{18}\begin{bmatrix}5-2-21&10-3-7&15-1-14\\-1-14+15&-2-21+5&-3-7+10\\-7+10-3&-14+15-1&-21+5-2\end{bmatrix}$$

$$=-\frac{1}{18}\begin{bmatrix}-18&0&0\\0&-18&0\\0&0&-18\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=I_3.$$

Example 4. If matrix $A=\begin{bmatrix}3 & 7\\ 2 & 5\end{bmatrix}$ and $B=\begin{bmatrix}6 & 8\\ 7 & 9\end{bmatrix}$ then prove that $(AB)^{-1}=B^{-1}A^{-1}$.

Solution : Here $|A|=\begin{vmatrix}3 & 7\\ 2 & 5\end{vmatrix}=1\neq 0$ (1)

$\therefore$ A^{-1} exists

and $|B|=\begin{vmatrix}6 & 8\\ 7 & 9\end{vmatrix}=-2\neq 0$ (2)

$\therefore$ B^{-1} exists

$\therefore$ $AB=\begin{bmatrix}3 & 7\\ 2 & 5\end{bmatrix}\begin{bmatrix}6 & 8\\ 7 & 9\end{bmatrix}=\begin{bmatrix}18+49 & 24+63\\ 12+35 & 16+45\end{bmatrix}$

$=\begin{bmatrix}67 & 87\\ 47 & 61\end{bmatrix}$ (3)

$\therefore$ $(AB)^{-1}=-\frac{1}{2}\begin{bmatrix}61 & -87\\ -47 & 67\end{bmatrix}$ (4)

$A^{-1}=\frac{1}{1}\begin{bmatrix}5 & -7\\ -2 & 3\end{bmatrix}$ (5)

and $B^{-1}=-\frac{1}{2}\begin{bmatrix}9 & -8\\ -7 & 6\end{bmatrix}$ (6)

$\because$ $B^{-1}A^{-1}=-\frac{1}{2}\begin{bmatrix}9 & -8\\ -7 & 6\end{bmatrix}\begin{bmatrix}5 & -7\\ -2 & 3\end{bmatrix}$

$=-\frac{1}{2}\begin{bmatrix}45+16 & -63-24\\ -35-12 & 49+18\end{bmatrix}$

$=-\frac{1}{2}\begin{bmatrix}61 & -87\\ -47 & 67\end{bmatrix}$ (7)

$\therefore$ from (4) and (7), $(AB)^{-1}=B^{-1}A^{-1}$. Hence Proved.

Example 5. If matrix $A=\begin{bmatrix}2 & 3\\ 1 & 2\end{bmatrix}$ then prove that $A^2-4A+I=0$, where $O=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$ and $I=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ and find A^{-1}.

Solution : $\because \quad A=\begin{bmatrix}2 & 3\\ 1 & 2\end{bmatrix}$

$$\therefore \quad A^2=\begin{bmatrix}2 & 3\\ 1 & 2\end{bmatrix}\begin{bmatrix}2 & 3\\ 1 & 2\end{bmatrix}=\begin{bmatrix}4+3 & 6+6\\ 2+2 & 3+4\end{bmatrix}=\begin{bmatrix}7 & 12\\ 4 & 7\end{bmatrix}$$

$$\therefore \quad A^2-4A+I=\begin{bmatrix}7 & 12\\ 4 & 7\end{bmatrix}-4\begin{bmatrix}2 & 3\\ 1 & 2\end{bmatrix}+\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}7 & 12\\ 4 & 7\end{bmatrix}+\begin{bmatrix}-8 & -12\\ -4 & -8\end{bmatrix}+\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}7-8+1 & 12-12+0\\ 4-4+0 & 7-8+1\end{bmatrix}$$

$$=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}=0\text{. Here } |A|=\begin{vmatrix}2 & 3\\ 1 & 2\end{vmatrix}=4-3=1\neq 0.$$

$\therefore \quad A^{-1}$ Exists

Now $\quad A^2-4A+I=0 \qquad \Rightarrow A^2-4A=-I \qquad \Rightarrow A(A-4I)=-I$

$\Rightarrow A^{-1}A(A-4I)=-A^{-1}I \qquad \Rightarrow (A^{-1}A)(A-4I)=-A^{-1} \qquad \Rightarrow I(A-4I)=-A^{-1}$

$\Rightarrow A-4I=-A^{-1} \qquad \Rightarrow A^{-1}=4I-A$

$$\Rightarrow A^{-1}=4\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}-\begin{bmatrix}2 & 3\\ 1 & 2\end{bmatrix}=\begin{bmatrix}4 & 0\\ 0 & 4\end{bmatrix}-\begin{bmatrix}2 & 3\\ 1 & 2\end{bmatrix}=\begin{bmatrix}2 & -3\\ -1 & 2\end{bmatrix}.$$

Exercise 5.1

1. For what value of x is the matrix $\begin{bmatrix}1 & -2 & 3\\ 1 & 2 & 1\\ x & 2 & -3\end{bmatrix}$ singular?

2. If matrix A is $\begin{bmatrix}1 & -1 & 2\\ 3 & 0 & -2\\ 1 & 0 & 3\end{bmatrix}$ then find $adjA$ and prove that $A\cdot(adjA)=|A|I_3=(adjA)\cdot A$.

3. Find the non-singualr matrix of the following:

(i) $\begin{bmatrix} 1 & 2 & 5 \\ 1 & -1 & -1 \\ 2 & 3 & -1 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$ (iii) $\begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix}$

4. If matrix $A = F(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$ then find A^{-1} and prove that

(i) $A^{-1}A = I_3$ (ii) $A^{-1} = F(-\alpha)$ (iii) $A\cdot(adjA) = |A|\,I = (adjA)\cdot A$

5. If $A = \frac{1}{9}\begin{bmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{bmatrix}$ then prove that $A^{-1} = A^T$

6. If matrix $A = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$ the prove that $A^{-1} = A^3$

7. If $A = \begin{bmatrix} 5 & 0 & 4 \\ 2 & 3 & 2 \\ 1 & 2 & 1 \end{bmatrix}$ and $B^{-1} = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$ then find $(AB)^{-1}$.

8. If $A = \begin{bmatrix} 1 & \tan\alpha \\ -\tan\alpha & 1 \end{bmatrix}$ then prove that $A^T A^{-1} = \begin{bmatrix} \cos 2\alpha & -\sin 2\alpha \\ \sin 2\alpha & \cos 2\alpha \end{bmatrix}$.

9. Prove that the matrix $A = \begin{bmatrix} 2 & -3 \\ 3 & 4 \end{bmatrix}$ satisfies the equation $A^2 - 6A + 7 = 0$ and find A^{-1} .

10. If matrix $A = \begin{bmatrix} -8 & 5 \\ 2 & 4 \end{bmatrix}$ then prove that $A^2 + 4A - 42I = 0$ then find A^{-1} .

5.06 Applications of Determinants

1. Area of a triangle

If the coordinates of vertices of a triangle are $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) then we know that

area of triangle $\Delta = \frac{1}{2}\left[x_1(y_2 - y_1) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\right]$ (1)

and $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = x_1\begin{vmatrix} y_2 & 1 \\ y_3 & 1 \end{vmatrix} - x_2\begin{vmatrix} y_1 & 1 \\ y_3 & 1 \end{vmatrix} + x_3\begin{vmatrix} y_1 & 1 \\ y_2 & 1 \end{vmatrix}$ (Expanding, along first column)

$$= x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) \quad (2)$$

from (1) and (2) $$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Thus area of triangle is $\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$.

Note: Since area is always positive hence the value of the determinant is always taken positive.

For Example : Find the area of the triangle if the vertices are $A(-3, 3)$, $B(2, 3)$ and $C(2, -2)$.

Solution : $$\Delta = \frac{1}{2}\begin{vmatrix} -3 & 3 & 1 \\ 2 & 3 & 1 \\ 2 & -2 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\left\{-3\begin{vmatrix} 3 & 1 \\ -2 & 1 \end{vmatrix} - 3\begin{vmatrix} 2 & 1 \\ 2 & 1 \end{vmatrix} + 1\begin{vmatrix} 2 & 3 \\ 2 & -2 \end{vmatrix}\right\}$$

$$= \frac{1}{2}\{-3(3+2) - 3(2-2) + 1(-4-6)\}$$

$$= \frac{1}{2}(-15 + 0 - 10)$$

$$= \frac{-25}{2} = -12.5 \text{ sq. Units}$$

$\because$ Area is positive therefore $\Delta = 12.5$ sq. units

2. Condition of collinearity of three points

If the points $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are collinear then the area of triangle ABC is zero

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \quad \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

For Example : Points $A(3,-2)$, $B(5,2)$ and $C(8,8)$ are collinear hence

$$\Delta = \frac{1}{2}\begin{vmatrix} 3 & -2 & 1 \\ 5 & 2 & 1 \\ 8 & 8 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\{3(2-8)+2(5-8)+1(40-16)\}$$

$$= \frac{1}{2}(-18-6+24) = 0$$

3. Equation of a line passing through two points

Let there be two points $A(x_1, y_1)$ and $B(x_2, y_2)$ and let $P(x, y)$, AB lies on a line passing through AB then P, A and B are collinear, if

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

which is the required equation.

For Example : Equation of line passing through $A(3,1)$ and $B(9,3)$ is $\begin{vmatrix} x & y & 1 \\ 3 & 1 & 1 \\ 9 & 3 & 1 \end{vmatrix} = 0$

$\Rightarrow$ $x(1-3)-y(3-9)+1(9-9)=0$

$\Rightarrow$ $-2x+6y=0$

$\Rightarrow$ $x-3y=0$

5.07 Solution of system of linear equations

If a given system of equations

$$a_{11}x + a_{12}y + a_{13}z = b_1$$

$$a_{21}x + a_{22}y + a_{23}z = b_2$$

$$a_{31}x + a_{32}y + a_{33}z = b_3$$

$b_1 = b_2 = b_3 = 0$ then it is said to be homogeneous otherwise it is called non-homogeneous

Here we shall find the solution of non-homogenous system of linear equations.

1. Cramer's Rule:

(i) Solution of system of linear equations of two variables

System of linear equation with two variables

$$a_1x+b_1y=c_1 \qquad (1)$$

$$a_2x+b_2y=c_2 \qquad (2)$$

solving through Cramer's rule

$$x=\frac{\Delta_1}{\Delta}, \qquad y=\frac{\Delta_2}{\Delta}$$

or $$\frac{x}{\Delta_1}=\frac{y}{\Delta_2}=\frac{1}{\Delta},\ \Delta\neq 0 \qquad \text{(Symmetric form)}$$

where $\Delta=\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$, $\Delta_1=\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$ and $\Delta_2=\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$

Proof : $\because$ $$\Delta=\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

$\therefore$ $$x\Delta=x\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}=\begin{vmatrix} a_1x & b_1 \\ a_2x & b_2 \end{vmatrix}$$

$\Rightarrow$ $$x\Delta=\begin{vmatrix} a_1x+b_1y & b_1 \\ a_2x+b_2y & b_2 \end{vmatrix}=\Delta_1 \qquad (\text{operation } C_1\to C_1+yC_2)$$

$\Rightarrow$ $$x\Delta=\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}=\Delta_1 \qquad (\text{using equation (1) and (2)})$$

similarly $$y\Delta=\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}=\Delta_2$$

$$x=\frac{\Delta_1}{\Delta} \text{ and } y=\frac{\Delta_2}{\Delta}, \text{ where } \Delta\neq 0$$

Special case : This equation represents two equations of straight line

(A) If $\dfrac{a_1}{a_2}\neq\dfrac{b_1}{b_2}$ then solution of the equation is unique and the equation is consistent and independent

(B) If $\dfrac{a_1}{a_2}=\dfrac{b_1}{b_2}\neq\dfrac{c_1}{c_2}$ then there is no solution and the equation is inconsistent.

(C) If $\dfrac{a_1}{a_2}=\dfrac{b_1}{b_2}=\dfrac{c_1}{c_2}$ then there are infinite solutions and the equation is consistent but not independent

(ii) Solution of system of linear equation for three variables

System of equations with three variables

$$a_1x + b_1y + c_1z = d_1 \quad (1)$$

$$a_2x + b_2y + c_2z = d_2 \quad (2)$$

$$a_3x + b_3y + c_3z = d_3 \quad (3)$$

Solving by Cramer's rule $x = \frac{\Delta_1}{\Delta},\ y = \frac{\Delta_2}{\Delta},\ z = \frac{\Delta_3}{\Delta}$

or $\frac{x}{\Delta_1} = \frac{y}{\Delta_2} = \frac{z}{\Delta_3} = \frac{1}{\Delta}$; $\Delta \neq 0$ [symmetric form]

where $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, $\Delta_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$, $\Delta_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$ and $\Delta_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$

Proof : $\because$ $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$\therefore$ $x\Delta = x\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

or $x\Delta = \begin{vmatrix} a_1x + b_1y + c_1z & b_1 & c_1 \\ a_2x + b_2y + c_2z & b_2 & c_2 \\ a_3x + b_3y + c_3z & b_3 & c_3 \end{vmatrix}$ $(C_1 \to C_1 + yC_2 + zC_3)$

or $x\Delta = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} = \Delta_1$ [using equation (1), (2) and (3)]

Similarly $y\Delta = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} = \Delta_2$ and $z\Delta = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix} = \Delta_3$

$\therefore$ $x = \frac{\Delta_1}{\Delta},\ y = \frac{\Delta_2}{\Delta}$ and $z = \frac{\Delta_3}{\Delta}$ if $\Delta \neq 0$

Special case :

(i) If $\Delta \neq 0$ then equation is consistent and the solution is unique.

(ii) If $\Delta = 0$ and $\Delta_1 = \Delta_2 = \Delta_3 = 0$ then system of equations can be consistent or inconsistent, if it is consistent then the solution are infinite.

(iii) If $\Delta = 0$ and amongst $\Delta_1, \Delta_2, \Delta_3$ any one is non-zero then equations are inconsistent with no solution.

2. Solution of system of linear equations using matrix method:

Consider the system of equations

$$\left.\begin{array}{l} a_{11}x + a_{12}y + a_{13}z = b_1 \\ a_{21}x + a_{22}y + a_{23}z = b_2 \\ a_{31}x + a_{32}y + a_{33}z = b_3 \end{array}\right\} \qquad (1)$$

The above equations can be written in a matrix form

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \qquad (2)$$

or $$AX = B \qquad (3)$$

where $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

If $|A| \neq 0$ then from equation (3)

$$AX = B$$

$\Rightarrow$ $$A^{-1}(AX) = A^{-1}B$$

$\Rightarrow$ $$(A^{-1}A)X = A^{-1}B$$

$\Rightarrow$ $$I\ X = A^{-1}B$$

$\Rightarrow$ $$X = A^{-1}B$$

Note: (i) $|A| \neq 0$, then A^{-1} exists

(ii) $|A| = 0$, then A^{-1} does not exist, that does not mean the equation cannot be solved.

Example : $x + 3y = 5$

$2x + 6y = 10,$

Here $|A| = \begin{vmatrix} 1 & 3 \\ 2 & 6 \end{vmatrix} = 0$ but it will have infinite solutions.

Illustrative Examples

Example 6. Find the area of the triangle whose vertices are $A(2,3)$, $B(-5,4)$ and $C(4,3)$.

Solution : Area of triangle ABC

$$\Delta = \frac{1}{2}\begin{vmatrix} 2 & 3 & 1 \\ -5 & 4 & 1 \\ 4 & 3 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\{2(4-3)+5(3-3)+4(3-4)\}$$

$$= \frac{1}{2}(2+0-4)$$

$$= -1$$

= 1 (numerical value) square units

Example 7. If points $(x,-2)$, $(5,2)$, $(8,8)$ are collinear then find the value of x.

Solution : $\because$ Given points $(x,-2)$, $(5,2)$ and $(8,8)$ are collinear

$$\therefore \quad \begin{vmatrix} x & -2 & 1 \\ 5 & 2 & 1 \\ 8 & 8 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \quad x(2-8)+2(5-8)+1(40-16)=0$$

$$\Rightarrow \quad -6x-6+24=0$$

$$\Rightarrow \quad -6x+18=0$$

$$\Rightarrow \quad x=3.$$

Example 8. Prove that $[bc, a(b+c)]$, $[ca, b(c+a)]$ and $[ab, c(a+b)]$ are collinear.

Solution : Three points are collinear

$$\therefore \quad \begin{vmatrix} bc & a(b+c) & 1 \\ ca & b(c+a) & 1 \\ ab & c(a+b) & 1 \end{vmatrix} = \begin{vmatrix} bc+ab+ca & a(b+c) & 1 \\ ca+bc+ab & b(c+a) & 1 \\ ab+ca+bc & c(a+b) & 1 \end{vmatrix} \qquad (C_1 \to C_1 + C_2)$$

$$= (ab+bc+ca)\begin{vmatrix} 1 & a(b+c) & 1 \\ 1 & b(c+a) & 1 \\ 1 & c(a+b) & 1 \end{vmatrix}$$

$$= (ab+bc+ca).0 \qquad (\because \text{two equal columns})$$

$$= 0$$

Thus given points are collinear

Example 9. Find the equation of line joining the points $A(4, 3)$ and $B(-5, 2)$ also find the value of k if the area fo the triangle ABC is 2 Sq. units where, $C(k, 0)$.

Solution : Let $P(x, y)$ be any point on AB then area of triangle ABC = 0

$$\Rightarrow \quad \frac{1}{2}\begin{vmatrix} 4 & 3 & 1 \\ -5 & 2 & 1 \\ x & y & 1 \end{vmatrix} = 0$$

$$\Rightarrow \quad \frac{1}{2}\left[4(2-y)-3(-5-x)+1(-5y-2x)\right]=0$$

$$\Rightarrow \quad 8-4y+15+3x-5y-2x=0$$

$$\Rightarrow \quad x-9y+23=0.$$

which is the required equation of AB

Now area of triangle ABC = 2 Sq. units

$$\Rightarrow \quad \frac{1}{2}\begin{vmatrix} 4 & 3 & 1 \\ -5 & 2 & 1 \\ k & 0 & 1 \end{vmatrix} = \pm 2$$

$$\Rightarrow \quad \frac{1}{2}\left[4(2-0)-3(-5-k)+1(0-2k)\right]=\pm 2$$

$$\Rightarrow \quad \frac{1}{2}\left[8+15+3k-2k\right]=\pm 2$$

$$\Rightarrow \quad 23+k=\pm 4$$

$$\Rightarrow \quad k=\pm 4-23$$

$$\Rightarrow \quad k=-19, -27$$

Example 10. If the solution of two below given equation is possible then solve using the Cramer's rule.

(i) $2x-3y=3$
$2x+3y=9$

(ii) $x+2y=5$
$2x+4y=10$

Solution : (i)

$$2x-3y=3$$
$$2x+3y=9$$

Here $\Delta = \begin{vmatrix} 2 & -3 \\ 2 & 3 \end{vmatrix} = 6+6=12 \neq 0$, $\Delta_1 = \begin{vmatrix} 3 & -3 \\ 9 & 3 \end{vmatrix} = 9+27=36 \neq 0$ and $\Delta_2 = \begin{vmatrix} 2 & 3 \\ 2 & 9 \end{vmatrix} = 18-6=12 \neq 0$

$\because \quad \Delta \neq 0,\ \Delta_1 \neq 0,\ \Delta_2 \neq 0$

$\therefore$ Equation is consistent and independent so its solution is finite.

Now using Cramer's rule

$$x=\frac{\Delta_1}{\Delta}=\frac{36}{12}=3, \qquad y=\frac{\Delta_2}{\Delta}=\frac{12}{12}=1$$

$\Rightarrow$ $x=3,\ y=1$.

(ii) $x+2y=5$
$2x+4y=10$

$\Rightarrow$ $\Delta=\begin{vmatrix}1 & 2\\ 2 & 4\end{vmatrix}=4-4=0$, $\Delta_1=\begin{vmatrix}5 & 2\\ 10 & 4\end{vmatrix}=20-20=0$ and $\Delta_2=\begin{vmatrix}1 & 5\\ 2 & 10\end{vmatrix}=10-10=0$

$\because$ $\Delta=0,\ \Delta_1=0,\ \Delta_2=0$

$\therefore$ Equation is inconsistent so its solution is infinite.

Let $y=k$ then $x+2k=5 \Rightarrow x=5-2k$ therefore $x=5-2k,\ y=k$ are the solutions where k is a real number

Example 11. Prove that the system of equations is inconsistent with no solution.

$$x+y+z=2$$
$$x+2y+3z=5$$
$$2x+3y+4z=11.$$

Solution : Let $\Delta=\begin{vmatrix}1 & 1 & 1\\ 1 & 2 & 3\\ 2 & 3 & 4\end{vmatrix}=1(8-9)-1(4-6)+1(3-4)=-1+2-1=0.$

$$\Delta_1=\begin{vmatrix}2 & 1 & 1\\ 5 & 2 & 3\\ 11 & 3 & 4\end{vmatrix}=2(8-9)-1(20-33)+1(15-22)=-2+13-7=4\neq 0.$$

$$\Delta_2=\begin{vmatrix}1 & 2 & 1\\ 1 & 5 & 3\\ 2 & 11 & 4\end{vmatrix}=1(20-23)-2(4-6)+1(11-10)=-13+4-1=-8\neq 0.$$

$$\Delta_3=\begin{vmatrix}1 & 1 & 2\\ 1 & 2 & 5\\ 2 & 3 & 11\end{vmatrix}=1(22-15)-1(11-10)+2(3-4)=7-1-2=4\neq 0.$$

$\because$ $\Delta=0$ and $\Delta_1\neq 0, \Delta_2\neq 0, \Delta_3\neq 0$.

$\therefore$ system of equations is inconsistent with no solution.

Example 12. Solve the following system of equations using Cramer's rule

$$x+y+z=9$$
$$2x+5y+7z=52$$
$$2x+y-z=0$$

Solution : Here $\Delta=\begin{vmatrix}1 & 1 & 1\\ 2 & 5 & 7\\ 2 & 1 & -1\end{vmatrix}=1(-5-7)-1(-2-14)+1(2-10)=-12+16-8=-4\neq 0$.

$$\Delta_1=\begin{vmatrix}9 & 1 & 1\\ 52 & 5 & 7\\ 0 & 1 & -1\end{vmatrix}=9(-5-7)-1(-52-0)+1(52-0)=-108+52+52=-4\neq 0.$$

$$\Delta_2=\begin{vmatrix}1 & 9 & 1\\ 2 & 52 & 7\\ 2 & 0 & -1\end{vmatrix}=1(-52-0)-9(-2-14)+(0-104)=-52+144-104=-12\neq 0.$$

$$\Delta_3=\begin{vmatrix}1 & 1 & 9\\ 2 & 5 & 52\\ 2 & 1 & 0\end{vmatrix}=1(0-52)-1(0-104)+9(2-10)=-52+104-72=-20\neq 0.$$

sign Cramer Rule

$$x=\frac{\Delta_1}{\Delta}=\frac{-4}{-4}=1,\qquad y=\frac{\Delta_2}{\Delta}=\frac{-12}{-4}=3,\qquad z=\frac{\Delta_3}{\Delta}=\frac{-20}{-4}=5$$

$\therefore$ $x=1,\ y=3,\ z=5$.

Example 13. Solve the system of equation using matrix inverse method.

$$5x-3y=2$$
$$x+2y=3.$$

Solution : Matrix form of the equation

$$\begin{bmatrix}5 & -3\\ 1 & 2\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}2\\ 3\end{bmatrix}$$

i.e. $AX=B$

where $A=\begin{bmatrix}5 & -3\\ 1 & 2\end{bmatrix},\ x=\begin{bmatrix}x\\ y\end{bmatrix}$ तथा $B=\begin{bmatrix}2\\ 3\end{bmatrix}$

$\because$ $|A|=\begin{vmatrix}5 & -3\\ 1 & 2\end{vmatrix}=10+3=13\neq 0$

$\therefore$ A^{-1} exists

$$A^{-1} = \frac{adjA}{|A|} = \frac{1}{13}\begin{bmatrix} 2 & 3 \\ -1 & 5 \end{bmatrix}$$

$\Rightarrow$ $$X = A^{-1}B = \frac{1}{13}\begin{bmatrix} 2 & 3 \\ -1 & 5 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$= \frac{1}{13}\begin{bmatrix} 4+9 \\ -2+15 \end{bmatrix} = \frac{1}{13}\begin{bmatrix} 13 \\ 13 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\Rightarrow$ $$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\Rightarrow$ $x = 1,\ y = 1$.

Example 14. Write the following system of equations in matrix form

$$2x - y + 3z = 9$$
$$x + y + z = 6$$
$$x - y + z = 2.$$

If $A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$ then find A^{-1} and solve the equations.

Solution : $\because AX = B$, where $A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 9 \\ 6 \\ 2 \end{bmatrix}$

Matrix form of the equation is

$$\begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 2 \end{bmatrix}$$

here $$|A| = \begin{vmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix} = 2(1+1) + 1(1-1) + 3(-1-1) = 4 + 0 - 6 = -2 \neq 0$$

$\therefore$ A^{-1} exists

$\therefore$ $$A^{-1} = \frac{adjA}{|A|} = -\frac{1}{2}\begin{bmatrix} 2 & -2 & -4 \\ 0 & -1 & 1 \\ -2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 2 \\ 0 & 1/2 & -1/2 \\ 1 & 1/2 & -3/2 \end{bmatrix}$$

$\because \qquad X = A^{-1}B$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} 2 & -2 & -4 \\ 0 & -1 & 1 \\ -2 & 1 & 3 \end{bmatrix}\begin{bmatrix} 9 \\ 6 \\ 2 \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} 18-12-8 \\ 0-6+2 \\ -18+6+6 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} -2 \\ -4 \\ -6 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$\Rightarrow \qquad x = 1,\ y = 2,\ z = 3.$

Example 15. If $A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{bmatrix}$ then find AB and solve the following equations

$x - y = 3; \quad 2x + 3y + 4z = 17, \quad y + 2z = 7$.

Solution :

$$AB = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{bmatrix} = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

$$= 6\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 6I_3$$

$$\Rightarrow \quad A \cdot \left(\frac{1}{6}B\right) = I_3$$

$$\Rightarrow \quad A^{-1} = \frac{1}{6}B = \frac{1}{6}\begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{bmatrix} \qquad (1)$$

Now matrix form of the given equation

$$\begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 17 \\ 7 \end{bmatrix}$$

$\Rightarrow \qquad AX = C$

$\Rightarrow \qquad X = A^{-1}C$

$$\Rightarrow \qquad X=\frac{1}{6}\begin{bmatrix}2 & 2 & -4\\ -4 & 2 & -4\\ 2 & -1 & 5\end{bmatrix}.\begin{bmatrix}3\\ 17\\ 7\end{bmatrix}$$

$$\Rightarrow \qquad \begin{bmatrix}x\\ y\\ z\end{bmatrix}=\frac{1}{6}\begin{bmatrix}6+34-28\\ -12+34-28\\ 6-17+35\end{bmatrix}=\frac{1}{6}\begin{bmatrix}12\\ -6\\ 24\end{bmatrix}=\begin{bmatrix}2\\ -1\\ 4\end{bmatrix}$$

$$\Rightarrow \qquad x=2,\ \ y=-1,\ \ z=4\,.$$

Example 16. Solve the following system of equations

$$\begin{bmatrix}3 & 0 & 3\\ 2 & 1 & 0\\ 4 & 0 & 2\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}8\\ 1\\ 4\end{bmatrix}+\begin{bmatrix}2y\\ z\\ 3y\end{bmatrix}$$

Solution : Given system of equation is

$$\begin{bmatrix}3 & 0 & 3\\ 2 & 1 & 0\\ 4 & 0 & 2\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}8\\ 1\\ 4\end{bmatrix}+\begin{bmatrix}2y\\ z\\ 3y\end{bmatrix}$$

$$\Rightarrow \qquad \begin{bmatrix}3x+3z\\ 2x+y\\ 4x+2z\end{bmatrix}=\begin{bmatrix}8+2y\\ 1+z\\ 4+3y\end{bmatrix}$$

$$\therefore \qquad \left.\begin{aligned}3x+3z=8+2y \ &\Rightarrow \ 3x-2y+3z=8\\ 2x+y=1+z \ &\Rightarrow \ 2x+y-z=1\\ 4x+2z=4+3y \ &\Rightarrow \ 4x-3y+2z=4\end{aligned}\right\} \qquad (1)$$

Matrix form of the given equations is (1)

$$\begin{bmatrix}3 & -2 & 3\\ 2 & 1 & -1\\ 4 & -3 & 2\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}8\\ 1\\ 4\end{bmatrix}$$

i.e. $\qquad AX=B$

$\Rightarrow \qquad X=A^{-1}B$

$$=-\frac{1}{17}\begin{bmatrix}-1 & -5 & -1\\ -8 & -6 & 9\\ -10 & 1 & 7\end{bmatrix}\begin{bmatrix}8\\ 1\\ 4\end{bmatrix} \qquad \left[\because A^{-1}=\frac{adjA}{|A|}\right]$$

$$= -\frac{1}{17}\begin{bmatrix} -8-5-4 \\ -64-6+36 \\ -80+1+28 \end{bmatrix} = -\frac{1}{17}\begin{bmatrix} -17 \\ -34 \\ -51 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\Rightarrow \qquad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\Rightarrow \qquad x = 1,\ y = 2,\ z = 3 .$$

Exercise 5.2

1. Find the area of triangle using the determinats whose vertices are:
   (i) (2, 5), (–2, –3) and (6, 0)
   (ii) (3, 8), (2, 7) and (5, –1)
   (iii) (0, 0), (5, 0) and (3, 4)
2. Usign determinants find the area of the triangle with vertices (1, 4), (2, 3) and (–5, –3), are the given points collinear?
3. Find the value of k if the area of triangle is 35 Sq. units and the vertices are (k, 4) (2, –6) and (5, 4).
4. Usign determinants find the value of k if the points (k, 2 –2 k), (–k + 1, 2k) and (–4 – k, 6 –2k) are collinear.
5. If points (3, –2), (x, 2) and (8, 8) are collinear then find the vlaue of x using determinant.
6. Using determinants, find the equation of line passing through the points (3, 1) and (9, 3) and also find the area of the triangle if the third point is (–2, –4).
7. Solve the following system of equations using Cramer's rule.

   (i) $2x + 3y = 9$, $3x - 2y = 7$

   (ii) $2x - 7y - 13 = 0$, $5x + 6y - 9 = 0$

8. Prove that the following system of equations are incosnstent:

   (i) $3x + y + 2z = 3$, $2x + y + 3z = 5$, $x - 2y - z = 1$

   (ii) $x + 6y + 11 = 0$, $3x + 20y - 6z + 3 = 0$, $6y - 18z + 1 = 0$

9 Solve the equations using Cramer's rule:

   (i) $x + 2y + 4z = 16$, $4x + 3y - 2z = 5$, $3x - 5y + z = 4$

   (ii) $2x + y - z = 0$, $x - y + z = 6$, $x + 2y + z = 3$

10. Solve the equations using determinants :

(i) $6x+y-3z=5$
$x+3y-2z=5$
$2x+y+4z=8$

(ii) $\frac{2}{x}+\frac{3}{y}+\frac{10}{z}=4$
$\frac{4}{x}-\frac{6}{y}+\frac{5}{z}=1$
$\frac{6}{x}+\frac{9}{y}-\frac{20}{z}=2$

11. Solve the equations using matrix method:

(i) $2x-y=-2$
$3x+4y=3$

(ii) $5x+7y+2=0$
$4x+6y+3=0$

(iii) $x+y-z=1$
$3x+y-2z=3$
$x-y-z=-1$

(iv) $6x-12y+25z=4$
$4x+15y-20z=3$
$2x+18y+15z=10$

12. If $A=\begin{bmatrix}1 & -2 & 0\\ 2 & 1 & 3\\ 0 & -2 & 1\end{bmatrix}$ then find A^{-1} and solve the system of equations:

$x-2y=10, \quad 2x+y+3z=8, \quad -2y+z=7.$

13. Find the product of matrices $\begin{bmatrix}-4 & 4 & 4\\ -7 & 1 & 3\\ 5 & -3 & -1\end{bmatrix}$ and $\begin{bmatrix}1 & -1 & 1\\ 1 & -2 & -2\\ 2 & 1 & 3\end{bmatrix}$ and solve the system of equations using the above product

$$x-y+z=4$$
$$x-2y-2z=9$$
$$2x+y+3z=1\cdot$$

14. Find the inverse of the matrix $\begin{bmatrix}1 & -1 & 1\\ 2 & 1 & -3\\ 1 & 1 & 1\end{bmatrix}$ and with the help of this solve the system of equations

$$\begin{bmatrix}1 & 0 & 1\\ 2 & 1 & 0\\ 0 & 1 & 1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\frac{1}{2}\begin{bmatrix}2y\\ 6z\\ -2x\end{bmatrix}+2\begin{bmatrix}2\\ 0\\ 1\end{bmatrix}$$

15. If the side of an equilateral triangle is a and vertices are (x_1, y_1), (x_2, y_2) and (x_3, y_3) then prove that $\begin{vmatrix}x_1 & y_1 & 2\\ x_2 & y_2 & 2\\ x_3 & y_3 & 2\end{vmatrix}^2=3a^4$

Miscellaneous Exercise - 5

1. If $A=\begin{bmatrix}1 & -1\\ 2 & 3\end{bmatrix}$ then find A^{-1}.

2. If $A=\begin{bmatrix}0 & 1 & 1\\ 1 & 0 & 1\\ 1 & 1 & 0\end{bmatrix}$ then find A^{-1}.

3. If Matrix $A=\begin{bmatrix}1 & -2 & 3\\ 1 & 2 & 1\\ x & 2 & -3\end{bmatrix}$ is a singualr matrix then find the vlaue of x

4. Solve the equations using Cramer's rule

   (i) $2x-y=17$
   $3x+5y=6.$

   (ii) $3x+ay=4$
   $2x+ay=2, \quad a\neq 0$

   (iii) $x+2y+3z=6$
   $2x+4y+z=7$
   $3x+2y+9z=14.$

5. Solve the equations using Cramer's rule and show that the equation are inconsistent:

   (i) $2x-y=5$
   $4x-2y=7$

   (ii) $x+y+z=1$
   $x+2y+3z=2$
   $3x+4y+5z=3$

6. Find the matrix A of order 2 if

$$A\begin{bmatrix}1 & -2\\ 1 & 4\end{bmatrix}=\begin{bmatrix}6 & 0\\ 0 & 6\end{bmatrix}$$

7. If $A=\begin{bmatrix}-8 & 5\\ 2 & 4\end{bmatrix}$ then prove that that $A^2+4A-42I=0$ and using this find A^{-1}.

8. If $A=\begin{bmatrix}2 & 3\\ 5 & -2\end{bmatrix}$ then prove that $A^{-1}=\frac{1}{19}A$.

9. If $A=\begin{bmatrix}1 & 3 & 3\\ 1 & 4 & 3\\ 1 & 3 & 4\end{bmatrix}$ then find A^{-1} and show that $A^{-1}A=I_3$.

10. If $A=\begin{bmatrix}1&2&2\\2&1&2\\2&2&1\end{bmatrix}$ then prove that $A^2-4A-5I=0$ and using this find A^{-1}.

11. Solve the following system of equations using the matrix method.

(i) $5x-7y=2$, $7x-5y=3$

(ii) $3x+y+z=3$, $2x-y-z=2$, $-x-y+z=1$

(iii) $x+2y-2z+5=0$, $-x+3y+4=0$, $-2y+z-4=0$

12. Find the area triangle ABC for the vertices given below:

(i) $A(-3,5), B(3,-6), C(7,2)$

(ii) $A(2,7)\ B(2,2)\ C(10,8)$

13. If the points $(2,-3),(\lambda,-2)$ and $(0,5)$ are collinear then find the vlaue of λ.

14. Find the matrix A where

$$\begin{bmatrix}1&2\\2&3\end{bmatrix}A\begin{bmatrix}4&7\\3&5\end{bmatrix}=\begin{bmatrix}1&0\\0&1\end{bmatrix}$$

15. If $A=\begin{bmatrix}1&1&1\\1&2&-3\\2&-1&3\end{bmatrix}$ then find A^{-1} and using this solve the equations:

$$x+y+2z=0,\quad x+2y-z=9,\quad x-3y+3z=-14$$

16. If $A=\begin{bmatrix}a&b\\c&\frac{1+bc}{a}\end{bmatrix}$ then find A^{-1} and solve that $aA^{-1}=\left(a^2+bc+1\right)I-aA$.

17. Solve the system of equations using determinants

$$x+y+z=1$$
$$ax+by+z=k$$
$$a^2x+b^2y+c^2z=k^2.$$

18. If $A=\begin{bmatrix}1&2&-3\\2&3&2\\3&-3&-4\end{bmatrix}$ then find A^{-1} then using this solve the following system of equations

$$x+2y-3z=-4,\quad 2x+3y+2z=2,\quad 3x-3y-4z=11.$$

19. If $\begin{bmatrix}1&-4\\3&-2\end{bmatrix}X=\begin{bmatrix}-16&-6\\7&2\end{bmatrix}$ then find the value of X.

20. If the system of equations have infinite solutions then find the vlaues of a and b

$$2x + y + az = 4$$
$$bx - 2y + z = -2$$
$$5x - 5y + z = -2.$$

Important Points

1. **Singular matrix:** A Square matrix A, whose $|A| = 0$
2. **Non-Singular matrix:** A Square matrix A, whose $|A| \neq 0$
3. **Adjoint of a matrix:** Adjoint of a matrix is a transpose of a matrix, obtained by co-factors of elements of |A| adjoint of the matrix A is written as *adjA*
4. **Inverse of a matrix:** If a square matrix is non-singualr i.e. $|A| \neq 0$ then $A^{-1} = \frac{adjA}{|A|}$
5. **Important theorems:**
   (i) For a matrix A to be non-singualr $|A| \neq 0$
   (ii) If A is a matrix of order n then $A.(adjA) = |A|\, I_n = (adjA).A$
   (iii) $(AB)^{-1} = B^{-1}A^{-1}$
   (iv) $(A^T)^{-1} = (A^{-1})^T$
6. For variables x, y, z the system of equations are

$$\left.\begin{aligned} a_{11}x + a_{12}y + a_{13}z = b_1 \\ a_{21}x + a_{22}y + a_{23}z = b_2 \\ a_{31}x + a_{32}y + a_{33}z = b_3 \end{aligned}\right\} \quad (1)$$

the solutions can be found out using the determinants or matrix method

(i) Cramer's rule usign determinants

For the above equation (1)

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}, \Delta_1 = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}, \quad \Delta_2 = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix} \text{ and } \Delta_3 = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix} \text{ then}$$

Case-I: when $\Delta \neq 0$ then solution is unique $\frac{x}{\Delta_1} = \frac{y}{\Delta_2} = \frac{z}{\Delta_3} = \frac{1}{\Delta}$

Case-II: when $\Delta = 0$ and $\Delta_1 \neq 0$ or $\Delta_2 \neq 0$ or $\Delta_3 \neq 0$ then there will be infinite solutions

Case-III: when $\Delta = 0$ and $\Delta_1 = \Delta_2 = \Delta_3 = 0$ then there will be infinite solutions

(ii) Matrix method: $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

i.e. $AX = B$

$\Rightarrow \quad X = A^{-1}B$, where $A^{-1} = \dfrac{adjA}{|A|}$.

Answers

Exercise 5.1

1. $x = -1$ 2. $\begin{bmatrix} 0 & 3 & 2 \\ -11 & 1 & 8 \\ 0 & -1 & 3 \end{bmatrix}$ 3. (i) $\frac{1}{27}\begin{bmatrix} 4 & 17 & 3 \\ -1 & -11 & 6 \\ 5 & 1 & -3 \end{bmatrix}$; (ii) $\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$; (iii) $\begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix}$

4. $\begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$ 7. $\begin{bmatrix} -2 & 19 & -27 \\ -2 & 18 & -25 \\ -3 & 29 & -42 \end{bmatrix}$ 10. $\frac{1}{42}\begin{bmatrix} -4 & 5 \\ 2 & 8 \end{bmatrix}$

Exercise 5.2

1. (i) 26 Sq. Units; (ii) 11 / 2 Sq. Units; (iii) 10 Sq. Units 2. 13 / 2 Sq. Units, No

3. $x = -2, 12$ 4. $k = -1, 1/2$ 5. $x = 5$ 6. $x - 3y = 0$, 10 Sq. Units

7. (i) $x = 3, y = 1$ (ii) $x = 3, y = -1$ 9. (i) $x = 2, y = 1, x = 3$; (ii) $x = 2, y = -1, z = 3$

10. (i) $x = 1, y = 2, z = 1$; (ii) $x = 2, y = 3, z = 5$

11. (i) $x = \frac{-5}{11}, y = \frac{12}{11}$; (ii) $x = \frac{9}{2}, y = \frac{-7}{2}$; (iii) $x = 2, y = 1, z = 2$; (iv) $x = \frac{1}{2}, y = \frac{1}{3}, z = \frac{1}{5}$

12. $A^{-1} = \frac{1}{11}\begin{bmatrix} 7 & 2 & -6 \\ -2 & 1 & -3 \\ -4 & 2 & 5 \end{bmatrix}$; $x = 4, y = -3, z = 1$ 13. $\begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix}, x = 3, y = -2, z = -1$

14. $\frac{1}{10}\begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}, x = 2, y = -1, z = 1$

Miscellaneous Exercise - 5

1. $\frac{1}{5}\begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}$ 2. $\frac{1}{2}\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$ 3. $x = -1$

4. (i) $x = 7,\ y = -3$; (ii) $x = 2,\ y = \frac{-2}{a}$; (iii) $x = y = z = 1$

6. $\begin{bmatrix} 4 & 1 \\ -1 & 1 \end{bmatrix}$ 7. $\frac{1}{42}\begin{bmatrix} -4 & 5 \\ 2 & 8 \end{bmatrix}$ 9. $\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$ 10. $A^{-1} = \frac{1}{5}2\begin{bmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{bmatrix}$

11. (i) $x = \frac{11}{24}, y = \frac{1}{24}$; (ii) $x = 1,\ y = -1,\ z = 1$; (iii) $x = 1,\ y = -1,\ z = 2$

12. (i) 46 Sq. Units; (ii) 20 Sq. Units 13. $\lambda = \frac{7}{4}$ 14. $A = \begin{bmatrix} 21 & -29 \\ -13 & 18 \end{bmatrix}$

15. $A^{-1} = -\frac{1}{11}\begin{bmatrix} 3 & 4 & 5 \\ 9 & -1 & -1 \\ 5 & -3 & -1 \end{bmatrix}, x = 1,\ y = 3,\ z = -2$ 16. $A^{-1} = \begin{bmatrix} \frac{1+bc}{a} & -b \\ -c & a \end{bmatrix}$

17. $x = \frac{(c-k)(k-b)}{(c-a)(a-b)},\ y = \frac{(k-c)(a-k)}{(b-c)(a-b)},\ z = \frac{(b-k)(k-a)}{(b-c)(c-a)}$

18. $A^{-1} = \frac{1}{67}\begin{bmatrix} -6 & 17 & 13 \\ 14 & 5 & -8 \\ -15 & 9 & -1 \end{bmatrix}, x = 3,\ y = -2,\ z = 1$ 19. $X = \begin{bmatrix} 6 & 2 \\ 11/2 & 2 \end{bmatrix}$ 20. $a = -2,\ b = 1$